A New Class of Index Numbers for International Price comparisons

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Abstract

In this paper, we introduce a new class of index numbers for international price comparisons. We prove the existence and uniqueness of the new price index. We then propose a stochastic approach to the Ikle (1972) and the new system of index numbers. The advantage of the stochastic approach is that we can derive standard errors for the estimates of the purchasing power parities (PPPs). The PPPs and the parameters of the stochastic model are estimated using a weighted maximum likelihood procedure. Finally we estimate PPPs and their standard errors for OECD countries using the proposed methods.

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1. Introduction

There is enormous interest for reliable comparisons of real incomes between countries. In order to make incomes comparable across we need to deflate them with appropriate price indices. For many reasons exchange rates are not considered appropriate. Instead a system of spatial price index numbers for different countries usually referred to as purchasing power parities (PPPs) of currencies of different countries is used. A range of methods have been proposed in the literature to compute purchasing power parities by different authors. Some of the more popular ones are Geary-Khamis (Khamis 1970), Ikle (1972), CPD (Rao 1990, 2004, 2005; Diewert, 2005), EKS (see e.g. Rao 2004). Balk (1996) has compared more than 10 different methods for calculation of PPPs. Alternative methods can be compared on the basis of at least three criteria: (i) how many of the test properties they satisfy (see Diewert 1986 or Balk 1996); (ii) whether they are superlative (i.e. there is a theoretical micro-foundation for them) or not; and (iii) whether we can associate the index to a stochastic model which makes it possible to calculate standard errors from it or not. In this paper, we propose a new class of index numbers which has the advantage that it can be easily incorporated into a stochastic model. We also show that an existing and well-known multilateral price index number system introduced by Ikle (1972) can be derived from a stochastic modeling approach.

This paper is organized as follows: In Section 2 we introduce a new method for computing of purchasing power parities and we show its relevance to Rao (1990) and Ikle (1972) methods. We prove the existence and uniqueness of the new price index in Section 3. In Section 4 we introduce a stochastic model incorporating the new system and we provide a maximum likelihood approach to estimate the model. In Section 4 we do the same for Ikle index. The advantage of the stochastic approach is that we can derive standard errors for the estimates of the purchasing power parities (PPPs). Finally we estimate PPPs and their standard errors for OECD countries using the proposed method.
2- Definitions

Let $p_{ij}$ and $q_{ij}$ represent the price and the quantity of the jth commodity in the ith country respectively where $j=1,...,M$ indexes the countries and $i=1,...,N$ indexes the commodities. We assume that all the prices are strictly positive and all the quantities are non-negative with the minimum condition that for each $i$ $q_{ij}$ is strictly positive for at least one $j$; and for each $j$ $q_{ij}$ is strictly positive for at least one $i$. Also define $PPP_j$ as purchasing power parity or the general price level in $j$-th country relative to a numeraire country and $P_i$ as the world average price for the ith commodity. We also need the following systems of weights $w_{ij}$ and $w_{ij}^*$ in defining different systems of index numbers. These weights are defined as

$$w_{ij} = \frac{p_{ij}q_{ij}}{\sum_{i=1}^{N} p_{ij}q_{ij}} \quad \text{and} \quad w_{ij}^* = \frac{w_{ij}}{\sum_{j=1}^{M} w_{ij}}$$

It is evident that $\sum_{i=1}^{N} w_{ij} = 1$ and $\sum_{j=1}^{M} w_{ij}^* = 1$.

