Equilibrium in Continuous-Time Financial Markets: Endogenously Dynamically Complete Markets

Robert M. Anderson
University of California at Berkeley
Department of Economics
549 Evans Hall #3880
Berkeley, CA 94720-3880 USA
anderson@econ.berkeley.edu

Roberto C. Raimondo
Department of Economics
University of Melbourne
Victoria 3010, Australia
raimondo@econ.unimelb.edu.au

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Abstract

We prove the existence of equilibrium in a continuous-time financial market in which the securities are potentially dynamically complete: the number of securities is one more than the number of independent sources of uncertainty. We derive dynamic completeness of the endogenously determined equilibrium prices entirely from mild exogenous assumptions on the endowments and utility functions of the agents, and the dividends of the securities. Our result is universal rather than generic: it follows from a mild exogenous nondegeneracy condition on the terminal security dividends. We find that the equilibrium prices, consumptions, and trading strategies are well-behaved functions of the stochastic process describing the evolution of information. We prove that equilibria of discrete approximations converge to equilibria of the continuous-time economy.

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1 Introduction

Virtually all of the work in continuous-time finance takes as given that the prices of securities follow an exogenously specified stochastic process. But prices of securities are in fact determined day by day and minute by minute by the balancing of supply and demand. A complete model of continuous-time trading requires the derivation of the pricing process as an equilibrium determined by more primitive data of the economy, in particular the agents’ information, utility functions, and endowments, and the securities’ dividend processes.

To date, the existence of equilibrium in continuous-time finance models has been established in the case of a single agent (Bick [12], He and Leland [37], Cox, Ingersoll and Ross [17], Duffie and Skiadis [33], Raimondo [66]), or of multiple agents with a complete set of Arrow-Debreu contingent claims (Mas-Colell and Richard [59] and Bank and Riedel [11]).

The assumption that there is a complete set of Arrow-Debreu contingent claims is clearly unrealistic as a description of actual security markets. However, if the number of securities is one greater than the number of independent sources of uncertainty, markets are potentially dynamically complete: it may be possible to replicate the Arrow-Debreu equilibrium through the rapid trading of the securities. Duffie and Zame [34] consider a model in which securities are specified by their dividend processes:

- They show existence of Arrow-Debreu equilibrium.
- They show that if the securities prices induced by the Arrow-Debreu equilibrium prices are dynamically complete, then there is a securities market equilibrium which implements the Arrow-Debreu equilibrium.
- They give a “spanning assumption” under which the securities prices induced by the Arrow-Debreu equilibrium price are dynamically complete, and hence a securities market equilibrium exists and it implements the Arrow-Debreu equilibrium.

1 The most widely studied price process is geometric Brownian motion, but other Itô Processes and Lévy Processes have also been extensively studied.

2 All of these papers except Raimondo [66] require one or more endogenous assumptions that are not expressed solely in terms of the primitives of the model.
This would be a complete solution to the problem if the spanning assumption were exogenous, but as we shall see, it is not. We make three main contributions:

- We replace the spanning assumption with a very general assumption expressed solely in terms of the primitives of the model.

- We obtain explicit formulas for the equilibrium pricing process and its dispersion matrix, each trader’s equilibrium securities wealth, and the dispersion matrix for each traders’ equilibrium trading strategy in terms of the equilibrium consumptions; each trader’s equilibrium trading strategy can then be calculated using linear algebra. These formulas are expressed in terms of the equilibrium consumptions, which are not known \textit{a priori}. However, for each equilibrium, there exist utility weights \( \lambda \) such that, at each node, the equilibrium consumptions maximize the weighted sum of the utilities of the agents. Thus, the key features of the equilibrium can be calculated explicitly from knowledge of the primitives of the model (endowments and utility functions of the individuals, and the dividends of the securities) and the equilibrium utility weights. This is potentially useful in studying comparative statics of equilibria in continuous-time financial markets.

- We prove that all key elements of equilibria of discrete approximations converge to the corresponding elements of equilibria of the continuous-time model.

The Duffie-Zame spanning assumption (A.3) is stated as follows: the martingales \( E \left( D_t^T | F_t \right) \) form a martingale generator where \( D \) is the cumulative dividend process of the securities. Their securities are assumed to pay their dividends in units of account in the budget set, at the Arrow-Debreu equilibrium prices. The Arrow-Debreu equilibrium prices are not part of the exogenous data of the economy; indeed, they are given by marginal utilities of consumption (unobservable) at the equilibrium consumptions (determined endogenously as part of the equilibrium).

One possible approach to filling the gap would be to show that for a generic set of economic primitives, the endogenous spanning assumption is satisfied. Genericity could be defined topologically or via relative shyness (Anderson and Zame [8]). This seems a promising approach, but no one has succeeded in carrying it out.
In this paper, we supply the missing piece in the Duffie-Zame program by replacing their endogenous spanning assumption by a quite general, exogenous assumption. Our model is quite similar to that of Duffie and Zame. The following are the key differences:

1. All of our securities pay dividends in units of consumption, not units of marginal utility. We establish effective dynamic completeness as a consequence of general exogenous assumptions.³

2. Like Duffie and Zame, our model is over a finite time interval \([0, T]\). In Duffie and Zame, the securities expire worthless at time \(T\). In our model, the securities pay a flow of dividends over the interval \([0, T]\) plus a lump dividend at time \(T\); the lump dividend may be thought of as the present value of dividends over the infinite future horizon.

3. Duffie and Zame [34], as well as Duffie [28] and Duffie and Huang [30], assume that the cumulative dividend process is an Itô Process. Let \(D_t\) be the cumulative dividend process so

\[
dD_t = a\, dt + \sigma\, dB_t
\]

with \(a \in \mathcal{L}\) and \(\sigma \in \mathcal{H}^2\). In our model, \(a\) is a function of a Brownian motion, and hence is an Itô Process, while \(\sigma\) is identically zero. Because they allow \(\sigma\) to be nonzero, their dividend processes are more general than ours.

4. Duffie and Zame assume that all securities are in zero net supply, which implies that there are no wealth effects arising from the securities dividends or the securities prices.⁴ By contrast, our model al-

³Our equilibrium prices are not quite dynamically complete. The information in our model is represented by a Brownian motion; the endowments, security dividends, and utility functions are all measurable in the Brownian filtration. However, the whole filtration in our model may strictly contain the Brownian filtration. The equilibrium prices are effectively dynamically complete, i.e. they permit replication of any contingent claim which is measurable in the terminal \(\sigma\)-algebra of the Brownian filtration; in particular, the equilibrium is Pareto optimal. But there may be other claims which are measurable in the larger filtration which cannot be replicated by trading strategies in the continuous-time model.

⁴Assuming that securities are in zero net supply is a harmless normalization provided that the securities dividends lie in the agents’ consumption sets. However, unless \(\sigma = 0\) in Equation (1), the dividends do not lie in the consumption set of the agents.
allows some securities in zero net supply and others in positive net supply, as in Breeden’s *Consumption-Based Capital Asset Pricing Model* [13], and the models of Merton [60], Lucas [57] and Cox Ingersoll and Ross [17]. Thus, our theorem can be applied to macroeconomic models which study the effects of household securities wealth on saving, investment and consumption decisions, and in particular to the equilibrium analysis of the Equity Premium Puzzle. The presence of wealth effects makes the existence problem more difficult, and we establish existence in this more difficult setting.

5. We allow a greater degree of state-dependence in our utility functions.

We now outline our model. The time period is the interval $[0, T]$. The uncertainty in the model is described by a $K$-dimensional Brownian motion $\beta$ on a probability space $\Omega$. We define $I(t, \omega) = (t, \beta(t, \omega))$. There are $J = K + 1$ securities; each is in net supply either zero or one. Security $j$ pays dividends (measured in consumption units) at a flow rate $A_j(t, \omega) = g_j(I(t, \omega))$ at times $t \in [0, T)$, and a lump dividend $A_j(T, \omega) = G_j(I(T, \omega))$ at time $T$. We assume that $g : \mathbb{R}^K \times [0, t] \to \mathbb{R}_+^K$ is an analytic function.

For example, $A_0$ could be a zero-coupon bond ($g_0(t, \omega) = 0$ for $t \in [0, T)$, $g_0(T, \omega)$ is constant), and $A_j(t, \omega) = e^{\sigma_j \beta(t, \omega)}$, where $\sigma_j$ is the $j$th row of a $J \times J$ matrix $\sigma$, for $j = 1, \ldots, J$.

There are $I$ agents. Agent $i$ has a flow rate of endowment $e_i(t, \omega) = f_i(I(t, \omega))$ at times $t \in [0, T)$, and a lump endowment $e_i(T, \omega) = F_i(I(T, \omega))$, where $f_i$ is analytic function. The utility functions are von-Neumann Morgenstern utility functions, expectations of analytic functions of the consumption and the process $I$. By allowing the felicity function to depend on $I$, we allow a considerable degree of state-dependence.

The assumption that the endowments, dividends and utility functions are analytic functions of time and the Brownian motion at times $t \in [0, T)$ is critical to our analysis. A $C^\infty$ function may be zero on an open set without being identically zero. By contrast, if a real analytic function on an open convex set in $\mathbb{R}^n$ is zero on a set of positive Lebesgue measure, it must be

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5 We believe that the principal results will go through if $I$ is a general Itô Process, but the analysis is more difficult and it is likely that the probability space and filtration will need to be expanded to obtain an equilibrium.

6 For the definition of an analytic function, and the properties of analytic functions we use in this paper, see Appendix A.
identically zero. A simple nondegeneracy condition on the terminal dividends guarantees that the determinant of dispensor matrix of the securities prices with respect to the Brownian motion has full rank except on a closed set of measure zero, but this is just the condition for dynamic completeness.

The assumption that the endowments and utility functions are analytic does not impose serious economic restrictions; for example, all conventional utility functions are in fact analytic. However, the dividend paid by an option is not analytic because of the kink that occurs when the stock price just equals the exercise price of the option on the exercise date. Moreover, shares of a limited liability corporation should not generally be thought of as analytic because they are, in effect, options to claim the corporation’s stream of earnings, at an exercise price equal to the corporation’s debt.\footnote{In particular, geometric Brownian Motion is problematic as a model of the price of shares in a limited liability corporation.} We require analyticity at the intermediate times \( t \in (0, T) \), and not at \( t = T \). Thus, our model allows us to include options and shares of limited liability corporations among our basic securities, provided that the exercise date is the terminal date \( T \). In addition, as long as the basic securities pay dividends which are analytic functions of \( I \) over \((0, T)\), there is no problem in using the equilibrium prices derived from the basic securities to price options or other derivatives on those basic securities with exercise date \( t \in (0, T) \). The equilibrium price of any security, analytic or not, is equal to the expected value of its future dividends, evaluated at the equilibrium consumption prices. Since we prove that the basic securities are essentially dynamically complete, any option or other derivative on the basic securities can be replicated by an admissible self-financing trading strategy on the basic securities. Thus, the equilibrium price of any security is also given by the equilibrium value of the portfolio specified by the replicating strategy at that node. For more detail on equilibrium pricing of options, see Anderson and Raimondo \[6, 7\].

Our starting point is a continuous-time model, on which we state our existence theorem, Theorem 2.1. We use a discretization procedure to construct a sequence of models; each has a finite number of trading dates and a finite event tree. In this construction, it is critical that the number of successor nodes of each nonterminal node equal \( K + 1 \). In the simplest \( K \)-dimensional random walk, each node has \( 2^K \) successors, but \( K + 1 \) securities cannot possibly be dynamically complete if each node has \( 2^K \) successor nodes and \( K > 1 \).
Endowments, dividends and utility functions are induced on the discrete economies from the specification of the continuous-time economy. These discrete economies are General Equilibrium Incomplete Markets (GEI) models; Magill and Quinzii [58] is an excellent reference to GEI models. Endowments and dividends are perturbed as necessary to ensure existence of a dynamically complete equilibrium, under the Duffie-Shafer Theorem [31, 32]. We can then state our convergence theorem, Theorem 3.1, that equilibria of the discrete approximations converge to equilibria of the continuous-time economy. The statements of Theorems 2.1 and 3.1 can be understood without any knowledge of nonstandard analysis.

Nonstandard analysis provides powerful tools to move from discrete to continuous time, and from discrete distributions like the binomial to continuous distributions like the normal; in particular, it provides the ability to transfer computations back and forth between the discrete and continuous settings. Our sequence of discrete approximations extends to a hyperfinite approximation, one which is infinite but has all the formal properties of finite approximations. In particular, the hyperfinite approximation has a GEI equilibrium which is dynamically complete in the hyperfinite model. We then use nonstandard analysis to produce a candidate equilibrium in the continuous-time model, show that the equilibrium in the hyperfinite model is infinitely close to the candidate equilibrium in the continuous-time model, verify that the candidate prices are dynamically complete, and are in fact equilibrium prices.

Anderson [2] provided a construction for Brownian motion and Brownian stochastic integration using nonstandard analysis. In nonstandard analysis, hyperfinite objects are infinite objects which nonetheless possess all the formal properties of finite objects. Anderson’s Brownian motion is a hyperfinite random walk which, using a measure-theoretic construction called Loeb measure, can simultaneously be viewed as being a standard Brownian motion in the usual sense of probability theory. While the standard stochastic integral is motivated by the idea of a Stieltjes integral, the actual standard definition of the stochastic integral is of necessity rather indirect because almost every path of Brownian motion is of unbounded variation, and Stieltjes integrals are only defined with respect to paths of bounded variation. However, a hyperfinite random walk is of hyperfinite variation, and hence a Stieltjes integral with respect to it makes perfect sense. Anderson showed that the standard stochastic integral can be obtained readily from this hyperfinite
This construction of Brownian motion has been used to answer a number of questions in stochastic processes, and we are able to make do in this paper with slight extensions of it. However, we anticipate that extending this work to the dynamically incomplete case, or the case of dividends with jumps, will require using subsequent work in nonstandard stochastic analysis, such as the work of Keisler [45] on stochastic differential equations with respect to Brownian motion; and the work by Albeverio and Herzberg [1], Hoover and Perkins [41], and Lindström [50, 51, 52, 53, 54, 55], on stochastic integration with respect to more general martingales.

The nonstandard theory of stochastic integration has previously been applied to option pricing in Cutland, Kopp and Willinger [19, 20, 21, 22, 23, 24, 25]. Those papers primarily concern convergence of discrete versions of options to continuous-time versions, and their methods can likely be used to establish convergence results for the option pricing formulas developed in Anderson and Raimondo [6]. Nonstandard analysis has also previously been applied to finance in Khan and Sun [46, 47] to relate the Capital Asset Pricing Model and Arbitrage Pricing Theory in a single-period setting.

We modify Anderson’s construction of the hyperfinite random walk to a random walk with branching number equal to $K + 1$ and extend the results on stochastic integration to that random walk. We show that equilibrium consumptions are nonzero at all times and states. Consequently, we can use the first order conditions to characterize the equilibrium prices. Then, we use the Loeb measure construction to produce a candidate equilibrium of the original continuous-time model. The Central Limit Theorem then allows us to explicitly describe the candidate equilibrium prices as integrals with respect to a normal distribution; however, with more than one agent, the prices depend on the terminal distributions of wealth, which are not described in closed form. We show that the hyperfinite equilibrium is infinitely close to the candidate equilibrium, which implies that the equilibria of the discrete approximations converge to candidate equilibria of the continuous-time model. Finally, in a process analogous to that first used in Brown and Robinson [14], we show that the candidate equilibrium is an equilibrium of the continuous-time economy.\footnote{The argument is more complicated here because our continuous-time economy is more complicated than the economy in [14].}
In the continuous-time economy, we show that the equilibrium prices, the volatility matrix of the securities prices; individuals’ consumptions and portfolio wealths; and the volatility matrix of individuals’ portfolios; are given by quite explicit analytic functions of time and the Brownian motion. The equilibrium trading strategies can be determined from these functions by linear algebra. The formulas are given in closed form as integrals, given the equilibrium consumptions of the agents. The equilibrium consumptions are not known \textit{a priori}. However, since the equilibrium is Pareto Optimal, there is a vector $\lambda$ of utility weights such that the equilibrium consumptions can be readily calculated from the primitives of the economy and the utility weights. Thus, the essential features of the equilibrium can be written in closed form provided that one knows the equilibrium utility weights. Even without knowledge of the equilibrium utility weights, the explicit nature of the formulas can potentially be used to establish general properties of equilibria, such as comparative statics results.

