RISK, NONCONVERGENCE AND CYCLES:
A TWO COUNTRY MODEL *

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Abstract

This paper investigates the effects of an international asset market on inequality of nations. The world consists of two homogenous countries which differ only in level of their capital stocks. Each country is represented by the standard overlapping generations model with an international market in which agents in two countries trade assets with stochastic dividends. Consumers can transfer wealth over time and across countries by holding international assets. Capital flows from the rich to the poor country and the poor country is better off while the rich country is worse off at the asymmetric steady state. However, the international asset market does not necessarily induce convergence of capital accumulation of both countries as credit demand is bounded above by the expectation in the asset market. Capital stock may fluctuate in both countries in the long run. The poor country needs sufficiently high capital stock initially to catch up with the rich country.

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1 Introduction

How does trading in an international asset market influence the capital accumulation of countries? The conventional wisdom suggests that international financial markets allocate the savings of the integrated economies to its most profitable use. Suppose the world consists of identical countries which differ only in level of their capital stocks. A standard neoclassical technology implies a capital flow from rich countries to poor countries so long until the rate of return in all countries are equalized. In fact, a perfect international capital market implies an immediate adjustment of per capita income across countries. However, as Lucas put it “Why doesn’t capital flow from rich to poor countries?” (see Lucas (1990)). In his paper he discusses why capital does not flow from rich to poor countries to the extent a standard neoclassical model would predict.

Responding to Lucas’ paradox, the neoclassical growth models have been revised to include mainly aspects of heterogeneity, human capital, income distribution and capital market imperfections (for a survey see Galor (1996)). These extended models show that the neoclassical framework with constant return to scale and diminishing marginal product is consistent with club convergence. In other words their economic system can be characterized by multiple, locally stable steady states. However, most of these models are closed economy models without explicitly modelled international capital markets. Notable exceptions are the one sector overlapping generations model modified to incorporate capital market imperfections by Boyd & Smith (1997) and Matsuyama (2004). They show that the unrestricted international financial flow precludes countries from converging.

Instead of assuming capital market imperfections the present paper modifies the overlapping generations model by introducing international assets agents can trade to transfer their wealth over time in addition to investment in the capital market. The asset market is modelled as in Böhm & Chiarella (2005) in which asset prices are determined explicitly by the interaction of the behavior of investors using a specific forecasting rule. The income stream is endogenous so that the factor prices are determined by their respective marginal products. Assets pay random dividends so that agents’ attitude towards risk plays a crucial role in the portfolio decision. Agents may sell assets short in the two

\footnote{It is well known that in the one sector overlapping generations model multiple steady states could emerge if the wage function is not a concave function of capital labor ratio. They do not rely on this result.}
country model. As asset demand is increasing in the capital stock, short selling implies a financial flow from the rich country to the poor country. Suppose the world consists of two identical countries which differ only in their stocks of capital. Then, the rate of return in the poor country is higher than in the rich country. This induces the rich country to hold more assets which in turn causes a reduction in the capital stock. The poor country in turn may sell short in the asset market which is equivalent to taking credit to invest in the capital market. However, this mechanism does not necessarily lead to convergence of per capita income across countries as the credit demand is bounded above by the expectations of traders in the asset market. There exist an asymmetric steady state in which capital flows from the rich to the poor country. In contrast to Boyd & Smith (1997) and Matsuyama (2004), the asymmetric steady state does not result from imperfections imposed in the capital market but from uncertainty caused by the randomness in the asset market. In other words, interactions of optimal behavior of agents with different income may generate an asymmetric steady state. The poor country is better off while the rich country is worse off than in an economy without an asset market. Furthermore, permanent economic fluctuations occur around the asymmetric steady state. The reminder of the paper is organized as follows. Section 2 defines the behavior of consumers and producers identical across countries. Section 3 and 4 describe the closed economy model and two country model in general. Section 5 analyzes the steady state properties of the closed economy model and the two country model using a quadratic production function. Section 6 complements the analysis of section 5 by numerical simulations. Section 7 concludes.

2 The Structure

Consider an overlapping generations economy evolving in discrete time. In addition to the markets for output, labor, and capital there is a market for assets. Each generation consisting of homogenous consumers lives for two periods and we assume that there is no population growth. All markets operate under perfect competition implying that the agents are price takers.
2.1 The Consumption Sector

The typical young consumer in period \( t = 0 \) supplies one unit of labor inelastically in the first period of his life time and receives labor income \( w \) in units of consumption good which is the numeraire good.\(^2\) His lifetime utility depends on old age consumption only. There is no storage possibilities for the consumption goods. He can transfer his wage income to the next period either by investing in production as capital or by purchasing assets. The young agent can not take credit in the capital market. In the second period of his life time when old, the agent receives the rate of return \( R_1 \) on his investment \( y \) and a random dividend \( \varepsilon_1 \) on his share holdings \( x \), which he resells in the market. The following assumptions characterize the consumers.