With the above notations, Rao (1990) defines a system for international price comparisons as follows

$$PPP_j = \prod_{i=1}^{N} \left( \frac{P_{ij}}{P_i} \right)^{w_{ij}}$$

$$P_i = \prod_{j=1}^{M} \left( \frac{P_{ij}}{PPP_j} \right)^{w_{ij}^*}$$

(2)

Following Balk (1996) another system proposed by Ikle (1972) can be written as :

$$\frac{1}{PPP_j} = \sum_{i=1}^{N} \left( \frac{p_{ij}w_{ij}}{P_i} \right)$$
\[
\frac{1}{P_i} = \sum_{j=1}^{M} \left( \frac{PPP_j}{P_{ij}} w_{ij}^* \right) \tag{3}
\]

Note that in Rao system, PPPs and world prices are defined as geometric means (Jevons type of price index) of some appropriate prices while in Ikle system harmonic means of the same prices have been used in a similar manner. Here, we propose a similar system of equations but using arithmetic means (Carli type of price index) as follows:

\[
PPP_j = \sum_{i=1}^{N} \left( \frac{p_{ij}}{P_i} w_{ij} \right) \\
P_i = \sum_{j=1}^{M} \frac{p_{ij}}{PPP_j} w_{ij}^* \tag{4}
\]

### 3- Existence and Uniqueness of the new Index

For both Rao and Ikle cases it has been shown that there are unique positive solutions for \( \mathbf{P} = (P_1, P_2, \ldots, P_N) \) and \( \mathbf{PPP} = (PPP_1, PPP_2, \ldots, PPP_M) \) in their systems (see Rao 1990 and Balk 1996). Following the same tradition we prove the existence and uniqueness of the new system. To do that, we use the following theorem from Nikaido (1968, page 170).

**Theorem (1):** Let

(i) functions \( G_i(x_1, x_2, \ldots, x_n) \) for \( i = 1, \ldots, n \) be homogenous of degree one;

(ii) \( G_i(.) \)s are defined for non-negative values of the arguments and are continuous except possibly at the origin;

(iii) for each k, \( x_k = y_k \) and \( x \geq y \) \( (i \neq k) \Rightarrow G_k(x_1, x_2, \ldots, x_n) \leq G_k(y_1, y_2, \ldots, y_n) \);

and

(iv) \( G_i(u_1, u_2, \ldots, u_n) > 0 \) \( (i = 1, \ldots, n) \) for some \( u_i \geq 0 \)

then the system of equations

\[
G_i(x_1, x_2, \ldots, x_n) = a_i \quad (i = 1, \ldots, n)
\]
has a unique positive solution if \( a_i > 0 \) \((i = 1, ..., n)\).

Before presenting the main theorem concerning the existence and uniqueness of the proposed index we can easily see that if \((P^*, PPP^*)\) is a solution to the system then 
\[
\left( \frac{P^*}{\lambda}, \lambda PPP^* \right)
\]
is also a solution. So we can normalize the system by setting \(P_N = 1\) and we can rewrite the system as

\[
PPP_j = \sum_{i=1}^{N-1} \left( \frac{p_{ij}}{P_i} w_{ij} \right) + p_{Nj}w_{Nj} \quad (j = 1, ..., M) \quad (5.1)
\]

\[
P_i = \sum_{j=1}^{M} \frac{p_{ij}}{PPP_j} w_{ij}^* \quad (i = 1, ..., N - 1) \quad (5.2)
\]

If we substitute \(P\)'s from (5.2) in the first set of above equations (5.1) we have

\[
PPP_j = \sum_{i=1}^{N-1} \frac{p_{ij}}{\sum_{j=1}^{M} \frac{p_{ij}}{PPP_j} w_{ij}^*} w_{ij} = p_{Nj}w_{Nj} \quad (j = 1, ..., M) \quad (6)
\]

Note that existence of a solution to (6) is equivalent to existence of a solution to the whole system (5.1) and (5.2). To prove that Let’s define

\[
f^j(P_1, P_2, ..., P_M) = \sum_{i=1}^{N-1} \left( \frac{p_{ij}}{P_i} w_{ij} \right)
\]

\[
h^i(PPP_1, PPP_2, ..., PPP_M) = \sum_{j=1}^{M} \frac{p_{ij}}{PPP_j} w_{ij}^*
\]

\[
G_j(PPP_1, PPP_2, ..., PPP_M) = PPP_j - \sum_{i=1}^{N-1} \frac{p_{ij}}{\sum_{j=1}^{M} \frac{p_{ij}}{PPP_j} w_{ij}^*} w_{ij}
\]
**Theorem:** (i) if \( p_{Nj}w_{Nj} > 0 \) and (ii) there is at least one \( \text{PPP} \geq 0 \) such that \( G_j(\text{PPP}) > 0 \) \((j = 1,\ldots,M)\), the system of equations (5) has a unique positive solution