It is important to emphasize that the primitives of the model, and the equilibrium, are analytic functions of time and the Brownian motion, not analytic functions of time. Brownian motion is almost surely nowhere differentiable, and almost surely of unbounded variation on every interval of time. Consequently, the equilibrium prices and trading strategies are nowhere differentiable, and of unbounded variation, as functions of time.

Nonstandard analysis is a conservative extension of conventional analysis, so the theorems presented here do not depend on any additional set theoretic or analytic axioms. In particular, the proofs presented here can be mechanically translated into standard proofs, but the resulting standard proofs would be exceedingly long and unintelligible. However, many of the key ideas in our arguments do have standard analogues, and we believe it would not be difficult to adapt them to an argument like that in Duffie and Zame [34]. Thus, we believe one could proceed as in Duffie and Zame and produce an Arrow-Debreu equilibrium using functional analytic arguments. Assuming that the primitives of the economy are given by analytic functions of the Brownian Motion at the intermediate dates $t \in (0, T)$, it should not be had to adapt our arguments to show that the equilibrium price dispersion matrix is analytic, and consequently to prove the spanning condition assumed by Duffie and Zame, and thus establish existence of equilibrium without the use of nonstandard analysis. We have chosen not to proceed in this direction, for the following reasons:
While our proof makes extensive use of nonstandard analysis, it is completely independent of knowledge of functional analysis.

As is often the case in nonstandard analysis, our theorems on the convergence of equilibria of discrete approximations to equilibria of the continuous time model are essentially immediate corollaries of the nonstandard existence proof. We believe that it would be extremely difficult to prove analogous convergence results using standard methods.

Our ultimate goal is to extend these methods to obtain existence of equilibria in continuous-time financial markets with dynamically incomplete securities. Since there is no known theorem asserting existence of equilibrium with more than one agent, an infinite-dimensional commodity space, and incomplete markets, there seems no hope of extending the Duffie-Zame approach to incomplete markets. Many steps in our argument work just as well for the dynamically incomplete case as for the dynamically complete case. While critical problems remain to be solved, our methods at least provide a way of attacking the dynamically incomplete case.

While nonstandard analysis plays a central role in the proof, we emphasize that the statements of the theorems are expressed entirely in terms of the standard continuous-time model and can be understood without any knowledge of nonstandard analysis.

2 The Model

In this Section we define the continuous-time model.

1. Trade and consumption occur over a compact time interval \([0, T]\), endowed with a measure \(\nu\) which agrees with Lebesgue measure on \([0, T]\) and such that \(\nu(\{T\}) = 1\).

2. The information structure is represented by a filtration \(\{\mathcal{F}_t : t \in [0, T]\}\) on a probability space \((\Omega, \mathcal{F}, \mu)\). A stochastic process \(X(t, \omega)\) is said to be adapted if, for all \(t\), \(X(t, \cdot)\) is measurable with respect to \(\mathcal{F}_t\).

3. There is a standard \(K\)-dimensional Brownian motion \(\beta = (\beta_1, \ldots, \beta_K)\) such that \(\beta_k\) is independent of \(\beta_{k'}\) if \(k \neq k'\) and such that the variance of
\[ \beta_k(t, \cdot) \text{ is } t \text{ and } \beta_k(t, \cdot) = E(\beta_k(T, \cdot) | \mathcal{F}_t) \]. Notice that we do not assume that \( \{\mathcal{F}_t : t \in [0, T]\} \) is the filtration generated by \( \beta \); in general, \( \mathcal{F}_t \) will contain more information than the history of \( \beta \) up to time \( t \).

4. Let \( \mathcal{I} \) be the \( \mathbb{R}^{K+1} \)-valued process

\[ \mathcal{I}(t, \omega) = (t, \beta(t, \omega)) \]

The endowments and utility functions of the agents, and the securities dividends, will be expressed in terms of \( \mathcal{I}(t, \omega) \). We believe the essential result holds if \( \mathcal{I}(t, \omega) \) is any Itô process such that the zeroth component is time and the Itô coefficients of \( \mathcal{I}_1, \ldots, \mathcal{I}_K \) with respect to \( \beta_1, \ldots, \beta_K \) are nonsingular for almost all \( (t, \omega) \in [0, T] \times \Omega \). Allowing \( \mathcal{I} \) to be a more general Itô Process would allow the dividends, endowments, and utility functions to depend on the history of the Brownian motion, not just its current value.

5. There are \( I \) agents \( i = 1, \ldots, I \). The endowment of the agent \( i \) is a process

\[ e_i(t, \omega) = \begin{cases} f_i(\mathcal{I}(t, \omega)) & \text{if } t \in [0, T) \\ F_i(\mathcal{I}(T, \omega)) & \text{if } t = T \end{cases} \]

where \( f_i : [0, T] \times \mathbb{R}^K \to \mathbb{R}_{++} \) is analytic and \( F_i : \{T\} \times \mathbb{R}^K \to \mathbb{R}_{++} \) is measurable. Note that since \( \nu([0, T)) = T \) and \( \nu(\{T\}) = 1 \), the endowment in period \( t \in [0, T) \) is interpreted as a rate of flow of endowment, while the endowment in period \( T \) is interpreted as a stock or lump. Let \( e(t, \omega) = \sum_{i=1}^{I} e_i(t, \omega) \) denote the aggregate endowment.

6. There are \( J + 1 = K + 1 \) tradable securities (indexed by \( j = 0, \ldots, J \)) which pay dividends

\[ A_j(t, \omega) = \begin{cases} g_j(\mathcal{I}(t, \omega)) & \text{if } t \in [0, T) \\ G_j(\mathcal{I}(T, \omega)) & \text{if } t = T \end{cases} \]

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9 The proof in the general case is considerably more complicated. Moreover, it appears that one can find only a weak equilibrium, in the sense of weak solutions of stochastic differential equations. It is well known that stochastic differential equations need not have solutions on the probability space and filtration on which they are defined. Instead, they possess weak solutions, i.e. solutions on a larger probability space and filtration.
where each $g_j : [0, T) \times \mathbb{R}^k \to \mathbb{R}_+$ is analytic; and each $G_j : \{T\} \times \mathbb{R}^k \to \mathbb{R}_+$ is a measurable function of $\beta$; satisfying the following growth and Lipschitz conditions:

$\exists r \in \mathbb{R} \forall t \in [0, T], x, h \in \mathbb{R}^k, |h| \leq 1$

$|f_i(t, x)| \leq r + e^{r|x|}$

$|F_i(T, x)| \leq r + e^{r|x|}$

$|g_j(t, x)| \leq r + e^{r|x|}$

$|G_j(T, x)| \leq r + e^{r|x|}$

$\frac{\partial f_i(t, x)}{\partial x} \leq r + e^{r|x|}$

$\frac{\partial F_i(T, x)}{\partial x} \leq r + e^{r|x|}$

$\frac{\partial}{\partial x} \left( \sum_{i=1}^{T} f_i(t, x) + \sum_{j=0}^{J} \eta_j g_j(t, x) \right) \leq r + e^{r|x|}$

Note that we do not require $G_j$ to be analytic, or even differentiable. Option payoffs are not differentiable at the strike price. Our formulation allows for the possibility that security $A_j$ is a derivative, such as an option with exercise date $T$, on another security which may or may not be traded. Our formulation also allows $A_j$ to be a stock in a limited liability corporation, since shares in limited liability corporations are in effect options to buy the earnings flow of the firm at an exercise price equal to the firm’s debt. As with the endowments, the dividend in period $t \in [0, T)$ is interpreted as a rate of flow, while the dividend in period $T$ is interpreted as a stock or lump.

7. The net supply of security $j$ is $\eta_j \in \{0, 1\}$; thus some securities may be in net supply one, while others are in net supply zero. We assume that there is some $m > 0$ such that the aggregate endowment plus dividend process satisfy the following condition

$e(t, \omega) + \sum_{j=0}^{J} \eta_j A_j(t, \omega) \geq m$

8. In continuous-time models, it is commonly assumed that the zeroth security is a money-market account, in other words, it is instantaneously
risk-free. Since we are determining securities prices endogenously, the assumption that a security is instantaneously risk-free is an endogenous assumption. For example, if we assume that $A_0$ is a bond (i.e. $G_0(t, \omega) = 0$ for $t < T$ and $G_0(T, \omega) = 1$, so the dividends of $A_0$ are risk-free), the Arrow-Debreu equilibrium price of $A_0$ will not be instantaneously risk-free except in degenerate situations. However, as long as the equilibrium securities prices are positive Itô Processes (and we shall show that they are), one is free to divide the securities price process and the consumption price process by the equilibrium price of the zero$^{th}$ security. Under this renormalization, relative prices are preserved and the price of the zero$^{th}$ security is identically one, which is obviously instantaneously risk-free. When one does this, the set of admissible self-financing trading strategies is left unchanged; see section 4.8 of Nielsen [62]. Consequently, the consumption processes that lie in the budget set remain invariant, and the renormalized prices are equilibrium prices. This motivates the form of our exogenous nondegeneracy condition on the terminal dividends of the securities. We assume that there is an open set $V \subset \mathbb{R}^K$ such that for $j = 1, \ldots, J$ and $i = 1, \ldots I$,

$$G_j, F_i \in C^1(V) \quad \text{and} \quad \exists x \in V \quad \text{rank} \begin{pmatrix} \frac{\partial (G_1/G_0)}{\partial \beta}(T,x) \\ \vdots \\ \frac{\partial (G_J/G_0)}{\partial \beta}(T,x) \end{pmatrix} = K$$

Note that if $A_0$ is a bond, the rank condition is equivalent to assuming that the matrix

$$\begin{pmatrix} \frac{\partial G_1}{\partial \beta}(T,x) \\ \vdots \\ \frac{\partial G_J}{\partial \beta}(T,x) \end{pmatrix}$$

has rank $K$.

9. Agent $i$ is initially endowed with deterministic security holdings $e_{iA} = (e_{iA_0}, \ldots, e_{iA_J}) \in \mathbb{R}^{J+1}_+$ satisfying

$$\sum_{i=1}^I e_{iA_j} = \eta_j$$
Note that the initial holdings are independent of the state $\omega$. Moreover, the initial security holdings are required to be nonnegative; without this restriction, there might be an agent who cannot make good on his/her initial short position, and hence no equilibrium would exist.

10. Given a measurable consumption function $c_i : [0, T] \times \Omega \to \mathbb{R}^{++}$, the utility function of the agent is

$$U_i(c) = E_\mu \left[ \int_0^T h_i(c_i(t, \cdot), I(t, \cdot)) dt + H_i(c_i(T, \cdot), I(T, \cdot)) \right]$$

where the functions $h_i : \mathbb{R}^+ \times ([0, T] \times \mathbb{R}^K) \to \mathbb{R} \cup \{-\infty\}$ and $H_i : \mathbb{R}^+ \times \{T\} \times \mathbb{R}^K \to \mathbb{R} \cup \{-\infty\}$ are analytic on $\mathbb{R}^+ \times ([0, T] \times \mathbb{R}^K)$ and $C^2$ on $\mathbb{R}^+ \times \{T\} \times \mathbb{R}^K$ respectively and satisfy

- $\lim_{c \to 0^+} \frac{\partial h_i}{\partial c} \bigg|_{(c, (t, x))} = \infty$ uniformly over $([0, T] \times \mathbb{R}^K)$
- $\lim_{c \to 0^+} \frac{\partial H_i}{\partial c} \bigg|_{(c, (T, x))} = \infty$ uniformly over $\{T\} \times \mathbb{R}^K$
- $\lim_{c \to -\infty} \frac{\partial h_i}{\partial c} = 0$ uniformly over $([0, T] \times \mathbb{R}^K)$
- $\lim_{c \to -\infty} \frac{\partial H_i}{\partial c} = 0$ uniformly over $\{T\} \times \mathbb{R}^K$
- $\lim_{c \to 0^+} h_i(c, (t, x)) = h_i(0, (t, x))$ uniformly over $([0, T] \times \mathbb{R}^K)$
- $\lim_{c \to 0^+} H_i(c, (T, x)) = H_i(0, (T, x))$ uniformly over $\{T\} \times \mathbb{R}^K$
- $\frac{\partial h_i}{\partial c} > 0$ on $\mathbb{R}^+ \times ([0, T] \times \mathbb{R}^K)$
- $\frac{\partial H_i}{\partial c} > 0$ on $\mathbb{R}^+ \times \{T\} \times \mathbb{R}^K$
- $\frac{\partial^2 h_i}{\partial c^2} < 0$ on $\mathbb{R}^+ \times ([0, T] \times \mathbb{R}^K)$
- $\frac{\partial^2 H_i}{\partial c^2} < 0$ on $\mathbb{R}^+ \times \{T\} \times \mathbb{R}^K$
- $\forall c > 0 \exists M \in \mathbb{R} \frac{\partial h_i}{\partial c} \bigg|_{(c, (t, x))} \leq M$ uniformly over $([0, T] \times \mathbb{R}^K)$
- $\forall c > 0 \exists M \in \mathbb{R} \frac{\partial H_i}{\partial c} \bigg|_{(c, (T, x))} \leq M$ uniformly over $\{T\} \times \mathbb{R}^K$

Note that these conditions are satisfied by all state-independent utility functions in the CARA and CRRA classes. Note also that we allow quite general state-dependence of the utility function, as long as the state-dependence enters through the process $I$. If the state-dependence were not measurable in the Brownian motions, there would be no hope of obtaining effective dynamic completeness with securities whose dividends are measurable with respect to the Brownian filtration.
In order to define the budget set of an agent, we need to have a way of calculating the capital gain the agent receives from a given trading strategy. In other words, we need to impose conditions on prices and strategies that ensure that the stochastic integral of a trading strategy with respect to a price process is defined. The essential requirements are that the trading strategy at time $t$ not depend on information which has not been revealed by time $t$, and the trading strategy times the variation in the price yields a finite integral. Specifically,

(a) A consumption price process is an Itô process $p_C(t, \omega)$.

(b) A securities price process is an Itô process $p_A = (p_{A_0}, \ldots, p_{A_J}) : \Omega \times [0, T] \rightarrow \mathbb{R}^{J+1}$ such that the associated cumulative gains process

$$\gamma_j(t, \omega) = p_{A_j}(t, \omega) + \int_0^t p_C(s, \omega) A_j(s, \omega) \, ds$$

is a martingale. Securities are priced cum dividend at time $T$.

(c) Given a securities price process $p_A$, an admissible trading strategy for agent $i$ is a process $z_i$ which is Itô integrable with respect to $\gamma$ (written $z_i \in L^2(\gamma)$) and such that $\int z_i \, d\gamma$ is a martingale.\[11\]

12. Given a securities price process $p_A$ and a consumption price process $p_C$, the budget set for agent $i$ is the set of all consumption plans $c_i$ such that there exists an admissible trading strategy so that $c_i$ and $t_i$ satisfy the budget constraint

$$p_A(t, \omega) \cdot z_i(t, \omega)$$

---

\[10\] In other words,

i. $z_i : [0, K] \times \Omega \rightarrow \mathbb{R}^{J+1}$

ii. $z_i(t, \cdot)$ is $\mathcal{F}_t$-measurable for all $t \in [0, T]$

iii. $z_i$ is measurable on the product $[0, T] \times \Omega$.

iv. If the Itô Process $\gamma$ is given by $d\gamma = \mu dt + \sigma d\beta$, then since $\gamma$ is a martingale, $\mu$ must be zero almost surely. Itô integrability with respect to $\gamma$ requires two conditions: that $z_i(\cdot, \omega) \cdot \mu(\cdot, \omega) \in L^2([0, T])$ almost surely, which is trivially satisfied; and that $z_i(\cdot, \omega) \cdot \sigma(\cdot, \omega) \in L^2([0, T])$ almost surely.

A stronger condition is that $z_i \in H^2(\gamma)$; the definition is the same except that we strengthen the condition that $z_i(\cdot, \omega) \cdot \sigma(\cdot, \omega) \in L^2([0, T])$ almost surely to $z_i \cdot \sigma \in L^2$.