**Assumption 1** Consumers have risk preferences over the mean \( \mu \) and the standard deviation \( \sigma \) of his future consumption/wealth described by a utility function

\[
U : \begin{cases}
\mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R} \\
(\mu, \sigma) \mapsto U(\mu, \sigma)
\end{cases}
\]

which is increasing in the mean \( \mu \) and decreasing in the standard deviation \( \sigma \).

Let \((x, y) \in \mathbb{R} \times \mathbb{R}_+\) denote a portfolio of assets and capital investment and let \( p \in \mathbb{R}_+ \) denote current price of risky assets in units of the numeraire commodity. The budget constraint requires

\[
w = px + y.
\]

The investor’s wealth in the following period \( t = 1 \) is given by

\[
W(w, p, x, R_1, p_1, \varepsilon_1) = R_1(w - px) + (p_1 + \varepsilon_1)x.
\]

When making the portfolio decision, next period’s rate of return on capital, equity price and dividend \((R_1, p_1, \varepsilon_1)\) are uncertain for young agents. It is assumed that they make point forecasts \((R^e, p^e)\) for the rate of return and the asset price. In addition, the following assumption is made about the expectation on the next period’s dividend payment \( \varepsilon_1 \).

\(^2\)For ease of notation the time index \( t \) will be suppressed unless necessary. Variables without time subscript refer to an arbitrary period \( t \) while subscript 1 refers to period \( t + 1 \) and \(-1\) to period \( t - 1 \).
Assumption 2  Consumers are endowed with a subjective probability distribution $\nu \in P(\mathbb{R}_+)$ for next period’s dividend payment parameterized by a pair $(\mathbb{E}_{\nu}[\varepsilon], \mathbb{V}_{\nu}[\varepsilon]) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ of expected value and variance. Then, for any asset portfolio $x \in \mathbb{R}$ the subjectively expected value of his future wealth can be expressed as

$$
\mathbb{E}_{\nu}[W(w, p, x, R^e, p^e, \cdot)] = \int_{\mathbb{R}_+} (R^e w + (p^e + \varepsilon - R^e p)x) \nu(d\varepsilon) = R^e w + (p^e + \mathbb{E}_{\nu}[\varepsilon] - R^e p)x
$$

with associated subjective variance

$$
\mathbb{V}_{\nu}[W(w, p, x, R^e, p^e, \cdot)] = \int_{\mathbb{R}_+} (W(w, p, x, R^e, p^e, \varepsilon) - \mathbb{E}_{\nu}(W(w, p, x, R^e, p^e, \cdot)))^2 \nu(d\varepsilon) = x^2 \mathbb{V}_{\nu}[\varepsilon]
$$

where $p^e + \mathbb{E}_{\nu}[\varepsilon] - R^e p$ is the expected risk premium. The young agent’s objective is to maximize the utility of next period consumption defined by

$$
\max_{x \in \mathbb{R}} \left\{ U \left( \mathbb{E}_{\nu}[W(w, p, x, R^e, p^e, \cdot)], \mathbb{V}_{\nu}[W(w, p, x, R^e, p^e, \cdot)]^{\frac{1}{2}} \right) \bigg| x \leq \frac{w}{p} \right\}
$$

which is utilizing equations (1) and (2) identical to

$$
\max_{x \in \mathbb{R}} \left\{ U \left( R^e w + (p^e + \mathbb{E}_{\nu}[\varepsilon] - R^e p)x, x \mathbb{V}_{\nu}[\varepsilon] \right) \bigg| x \leq \frac{w}{p} \right\}.
$$

### 2.2 The Production Sector

There is a single infinitely lived firm in the economy which uses labor $L = 1$ and capital $K$ to produce consumption goods. The aggregate production function is given by

$$
F(K, 1) + \varepsilon \bar{x},
$$

where $F$ is homogenous of degree one and $\varepsilon \bar{x}$ is stochastic profit paid to shareholders as dividend. Then the intensive form can be written as

$$
f(k) + \varepsilon \bar{x}.
$$

where $k = K$ as $L = 1$. In the overlapping generations structure the young generations are the shareholders of the firm. The labor and capital markets are assumed to be competitive such that the firm pays the wage $w(k) = f(k) - kf'(k)$ and the rate of return on capital investment $f'(k)$ according to the marginal product rule.
Assumption 3 The production function in the intensive form \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is \( C^2 \) and \( f'(k) > 0 > f''(k) \) and satisfies the Inada conditions \( \lim_{k \to \infty} f'(k) = 0 \) and \( \lim_{k \to 0} f'(k) = \infty \).