As we showed above the system (5.1) and (5.2) can be reduced to

\[
G_j(\text{PPP}) = PPP_j - \sum_{i=1}^{N-1} \sum_{j=1}^{M} \frac{p_{ij}}{\sum_{j=1}^{M} p_{ij}} w_{ij} = p_{Nj} w_{ij} \tag{7}
\]

It is easy to check that \( G_j \) satisfall the three conditions:

(i) \( G_j \) s is homogenous of degree one in \( \text{PPP} \)

(ii) \( G_j \) s are defined over the non-negative values and are continuous except at the origin

(iii) there is at least one \( \text{PPP} \geq 0 \) such that \( G_j \geq 0 \) (one of the theorem’s assumptions)

We can also show that

\[
\frac{\partial G_j}{\partial PPP_k} = -\sum_{i=1}^{N-1} \frac{\partial f_j}{\partial P_i} \frac{\partial h_i}{\partial PPP_k} \quad j \neq k
\]

It is easy to see that \( \frac{\partial f_j}{\partial P_i} \leq 0 \) and \( \frac{\partial h_i}{\partial PPP_k} \leq 0 \) therefore \( \frac{\partial G_j}{\partial PPP_k} \leq 0 \) \((j \neq k)\) which proves condition (iv) required from theorem (1).

As we see all the conditions cited in theorem (1) are satisfied. Therefore there is a unique positive \( \text{PPP} \) that solves (7).

\[\text{Q.E.D}\]

In the above theorem we have assumed that there is at least one \( \text{PPP} \geq 0 \) such that \( G_j(\text{PPP}) > 0 \) \((j = 1,\ldots,M)\). Our guess is that this condition is always satisfied but so far we haven’t been able to prove it. So this theorem is not a perfect existence theorem.
however it guarantees uniqueness of the solution and that is what is usually necessary for empirical applications.

4- Stochastic Approach to the New Index

To obtain the stochastic model incorporating the new index we follow Rao (2005) and Diewert (2005) postulate that the observed price of j-th commodity in i-th country, \( p_{ij} \), is the product of three components: the purchasing power parity (i.e. \( PPP_j \); the price level of the j-th commodity relative to other commodities (i.e. \( P_i \)) and a random disturbance term \( u_{ij} \) as follows

\[
p_{ij} = P_iPPP_j u_{ij} \tag{8}
\]

where \( u_{ij} \)'s are random disturbance terms which are independently and identically distributed. Rao (2005) has shown that Rao system (2) can be obtained as an estimator from the above model using a weighted least square argument after taking logs from both sides of the above equation. The same solution can be obtained by assuming a log-normal distribution for \( u_{ij} \) and using a maximum likelihood approach. Here we assume \( u_{ij} \)'s follows a gamma distribution as follows

\[
u_{ij} \sim \text{Gamma}(r,r) \tag{9}
\]

where \( r \) is a parameter to be estimated. We combine (8) and (9) to write

\[
\frac{p_{ij}}{P_iPPP_j} \sim \text{Gamma}(r,r) \tag{10}
\]

\footnote{One may notice the close association of the proposed model to what is known as a generalized linear model with gamma distribution. A generalized linear gamma regression may be defined as (see McCullagh and Nelder 1989) \( \frac{y_i}{x_i \beta} \sim \text{Gamma}(r,r) \). Our model is a nonlinear version of such a model.}
Our purpose here is to estimate parameters (i.e., $P_i$, $PPP_j$, and $r$) using a maximum likelihood procedure. From the definition of the gamma density function we can easily show that

$$p_{ij} \sim r^r \frac{p_{ij}^{r-1}}{\Gamma(r) \frac{P_i}{PPP_j}^r} e^{-r \frac{p_{ij}}{P_iPPP_j}}$$ \hspace{1cm} (11)$$