We also define $L^2 = L^2(\beta)$ and $H^2 = H^2(\beta)$. For more details, see Nielsen [62].

\[11\] The requirement that $\int z_i \, d\gamma$ be a martingale is the standard “admissibility” condition ruling out arbitrage strategies such as the doubling strategy of Harrison and Kreps [36].
= \( p_A(0, \omega) \cdot e_{iA}(\omega) + \int_0^t z_i \, d\gamma + \int_0^t p_C(s, \omega)(e_i(s, \omega) - c_i(s, \omega)) \, ds \)
for almost all \( \omega \) and all \( t \in [0, T) \)

0 = \( p_A(0, \omega) \cdot e_{iA}(0, \omega) + \int_0^T z_i \, d\gamma + \int_0^T p_C(s, \omega)(e_i(s, \omega) - c_i(s, \omega)) \, ds \\
+ p_C(T, \omega)(e_i(T, \omega) - c_i(T, \omega)) \)
for almost all \( \omega \)

13. Given a price process \( p \), the demand of the agent is a consumption plan and an admissible trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.

14. An equilibrium for the economy is a securities price process \( p_A \), a consumption price process \( p_c \), an admissible trading strategy \( z \) and a consumption plan \( c \) which lies in the demand set so that the securities and goods markets clear, i.e. for almost all \( \omega \)

\[
\sum_{i=1}^I z_{iA_j}(t, \omega) = \eta_j \text{ for } j = 0, \ldots, J \text{ and almost all } (t, \omega)
\]

\[
\sum_{i=1}^I c_i(t, \omega) = \sum_{i=1}^I e_i(t, \omega) + \sum_{j=0}^J \eta_j A_j(t, \omega) \text{ for almost all } (t, \omega)
\]

**Theorem 2.1** The continuous-time finance model just described has an equilibrium, which is Pareto optimal. Let \( p_A, p_c, c_i, \) and \( z_i \) denote the equilibrium securities prices, consumption prices, consumptions, and trading strategies.

- \( p_A \) is a function of \( I(t) \) given by

\[
p_A(t, \beta) = E \left( p_C(T, \beta(T))A(T, \beta(T)) + \int_t^T p_C(s, \beta(s))A(s, \beta(s)) \, ds \mid \beta(t) = \beta \right)
\]

- if we define \( \Sigma = \frac{\partial p_A}{\partial \beta} \) to be the volatility matrix of the securities prices, then

\[
\Sigma(t, \beta) = E \left( \frac{p_C(T, \beta(T))A(T, \beta(T))(\beta(T) - \beta)}{T-t} \right)
\]

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\[ + \int_t^T \frac{p_C(s, \beta)A(s, \beta(s))(\beta(s) - \beta)}{s - t} \bigg| \beta(t) = \beta \bigg) \quad (5) \]

- \( c_1, \ldots, c_I \) are given as functions of \( \mathcal{I}(t, \omega) \) and there exists a unique \( \lambda \in \mathbb{R}_+^I \) with \( \sum_{i=1}^I \lambda_i = 1 \) such that for all \( (t, \beta) \), \( c_1(t, \beta), \ldots, c_I(t, \beta) \) solve the problem

\[
\max \left\{ \sum_{i=1}^I \lambda_i h_i(c_i) : \sum_{i=1}^I c_i = \sum_{i=1}^I c_i(t, \beta) + \sum_{j=1}^J \eta_j A_j(t, \beta) \right\} \quad (6)
\]

for \( t \in [0, T) \) and the problem

\[
\max \left\{ \sum_{i=1}^I \lambda_i H_i(c_i) : \sum_{i=1}^I c_i = \sum_{i=1}^I c_i(t, \beta) + \sum_{j=1}^J \eta_j A_j(t, \beta) \right\} \quad (7)
\]

for \( t = T \);

- if we define \( \Sigma_i = z_i \Sigma \) to be the volatility matrix of agent \( i \)’s trading strategy, and \( W_i = z_i \cdot p_A \) to be agent \( i \)’s securities wealth, then each \( \Sigma_i \) and \( W_i \) has a continuous version (still denoted \( \Sigma_i \) and \( W_i \))

\[
\Sigma_i(t, \beta) = E \left( \frac{p_C(T, \beta(T)) (c_i(T, \beta(T)) - e_i(T, \beta(T))) (\beta(T) - \beta)}{T - t} \right.
\]

\[
+ \int_t^T \frac{p_C(s, \beta(S)) (c_i(s, \beta(s)) - e_i(s, \beta(s))) (\beta(s) - \beta)}{s - t} ds \bigg| \beta(t) = \beta \bigg) \quad (8)
\]

\[
W_i(t, \beta) = E \left( p_C(T, \beta(T)) (c_i(T, \beta(T)) - e_i(T, \beta(T))) \right.
\]

\[
+ \int_{s=t}^T p_C(s, \beta(s)) (c_i(s, \beta(s)) - e_i(s, \beta(s))) ds \bigg| \beta(t) = \beta \bigg) \quad (9)
\]

- \( p_A, c_1, \ldots, c_I, W_1, \ldots, W_I \in L^2; \Sigma, \Sigma_1, \ldots, \Sigma_I \in \mathcal{H}^2; z_1, \ldots, z_I \in \mathcal{H}^2(\gamma) \); \( c_i \) is uniformly bounded below, and \( p_C \) is uniformly bounded above;

- \( p_A, W_1, \ldots, W_I \) are functions of \( \mathcal{I}(t, \omega) \) which are continuous on \( [0, T] \times \mathbb{R}^K \) and analytic on \( (0, T) \times \mathbb{R}^K \);

- \( \Sigma, \Sigma_1, \ldots, \Sigma_I \) are functions of \( \mathcal{I}(t, \omega) \) which are continuous on \( [0, T] \times V \) and analytic on \( (0, T) \times \mathbb{R}^K \);
\begin{itemize}
\item $p_c$ and $c_i$ are given separately over $[0, T) \times \mathbb{R}^k$ and $\{T\} \times \mathbb{R}^K$ as functions of $I(t, \omega)$; the functions over $[0, T) \times \mathbb{R}^k$ are continuous on $[0, T) \times \mathbb{R}^K$, have a continuous extension to $[0, T] \times \mathbb{R}^K$, and are analytic on $(0, T) \times \mathbb{R}^K$, while the functions over $\{T\} \times \mathbb{R}^K$ are continuous in $\beta(T, \omega)$;
\item there is an open set $B$ of full measure zero in $(0, T) \times \mathbb{R}^K$ and analytic functions $Z_1, \ldots, Z_I$ with domain $B$ such that $z_i(t, \omega) = Z_i(I(t, \omega))$ whenever $I(t, \omega) \in B$;
\item the equilibrium prices are effectively dynamically complete: any integrable consumption process which is adapted to the Brownian filtration can be replicated by a unique admissible trading strategy.
\end{itemize}

\section{Discrete Approximations of the Continuous-Time Model}

In this section, we describe a process for taking a continuous-time model and discretizing it. We then state a theorem indicating that the equilibria of the discretized economy are close to those of the continuous-time economy. This result has two important consequences:

\begin{itemize}
\item It provides an effective computational method for computing equilibria of the continuous-time economy. All one has to do is compute, using standard algorithms, an equilibrium of a sufficiently fine discretization.
\item In practice, actual markets are discrete in a number of important ways. Prices are restricted to lie in a grid, trades are carried out in integral numbers of shares (and transaction costs are lower for large blocks of stock than for individual shares), and trades take a certain amount of time to execute. Continuous-time models are useful because pricing formulas can be expressed more cleanly in continuous time than in discrete time. However, in order to know that the formulas obtained from continuous-time models are applicable to real markets, we need to know that the behavior of large discrete models is close to the behavior of large continuous-time models.
\end{itemize}

Here is our discretization procedure:
1. Choose $n \in \mathbb{N}$. For $t \in [0, T]$, define $\hat{t} = \left\lfloor \frac{nt}{n} \right\rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$; in particular, $\hat{T} = \left\lfloor \frac{nT}{n} \right\rfloor$. Define $\Delta T = \frac{1}{n}$ and $T = \{0, \Delta T, 2\Delta T, \ldots, \hat{T}\}$. Define a measure $\hat{\nu}$ on $T$ is given by $\hat{\nu}(\{t\}) = \Delta T$ if $t < \hat{T}$ and $\hat{\nu}(\{\hat{T}\}) = 1$.

2. We prove by induction that we can choose $K + 1$ vectors $v_0, \ldots, v_K \in \mathbb{R}^K$ such that

$$v_j \cdot v_k = \begin{cases} K & \text{if } j = k \\ -1 & \text{if } j \neq k \end{cases}$$

$$\sum_{k=0}^{K} (v_k)_i (v_k)_j = \delta_{ij}(K + 1)$$

$$\sum_{k=0}^{K} v_k = 0$$

Suppose first $K = 1$; let $v_0 = (1), v_1 = (-1)$. Then $v_0 \cdot v_0 = v_1 \cdot v_1 = 1$ and $v_0 \cdot v_1 = -1; (v_0)_1 (v_0)_1 + (v_1)_1 (v_1)_1 = 1 + 1 = 2$; and $(v_0)_1 + (v_1)_1 = 1 - 1 = 0$.

Now suppose we have chosen $v_0, \ldots, v_K \in \mathbb{R}^K$ such that $v_j \cdot v_k = -1$ if $j \neq k$ and $v_k \cdot v_k = K; \sum_{k=0}^{K} (v_k)_i (v_k)_j = \delta_{ij}(K + 1)$; and $\sum_{k=0}^{K} v_k = 0$.

Let

$$\tilde{v}_k = \begin{cases} \left(\sqrt{\frac{K+2}{K+1}} (v_k)_1, \ldots, \sqrt{\frac{K+2}{K+1}} (v_k)_K, -\frac{1}{\sqrt{K+1}}\right) & \text{for } k = 1, \ldots, K + 1 \\ (0, \ldots, 0, \sqrt{K+1}) & \text{for } k = K + 2 \end{cases}$$

For $j, k \in \{1, \ldots, K + 1\}$,

$$\tilde{v}_j \cdot \tilde{v}_k = \frac{K + 2}{K + 1} v_j \cdot v_k + \frac{1}{K + 1}$$

$$= \begin{cases} \frac{K+2}{K+1} (-1) + \frac{1}{K+1} = -1 & \text{if } j \neq k \\ \frac{K+2}{K+1} K + \frac{1}{K+1} = \frac{K^2 + 2K + 1}{K+1} = K + 1 & \text{if } j = k \end{cases}$$

$$\tilde{v}_j \cdot \tilde{v}_{K+2} = -1$$

$$\tilde{v}_{K+2} \cdot \tilde{v}_{K+2} = K + 1$$
For $i, j \in \{1, \ldots, K\}$,
\[
\sum_{k=0}^{K+1} (\tilde{v}_k)_i (\tilde{v}_k)_j = \sum_{k=0}^{K} (\tilde{v}_k)_i (\tilde{v}_k)_j + 0
\]
\[
= \frac{K + 2}{K + 1} \sum_{k=2}^{K} (v_k)_i (v_k)_j
\]
\[
= \frac{K + 2}{K + 1} (K + 1) \delta_{ij}
\]
\[
= \delta_{ij} (K + 2)
\]

For $i \in \{1, \ldots, K\}$ and $j = K + 1$,
\[
\sum_{k=0}^{K+1} (\tilde{v}_k)_i (\tilde{v}_k)_j = \sum_{k=0}^{K} \sqrt{\frac{K + 2}{K + 1}} (v_k)_i \cdot 0 + 0 \cdot \sqrt{K + 1}
\]
\[
= 0
\]
\[
= \delta_{ij} (K + 2)
\]

The argument for $i = K + 1$ and $j \in \{1, \ldots, K\}$ is similar. Finally, for $i = j = K + 1$
\[
\sum_{k=0}^{K+1} (\tilde{v}_k)_i (\tilde{v}_k)_j = (K + 1) \cdot \frac{1}{K + 1} + (K + 1)
\]
\[
= 1 + (K + 1)
\]
\[
= K + 2
\]
\[
= \delta_{ij} (K + 2)
\]

For $j = 1, \ldots, K$,
\[
\sum_{k=0}^{K+1} (\tilde{v}_k)_j = \sqrt{\frac{K + 2}{K + 1}} \sum_{k=0}^{K} (v_k)_j = \sqrt{\frac{K + 2}{K + 1}} \cdot 0 = 0
\]

Moreover,
\[
\sum_{k=0}^{K+1} (\tilde{v}_k)_{K+1} = -\frac{K + 1}{\sqrt{K + 1}} + \sqrt{K + 1} = 0
\]

This shows that $\sum_{k=0}^{K+1} \tilde{v}_k = 0$.

Thus, by induction, we can choose vectors $v_0, \ldots, v_K \in \mathbb{R}^K$ with the specified properties.
3. Let
\[ \hat{\Omega} = \{ \omega : T \setminus \{0\} \to \{0, K\} \} \]
If \( s \in T \setminus \{0\} \), we write \( \omega_s = \omega(s) \).
The measure \( \hat{\mu} \) on \( \hat{\Omega} \) is given by
\[ \hat{\mu}(A) = \frac{|A|}{|\hat{\Omega}|} \]
for every \( A \in \hat{\mathcal{F}} \), the algebra of all subsets of \( \hat{\Omega} \). For \( t \in T \), \( \hat{\mathcal{F}}_t \) is the algebra of all subsets of \( \hat{\Omega} \) that respect the equivalence relation \( \omega \sim t \omega' \iff \omega_s = \omega'_s \) for all \( s \leq t \).

4. For \( s \in T \), define the random variable
\[ v_s(\omega) = v_{\omega_s} \]
If \( s \neq t \), the random variables \( v_s \) and \( v_t \) are independent. Moreover, for each \( \ell \in \{1, \ldots, K\} \), the random variable \( (v_s)_\ell \) has mean zero and standard deviation one. Define \( \hat{\beta} : T \times \hat{\Omega} \to \mathbb{R}^K \) by
\[ \hat{\beta}(t, \omega) = \sum_{0 < s \leq t, s \in T} v_s(\omega) \sqrt{\Delta T} \]
\( \hat{\beta} \) is a \( K \)-dimensional random walk. Note that \( \hat{\beta}(t, \cdot) \) has variance-covariance matrix \( tI \), where \( I \) is the \( K \times K \) identity matrix. Define
\[ \hat{I}(t, \omega) = (t, \hat{\beta}(t, \omega)) \]

5. Given a consumption plan \( \hat{c} : T \times \hat{\Omega} \to \mathbb{R}_+ \), the agent’s utility is
\[ \hat{U}_i(\hat{c}) = E_{\hat{\mu}} \left( \left( \Delta T \sum_{s \in T, s < T} h_i(\hat{c}(t, \omega), \hat{I}(t, \omega)) \right) + H_i(\hat{c}(\hat{T}, \omega), \hat{\beta}(\hat{T}, \omega)) \right) \]
6. We use the Duffie-Shafer Theorem [31, 32] to perturb the endowments and security dividends to ensure the existence of a Pareto Optimal equilibrium. Let \( \hat{e}_i(t, \omega) \geq 0, \quad |\hat{e}_i(t, \omega) - f_i(\hat{I}(t, \omega))| \leq (\Delta T)^2 \) for all \( t < \hat{T} \) and \( \hat{e}_i(\hat{T}, \omega) \geq 0, \quad |\hat{e}_i(\hat{T}, \omega) - F_i(\hat{I}(\hat{T}, \omega))| \leq (\Delta T)^2 \) denote the perturbed endowments. For all \( \omega \in \hat{\Omega} \), let \( \hat{A}(t, \omega) \) denote the perturbed dividends for all \( t < \hat{T} \), and \( \hat{A}(\hat{T}, \omega) \geq 0, \quad |\hat{A}(\hat{T}, \omega) - G(\hat{I}(\hat{T}, \omega))| \leq (\Delta T)^2 \). Notice that there is a positive constant \( m \)

\[
\hat{e}(t, \omega) + \sum_{j=0}^{J} \eta_j \hat{A}_j(t, \omega) \\
\geq \sum_{j=0}^{J} f_i(I(t, \omega)) + \sum_{j=0}^{J} \eta_j g_j(I(t, \omega)) - \frac{I + J + 1}{n^2} \\
\geq m - \frac{I + J + 1}{n^2} \\
\geq \frac{m}{2}
\]

for \( n \) sufficiently large.