Assumption 4 \( \varepsilon \) is an i.i.d random variable with finite first and second moments. We assume that the agents have an unbiased prediction for the first and second moments in the sense of Böhm & Wenzelburger (2002). This implies the first two moments of the subjective distribution coincides with the first two moments of the true distribution. It follows that

\[
E_{\nu}[\varepsilon] = \frac{1}{\nu} E[\varepsilon]
\]

where \( E[\varepsilon] \) is the mean value of the random variable \( \varepsilon \) and

\[
\nabla_{\nu}[\varepsilon] = \nabla[\varepsilon],
\]

where \( \nabla[\varepsilon] \) is the variance of the random variable \( \varepsilon \).

3 The Closed Economy model

We assume that the amount of risky assets is given exogenously in the economy.\(^3\) In the overlapping generation structure this implies that all the assets sold by old consumers are bought by young investors.

3.1 Temporary Equilibrium

Let the preference of an investor be given by the linear mean variance function of future wealth

\[
U(\mu, \sigma) = \mu - \frac{\alpha}{2} \sigma,
\]

where \( \alpha \) is usually interpreted as a measure of risk aversion. Then, the asset demand of the investor is given by

\[
x = \varphi(p^e, R^e, p, k) := \text{Min} \left( \frac{p^e + E_{\nu}[\varepsilon] - R^e p}{\alpha \nabla_{\nu}[\varepsilon]}, \frac{w(k)}{p} \right).
\]

\(^3\)We do not address the issue of how firms' decision to raise capital influence the economy but focus on the spill over effects of consumption decision on capital accumulation.
The price law is implicitly defined by the solution \( p = S(p^e, R^e, k) \) of

\[
\varphi(p^e, R^e, p, k) = \bar{x}
\]

where \( \bar{x} \) denotes the constant supply of the risky asset.

**Assumption 5** The risk adjusted expected cum-dividend price has to be greater than zero, i.e. \( p^e + c > 0 \) where \( c := \mathbb{E}_\nu[\varepsilon] - \alpha \bar{x} \mathbb{V}_\nu[\varepsilon] \) which can be interpreted as risk adjusted dividend payment.

**Proposition 1** Given assumption (5) there exist a unique positive price

\[
p = \text{Min} \left( S(p^e, R^e, k), \frac{w(k)}{\bar{x}} \right).
\]

The assertion in Proposition 1 is obvious as the asset demand function \( \varphi(p^e, R^e, p, k) \) is decreasing in \( p \) and \( \varphi(p^e, R^e, 0, k) = \frac{p^e + \mathbb{E}_\nu[\varepsilon]}{\alpha \mathbb{V}_\nu[\varepsilon]} \).

Note that in equilibrium there is no short sale in the asset market as the young consumers are homogenous. The capital investment is defined by the wage minus purchases of risky assets. We assume that the capital investment is reversible. Then the depreciated capital is paid back as a part of rate of return on capital investment. The evolution of capital is given by

\[
k_1 = w(k) - \bar{x}p.
\]

Equation (4) and (5) define the temporal equilibrium given expectations.

### 3.2 Expectations

Given the equation (5) for capital accumulation, the rate of return on capital at \( t = 1 \) is given by

\[
R_1 = R(k, p) := f'(w(k) - \bar{x}p) + 1 - \delta.
\]

The perfect prediction at \( t = 0 \) for the rate of return on capital at \( t = 1 \) requires

\[
R^e \overset{\perp}{=} R(k, p)
\]

The perfect predictor at \( t = 0 \) for the asset price in \( t = 1 \) is implicitly defined by the solution \( p^e = \Psi(p^e_{-1}, R^e, k) \) of

\[
p^e_{-1} = S(p^e, R^e, k).
\]
Substituting (7) into (8) we obtain a perfect predictor for the asset price consistent with a perfect prediction for the rate of return on capital defined in the following Proposition. Note that given the perfect prediction for the rate of return on capital, the asset demand now becomes dependent on wage income in general. This also implies that the price law is dependent wage income in general.

**Proposition 2** Suppose $p_{t-1}^e \in \mathbb{R}_+$. There exist a unique perfect predictor for asset price consistent with the perfect forecasting rule in the capital market defined by

$$p_{t-1}^e \mapsto \Psi(p_{t-1}^e, R(k, p), k)$$

if and only if $p_{t-1}^e \in \left(0, \frac{w(k)}{x}\right)$. The perfect predictor is positive if $c \leq 0$ or $c > 0$ and $p_{t-1}^e \in \left(h(k), \frac{w(k)}{x}\right)$ where $h(k)$ is implicitly defined by $\Psi(p_{t-1}^e, R(k, p), k) = 0$.

Proof is provided in Appendix.