Therefore the log of density function can be written as

$$\ln L_{ij} \propto r \ln r - \ln \Gamma(r) + (r-1) \ln p_{ij} - \ln P_i - r \ln PPP_j - r \frac{p_{ij}^r}{P_iPPP_j}$$ \hspace{1cm} (12)$$

We can proceed with this (log-) density function and obtain estimates of the parameters of interest using the standard maximum likelihood procedure but we would like to incorporate the weights into the model as well. Use of weights is consistent with standard index number approach of weighting price relatives by their expenditure shares. This is also the approach used by Rao (2005) where weighted least squares method is employed.

One way of doing this is to use the theory of maximum weighted likelihood estimation discussed in Hu (1997) and Hu and Zidek (2002). According to this theory under general conditions, estimates from maximization of a weighted likelihood are consistent and asymptotically normal as long as the weights are positive and sums to one. We define our weighted likelihood function as

$$WL = \prod_{i=1}^{N} \prod_{j=1}^{M} L_{ij}^{w_{ij}/M}$$ \hspace{1cm} (13)$$

and therefore the weighted log-likelihood function becomes

$$\ln WL = \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{w_{ij}}{M} \ln L_{ij}$$ \hspace{1cm} (14)$$

The likelihood function becomes
\[
\ln WL \propto (r - 1) \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln p_{ij} - r \sum_{i=1}^{N} \sum_{j=1}^{M} p_{ij} w_{ij} \ln P_{i} - r \sum_{i=1}^{N} \sum_{j=1}^{M} w_{ij} \ln PPP_{j} - \\
\sum_{i=1}^{N} \sum_{j=1}^{M} \frac{p_{ij} w_{ij}}{P_{i} PPP_{j}} + r \ln r \left( \sum_{i=1}^{N} \sum_{j=1}^{M} w_{ij} \right) - \ln \Gamma(r) \sum_{i=1}^{N} \sum_{j=1}^{M} w_{ij} \quad (15)
\]

Maximization of this likelihood function is not particularly difficult. The only potential problem is the presence of a gamma function in the likelihood function however most of the existing software such as LIMDEP and GAUSS can handle maximization of the functions containing gamma functions fairly easily.

We can also derive the first order conditions from maximization of the above likelihood function as follows

\[
\sum_{i=1}^{N} \sum_{j=1}^{M} w_{ij} \ln p_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln P_{i} - \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln PPP_{j} - \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{p_{ij} w_{ij}}{P_{i} PPP_{j}} + M + M \ln r - M \frac{\partial}{\partial r} \ln \Gamma(r) = 0
\]

From the above sets of equations we may obtain

\[
P_{i} - \sum_{j=1}^{M} \frac{P_{ij} w_{ij}^{*}}{PPP_{j}} = 0 \quad (16)
\]

\[
PPP_{j} - \sum_{i=1}^{N} \frac{P_{ij} w_{ij}}{P_{i}} = 0
\]

\[
\frac{\partial}{\partial r} \ln \Gamma(r) - \ln r = \frac{1}{M} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln p_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln P_{i} - \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln PPP_{j} - \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{P_{ij} w_{ij}}{P_{i} PPP_{j}} + M \right)
\]
As we can see the first two equations are the same as the system of equations we introduced as the new system and these equations do not depend upon the value of $r$.

5- Stochastic Approach to the Ikle Index

A similar procedure can be followed to obtain the stochastic model leading to Ikle index.