7. A securities price process is a function \( \hat{p}_A : T \times \hat{\Omega} \to \mathbb{R}^{J+1} \) which is adapted with respect to \( \{\hat{F}_t\}_{t \in T} \). We will price securities \textit{ex dividend} for \( t < \hat{T} \); it will be convenient to price securities \textit{cum dividend} at \( t = \hat{T} \).

8. A consumption price process is a function \( \hat{p}_C : T \times \hat{\Omega} \to \mathbb{R}_+ \) which is adapted with respect to \( \{\hat{F}_t\}_{t \in T} \).

9. Given a securities price process \( \hat{p}_A \) and a consumption price process \( \hat{p}_C \), the associated total gains process \( \hat{\gamma} \) is given by

\[
\hat{\gamma}_j(t, \omega) = \begin{cases} 
\sum_{s \in T, s \leq t} \hat{p}_C(s, \omega) \hat{A}_j(s, \omega) + \hat{p}_A_j(t, \omega) & \text{if } t < \hat{T} \\
\sum_{s \in T, s < t} \hat{p}_C(s, \omega) \hat{A}_j(s, \omega) + \hat{p}_A_j(t, \omega) & \text{if } t = \hat{T}
\end{cases}
\]

10. A trading strategy\(^{12}\) for agent \( i \) is a function \( \hat{z}_i : (T \cup \{-\Delta T\}) \times \hat{\Omega} \to \mathbb{R}^{J+1} \) which is adapted with respect to \( \{\hat{F}_t\}_{t \in T} \) such that \( \hat{z}_i(-\Delta T, \omega) = e_{iA} \) and \( \hat{z}_i(\hat{T}, \omega) = \hat{z}_i(\hat{T} - \Delta T, \omega) \).

\(^{12}\)In the discrete model, stochastic integrals with zero drift are automatically martingales, so we do not need to require admissibility as a separate assumption.
11. A consumption plan for agent $i$ is a function $\hat{c}_i : T \times \hat{\Omega} \to \mathbb{R}_+$. The budget set for agent $i$ is the set of all consumption plans $\hat{c}_i$ such that there exists a trading strategy $\hat{z}_i$ for which $\hat{c}_i$ satisfies the budget constraint

$$\left(\hat{c}(s, \omega) - \hat{e}_i(s, \omega) - \hat{z}_i(s) \cdot \hat{A}(s, \omega)\right) \cdot \hat{p}_C(s, \omega) = \left(\hat{z}_i(s) \cdot \hat{A}(s, \omega)\right) \cdot \hat{p}_A(s, \omega)$$

for all $s \in T$ and all $\omega \in \hat{\Omega}$. Note that since $\hat{z}_i$ is required to be adapted to $\{\hat{F}_t\}_{t \in \hat{T}}$, it follows that $\hat{c}_i$ is adapted to $\{\hat{F}_t\}_{t \in \hat{T}}$.

12. Given a security price $\hat{p}$ and a consumption price $\hat{p}_C$, the demand of the agent is a consumption plan and a trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.

13. An equilibrium for the economy is a security price process $\hat{p}$, a consumption price process $\hat{p}_C$, trading strategies $\hat{z}_i$ and consumption plans $\hat{c}_i$ which lies in the demand sets of the agents so that the securities and goods markets clear, i.e. for all $t \in T$ and all $\omega \in \hat{\Omega}$

$$\sum_{i=1}^{I} \hat{z}_i(t, \omega) = (\eta_0, \ldots, \eta_J)$$
$$\sum_{i=1}^{I} \hat{c}_i(t, \omega) = \sum_{i=1}^{I} \hat{e}_i(t, \omega) + \sum_{j=0}^{J} \eta_j \hat{A}_j(\omega, t)$$

14. The first order conditions imply that the total gains process $\hat{\gamma}$ is a vector martingale. Define

$$\hat{C}_i(t_0, \omega_0) = E \left( \left( \left( \hat{p}_C(\hat{T}, \cdot) \left( \hat{c}_i(\hat{T}, \cdot) - \hat{e}_i(\hat{T}, \cdot) \right) \right) \right) \right) + \Delta T \sum_{t=0}^{\hat{T}-\Delta T} \hat{p}_C(t, \cdot) \left( \hat{c}_i(t, \cdot) - \hat{e}_i(t, \cdot) \right) \left| (t_0, \omega_0) \right)$$

From the definition, $\hat{C}_i$ is a martingale. For each node $(t, \omega) \in T \times \hat{\Omega}$, there is a unique $(K+1) \times K$ matrix $\check{\sigma}(t, \omega)$ and unique row vectors

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13 We take $\hat{z}_i(-\Delta T, \omega) = e_{iA}(0)$, so that agent enters period 0 holding the securities with which s/he is endowed. Since securities are priced cum dividend at $t = \hat{T}$, we require that $\hat{z}_i(\hat{T}, \omega) = \hat{z}_i(\hat{T} - \Delta T, \omega)$. 

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\( \hat{\sigma}_1(t, \omega), \ldots, \hat{\sigma}_I(t, \omega) \in \mathbb{R}^K \) such that

\[
\begin{align*}
\hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega) &= \hat{\sigma}(t, \omega) \left( \hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega) \right) \\
\hat{C}_1(t + \Delta T, \omega) - \hat{C}_1(t, \omega) &= \hat{\sigma}_1(t, \omega) \left( \hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega) \right) \\
&\vdots \\
\hat{C}_I(t + \Delta T, \omega) - \hat{C}_I(t, \omega) &= \hat{\sigma}_I(t, \omega) \left( \hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega) \right)
\end{align*}
\]

(10)
at the first \( K \) of the \( K + 1 \) successor nodes to \( (t, \omega) \). Since \( \hat{\beta}, \hat{\gamma}, \) and \( \hat{C}_1, \ldots, \hat{C}_I \) are martingales, Equation (10) also holds at the \((K + 1)\)st successor node. Thus, the processes \( \hat{\sigma} \) and \( \hat{\sigma}_1, \ldots, \hat{\sigma}_I \) are adapted. Notice that

\[
\hat{p}_A(t + \Delta T, \omega) - \hat{p}_A(t, \omega) \\
= \hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega) - \hat{p}_C(t, \omega)\hat{A}(t, \omega) \\
= \hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega) + O(\Delta T) \\
= \hat{\sigma}(t, \omega) \left( \hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega) \right) + o\left(\sqrt{\Delta T}\right)
\]

The equilibrium trading strategies \( \hat{z}_i(t, \omega) \) are the unique \( 1 \times (K + 1) \) row vectors satisfying the budget constraint and the equation

\[
\hat{\sigma}_i(t, \omega) = \hat{z}_i(t, \omega)\hat{\sigma}(t, \omega)
\]

**Theorem 3.1** Given a continuous-time economy satisfying the assumptions of Theorem 2.1, let \( \hat{p}_{nA}, \hat{p}_{nC}, \hat{c}_{ni} \) and \( \hat{z}_{ni} \) denote any equilibrium securities prices, consumption prices, consumptions and trading strategies of the discretized sequence of economies just described. Let \( \hat{\lambda}_n \) denote the utility weights maximized at that equilibrium. Then there are equilibrium securities prices \( p_{nA} \), consumption prices \( p_{nC} \), consumptions \( c_{ni} \), trading strategies \( z_{ni} \) and weights \( \lambda_n \) of the continuous-time economy satisfying the conclusions of Theorem 2.1, in particular Equations (4, 5, 6, 7, 8 and 9) such that

\[
\begin{align*}
|\hat{\lambda}_n - \lambda_n| &\to 0 \\
\left\| \hat{p}_{nA} - p_{nA} \circ \hat{T} \right\|_2 &\to 0 \\
\max_{t \in \mathbb{T}_n} \left| \hat{p}_{nA}(t, \cdot) - p_{nA}(\hat{T}(t, \cdot)) \right| &\to 0 \text{ in probability}
\end{align*}
\]

(11)
\[ \| \hat{p}_{nc} - p_{nc} \circ \hat{I} \|_2 \to 0 \quad (14) \]

\[ \max_{t \in T_n} | \hat{p}_{nc}(t, \cdot) - p_{nc}(\hat{I}(t, \cdot)) | \to 0 \text{ in probability} \quad (15) \]

\[ \| \hat{\sigma}_n - \Sigma_n \circ \hat{I} \|_2 \to 0 \quad (16) \]

\[ \max_{t \in T_n} | \hat{\sigma}_n(t, \cdot) - \Sigma_n(\hat{I}(t, \cdot)) | \to 0 \text{ in probability} \quad (17) \]

\[ \| \hat{c}_{ni} - c_{ni} \circ \hat{I} \|_2 \to 0 \quad (18) \]

\[ \max_{t \in T_n} | \hat{c}_{ni}(t, \cdot) - c_{ni}(\hat{I}(t, \cdot)) | \to 0 \text{ in probability} \quad (19) \]

\[ \| \hat{\sigma}_{ni} - \Sigma_{ni} \circ \hat{I} \|_2 \to 0 \quad (20) \]

\[ \max_{t \in T_n} | \hat{\sigma}_{ni}(t, \cdot) - \Sigma_{ni}(\hat{I}(t, \cdot)) | \to 0 \text{ in probability} \quad (21) \]

\[ \| \hat{z}_{ni} - z_{ni} \circ \hat{I} \|_2 \to 0 \quad (22) \]

\[ \max_{t \in T_n} | \hat{z}_{ni}(t, \cdot) - z_{ni}(\hat{I}(t, \cdot)) | \to 0 \text{ in probability} \quad (23) \]

\[ \| \hat{z}_{ni} \cdot \hat{p}_{An} - W_{ni} \circ \hat{I} \|_2 \to 0 \quad (24) \]

\[ \max_{t \in T_n} | \hat{z}_{ni}(t, \cdot) \cdot \hat{p}_{An} - W_{ni}(\hat{I}(t, \cdot)) | \to 0 \text{ in probability} \quad (25) \]

### 4 Proofs of Theorems 2.1 and 3.1

Up to now, all of our definitions and results have been stated without any reference to nonstandard analysis. Our proof makes extensive use of nonstandard analysis, in particular Anderson’s construction of Brownian Motion and the Itô Integral ([2]). It is beyond the scope of this paper to develop these methods; see Appendix B for material on nonstandard stochastic calculus, Anderson [4] and Hurd and Loeb [42] are references to nonstandard analysis.

We construct our probability space, filtration and Brownian Motion following Anderson’s construction [2]. Choose \( n \in \ast \mathbb{N} \setminus \mathbb{N} \). Using this hyperfinite \( n \), define \( T, \hat{t}, \hat{T}, \hat{\nu}, \hat{\mu}, \hat{\Omega}, \hat{\beta}, \hat{F}, \hat{F}_t, \hat{U}_i \) and so on exactly as they were defined in Section 3. By the Transfer Principle, the economy has an equilibrium, and the equilibrium is Pareto optimal.

Let \( (T, L(\hat{\nu})) \) denote the complete Loeb measure generated by \( \hat{\nu} \) on \( T \). For \( B \subset [0, T] \), let \( st^{-1}(B) = \{ t \in T : st \in B \} \). For any Lebesgue measurable set \( B \subset [0, T] \), \( st^{-1}(B) \) is Loeb measurable and \( L(\hat{\nu})(st^{-1}(B)) = \nu(B) \).
Let \((\Omega, \mathcal{F}, L(\hat{\mu}))\) be the (complete) Loeb measure generated by \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})\) (Loeb [56]). Although \((\Omega, \mathcal{F}, L(\hat{\mu}))\) is generated by a nonstandard construction, Loeb showed that it is a probability space in the usual standard sense.

Let \(F_t = \{B \in \mathcal{F}: L(\hat{\mu})(B \Delta C) = 0 \text{ for some } C \text{ which respects the equivalence relation } \omega \sim \omega' \Leftrightarrow \omega \sim_s \omega' \text{ for all } s \simeq t\}\)

Let \(\beta: [0,T] \times \Omega \to \mathbb{R}^K\) be defined by \(\beta(t,\omega) = \circ (\hat{\beta}(\hat{t},\omega))\). \(\beta\) is a \(K\)-dimensional Brownian motion in the usual standard sense, and \(\beta(t,\cdot) = E(\beta(T,\cdot) | \mathcal{F}_t)\).\(^{14}\) Let \(I: [0,T] \times \Omega \to \mathbb{R}^{K+1}\) be defined by

\[I(t,\omega) = (t, \beta(t,\omega))\]

Since the equilibrium is Pareto optimal, the marginal utility of consumption is infinite at zero, and the aggregate consumption is strictly positive at each node, the equilibrium consumptions of all agents are strictly positive at each node. Let \(\Delta\) be the open \(I - 1\)-dimensional simplex in \(\mathbb{R}^I^{++}\). Pareto optimality implies that there exists \(\hat{\lambda} = (\hat{\lambda}_1,\ldots,\hat{\lambda}_I)\) in \(^*\Delta\) such that at each

---

\(^{14}\)Anderson [2] proved that \(\beta\) is a standard Brownian motion provided that \(\hat{\beta}\) moves up or down \(1/\sqrt{n}\) independently in each coordinate at every node; this would require each node in the tree for \(\hat{\beta}\) to have \(2^K\) successor nodes, precluding dynamic completeness in the hyperfinite model for \(K > 1\). Neither Anderson nor Keisler [45] quite covers the random walk \(\hat{\beta}\) considered here because the coordinates of \(\hat{\beta}\) are uncorrelated but not independent. However, the coordinates of \(\beta\) are independent. To see this, fix \(x \in \mathbb{R}^K\). Then \(\{x \cdot v_s : s \in T\}\) is a family of IID random variables with standard distribution, mean zero and finite variance \(\sigma_x\) so Anderson’s Theorem 21 implies that \(x \cdot \beta(t,\cdot)\) is Normal mean zero variance \(t\sigma_x\). Since \(x \cdot \beta(t,\cdot)\) is Normal for all \(x \in \mathbb{R}^K\), it is well known that \(\beta(t,\cdot)\) is system Normal (see, for example, Bryc [15], Theorem 2.2.4). Since \(\beta(t,\cdot)\) is system Normal with variance-covariance matrix \(tI\), where \(I\) is the \(K \times K\) identity matrix, the components of \(\beta\) are independent. Each component of the random walk \(\hat{\beta}\) is a hypermartingale (martingale with respect to the hyperfinite filtration), so satisfies the \(S\)-continuity property by Keisler’s continuity theorem for hypermartingales, and it follows from Anderson’s proof that \(\beta\) is almost surely continuous. Anderson’s proof that \(\beta\) has independent increments goes through without change in the present setting. Anderson used a slightly smaller filtration than the one considered here, while Keisler used the filtration considered here.
node \((t, \omega)\), there is a positive constant \(\hat{\mu}(t, \omega)\) such that

\[
\hat{\lambda}_1 \left( \frac{\partial h_1}{\partial c} (c_1(t, \omega), \hat{I}(t, \omega)) \right) = \cdots = \hat{\lambda}_I \left( \frac{\partial H_I}{\partial c} (c_I(t, \omega), \hat{I}(t, \omega)) \right) = \hat{\mu}(t, \omega) \quad \text{for } t < \hat{T}.
\]

Let \(\hat{c}(t, \omega) = \sum_{i=1}^I \hat{c}_i(t, \omega)\). By the analytic implicit function theorem\(^{15}\), there exist standard analytic functions \(\hat{\pi}, \hat{\Pi}, \hat{\psi}_i, \hat{\Psi}_i : \Delta \times \mathbb{R}_{++} \times ([0, \hat{T}] \times \mathbb{R}^K) \to \mathbb{R}\) such that

\[
\hat{\mu}(t, \omega) = \hat{\pi}(\hat{\lambda}, \hat{c}(t, \omega), \hat{I}(t, \omega)) \quad \text{for } t < \hat{T},
\]

\[
\hat{\Pi}(\hat{T}, \omega) = \hat{\Pi}(\hat{\lambda}, \hat{c}(\hat{T}, \omega), \hat{I}(\hat{T}, \omega)),
\]

\[
\hat{c}_i(t, \omega) = \hat{\psi}_i(\hat{\lambda}, \hat{c}(t, \omega), \hat{I}(t, \omega)) \quad \text{for } t < \hat{T},
\]

\[
\hat{c}_i(\hat{T}, \omega) = \hat{\Psi}_i(\hat{\lambda}, \hat{c}(\hat{T}, \omega), \hat{I}(\hat{T}, \omega)).
\]