Given perfect prediction for the next period’s asset price and the rate of return, a price map along which a perfect point prediction is guaranteed is given by

$$p_t = \Psi(p, k) := p(f'(w(k) - \bar{x}p) + 1 - \delta) - c.$$  \hfill (10)

Equations (5) and (10) define the dynamical system for the closed economy model under rational expectations.

## 4 Two country model

In this section we assume that the world economy consists of two countries inhabited by homogenous consumers. The production technologies in both countries are assumed to be also identical making two countries distinguished only by the stock of capital. The asset markets of two countries are integrated into an international market while there exist capital markets in both countries. We assume that when young consumers buy assets in the international market, they do not distinguish between assets of two countries. We also assume that consumers can not invest in the capital market abroad. In other words, we rule out foreign direct investment. Therefore, the only channel between capital accumulation in each country is the international asset market. Since young consumers can be shareholders of foreign firms, capital investment is no longer
independent of asset demand as it was in the closed economy model where constant stocks of assets were transferred from old to young within an country. In turn, the asset demand is dependent on wage income given rational expectation generating a feedback effect between asset demand and capital accumulation in both countries. Different wage incomes in both countries enable short selling at equilibrium in the international asset market. In such an equilibrium, the international asset market serves as an international credit market inducing trading of consumption commodities across countries.

4.1 Temporary Equilibrium in the International Asset Market

Suppose there exist international assets composed of assets in two countries which pay dividend of

\[ d = \frac{\varepsilon + \varepsilon}{2}. \] (11)

Since the productivity shocks in two countries are both i.i.d. random variables drawn from the same distribution, the first and second moment of \( d \) will be

\[ \mathbb{E}[d] = \mathbb{E}\left[\frac{\varepsilon + \varepsilon}{2}\right] = \mathbb{E}[\varepsilon] \] (12)

and

\[ \mathbb{V}[d] = \mathbb{V}\left[\frac{\varepsilon + \varepsilon}{2}\right] = \mathbb{V}[\varepsilon] \] (13)

respectively. Assuming unbiased predictions for the first and second moments as before, we obtain the asset demand function of young consumers at \( t = 0 \) given by\(^4\)

\[ x = \varphi(p^e, R^e, p, k) := \min\left(\frac{p^e + \mathbb{E}[\varepsilon] - R^e p}{a \mathbb{V}[\varepsilon]}, \frac{w(k)}{p}\right). \] (14)

The evolution of capital in each country is now dependent on asset demand and is given by

\[ k_1 = w(k) - xp. \] (15)

Then, the rate of return on capital in \( t = 1 \) in each country is given by

\[ R_1 = R(k, x, p) := f'(w(k) - xp) + 1 - \delta. \] (16)

\(^4\)For ease of notation we suppress the superscript \( i = 1, 2 \) denoting the individual country unless necessary.
Substituting equation (16) into equation (14) we implicitly obtain the asset demand consistent with a perfect foresight for the rate of return on capital in $t = 1$. If the budget constraint is binding, the asset demand is independent of expectations and given by $\frac{w(k)}{p}$. If the budget constraint is not binding, the asset demand is implicitly defined by the solution $x = x(p^x, k, p)$ of

$$x = \frac{p^x + E_\nu[\varepsilon] - R(k, x, p)p}{\alpha \nu_\nu[\varepsilon]}.$$

(17)

The following Proposition characterizes the asset demand function consistent with perfect foresight for the rate of return on capital.

**Proposition 3** The asset demand is decreasing in $p$ i.e., $\frac{\partial}{\partial p} x(p^x, k, p) \leq 0$ if and only if $E_{R,x}(k, x, p) \leq -1$ where $E_{R,x}(k, x, p)$ denotes the elasticity of expected rate of return on capital investment with respect to asset demand defined by

$$E_{R,x}(k, x, p) := \frac{\partial R(k, x, p)}{\partial x} \frac{x}{R(k, x, p)} = - \frac{f''(w(k) - xp)x}{f'(w(k) - xp) + (1 - \delta)}.$$ 

(18)

Furthermore, $E_{R,x}(k, x, p) > -1$ implies $p \in P$ where $P := \{p \in \mathbb{R}_+ | x(p^x, p, k) < 0\}$.

Proof is provided in Appendix.

The Proposition 3 states that the asset demand can be increasing in price if there is short selling in the market. To summarize the asset demand of young consumers is given by

$$x = \varphi(p^x, p, k) := \min \left( x(p^x, p, k), \frac{w(k)}{p} \right).$$

(19)

Setting the demand of two countries equal to the total supply of assets, the price law is implicitly defined by the solution $p = S(p^x, k^1, k^2)$ of,

$$\varphi(p^x, p, k^1) + \varphi(p^x, p, k^2) = 2\bar{r}.$$ 

(20)

**Proposition 4** There exist a unique market clearing price defined implicitly by equation (20) if and only if $\frac{\partial}{\partial p} \varphi(p^x, k^1, p) + \frac{\partial}{\partial p} \varphi(p^x, k^2, p) < 0$. Otherwise, there exist multiple market clearing prices. Furthermore, $\varphi(p^x, k^1, p) < \varphi(p^x, k^2, p)$ implies $k^1 < k^2$ in equilibrium.