Define

$$\frac{1}{p_{ij}} = \frac{1}{P_{PPP}} u_{ij}$$  \hspace{1cm} (17)

where $u_{ij}$'s are random disturbance terms which are independently and identically and as before they are assumed to follow a gamma distribution

$$u_{ij} \sim Gamma(r, r)$$  \hspace{1cm} (18)

where $r$ is a parameter to be estimated. Model in equation (17) differs from the model in equation (4) mainly in the specification of the disturbance term and how it enters the equation. One of the possible advantages of this model is that we do not have the inverse relationship between variance of $p_{ij}$ and $w_{ij}$. We combine (17) and (18) to write

$$\frac{1}{p_{ij}} \propto \frac{r^r}{\Gamma(r)} (P_{PPP})^r \frac{P_{PPP}}{p_{ij}} - r e^{-r}$$  \hspace{1cm} (19)

Following the same procedure as we used in section (4) we may obtain the likelihood function as

$$\ln L \propto -(r-1) \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln p_{ij} + r \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln P_{i} + r \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln P_{PPP} -$$

$$r \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{P_{PPP}}{p_{ij}} w_{ij} + r \ln r \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} - \ln \Gamma(r) \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}$$  \hspace{1cm} (20)

Taking derivative with respect to $PPP$ and $P$ yields the Ikle system of equations
\[
\frac{1}{PPP_j} = \sum_{i=1}^{N} \left( \frac{P_i}{P_{ij}} \right) w_{ij}
\]

\[
\frac{1}{P_i} = \sum_{j=1}^{M} \left( \frac{PPP_j}{P_{ij}} \right) w_{ij}^*
\]

(21)

Thus the difference between Ikle and our newly proposed system in this paper is in the specification of the disturbance term.

6- Application to OECD countries

In this section, we apply the methods proposed in this paper to the OECD data from 1996. The price information that we have is in the form of PPPs at the basic heading level for 158 basic headings, with US dollar used as the numeraire currency. The estimates of PPPs based on the new index, Ikle’s and the standard CPD for 24 OECD countries along with their standard errors are presented in the following table.

<table>
<thead>
<tr>
<th>Country</th>
<th>MLE Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CPD</td>
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<tr>
<td></td>
<td>PPP</td>
</tr>
<tr>
<td>GER</td>
<td>1.878</td>
</tr>
<tr>
<td>FRA</td>
<td>6.067</td>
</tr>
<tr>
<td>ITA</td>
<td>1419</td>
</tr>
<tr>
<td>NLD</td>
<td>1.909</td>
</tr>
<tr>
<td>BEL</td>
<td>35.3</td>
</tr>
<tr>
<td>LUX</td>
<td>33.35</td>
</tr>
<tr>
<td>UK</td>
<td>0.5996</td>
</tr>
<tr>
<td>IRE</td>
<td>0.633</td>
</tr>
<tr>
<td>DNK</td>
<td>8.481</td>
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<tr>
<td>GRC</td>
<td>179.5</td>
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<tr>
<td></td>
<td></td>
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<tr>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>SPA</td>
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<tr>
<td>PRT</td>
<td>125.4</td>
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<td>12.71</td>
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<tr>
<td>SUI</td>
<td>2.037</td>
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<tr>
<td>SWE</td>
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</tr>
<tr>
<td>FIN</td>
<td>6.12</td>
</tr>
<tr>
<td>ICE</td>
<td>86.15</td>
</tr>
<tr>
<td>NOR</td>
<td>8.751</td>
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<td>TUR</td>
<td>6251</td>
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<td>AUS</td>
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<td>JAP</td>
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</tr>
<tr>
<td>CAN</td>
<td>1.16</td>
</tr>
<tr>
<td>USA</td>
<td>1.0</td>
</tr>
</tbody>
</table>

For CPD estimates we have used the weighted least squares methodology as explained in Rao (2005). For Ikle and the new index we used a maximum likelihood approach as explained in previous sections. Results shown in the table clearly demonstrate the feasibility and comparability of the new approaches to the estimation of PPPs. As it can be seen, PPPs and their standard errors based on CPD and the new index are almost indistinguishable which is not unexpected because of the robustness of the CPD model to distributional assumptions. However there are fairly significant differences between these two sets of estimates and estimates based on Ikle model which needs to be further investigated.
References