Let \(\lambda \in \Delta, \pi : \mathbb{R}_{++} \times \mathbb{R}^K \times [0, \hat{T}] \to \mathbb{R}, \Pi : \mathbb{R}_{++} \times \mathbb{R}^K \to \mathbb{R}, \psi_i : \mathbb{R}_{++} \times \mathbb{R}^K \times [0, \hat{T}] \to \mathbb{R}\) and \(\Psi_i : \mathbb{R}_{++} \times \mathbb{R}^K \to \mathbb{R}\) be defined by

\[
\lambda = \circ \hat{\lambda},
\]

\[
\pi(c, \mathcal{I}) = \hat{\pi}(\lambda, c, \mathcal{I}),
\]

\[
\Pi(c, \beta) = \hat{\Pi}(\lambda, c, \beta),
\]

\[
\psi_i(c, \mathcal{I}) = \hat{\psi}_i(\lambda, c, \mathcal{I}),
\]

\[
\Psi_i(c, \beta) = \hat{\Psi}_i(\lambda, c, \beta).
\]

Because \(\hat{\pi}\) and \(\hat{\Pi}\) are standard analytic functions, \(\pi\) and \(\Pi\) are standard analytic functions. Because aggregate consumption is uniformly bounded away from zero by a noninfinitesimal amount, \(\hat{\pi}, \hat{\Pi}, \pi\) and \(\Pi\) are uniformly bounded above by a finite number. Let \(c(t, \omega) = \circ \hat{c}(t, \omega), c_i(t, \omega) = \circ \hat{c}_i(t, \omega)\). Since \(c(t, \omega)\) and \(\psi_i(c, \beta, t)\) are analytic functions for \(t \in [0, \hat{T}]\), each \(c_i(t, \omega)\) is an analytic function of \(\mathcal{I}(t, \omega)\). Each \(c_i(T, \omega)\) is an analytic function of \(c(T, \omega)\); however, since \(c(T, \omega)\) is a continuous but not necessarily analytic function of \(\mathcal{I}(T, \omega)\), \(c_i(T, \omega)\) is a continuous but not necessarily analytic function of \(\mathcal{I}(T, \omega)\).

Note that for every \(t \in \mathcal{T}\) and \(\omega \in \hat{\Omega}\),

\[
\hat{c}(t, \omega) = \hat{e}(t, \omega) + \sum_{j=0}^J \eta_j \hat{A}_j(t, \omega) \geq \frac{m}{2}.
\]

\(^{15}\) The statement is given in Theorem A.2 in Appendix A.
where $m$ is standard and positive. Therefore, from the assumptions on the utility functions, there exists a finite number $M$ such that $\hat{\mu}(t, \omega) \leq M$ for all $(t, \omega)$. If we set $\hat{p}_C(t, \omega) = \hat{\mu}(t, \omega)$, $\hat{p}_C$ gives the Arrow-Debreu prices of consumption. Let

$$p_C(t, \omega) = \circ \hat{p}_C(t, \omega)$$

$p_C$ is an analytic function of $\mathcal{I}(t, \omega)$ on $(0, T) \times \mathbb{R}^K$, and has a continuous extension to $[0, T] \times \mathbb{R}^K$, but this continuous extension need not agree with $p_C(T, \cdot)$; $p_C(T, \cdot)$ is continuous on $\mathbb{R}^K$.

In the following calculation,

- Equations (26) and (30) follow because $\hat{p}_A(\hat{T}, \cdot) = \hat{p}_C(\hat{T}, \omega) A(\hat{T}, \omega)$, $\hat{p}_C$ is uniformly bounded above by a standard number, and $A_j(t, \omega) \leq r + e^{s|\hat{\beta}(t, \omega)|} + O((\Delta T)^2)$, so $\hat{p}_A(\hat{T}, \cdot)$ and $\hat{p}_C(\cdot, \cdot) \hat{A}(\cdot, \cdot)$ are $SL^2$ by Proposition 3.2 of Raimondo [66]; $\hat{\beta}$ is $SL^2$ so $\hat{p}_C \hat{A} \hat{\beta}$ is $SL^1$ (Anderson [2]); and the internal integrands are $S$-continuous functions of $\hat{I}(t, \omega)$ whenever $\hat{I}(t, \omega)$ is finite.

- Equations (27) and (31) follow because $\hat{I}(\cdot, \omega)$ is almost surely $S$-continuous, and for any such $\omega$ and any $s \geq t$ (including $s = T$), the conditional distribution $\hat{I}(s, \omega)$ is the same given $(\circ t, \omega)$ as it is given $(t, \omega)$.

- In Equation (28), we write $\hat{p}_C \hat{A} \left( \hat{\beta}(t, \omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T} \right)$ as an abbreviation for

\[
\hat{p}_C \left( s, \hat{\beta}(t, \omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T} \right) \hat{A} \left( s, \hat{\beta}(t, \omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T} \right)
\]

\[
\circ \hat{p}_A(t, \omega) = \circ E \left( \left( \hat{p}_A(\hat{T}, \cdot) + \Delta T \sum_{s=t+\Delta T}^{T} \hat{p}_C(s, \cdot) \hat{A}(s, \cdot) \right) \big| (t, \omega) \right) = \circ E \left( \hat{C} \hat{A} \left( \hat{\beta}(t, \omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T} \right) \hat{A} \left( \hat{\beta}(t, \omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T} \right) \big| (t, \omega) \right)
\]
\[ + \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \Delta T \sum_{s=t+\Delta T}^{T-\Delta T} \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \right) \]

\[ + E \left( \Pi(c(T, \cdot), \hat{I}(T, \cdot))G(\hat{I}(T, \cdot))(t, \omega) \right) + E \left( \int_{s=t}^{T} \pi(c(s, \cdot, \hat{I}(s, \cdot)), g(\hat{I}(s, \cdot))) ds (t, \omega) \right) \]

\[ + E \left( \Pi(c(T, \cdot), \hat{I}(T, \cdot))G(\hat{I}(T, \cdot)) + \int_{s=t}^{T} \pi(c(s, \cdot, \hat{I}(s, \cdot)), g(\hat{I}(s, \cdot))) ds \right) \left( t, \omega \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ = E \left( \Pi(c(T, \cdot), \hat{I}(T, \cdot))G(\hat{I}(T, \cdot))(t, \omega) \right) + E \left( \int_{s=t}^{T} \pi(c(s, \cdot, \hat{I}(s, \cdot)), g(\hat{I}(s, \cdot))) ds (t, \omega) \right) \]

\[ + E \left( \Pi(c(T, \cdot), \hat{I}(T, \cdot))G(\hat{I}(T, \cdot)) + \int_{s=t}^{T} \pi(c(s, \cdot, \hat{I}(s, \cdot)), g(\hat{I}(s, \cdot))) ds \right) \left( t, \omega \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]

\[ \sum_{s=t+\Delta T}^{T-\Delta T} \epsilon \left( \hat{c}(s, \cdot, t, \cdot) \epsilon \left( \hat{I}(s, \cdot) \right) + O \left( \Delta T^2 \right) \right) \]
This shows that \( \hat{p}_A \) is an \( S \)-continuous function of \( (t, \hat{\beta}) \in st^{-1} \left( [0, T] \times \mathbb{R}^K \right) \), and that \( \hat{\sigma} \) is an \( S \)-continuous function of \( (t, \hat{\beta}) \in st^{-1} \left( [0, T] \times \mathbb{R}^K \right) \). Let

\[
\hat{p}_A(t, \omega) = \sigma(t, \omega)
\]

Letting \( \Sigma : [0, T) \times \mathbb{R}^K \to \mathbb{R}^{K+1} \times \mathbb{R}^K \) be given by

\[
\Sigma(t, \beta) = E \left( \frac{p_C(T, \beta(T)) A(T, \beta(T)) \beta(T) - \beta(t)}{T - t} \right) + \int_t^T \frac{p_C(s, \beta(s)) A(s, \beta(s)) \beta(s) - \beta(t)}{s - t} ds \bigg| \beta(t) = \beta(t)
\]

we have

\[
\hat{\sigma}(t, \omega) = \Sigma(\mathcal{T}(t, \omega))
\]

for every \((t, \omega)\) such that \( \mathcal{T}(t, \omega) \) is finite. We claim that \( \Sigma \) is an analytic function of \((t, \beta) \in (0, T) \times \mathbb{R}^K\). The expression for \( \Sigma \) is the sum of two terms:

- The first term can be rewritten as

\[
E \left( \frac{p_A(T, \beta(T)) \beta(T) \beta(t, \omega) = \beta) - \beta(t)}{T - t} \right) - \beta E \left( p_A(T, \beta(T)) \beta(T) \beta(t, \omega) = \beta) - \beta(t) \right)
\]

The growth condition on \( A \) (Equation (2)) and the fact that consumption is uniformly bounded away from zero, hence \( P_C \) is uniformly bounded above, imply that \( p_A = Ap_C \leq r + e^{r|\beta|} \) for some \( r \in \mathbb{R} \), so the integrands \( p_A(T, \beta(T)) \beta(T) \) and \( p_A(T, \beta(T)) \) satisfy the growth condition in the hypotheses of Theorem A.4, so the conditional expectations are jointly analytic over \((0, T) \times \mathbb{R}^K\), and thus the whole first term is jointly analytic.
• The denominator \( s - t \) in the second term appears problematic,\(^{16}\) so we redo the calculation of the second term, using the fact that the dividends and endowments are analytic for \( t < T \):

\[
\Delta T \sum_{s=t+\Delta T}^{T-\Delta T} \left( E \left( \left\{ \hat{p}_C(s, \cdot) \hat{A}(s, \cdot) \right\} \right) | (t + \Delta T, \omega) \right) - E \left( \left\{ \hat{p}_C(s, \cdot) \hat{A}(s, \cdot) \right\} \right) | (t, \omega) \right) \\
= \Delta T \sum_{s=t+\Delta T}^{T-\Delta T} \left( E \left( \left\{ \ast \hat{\theta} (\hat{\lambda}(s, \cdot), \hat{\omega}(s, \cdot)) \hat{A}(s, \cdot) \right\} \right) | (t + \Delta T, \omega) \right) - E \left( \left\{ \ast \hat{\theta} (\hat{\lambda}(s, \cdot), \hat{\omega}(s, \cdot)) \hat{A}(s, \cdot) \right\} \right) | (t, \omega) \right) \\
= \Delta T \sum_{s=t+\Delta T}^{T-\Delta T} \left( E \left( \left\{ \ast \hat{\theta} (\hat{\lambda}(s, \cdot), \hat{\omega}(s, \cdot)) \hat{g}(\hat{\omega}(s, \cdot)) + O \left( (\Delta T)^2 \right) \right\} \right) | (t + \Delta T, \omega) \right) \\
= \Delta T \sum_{s=t+\Delta T}^{T-\Delta T} E \left( \left\{ \hat{\theta} (\hat{\lambda}(s, \cdot), \hat{\omega}(s, \cdot)) \hat{g}(\hat{\omega}(s, \cdot)) \right\} \right) | (t + \Delta T, \omega) \right) \\
= \int_t^T E \left( \left( \hat{\theta} (\hat{\lambda}(s, \cdot), \hat{\omega}(s, \cdot)) \hat{g}(\hat{\omega}(s, \cdot)) \right) \right) | (t, \omega) \right) ds \\
\]

\(^{16}\)An intuitive way way to see why the denominator does not cause problems is to note that for \( s \) close to \( t \), we have

\[
E \left( \frac{\hat{p}_C(s, \beta(s)) \hat{A}(s, \beta(s)) | (s-t) \right) \left( \beta(s) - \beta(t) \right) \right) | (t, \beta(t)) \right) \approx E \left( \frac{\partial \hat{p}_C A}{\partial t} \right) | (t, \beta(t)) \left( \beta(s) - \beta(t) \right) | (t, \beta(t)) \right) \\
= \frac{\partial \hat{p}_C A}{\partial t} \left( \beta(s) - \beta(t) \right) | (t, \beta(t)) + \frac{\partial \hat{p}_C A}{\partial \beta} \left( \beta(s) - \beta(t) \right) | (t, \beta(t)) \right) \\
= \frac{\partial \hat{p}_C A}{\partial \beta} \left( \beta(s) - \beta(t) \right) | (t, \beta(t)) \right) \\

30
which is analytic by Proposition 2.2.3 of Krantz and Parks [49].

Thus, $\Sigma$ is analytic on $(0, T) \times \mathbb{R}^K$.

Extend $\Sigma$ to $\{T\} \times V$ by

$$
\Sigma(T, \beta) = \frac{\partial p_A(T, \beta)}{\partial \beta}
$$

We claim that $\Sigma$ is continuous on $[0, T] \times V$. Notice that the second term
in the definition of $\Sigma$ tends to zero as $t \to T$, uniformly over $\beta$ ranging
over compact subsets of $\mathbb{R}^K$, so we can restrict attention to the first term.

Suppose $\beta_0 \in V$. Fix $\varepsilon > 0$. Since $A(T, \cdot)$ is $C^1$ on $V$ and $p_C(T, \cdot)$ is analytic
in $\mathbb{R}^K$, $p_A(T, \cdot) = A(T, \cdot)p_C(T, \cdot)$ is $C^1$ on $\{T\} \times V$. Since $V$ is open, we may
find $\delta > 0$ such that $B(\beta_0, 2\delta) \subset V$ and

$$
\beta, y \in B(\beta_0, 2\delta) \Rightarrow \left\| \frac{\partial p_A}{\partial \beta}(T, y) - \frac{\partial p_A}{\partial \beta}(T, \beta) \right\| < \frac{\varepsilon}{3}
$$

For any $\beta \in B(\beta_0, \delta)$ and $y \in B(\beta, \delta)$, we have $y \in B(\beta_0, 2\delta)$. Using the
Mean Value Theorem one component at a time, we find that

$$
\left| p_A(T, y) - p_A(T, \beta) - \frac{\partial p_A}{\partial \beta}(T, \beta) (y - \beta) \right| < \frac{\varepsilon|y - \beta|}{3}
$$

so

$$
\left\| \left( p_A(T, y) - p_A(T, \beta) - \frac{\partial p_A}{\partial \beta}(T, \beta) (y - \beta) \right) (y - \beta)' \right\| < \frac{\varepsilon|y - \beta|^2}{3}
$$

Given that $p_C$ is uniformly bounded above and $|A_j(T, y)| \leq r + e^{|y|}$, and
$\left\| \frac{\partial p_A}{\partial \beta}(T, \beta) \right\|$ is uniformly bounded over $\beta \in B(\beta_0, \delta)$, there is a constant $\bar{r} \in \mathbb{R}$
such that for every $\beta \in B(\beta_0, \delta)$, for all $y \in \mathbb{R}^K$,

$$
\left\| \left( p_A(T, y) - p_A(T, \beta) - \frac{\partial p_A}{\partial \beta}(T, \beta) (\beta - y) \right) (\beta - y)' \right\| \leq \bar{r} + e^{||y||}
$$

Find $t_0 < T$ such that for all $t \in [t_0, T)$,

$$
\frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K \setminus B(\beta, \delta)} \frac{\bar{r} + e^{||y||}}{T-t} e^{-|y-\beta|^2/2(T-t)} dy < \frac{\varepsilon}{3}
$$
If $t \in [t_0, T)$ and $\beta \in B(\beta_0, \delta)$,

\[
\left| \frac{E(p_A(T, \beta(T, \cdot))(\beta(T, \cdot) - \beta(t, \cdot))| \beta(t, \omega) = \beta)}{T - t} - \frac{\partial p_A}{\partial \beta}_{(T, \beta_0)} \right|
\]

\[
= \left| \frac{1}{(2\pi)^{K/2}} \int_{\mathbb{R}^K} \frac{p_A(T, \beta + \sqrt{T - t}x) \sqrt{T - t}x^t}{T - t} e^{-|x|^2/2} dx - \frac{\partial p_A}{\partial \beta}_{(T, \beta_0)} \right|
\]

\[
= \left| \frac{1}{(2\pi(T - t))^{K/2}} \int_{\mathbb{R}^K} \frac{p_A(T, y)(y - \beta)^t}{T - t} e^{-|y - \beta|^2/(2(T-t))} dy - \frac{\partial p_A}{\partial \beta}_{(T, \beta_0)} \right|
\]

\[
\leq \left| \frac{1}{(2\pi(T - t))^{K/2}} \int_{\mathbb{R}^K} \frac{\partial p_A}{\partial \beta}_{(T, \beta)} (y - \beta)(y - \beta)^t}{T - t} e^{-|y - \beta|^2/(2(T-t))} dy - \frac{\partial p_A}{\partial \beta}_{(T, \beta_0)} \right|
\]

\[
+ \frac{1}{2(T - t)^{K/2}} \int_{\mathbb{R}^K} \frac{p_A(t, y)(y - \beta)^t}{T - t} e^{-|y - \beta|^2/(2(T-t))} dy \right|
\]

\[
+ \frac{1}{(2\pi(T - t))^{K/2}} \int_{\mathbb{B}(\beta, \delta)} \frac{\varepsilon(y - \beta)(y - \beta)^t}{3(T - t)} e^{-|y - \beta|^2/(2(T-t))} dy
\]

\[
+ \frac{1}{(2\pi(T - t))^{K/2}} \int_{\mathbb{R}^K \setminus \mathbb{B}(\beta, \delta)} \frac{\bar{r} + e^{|y|}}{T - t} e^{-|y - \beta|^2/(2(T-t))} dy
\]

\[
= \left| \frac{\partial p_A}{\partial \beta}_{(T, \beta)} - \frac{\partial p_A}{\partial \beta}_{(T, \beta_0)} \right| + 0 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}
\]

\[
< \frac{\varepsilon}{3} + 0 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]

which shows that $\Sigma$ is continuous on $[0, T] \times V$.