Proof is provided in Appendix.
Given results in Proposition 3, the Proposition 4 implies that there may exist multiple market clearing asset prices if there is short selling in the market. In equilibrium, it is only the poor country that sells short. Equations (15), (19) and the price law \( p = S(p^e, k^1, k^2) \) defined by equation (20) define the temporal equilibrium at \( t = 0 \) given expectation for asset price at \( t = 1 \).

### 4.2 Expectations and Dynamical System

The perfect predictor must satisfy

\[
p^e_{-1} \equiv S(p^e, k^1, k^2)
\]

**Proposition 5** Suppose \((p^e, k^1, k^2) \in \mathbb{R}^3_+\). There exists a unique perfect predictor consistent with the perfect forecast for the rate of return on capital

\[
p^e_{-1} \mapsto \Psi(p^e_{-1}, k^1, k^2)
\]

if and only if \( p^e_{-1} \in \left(0, \frac{w(k^1) + w(k^2)}{2\varepsilon} \right) \). The perfect predictor is positive if \( c \leq 0 \) and if \( c > 0 \) and \( p^e_{-1} \in \left(0, \frac{w(k^1) + w(k^2)}{2\varepsilon} \right) \setminus P^e \) where \( P^e := (0, p^1) \cup (p^2, p^3) \) and \( \{p^1, p^2, p^3\} = \{p^e_{-1} \in \mathbb{R}_+ \mid \Psi(p^e_{-1}, k^1, k^2) = 0\} \).

Proof: Let \( G(p^e, p) := \varphi(p^e, k^1, p) + \varphi(p^e, k^2, p) - 2\bar{x} \cdot \frac{\partial}{\partial p} \Psi(p^e, k^1, k^2) = -\frac{\partial G(p^e, p)}{\partial p} \).

From Proposition 3 we know that \( \frac{\partial G(p^e, p)}{\partial p} \) may be positive for some interval whereas \( \frac{\partial G(p^e, p)}{\partial p^e} \) \( > 0 \). Hence, \( \frac{\partial}{\partial p} \Psi(p^e, k^1, k^2) < 0 \) for that interval. For \( c > 0 \) the predictor may be negative for some interval as \( \Psi(0, k^1, k^2) = -c \).

The perfect forecasting rule for the rate of return in both countries is implicitly defined by solutions \( R^{ie} = R^i(k^1, k^2, p), \forall i = 1, 2 \) of

\[
R^{1e} = f' \left( w(k^1) - p \left( \frac{R^{1e} - R^{2e}}{2\alpha V_{r^e} (\bar{x})} \cdot p \right) \right) + 1 - \delta \\
R^{2e} = f' \left( w(k^2) - p \left( \frac{R^{2e} - R^{1e}}{2\alpha V_{r^e} (\bar{x})} \cdot p \right) \right) + 1 - \delta.
\]

These equations have a unique solutions as the left hand side is positive for \( R^{ie} = 0 \), decreasing with respect to \( R^{ie} \) and goes to zero as \( R^{ie} \to \infty \), \( \forall i = 1, 2 \).
The dynamical system of the two country model under rational expectations is characterized by

$$
k^1_1 = \Phi(k^1, k^2, p) := w(k^1) - p\left(\bar{x} - \frac{R^1(k^1, k^2, p) - R^2(k^1, k^2, p)}{2\sigma V[\varepsilon]} \cdot p\right)
$$

$$
k^2_1 = \Phi(k^2, k^1, p) := w(k^2) - p\left(\bar{x} - \frac{R^2(k^1, k^2, p) - R^1(k^1, k^2, p)}{2\sigma V[\varepsilon]} \cdot p\right)
$$

$$
p^1 = \Psi(p, k^1, k^2) := \frac{R^1(k^1, k^2, p) + R^2(k^1, k^2, p)}{2} \cdot p - c.
$$

(23)

**Proposition 6** There exist unique positive symmetric steady states with rational expectations which coincides with the unique positive steady states of the closed economy.

The proof follows directly from equations (22) and (22) as well as equations in (23).