Now, we show that the pricing process is dynamically complete. Let $\Sigma_j$ ($j = 0, \ldots, J$) denote the $j^{th}$ row of $\Sigma$, and let $\Sigma'(t, \beta)$ be the $K \times K$ matrix whose $j^{th}$ row ($j = 1, \ldots, J$) is

\[
\Sigma_j'(t, \beta) = \frac{p_{A_0}(t, \beta) \Sigma_j(t, \beta) - p_{A_1}(t, \beta) \Sigma_0(t, \beta)}{(p_{A_0}(t, \beta))^2}
\]

\[
= \frac{\partial p_{A_1}}{\partial \beta}
\]

Notice that

\[
\text{rank } \Sigma'(t, \beta) \leq \text{rank } \Sigma(t, \beta)
\] (33)
Let

$$B = \{ \mathcal{I} \in [0, T] \times \mathbb{R}^K : \det \Sigma'(\mathcal{I}) = 0 \}$$

Suppose that $B$ has positive Lebesgue measure. Then $\det \Sigma'(\mathcal{I}) = 0$, for every $\mathcal{I} \in B$. The determinant is a polynomial function of the entries of the matrix, hence is an analytic function of $\mathcal{I} \in \mathbb{R}^K \times [0, T)$, so $\det \Sigma'(\mathcal{I})$ must be identically zero on $[0, T)$, since it is continuous on $[0, T]$, it is identically zero on $[0, T]$. Using the nondegeneracy assumption (Equation (3)), choose $\omega$ such that $\hat{\beta}(\cdot, \omega)$ is finite and

$$\det \left( \begin{array}{c} \frac{\partial (G_{j}/G_0)}{\partial \beta} |_{I(T, \omega)} \\
\vdots \\
\frac{\partial (G_j/G_0)}{\partial \beta} |_{I(T, \omega)} 
\end{array} \right) \neq 0$$

Since the securities prices are equilibrium prices, they must be arbitrage-free, so we have for $j = 1, \ldots, J$,

$$\frac{p_{A_j}(T, \omega)}{p_{A_0}(T, \omega)} \approx \frac{\hat{p}_{A_j}(\hat{T}, \omega)}{\hat{p}_{A_0}(\hat{T}, \omega)}$$

$$= \frac{\hat{A}_j(\hat{T}, \omega)}{\hat{A}_0(\hat{T}, \omega)}$$

$$= \frac{*G_j(\hat{T}, \hat{\beta}(\hat{T}, \omega)) + O((\Delta T)^2)}{*G_0(\hat{T}, \hat{\beta}(\hat{T}, \omega)) + O((\Delta T)^2)}$$

$$\approx \frac{*G_j(\hat{T}, \hat{\beta}(\hat{T}, \omega))}{*G_0(\hat{T}, \hat{\beta}(\hat{T}, \omega))}$$

$$\approx \frac{G_j(T, \beta(T, \omega))}{G_0(T, \beta(T, \omega))}$$

\[17\text{See Theorem A.3 in Appendix A.}\]
\[ \det \Sigma'(T, \beta(T, \omega)) = \det \begin{pmatrix} \frac{\partial (p_{A_1}/p_{A_0})}{\partial \beta} \bigg|_{I(T, \omega)} & \vdots & \frac{\partial (p_{A_J}/p_{A_0})}{\partial \beta} \bigg|_{I(T, \omega)} \\ \vdots & \ddots & \vdots \\ \frac{\partial (G_1/G_0)}{\partial \beta} \bigg|_{I(T, \omega)} & \vdots & \frac{\partial (G_J/G_0)}{\partial \beta} \bigg|_{I(T, \omega)} \end{pmatrix} \neq 0 \]

a contradiction which proves that \( B \) is a set of measure zero.

If we let \( B_t = \{ \beta \in \mathbb{R}^K : (t, \beta) \in B \} \) denote the \( t \)-section of \( B \), then by Fubini’s Theorem, \( \lambda(\{ t : B_t \text{ has positive Lebesgue measure} \}) = 0 \). Since the distribution of \( \beta(t, \cdot) \) is absolutely continuous with respect to Lebesgue measure,

\[
(\lambda \times L(\hat{\mu})) (\{ (t, \omega) : (I(t, \omega)) \in B \}) = \int_{[0,T]} L(\hat{\mu}) (\{ \omega : (I(t, \omega)) \in B \}) \, d\lambda \\
= \int_{[0,T]} L(\hat{\mu}) (\{ \omega : \beta(t, \omega) \in B_t \}) \, d\lambda \\
= 0
\]

Therefore, the normalized securities price process \( p_A/p_{A_0} \) is essentially dynamically complete ([62], Theorem 5.6). Since \( p_A \) is a positive Itô Process, the renormalization does not change the set of admissible self-financing trading strategies ([62], Section 4.8). Consequently, the securities price process \( p_A \) is essentially dynamically complete; and the budget sets under \( p_A \) are the same as the budget sets under \( p_A/p_{A_0} \).

We now derive the formulas for the hyperfinite equilibrium trading strategies, and show that they are sufficiently regular to extract a candidate trading strategy in continuous time. By the same arguments we used to derive the formulas for \( \hat{p}_A \) and \( \hat{\sigma} \), we have

\[
\hat{z}_i(t, \omega) \cdot \hat{p}_A(t, \omega) = E \left( \Pi(c(T, \cdot), I(T, \cdot)) \left( \Psi_i(c(T, \cdot), I(T, \cdot)) - F_i(I(T, \cdot)) \right) \\
+ \int_{\tilde{t}}^{T} \pi(c(s, \cdot), I(s, \cdot)) (\psi_i(c(s, \cdot), I(s, \cdot)) - f_i(I(s, \cdot))) \, ds \right) \bigg|^{c(t, \omega)} \]

\[
\hat{C}_i(t + \Delta T, \omega) - \hat{C}_i(t, \omega)
\]

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Thus, there are standard functions \( \Sigma_1, \ldots, \Sigma_I : [0, T] \times \mathbb{R}^K \to \mathbb{R}^K \), and 
\( W_1, \ldots, W_I : [0, T] \times \mathbb{R}^K \to \mathbb{R}^K \) defined by
\[
\Sigma_i(t, \beta) = E \left( \frac{\Pi(c(T, \cdot), \mathcal{I}(T, \cdot)) (\Psi_i(c(T, \cdot), \mathcal{I}(T, \cdot)) - F_i(\mathcal{I}(T, \cdot))) (\beta(T, \omega) - \beta(0, t, \omega))}{T - t} \right. \\
\left. + \int_t^T \pi(c(s, \cdot), \mathcal{I}(s, \cdot)) (\psi_i(c(s, \cdot), \mathcal{I}(s, \cdot)) - f_i(\mathcal{I}(s, \cdot))) (\beta(s, \cdot) - \beta(0, t, \cdot)) \, ds \right| (\beta(t) = \beta)
\]
\[
W_i(t, \beta) = E \left( \Pi(c(T, \cdot), \mathcal{I}(T, \cdot)) (\Psi_i(c(T, \cdot), \mathcal{I}(T, \cdot)) - F_i(\mathcal{I}(T, \cdot))) \right. \\
\left. + \int_{s=t}^T \pi(c(s, \cdot), \mathcal{I}(s, \cdot)) (\psi_i(c(s, \cdot), \mathcal{I}(s, \cdot)) - f_i(\mathcal{I}(s, \cdot))) \, ds \right| (\beta(t) = \beta)
\]
which are continuous on \([0, T]\) and analytic on \((0, T)\). Summing up, we have
\[
\begin{align*}
\hat{p}_A(t, \omega) &\approx p_A(0, t, \omega) \\
\hat{p}_e(t, \omega) &\approx p_e(0, t, \omega) \\
\hat{\sigma}(t, \omega) &\approx \Sigma_i (\mathcal{I}(0, t, \omega)) \\
\hat{c}_i(t, \omega) &\approx c_i(0, t, \omega) \\
\hat{z}_i(t, \omega) \cdot \hat{\sigma}(t, \omega) &\approx \hat{\sigma}_i(t, \omega) \\
\hat{z}_i(t, \omega) \cdot \hat{p}_A(t, \omega) &\approx W_i (\mathcal{I}(0, t, \omega))
\end{align*}
\]
for every \((t, \omega)\) such that \(\hat{\mathcal{I}}(t, \omega)\) is finite. But for \(L(\hat{\mu})\)-almost all \(\omega, \hat{\mathcal{I}}(t, \omega)\) is \(S\)-continuous (see footnote 14), so for \(L(\hat{\mu})\)-almost all \(\omega\), we have for every
and this implies Equations (13, 15, 17, 19, 23) in the statement of Theorem 3.1.

We have already noted that \( \hat{p}_C \) is uniformly bounded, hence \( SL^2 \), and that \( \hat{p}_A \) is \( SL^2 \). From Equations (29) and (32), the Mean Value Theorem, the growth conditions (Equation (2)) and the fact that 
\[
\partial_{c_i} \left| \sigma_i(t, \omega) \right| \leq \left| \frac{\partial c_i}{\partial \beta} \right|, 
\]
imply that, regardless of whether \( \hat{I}(t, \omega) \) is finite, \( |\hat{\sigma}(t, \omega)| \leq r + e^{|\hat{\beta}(t, \omega)|} \) and \( |\hat{\sigma}_i(t, \omega)| \leq r + e^{|\beta(t, \omega)|} \), which implies by Proposition 3.2 of Raimondo [66] that \( \hat{\sigma}, \hat{\sigma}_1, \ldots, \hat{\sigma}_T \in SL^2 \). \( \hat{z}_i \cdot \hat{p}_A \) is bounded below by minus the value (at \( \hat{p}_c \)) of \( i \)'s future endowment, and above by the value of total market future consumption, hence \( \hat{z}_i \cdot \hat{p}_A(t, \omega) \leq r + e^{|\hat{\beta}(t, \omega)|+T} \), so \( \hat{z}_i \cdot \hat{p}_A \) is \( SL^2 \). \( \hat{c}_i(t, \omega) \leq r + e^{|\hat{\beta}(t, \omega)|} \), so \( \hat{c}_i \) is \( SL^2 \).

\[
\hat{p}_{A_j}(t, \omega) \leq r + e^{|\hat{\beta}(t, \omega)|+T} \quad \Rightarrow \quad p_{A_j}(\hat{I}(t, \omega)) \leq r + e^{|\hat{\beta}(t, \omega)|+T} \\
\Rightarrow \quad *p_{A_j}(\hat{I}(t, \omega)) \leq r + e^{|\hat{\beta}(t, \omega)|+T} \\
\Rightarrow \quad *p_{A_j} \circ \hat{I} \in SL^2
\]

Similarly, \( *p_c \circ \hat{I}, *\Sigma \circ \hat{I}, *c_i \circ \hat{I}, *\Sigma_i \circ \hat{I}, *W_i \circ \hat{I} \in SL^2 \). This, together with Equation (35) implies Equations (12, 14, 16, 18, 22, 24) in the statement of Theorem 3.1. We still need to establish Equations (20, 21) and show that \( p_A, p_c, c_i, z_i \) form an equilibrium of the continuous-time economy.

By Theorem B.3, for all \( t \in [0, T] \)
\[
p_A(t, \omega) = \circ \hat{p}_A(t, \omega)
\]

\[\text{In Equation (35), when we write } *p_A(\hat{I}(t, \omega)), \text{ we mean } p_A \text{ is an analytic function of } \hat{I}(t, \omega) \in [0, T] \times \mathbb{R}^K; \text{ take the nonstandard extension of this analytic function defined on } [0, T] \times \mathbb{R}^K, \text{ and evaluate it at } \hat{I}(t, \omega). \text{ The analogous definition is used for } *p_c, *\Sigma, *\Sigma_1, \ldots, *\Sigma_t, *W_1, \ldots, *W_I.\]
\[
\begin{align*}
\gamma(t, \omega) &= \gamma(0, \omega) + \int_0^t \sigma d \beta - \int_0^t \sum_{s \leq i, s < T} \hat{p}_C(s, \omega) \hat{A}(s, \omega) ds \\
&= \gamma(0, \omega) + \int_0^t \sigma d \beta - \int_0^t \sum_{s \leq i, s < T} \hat{p}_C(s, \omega) \hat{A}(s, \omega) ds
\end{align*}
\]

Given any hyperfinite trading strategy \( \hat{z}_i \) and \( t \in T \), we have

\[
\int_0^t \hat{z}_i d \hat{\gamma} = \sum_{s \in T, s < t} \hat{z}_i(s, \omega) \cdot (\hat{\gamma}(s + \Delta T, \omega) - \hat{\gamma}(s, \omega))
\]

Now, we consider the form of the equilibrium trading strategies \( \hat{z}_i \). \( \hat{\beta}(t, \omega) \) is finite and \( \det \Sigma'(t, \beta(t, \omega)) \neq 0 \) at (Loeb) almost every node; fix such a node \((t_0, \omega_0)\). Then \( \hat{\beta}(t_0 + \Delta T, \omega_0) \) is also finite. From the growth condition on dividends, it follows that \( \hat{A}(t_0 + \Delta T, \omega_0) \) is finite. We can assume without loss of generality that \( \omega_0(t + \Delta T) = 0 \); choose \( \omega_1, \ldots, \omega_K \) such that \( \omega_k(s) = \omega_0(s) \) for \( s \leq t_0 \) and \( \omega_k(t + \Delta t) = k \). Since \( \hat{z}_i \) finances \( \hat{c}_i \), we have for \( k = 0, \ldots, K \)

\[
\hat{\sigma}_i(t_0, \omega_0) \left( \hat{\beta}(t_0 + \Delta T, \omega_k) - \hat{\beta}(t_0, \omega_0) \right)
= \hat{C}_i(t_0 + \Delta T, \omega_k) - \hat{C}_i(t_0, \omega_0)
\]

Since \( \sum_{k=0}^K v_k = 0 \) and \( \sum_{k=0}^K \hat{C}_i(t_0 + \Delta T, \omega_k) = (K + 1) \hat{C}_i(t_0, \omega_0) \), the preceding equation for \( k = 0 \) must be satisfied whenever the equations for \( k = 1, \ldots, K \) are satisfied. Since \( \hat{z}_i(t_0, \omega_0) \) must finance the future consumption, it satisfies
the following form of the budget constraint:

\[
\hat{p}_A(t_0, \omega_0) \cdot \hat{z}_i(t_0, \omega_0) = E \left( \hat{\rho}_C(T, \cdot) (\hat{e}_i(T, \cdot) - \hat{\rho}_i(T, \cdot)) + \Delta T \sum_{s=t+\Delta T} \hat{\rho}_C(s, \cdot) (\hat{e}_i(s, \cdot) - \hat{\rho}_i(s, \cdot)) \right) \bigg| (t_0, \omega_0)
\]

\[
\simeq W_i (\mathcal{I}(\omega t_0, \omega_0))
\]

The matrix

\[
(\hat{\sigma}(t_0, \omega_0))(v_1 \cdots v_K) \simeq \Sigma (\mathcal{I}(\omega t_0, \omega_0))(v_1 \cdots v_K)
\]

is \((K+1) \times K\) and has rank \(K\) since \(\Sigma (\mathcal{I}(\omega t, \omega))\) has rank \(K\) by Equation (33) and \(v_1, \ldots, v_K\) are linearly independent. When we add \(\hat{p}_A(t_0, \omega_0)\) as the \((K+1)^{st}\) column, we obtain the \((K+1) \times (K+1)\) matrix

\[
((\hat{\sigma}(t_0, \omega_0))(v_1 \cdots v_K) | \hat{p}_A(t_0, \omega_0)) \simeq (\Sigma (\mathcal{I}(\omega t_0, \omega_0))(v_1 \cdots v_K) | p_A(\omega t_0, \omega_0))
\]

If \(p_A(\omega t_0, \omega_0)\) lies in the span of the first \(K\) columns, then there is a vector \(x \in \mathbb{R}^K\) such that \(\Sigma (\mathcal{I}(\omega t_0, \omega_0)) x = p_A(\omega t_0, \omega_0)\), so (since a change in the price vector parallel to the price vector leaves normalized prices unchanged) \(\Sigma (\mathcal{I}(\omega t_0, \omega_0)) x = 0\), contradiction. Accordingly, the matrix

\[
(\Sigma (\mathcal{I}(\omega t_0, \omega_0))(v_1 \cdots v_K) | p_A(\omega t_0, \omega_0))
\]

has rank \(K+1\), and so has a unique \((K+1) \times (K+1)\) inverse matrix \(M (\mathcal{I}(\omega t_0, \omega_0))\) whose coefficients are given by a standard analytic function of \(\mathcal{I}(\omega t_0, \omega_0)\). Therefore, the matrix

\[
((\hat{\sigma}(t_0, \omega_0))(v_1 \cdots v_K) | \hat{p}_A(t_0, \omega_0))
\]

is also invertible, and its inverse \(\hat{M}(t_0, \omega_0)\) satisfies

\[
\hat{M}(t_0, \omega_0) \simeq M (\mathcal{I}(\omega t_0, \omega_0))
\]

so

\[
\hat{z}_i(t_0, \omega_0)
\]
\[
\begin{align*}
\mathbf{v}^T &= \\
&= 
\begin{pmatrix}
\sqrt{\Delta T} \left( \hat{C}_i(t_0 + \Delta T, \omega_1) - \hat{C}_i(t_0, \omega_0) \right) \\
\vdots \\
\sqrt{\Delta T} \left( \hat{C}_i(t_0 + \Delta T, \omega_K) - \hat{C}_i(t_0, \omega_0) \right) \\
\hat{p}_A(t_0, \omega_0) \cdot \hat{z}_i(t_0, \omega)
\end{pmatrix}
\begin{pmatrix}
\hat{M}(t_0, \omega_0) + O((\Delta T)^{3/2})
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{r}^T &= \\
&= 
\begin{pmatrix}
\sqrt{\Delta T} \left( \hat{\sigma}_i(t_0, \omega_0) \left( \hat{\beta}(t_0 + \Delta T, \omega_1) - \hat{\beta}(t_0, \omega_0) \right) \right) \\
\vdots \\
\sqrt{\Delta T} \left( \hat{\sigma}_i(t_0, \omega_0) \left( \hat{\beta}(t_0 + \Delta T, \omega_K) - \hat{\beta}(t_0, \omega_0) \right) \right) \\
\hat{p}_A(t_0, \omega_0) \cdot \hat{z}_i(t_0, \omega)
\end{pmatrix}
\begin{pmatrix}
\hat{M}(t_0, \omega_0)
\end{pmatrix}
\end{align*}
\]

\[
\mathbf{v}^T = \\
= 
\begin{pmatrix}
\hat{\sigma}_i(t_0, \omega_0) v_1 \\
\vdots \\
\hat{\sigma}_i(t_0, \omega_0) v_K \\
\hat{p}_A(t_0, \omega_0) \cdot \hat{z}_i(t_0, \omega)
\end{pmatrix}
\begin{pmatrix}
\hat{M}(\mathcal{I}(\hat{t}_0, \omega))
\end{pmatrix}
\]

so there are standard analytic functions \(Z_1, \ldots, Z_I : \left( [0, T] \times \mathbb{R}^K \right) \setminus B \to \mathbb{R}^{K+1}\) such that

\[
\hat{z}_i(t_0, \omega) \simeq Z_i(\mathcal{I}(\hat{t}_0, \omega)) \simeq *Z_i(\hat{I}(t_0, \omega))
\]

whenever

\[
\mathcal{I}(\hat{t}_0, \omega_0) \in \left( [0, T] \times \mathbb{R}^K \right) \setminus B
\]

which proves Equation (21) in the statement of Theorem 3.1. Define \(z_i(t, \omega) = \Diamond \hat{z}_i(t, \omega)\); \(\hat{z}_i \hat{\sigma}\) is an SL\(^2\) lifting of \(z_i \sigma\), which belongs to \(\mathcal{H}^2\).

Now, we show show that the prices \(p_A, p_C\), consumptions \(c_1, \ldots, c_I\) and trading strategies \(z_1, \ldots, z_I\) form an equilibrium for the Loeb continuous time economy generated by the hyperfinite model and induce an equilibrium for the original continuous-time model. For all \(t \in [0, T)\),

\[
p_A(t, \omega) \cdot z_i(t, \omega) - p_A(0, \omega) \cdot c_i \omega) \\
- \int_0^t z_i d\gamma - \int_0^t p_C(s, \omega) (c_i(s, \omega) - c_i(s, \omega)) ds
\]

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\[ p_A(t, \omega) \cdot z_i(t, \omega) - p_A(0, \omega) \cdot e_iA(\omega) \]
\[ - \int_0^t z_i \sigma \, d\beta - \int_0^t p_C(s, \omega) (e_i(s, \omega) - c_i(s, \omega)) \, ds \]
\[ \simeq \hat{p}_A(\hat{t}, \omega) \cdot \hat{z}_i(\hat{t}, \omega) - \hat{p}_A(0, \omega) \cdot \hat{e}_iA(\omega) \]
\[ - \int_0^\hat{t} \hat{z}_i \hat{\sigma} \, d\hat{\beta} - \int_0^\hat{t} \hat{p}_C(s, \omega) (\hat{e}_i(s, \omega) - \hat{c}_i(s, \omega)) \, ds \]
\[ = \hat{p}_A(\hat{t}, \omega) \cdot \hat{z}_i(\hat{t}, \omega) - \hat{p}_A(0, \omega) \cdot \hat{e}_iA(\omega) \]
\[ - \int_0^\hat{t} \hat{z}_i \hat{d}\hat{\gamma} - \int_0^\hat{t} \hat{p}_C(s, \omega) (\hat{e}_i(s, \omega) - \hat{c}_i(s, \omega)) \, ds \]
\[ = 0 \]

since the trading strategy \( z_i \) finances the consumption \( c_i \). Thus, the trading strategy \( z_i \) finances the consumption \( c_i \).

Suppose that \( c'_i \) is any consumption in the budget set of agent \( i \); we need to show that \( U_i(c'_i) \leq U_i(c_i) \). Since \( c'_i \) is in the budget set, there is an admissible self-financing trading strategy \( z'_i \) which finances it. Since we do not require that \( z'_i \) be adapted to the Brownian filtration, only to the larger Loeb filtration, \( c'_i \) is not necessarily adapted to the Brownian filtration. Since \( z'_i \) is admissible, the capital gain process it generates is a martingale; since in addition the growth condition on endowments implies that \( e_i \) is integrable, \( p_c c'_i \) is integrable. Let \( \bar{c}_i \) be the unique process so that \( p_c \bar{c}_i \) equals the conditional expectation of \( p_c c'_i \) with respect to the Brownian filtration; since \( p_c \) is adapted to the Brownian filtration, so is \( \bar{c}_i \). Since the securities price process is effectively dynamically complete, there is an admissible self-financing trading strategy \( \bar{z}_i \) which finances \( \bar{c}_i \). Since the capital gains process of both \( z'_i \) and \( \bar{z}_i \) are martingales,

\[ p_A(0) \cdot \bar{z}_i(0) = p_A(0) \cdot z'_i(0) = p_A(0) \cdot e_iA \]

Thus, \( \bar{c}_i \) lies in the budget set of agent \( i \). \( p_c c'_i \) is a mean-preserving spread of \( p_c \bar{c}_i \). Since the endowments and utility functions are measurable with respect to the Brownian filtration, \( U_i(c'_i) \geq U_i(c'_i) \). So without loss of generality, we may assume that \( c'_i \) is adapted to the Brownian filtration.

\( c'_i \) need not be uniformly bounded away from zero. However, since the securities price process \( p_A \) is essentially dynamically complete, and the total gains process of an admissible trading strategy is a martingale, a consumption process \( \bar{c}_i \) which is adapted to the Brownian filtration lies in the budget set.
if and only if

\[
E \left( p_C(T, \cdot) \cdot \bar{c}_i(T, \cdot) + \int_0^T p_C(s, \cdot) \cdot \bar{c}_i(s, \cdot) \right) \\
\leq p_A(0) \cdot e_{iA} + E \left( p_C(T, \cdot) \cdot e_i(T, \cdot) + \int_0^T p_C(s, \cdot) \cdot e_i(s, \cdot) \right)
\]

Since \( p_C \) is uniformly bounded above, and \( U_i \) satisfies an Inada condition, we can find \( \bar{c}_i \) which lies in the budget set and is uniformly bounded away from zero, such that \( U_i(\bar{c}_i) \geq U_i(c'_i) \). Thus, without loss of generality, we may assume that \( c'_i \) is uniformly bounded away from zero.

Find \( \hat{c}'_i \) such that

\[
\circ \hat{c}'_i(t, \omega) = c'_i(\circ t, \omega)
\]

almost everywhere, \( \hat{c}'_i \) is uniformly bounded away from zero by a non-infinitesimal amount, and \( \hat{p}_C \hat{c}'_i \in SL^1(T \times \Omega) \). Then

\[
E \left( \hat{p}_c(\hat{T}, \cdot)\hat{c}'_i(T, \cdot) + \Delta T \sum_{t=0}^{T-\Delta T} \hat{p}_c(s, \cdot)\hat{c}'_i(s, \cdot) \right) \\
\cong E \left( p_c(T, \cdot)\hat{c}'_i(T, \cdot) + \int_0^T p_c(s, \cdot)\hat{c}'_i(s, \cdot) \, ds \right) \\
\leq p_A(0) \cdot e_{iA} + E \left( p_c(T, \cdot)e_i(T, \cdot) + \int_0^T p_c(s, \cdot)e_i(s, \cdot) \, ds \right) \\
\cong \hat{p}_A(0) \cdot \hat{e}_{iA} + E \left( \hat{p}_c(\hat{T}, \cdot)\hat{e}_i(T, \cdot) + \Delta T \sum_{s=0}^{\hat{T}-\Delta T} \hat{p}_c(s, \cdot)\hat{e}_i(s, \cdot) \right)
\]

so by subtracting a constant infinitesimal from \( \hat{c}'_i \) we can assume without loss of generality that

\[
E \left( \hat{p}_c(\hat{T}, \cdot)\hat{c}'_i(T, \cdot) + \Delta T \sum_{t=0}^{T-\Delta T} \hat{p}_c(s, \cdot)\hat{c}'_i(s, \cdot) \right) \\
= \hat{p}_A(0) \cdot \hat{e}_{iA} + E \left( \hat{p}_c(\hat{T}, \cdot)\hat{e}_i(T, \cdot) + \Delta T \sum_{s=0}^{\hat{T}-\Delta T} \hat{p}_c(s, \cdot)\hat{e}_i(s, \cdot) \right)
\]

Since the hyperfinite securities price process \( \hat{p}_A \) is internally dynamically complete, there is a unique trading strategy \( \hat{z}'_i \) that finances \( \hat{c}'_i \), so \( \hat{c}'_i \) lies in
the budget set of agent $i$, so

$$\hat{U}_i(\hat{c}_i') \leq \hat{U}_i(\hat{c}_i)$$

Since $\hat{c}_i'(t, \omega) \simeq c_i'(t, \omega)$ almost everywhere, and consumption is bounded away from zero by a noninfinitesimal, utility at each node is bounded below, so

$$\circ \hat{U}_i(\hat{c}_i') \geq U_i(c_i)$$

Since $\hat{c}_i$ is a feasible allocation in the hyperfinite economy, the growth condition on endowments and dividends implies that $\hat{c}_i$ is $SL^1$, which implies that $c_i$ is $L^1$; since consumption is bounded away from zero, marginal utility is bounded, and so $\hat{U}_i \circ \hat{c}_i$ is $SL^1$, and $\hat{U}_i \circ c_i$ is $L^1$, so

$$\hat{U}_i(\hat{c}_i) \simeq U_i(c_i)$$

which proves Equation (20) in the statement of Theorem 3.1. Therefore,

$$U_i(c_i') \leq \circ \hat{U}_i(\hat{c}_i') \leq \circ \hat{U}_i(\hat{c}_i) = U_i(c_i)$$

which shows that $c_i$ is in agent $i$'s demand set. This completes the proof that the prices $p_c$ and $p_A$, trading strategies $z_1, \ldots, z_I$, and consumptions $c_1, \ldots, c_I$ form an equilibrium of the Loeb continuous time economy.

Now, consider the original continuous-time economy specified in the statement of Theorem 2.1. Since the equilibrium prices, consumptions and (except for a set of measure zero) trading strategies of the Loeb economy are given by analytic, hence continuous, functions of $I(t, \omega)$ and $\beta(T, \omega)$, we can use these functions to define prices, consumptions and trading strategies for the original continuous-time economy. The consumptions lie in the budget sets of the agents. Since we do not require the filtration $\{F_t\}$ to be the Brownian filtration, the budget set may contain consumptions which are not adapted to the Brownian filtration. However, we may without loss of generality restrict attention to elements of the budget set which are adapted to the Brownian filtration; the argument is exactly the same as the argument for the same fact in the Loeb continuous-time economy. Consequently, the consumptions induced in the original economy lie in the demand sets, and thus the induced prices, trading strategies, and consumptions form an equilibrium of the original continuous-time economy. This completes the proof of Theorem 2.1 and Theorem 3.1.
A Real Analytic Functions of Several Variables

In this Appendix, we summarize the results on real analytic functions of several variables used in our proof.

Definition A.1 Let $D \subset \mathbb{R}^n$. A function $F: D \to \mathbb{R}^m$ is real analytic if there is an open set $U$ with $D \subset U \subset \mathbb{R}^n$ and an extension $\tilde{F}$ of $F$ to $U$ such that, for every $x_0 \in U$, there is a power series $G_{x_0}(x)$ centered at $x_0$ with a positive radius of convergence $\delta_{x_0}$ such that $\tilde{F}(x) = G_{x_0}(x)$ whenever $x \in U$ and $|x - x_0| < \delta_{x_0}$.

Theorem A.2 (The Analytic Implicit Function Theorem) Suppose $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ are open, $F: X \times Y \to \mathbb{R}^n$ is real analytic, $x_0 \in X, y_0 \in Y, F(x_0, y_0) = 0$, and
\[
\det \frac{\partial F}{\partial x} \neq 0
\]
at $(x_0, y_0)$. Then there are neighborhoods $U$ of $x_0$ and $V$ of $y_0$ and a real analytic function $\phi: V \to U$ such that for all $y \in V$,
\[
F(\phi(y), y) = 0
\]
Moreover, if $x \in U, y \in V$, and $F(x, y) = 0$, then $x = \phi(y)$.