## 5 Quadratic Production Function Case

As it is shown in the previous section we can not globally derive an unbiased predictor for the asset price when the budget constraints are binding. Even for the unconstrained asset demand an unbiased predictor can only be derived globally when $c \leq 0$. This is a general feature of the CAPM models as in Böhm, Deutscher & Wenzelburger (2000) and Böhm & Chiarella (2005) for a given positive return on riskless assets. To investigate the existence of steady states under rational expectation\(^5\) and their stability properties we use a specific production function. The quadratic production function has a technically convenient property that the first derivative is a linear function.\(^6\) This linearity of the first derivative allows us to obtain a closed form solution of the model. Let the quadratic production function be

$$
f(k) = \begin{cases} 
Ak(2d - k) & \text{if } k < d \\
Ad^2 & \text{if } k \geq d.
\end{cases}
$$

(24)

Figure 1 illustrates the quadratic production function with the associated wage function. Notice that the wage function is not globally concave. This is crucial for the existence of multiple steady states as we see later.

\(^5\)More specifically by rational expectation we mean an unbiased prediction and/or a perfect prediction whenever available.

\(^6\)Day (1983) was one of the first to exploit the property of this function.
5.1 Dynamics in the Closed Economy

Using the quadratic production function we can solve explicitly for the dynamical system under rational expectation given by

\[
p_1 = \Psi(p, k) := p(f'(w(k) - \bar{\bar{x}}) + 1 - \delta) - c \quad (25)
\]

\[
k_1 = \Phi(p, k) := w(k) - \bar{x}p. \quad (26)
\]

The following Proposition characterizes the stability property of all steady states.

**Proposition 7** If \( c > 0 \), there exist at most two steady states. Both of them are unstable and \( k < d \). If \( c \leq 0 \), there exist two steady states if \( (Ad^2 - d)\delta > -c\bar{x} \). One is unstable and \( k < d \). The other is stable and \( k \geq d \).

Proof is provided in Appendix.

5.2 Dynamics in the Two Country Model

The asset demand and the unbiased predictor is derived for the quadratic production function case in the Appendix. From Proposition 6 we know that the symmetric steady state of the two country model is identical to the steady state of the closed economy model. So the existence of the symmetric steady state is already characterized by Proposition 7. The following Proposition gives a condition of when two countries convergence to the symmetric steady state.
Proposition 8 There exist a positive symmetric steady state $k^1 = k^2 = Ad^2 - \frac{c}{\delta}$ for $c \leq 0$ and $\delta(Ad^2 - d) > -c\bar{x}$. If $k^1 = k^2$ or $k^1, k^2 > d$ the world economy converges to this symmetric steady state from around its neighborhood.

Proof is provided in Appendix. □

Note that there are no financial flows between countries at symmetric steady states.

Definition 1 We call an asymmetric steady state an inventive compatible trading steady state if the budget constraint is not binding for the asset demand in both rich and poor country and if there are financial flows between both countries.

Proposition 9 There exist incentive compatible trading asymmetric steady states in which $k^2 < d < k^1$ and $x^2 < 0 < x^1$.

The Proposition implies $w(k^1) > I(k^1) > I(k^2) > w(k^2)$ at the asymmetric steady states where $I(k^i) := w(k^i) - px^i, \forall i = 1, 2$ denotes the capital investment in each countries. This means the poor country requires external finance from the rich country in form of short selling in the international asset market for its capital investment.

Proposition 10 The poor country is better off while the rich country is worse off at the incentive compatible trading steady state than in economy without an asset market.

Proof is provided in Appendix. □

6 Numerical Simulation

The quadratic production function is used throughout the numerical analysis. To obtain an unbiased prediction for the first and second moment of the next period’s dividend payment, the following assumption is made about the random variable $\varepsilon$.

Assumption 6 We assume that the random variable $\varepsilon$ has a uniform distribution on interval $[a, b]$. The probability density function for a continuous uniform distribution on interval $[a, b]$ is

$$P(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon < a \\ \frac{1}{b-a} & \text{if } a \leq \varepsilon \geq b \\ 0 & \text{if } \varepsilon > b \end{cases}$$

(27)
with a mean \( \frac{a+b}{2} \) and a variance \( \frac{(b-a)^2}{12} \).

The first general observation is that the dynamical system converges to a symmetric steady state for a wide range of parameter sets for relatively low \( a \mathbb{V}_\rho [\varepsilon] \). This means that if the young consumers hypothesize a relatively low risk, the international asset market induces equalizing forces for the capital accumulation in both countries. Especially, a zero risk induces the rate of return on capital investment in both countries to be identical. In other words a zero risk implies that the non arbitrage condition has to hold between the expected return in the international asset market and the expected return in the domestic capital market. This means that the dynamical system for zero risk degenerates to the dynamical system of a closed economy. Therefore, in this section we choose belief of a relatively high risk \( (\mathbb{V}_\rho [\varepsilon], \alpha) = (0.25, 3) \) along with \( \bar{x} = 1, A = 0.5, d = 3.2, \mathbb{E}_\rho [\varepsilon] = 0.5, p_0 = 0.4, k_1 = 1, k_0 = 3 \) as the standard parameter set to analyze the dynamic behavior of the system. Figure 2 shows a bifurcation diagram with respect to the depreciation rate \( \delta \). It shows that the asymmetric steady state \( k^1 < d < k^2 \) described in proposition 9 is stable for an open parameter set. In other words there exist an open parameter set for which the incentive compatible trading steady state is stable.