Proof: The conventional statement of the Implicit Function Theorem establishes all the claims except the claim that the implicit function $\phi$ is real analytic. For this, see Theorem 2.3.5 of Krantz and Parks [49].

Theorem A.3 Let $U \subset \mathbb{R}^n$ be convex, $F: U \to \mathbb{R}$ real analytic. If $\{x \in U : F(x) = 0\}$ has positive Lebesgue measure, then $F$ is identically zero on $U$.

Proof: We may assume without loss of generality that $U$ is open. Łojasiewicz’s Structure Theorem for Varieties (Theorem 6.3.3 of Krantz and Parks [49]) states that if $U$ is an open set in $\mathbb{R}^n$ and $F: U \to \mathbb{R}$ is analytic, then for every $x_0 \in U$, there exists a neighborhood $V_{x_0}$ of $x_0$ such that either $F(x) = 0$ for all $x \in V_{x_0}$ or $\{x \in V_{x_0} : F(x) = 0\}$ is a finite union of algebraic varieties of dimension $< n$. If $F(x) = 0$ for all $x \in V_{x_0}$ and $y \in U$, there is a ray that passes through $V_{x_0}$ and through $y$; the restriction of $F$ to the ray is
an analytic function of a single variable, and it vanishes on an interval (the intersection of the ray with the set $V_0$); since it is well known that an analytic function of one variable that vanishes on an interval is identically zero, we must have $F(y) = 0$ for all $y \in U$ and we are done. On the other hand, if $\{x \in V : F(x) = 0\}$ is a finite union of algebraic varieties of dimension $< n$, $\{x \in V_{x_0} : F(x) = 0\}$ has Lebesgue measure zero. There is a countable collection $\{x_n : n \in \mathbb{N}\}$ such that $\bigcup_{n \in \mathbb{N}} V_{x_n} \supset U$, so $\{x \in U : F(x) = 0\}$ has Lebesgue measure zero.

Finally, we show that the conditional expectation of a function of $G(\beta(T))$, conditional on $(t, \beta(t))$ is, under mild hypotheses on $G$, an analytic function of $(t, \beta(t))$. This fact is widely known among probabilists, but we were unable to find a specific reference that shows that the analyticity is joint in $(t, \beta(t))$, and we need the joint analyticity. Therefore, we provide a proof.

**Theorem A.4** Suppose $F$ is measurable on $\mathbb{R}^K$ and there exists $r \in \mathbb{R}$ such that

$$|F(x)| \leq r + e^{r|x|}$$

Let $\beta$ be a standard $K$-dimensional Brownian Motion, and let

$$G(t, \beta) = E(F(\beta(T))|\beta(t) = \beta)$$

Then $G(t, \beta)$ is an analytic function of $(t, \beta) \in (0, T) \times \mathbb{R}^K$.

**Proof:** Fix $t < T$. Then

$$G(t, \beta) = \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} F(\beta + x)e^{-|x|^2/(2(T-t))} \, dx$$

$$= \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} F(y)e^{-|y-\beta|^2/(2(T-t))} \, dy$$

$$= \frac{e^{-|\beta|^2/(2(T-t))}}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} e^{\beta \cdot y/(T-t)}F(y)e^{-|y|^2/(2(T-t))} \, dy$$

$$= \frac{e^{-|\beta|^2/(2(T-t))}}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} \sum_{k=0}^{\infty} \left( \frac{\beta \cdot y}{(T-t)} \right)^k e^{-|y|^2/(2(T-t))} \, dy$$

$$= \frac{e^{-|\beta|^2/(2(T-t))}}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} \sum_{k_1+\ldots+k_K=k} \frac{1}{(T-t)^{k_1}k_1! \ldots (T-t)^{k_K}k_K!} \sum_{k_1+\ldots+k_K=k}$$

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provided we can apply Fubini’s Theorem; we will justify this in a moment.

\[
\begin{align*}
\int_{\mathbb{R}^K} \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} F(y) e^{-|y|^2/2(T-t)} dy \\
\leq \frac{2^K}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} \frac{(r + e^{|y|}) |y|^{k_1} \cdots y^{k_K}}{k_1! \cdots k_K!} e^{-|y|^2/2(T-t)} dy \\
= \frac{2^K r}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} \frac{y^{k_1} \cdots y^{k_K}}{k_1! \cdots k_K!} e^{-|y|^2/2(T-t)} dy \\
+ \frac{2^K}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}_+^K} \frac{y^{k_1} \cdots y^{k_K}}{k_1! \cdots k_K!} e^{-|y|^2/2(T-t)} dy \\
\leq \frac{2^K}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} \frac{(4(T-t))^k e^{4r(T-t)}}{k_1! \cdots k_K!} e^{-|y|^2/2(T-t)} dy \\
+ \frac{2^K}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}_+^K} \frac{y^{k_1} \cdots y^{k_K}}{k_1! \cdots k_K!} e^{-|y|^2/4(T-t)} dy \\
\leq \frac{r(T-t)^{k/2}}{\sqrt{k_1! \cdots k_K!}} + \frac{(4(T-t))^k e^{4r(T-t)}}{k_1! \cdots k_K!} + \frac{r(2(T-t))^{k/2}}{\sqrt{k_1! \cdots k_K!}}
\end{align*}
\]

\[\text{(36)}\]
which shows that the power series for $G$ given in Equation (36) has infinite radius of convergence and justifies the application of Fubini’s Theorem. Therefore, for fixed $t < T$, $G(t, \beta)$ is an analytic function of $\beta \in \mathbb{R}^K$. For any $s \in (0, t)$, we have

$$G(s, \beta) = \frac{1}{(2\pi(t-s))^{K/2}} \int_{\mathbb{R}^K} G(t, \beta + x)e^{-|x|^2/2(t-s)} dx$$

The right side is an integral with respect to $x$ of an analytic function of $(s, \beta, x)$, and hence $G$ is analytic on $(0, t) \times \mathbb{R}^K$. Since this is true for every $t < T$, $G$ is analytic on $(0, T) \times \mathbb{R}^K$.

**B Nonstandard Stochastic Integration**

In order to show that the equilibrium of the hyperfinite economy generates an equilibrium of the standard continuous-time economy, one needs to show that capital gains are the same in the two settings. Capital gains are given by Stieltjes Integrals with respect to securities prices in the hyperfinite setting and by Itô Integrals with respect to securities prices in the continuous-time setting.

Anderson [2] showed that the Itô Integral with respect to Brownian Motion is the standard part of a Stieltjes Integral with respect to a hyperfinite random walk. Anderson’s theorem covers hyperfinite random walks which move independently in each component by an amount $\pm 1/\sqrt{n}$. In that random walk, each node in the tree has $2^K$ successor nodes. As discussed above, in order to obtain dynamic completeness in the hyperfinite model, we need to use a random walk in which each node has $K + 1$ successor nodes. Thus, Anderson’s theorem does not cover the case considered here.

Lindstrom [50, 51, 52, 53] showed that the stochastic integral with respect to a square integrable martingale is the standard part of a Stieltjes Integral with respect to a hyperfinite $SL^2$ martingale. Lindstrom’s theorem is limited to one-dimensional martingales. Because the components of a vector Brownian Motion are uncorrelated, a process is Itô Integrable with respect to a vector Brownian Motion if and only if it is integrable with respect to each component. However, the components of a vector martingale can be correlated and consequently, a process can be integrable with respect to a vector
martingale even if it is not integrable with respect to the individual components. This fact has economic significance. If two components of the vector martingale are instantaneously nearly perfectly correlated at some point, then the equilibrium trading strategy may well require taking a nonstandard infinite long position in one security and a nonstandard infinite short position in the other. In both the hyperfinite and continuous-time model, the capital gain is well-defined and finite when computed with respect to the vector martingale. However, the hyperfinite capital gain may be a positive nonstandard infinite number in one component and a negative nonstandard infinite number in the other components; they add up to a well-defined finite integral when both components are considered. The continuous-time capital gain may be undefined with respect to the two components when considered separately, but well-defined and finite when the integral is computed with respect to the vector martingale.

Thus, we need to extend either Anderson’s theorem or Lindstrom’s theorem. The more general approach would be to extend Lindstrom’s theorem to vector martingales; such an extension is probably needed to tackle the dynamically incomplete markets case. However, it is considerably easier, and sufficient for our purposes in this paper, to extend Anderson’s theorem to the particular kind of random walk considered here. This is the approach we follow.

For definitions of standard terms in stochastic integration (such as $H^2$ and $L^2$), see Nielsen [62]. For definitions of nonstandard terms such as $SL^2$ lifting, see Anderson [2].

**Theorem B.1** Let $\hat{\beta}$ be the hyperfinite random walk defined above, and $\beta = \circ \hat{\beta}$ the standard Brownian Motion it generates. Suppose $Z \in H^2$. Then there is an $SL^2$ lifting $\hat{Z}$ of $Z$. Given any $SL^2$ lifting $\hat{Z}$ of $Z$, for every $t \in T$ we have

$$\circ \int_0^t \hat{Z}d\hat{\beta} = \int_0^t Zd\beta$$

**Proof:** Lemma 31 in Anderson [2] proves the existence of an $SL^2$ lifting; the hyperfinite probability space is slightly different from the one considered here, but the proof goes through without change.

Theorem 33 in Anderson [2] shows that, with respect to the hyperfinite random walk and Brownian Motion considered there, the Itô Integral is the standard part of the hyperfinite Stieltjes Integral. The proof of Theorem
33 depends on the specific form of the hyperfinite random walk only in establishing the Itô Isometry, so we show that the Itô Isometry holds for the random walk \( \hat{\beta} \). \( \hat{Z} \) may be a \( 1 \times 1 \) scalar process, \( 1 \times K \) vector process, or a \( (J + 1) \times K \) matrix process. The proofs in the vector and matrix cases are virtually identical apart from notation, while the proof in the scalar case is easier, so we assume that \( \hat{Z} \) is a \( 1 \times K \) vector process with \( k^{th} \) component \( \hat{Z}_k \).

\[
\left\| \int_0^t \hat{Z} \, d\hat{\beta} \right\|_2^2 = \left\| \int_0^t \hat{Z}_k \, d\hat{\beta}_k \right\|_2^2 = \left\| \sum_{k=1}^K \sum_{s \in T, s < t} \hat{Z}_k(s, \cdot) (v_{s+\Delta T})_k \sqrt{\Delta T} \right\|_2^2 = \sum_{k=1}^K \sum_{s \in T, s < t} \left\| \hat{Z}_k(s, \cdot) (v_{s+\Delta T})_k \sqrt{\Delta T} \right\|_2^2
\]

(because the terms \( \hat{Z}_k(s, \omega)v_{s+\Delta T} \) are uncorrelated across \( s, k \))

\[
= \sum_{k=1}^K \sum_{s \in T, s < t} \Delta T \left\| \hat{Z}_k(s, \cdot) \right\|_2^2 = \sum_{k=1}^K \left\| \hat{Z}_k \{ s \in T : s < t \} \right\|_2^2 = \left\| \hat{Z} \{ s \in T : s < t \} \right\|_2^2
\]
which establishes the Itô Isometry.

Since we have already shown that \( \beta \) is a standard Brownian Motion, the rest of the proof goes through unchanged.

**Definition B.2** Suppose \( Z \in \mathcal{L}^2 \). An \( \mathcal{S}\mathcal{L}^2 \) lifting of \( Z \) is an internal nonanticipating process \( \hat{Z} \) such that \( \hat{Z}(\cdot, \omega) \in SL^2(\mathcal{T} \setminus \{ \hat{T} \}) \) for almost all \( \omega \) and such that \( ^\circ \hat{Z}(t, \omega) = Z(\circ t, \omega) \) for almost all \( (t, \omega) \in \left( \mathcal{T} \setminus \{ \hat{T} \} \right) \times \Omega \).

**Theorem B.3** Let \( \hat{\beta} \) be the hyperfinite random walk defined above, and \( \beta = \circ \hat{\beta} \) the standard Brownian Motion it generates. Suppose \( Z \in \mathcal{L}^2 \). Then there
exists an $SL^2$ lifting of $Z$. If $\hat{Z}$ is any $SL^2$ lifting of $Z$, then for every $t \in T$ we have

$$\circ \int_0^t \hat{Z} d\beta = \int_0^t Z d\beta$$

**Proof:** Let $f(\omega) = \|Z(\cdot, \omega)\|_2$ and find $\bar{f}$ internal such that $\circ \bar{f}(\omega) = f(\omega)$ $L(\hat{\mu})$-almost surely. Let $\bar{Z}$ be an internal nonanticipating process such that $\circ \bar{Z}(t, \omega) = Z(\circ t, \omega)$ $L(\hat{\mu}) \times L(\hat{\nu})$-almost everywhere, so for $L(\hat{\mu})$-almost all $\omega$, $\circ \bar{Z}(\cdot, \omega) = Z(\circ t, \omega)$ for $\nu$-almost all $t \in T \setminus \{\hat{T}\}$. For $m \in \mathbb{N}$, let

$$(\bar{Z}_m)_{ij}(t, \omega) = \max \{ -m, \min \{ m, \bar{Z}_{ij}(t, \omega) \} \}$$

$$\bar{f}_m(\omega) = \|\bar{Z}_m(\cdot, \omega)\|_2$$

For all $\omega$, we have $\bar{f}_{m+1}(\omega) \geq \bar{f}_m(\omega)$. For $L(\hat{\mu})$ almost all $\omega$, we have $\lim_{m \in \mathbb{N}, n \to \infty} \circ \bar{f}_m(\omega) = f(\omega) < \infty$. Therefore, there exists $m_0 \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $m \geq m_0$

$$\hat{\mu} \left( \left\{ \omega : |\bar{f}_m(\omega) - \bar{f}(\omega)| < \frac{1}{m} \right\} \right) > 1 - \frac{1}{m}$$

so we may find $m \in \mathbb{N} \setminus \mathbb{N}$ such that

$$\hat{\mu} \left( \left\{ \omega : |\bar{f}_m(\omega) - \bar{f}(\omega)| < \frac{1}{m} \right\} \right) > 1 - \frac{1}{m}$$

so $\circ \bar{f}_m(\omega) = \bar{f}(\omega)$ $L(\hat{\mu})$-almost surely; for any such $\omega$, $\bar{Z}_m(\cdot, \omega) \in SL^2$ (Anderson [2], Theorem 11), so if we define $\hat{Z} = Z_m$, $\hat{Z}$ is an $SL^2$ lifting of $Z$.

Now, suppose $\hat{Z}$ is any $SL^2$ lifting of $Z$. For $m \in \mathbb{N}$, define the internal stopping time

$$\hat{\tau}_m(\omega) = \max \left\{ t \in T : \int_0^t \|\hat{Z}(s, \omega)\|^2_2 d\hat{\nu} \leq m \right\}$$

and define

$$\hat{Z}_m(t, \omega) = \left\{ \begin{array}{ll} \hat{Z}(t, \omega) & \text{if } t \leq \tau_m(\omega) \\ 0 & \text{if } t > \tau_m(\omega) \end{array} \right.$$ 

Let $Z_m(t, \omega) = \circ \hat{Z}_m(t, \omega)$. For $L(\hat{\mu})$-almost all $\omega$, there exists $m(\omega)$ such that $\tau_m(\omega) = \hat{T}$, in which case $\hat{Z}_m(\cdot, \omega) = \hat{Z}(\cdot, \omega)$ and therefore

$$\int_0^t \hat{Z}_m \circ d\beta = \int_0^t \hat{Z}(\cdot, \omega) \circ d\beta$$

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By the definition of the standard stochastic integral (see, for example, page 96 of Steele [67]) and Theorem B.1,

\[
\int_0^t Z \, d\beta = \lim_{m \to \infty} \int_0^t Z_m \, d\beta = \lim_{m \to \infty} \int_0^t \hat{Z}_m \, d\hat{\beta} = \int_0^t \hat{Z} \, d\hat{\beta}
\]

References