![Bifurcation Diagram](image)

**Figure 2: Bifurcation Diagram**

**Proposition 11** The incentive compatible trading asymmetric steady states \( k^2 < d < k^1 \) go through a supercritical Neimark-Sacker bifurcation.

Proof is provided in the Appendix. \( \square \)

The steady state \( k^2 < d < k^1 \) loses its stability and a closed invariant curve appears which is shown in Figure 3 (a). Figure 3 (b) shows the corresponding time series. The
dynamical system switches from the case where $k^2 < d < k^1$ to the case where $k^1, k^2 < d$ when the invariant curve around $k^1$ touches $d$ and become unstable.

Figure 3: $\delta = 0.575$

Figure 4 shows the basin of attraction for the asymmetric steady states at $\delta = 0.625$. Note that $c = 0.25 > 0$ for our standard parameter set.

Figure 4: Basin of attraction $\delta = 0.625$

We know from proposition 8 that there exist no positive symmetric steady state for $c > 0$ and there exist a positive stable symmetric steady state for $c \leq 0$. The white color depicts the initial values for which the dynamical system explodes. This suggests that the risk adjusted expected dividend $c$ plays a crucial role on the stability of steady states. Intuitively the risk premium has to be positive for a country to invest in the international asset market. Therefore, ceteris paribus the stability of the asymmetric steady state requires a high expected dividend. Figure 5 compares the capital accumulation law (47) with the capital accumulation law at the incentive compatible trading steady state. The figure shows that the non concavity of the wage function together with short selling is
responsible for the existence of the asymmetric steady state. The horizontal green line depicts the map of the rich country while the other green curve depicts the map of the poor country. The positive intercept of the map at $k = 0$ arises from short selling and the convexity from the wage function.

![Figure 5: Time Map: $A = 0.5, d = 3.2, \alpha = 1, x = 1, \sigma = 0.5$.](image)

7 Concluding Remarks

The neoclassical growth model with a perfect international financial market predicts immediate convergence of capital stocks across countries given common production technology world wide. One way of resolving this paradox is to introduce some kind of imperfections in the financial market. In the present paper we ruled out foreign direct investment to focus on the role of international asset markets for the convergence of per capita income across countries. We introduced an international asset market in which the equilibrium price is determined endogenously by the demand behavior of consumers in two countries. If the consumers are not allowed to go short in the international asset market, the market only serves to transfer their wealth over time not over countries. However, short selling in the international asset market induces capital flows between countries by enabling the young consumers in two countries to trade real commodities between them. In the two country model the poor country sells short, which implies capital flows from the rich to the poor country. We would expect the two countries to converge to a symmetric steady state through this mechanism. However, as the consumers trade capital across two countries only through the international asset market, their credit demand is bounded above by the risk adjusted price expectation in the
asset market. In this way credit taking is not restricted by the wealth of the country as in the models with financial market imperfections but by the risk adjusted price expectations of consumers in the asset market. This together with a non-concave wage function generates asymmetric steady states in which capital flows from the rich to the poor country. Will an international asset market bring about convergence? The present model predicts only if the countries have sufficiently high capital stock initially.

Appendix

Proof of Proposition 2

Let \( G(p^e, k, p, c) := \varphi(q^e, k, p) - \bar{x} \). By Implicit Function Theorem, \( \frac{\partial}{\partial p^e} \Psi(p^e-1, c, k) = -\frac{G_p}{G_{p^e}} \). \( G_p < 0 \) and \( G_{p^e} > 0 \) if \( \varphi(q^e, p, k) = \frac{q^e - R(k)}{\alpha V_e[c]} \). If \( \varphi(q^e, p, k) = \frac{w(k)}{p} \), \( p = \frac{w(k)}{x} \) and \( G_{p^e} = 0 \). Furthermore, \( \Psi(0, k_t) = -c \). \( \Box \)

Proof of Proposition 3

There exist a unique solution to the equation (17) since the right hand side is positive for \( p = 0 \) and is decreasing in \( x \). If \( p = 0 \), \( x(q^e, k, p) = \frac{q^e}{\alpha V_e[c]} \). Let \( G(x, p) := x - \frac{q^e - R(k) - xp}{\alpha V_e[c]} \). By the Implicit Function Theorem \( \frac{\partial x}{\partial p} = -\frac{G_p(x, p)}{G_x(x, p)} \) where \( G_p(x, p) = \frac{f''(w(k)-px)}{\alpha V_e[c]} - xp \cdot \frac{f''(w(k)-px)}{\alpha V_e[c]} \), \( G_x(x, p) = 1 - p^2 \cdot \frac{f''(w(k)-px)}{\alpha V_e[c]} > 0 \). \( \Box \)

Proof of Proposition 4

The demand function is increasing in \( k \). This implies there exist a critical value of \( p \) at which the country with less capital stock first start selling short. If the other country start selling short, the aggregate demand is negative and there does not exist a supporting price. \( \Box \)
Proof of Proposition 7

For $k \geq d$, the steady state is defined by

\[ p = -\frac{c}{\delta}, \quad (28) \]
\[ p = \frac{Ad^2 - k}{\bar{x}}. \quad (29) \]

This excludes any positive steady states $(p, k)$ where $k > d$ and $p > 0$ for $c > 0$. If $c \leq 0$ there exist a unique positive steady state $(p, k)$ if \( \frac{Ad^2 - d}{\bar{x}} > -\frac{c}{\bar{x}} \). The system in the neighborhood of the steady state is given by

\[ p_1 = \Psi(p, k) := (1 - \delta)p - c, \quad (30) \]
\[ k_1 = \Phi(p, k) := Ad^2 - \bar{x}p. \quad (31) \]

The Jacobian is

\[ J(p, k) = \begin{pmatrix} 1 - \delta & 0 \\ -\bar{x} & 0 \end{pmatrix}. \quad (32) \]

The determinant is zero and the trace is $1 - \delta$. The eigenvalues are 0 and $1 - \delta$. Thus the steady state where $k > d$ is stable.

For $k < d$, the steady state is defined by

\[ p = p(2A(d - k) + 1 - \delta) - c, \quad (33) \]
\[ p = \frac{Ad^2 - k}{\bar{x}}. \]

The system in the neighborhood of the steady state is given by

\[ p_1 = \Psi(p, k) := p\left(f'(Ak^2 - \bar{x}p) + 1 - \delta \right) - c, \quad (34) \]
\[ k_1 = \Phi(p, k) := Ak^2 - \bar{x}p. \quad (35) \]

The figure 6 shows there exist at most two steady states if $c > 0$ and there exists always one steady state if $c \leq 0$ and $\delta(Ad^2 - d) > -c\bar{x}$.

The Jacobian is

\[ J(p, k) = \begin{pmatrix} 2A(d - k + \bar{x}p) + 1 - \delta & -4A^2pk \\ -\bar{x} & 2Ak \end{pmatrix}. \quad (36) \]

The determinant is $4A^2k(d - k) + 2Ak(1 - \delta) > 0$ and the trace is $2A(d + \bar{x}p) > 0$. The trace can be rewritten as $2A(d - k) + 2A^2k^2 + 1 - \delta$. We know that at positive steady states $Ak < 1$. This implies that the trace is always greater than 2 at any positive steady states. \( \square \)
Derivation of the asset demand in the two country model

The unconstrained asset demand function of each country with the perfect foresight for interest factor is implicitly defined by the solution $x(p^e, k, p)$ of

$$x = \frac{q^e - (f'(w(k) - px) + 1 + \delta)p}{\alpha \mathbb{E}[\varepsilon]}.$$ 

The asset demand function of each country can be defined as the following.

If $w(k) < d$,

$$x = \varphi(p, p^e, k) := \begin{cases} \frac{q^e - R(w(k))p}{\alpha \mathbb{E}[\varepsilon] + 2Ap} & \text{if } q^e < b(k, p) \\ \frac{w(k)}{p} & \text{if } q^e \geq b(k, p) \end{cases} \quad (37)$$

where $R(k) := 2A(d - k) + 1 - \delta$ with slight abuse of notation and $b(k, p) := \frac{w(k)\alpha \mathbb{E}[\varepsilon]}{p} + 2Adp + (1 - \delta)p > 0$ denotes the expected cum dividend price above which the budget constraint is binding.
If \( w(k) \geq d \),

\[
    x = \varphi(p, p^e, k) := \begin{cases}
        \frac{q^e - (1 - \delta)p}{\alpha V_e[e]} & \text{if } q^e < z(k, p) \\
        \frac{q^e - R(w(k))p}{\alpha V_e[e] + 2A p^2} & \text{if } z(k, p) \leq q^e < b(k, p) \\
        \frac{w(k)}{p} & \text{if } b(k, p) \leq q^e
    \end{cases}
\]

where \( z(k, p) := \frac{\alpha V_e[e](w(k) - d)}{p} + (1 - \delta)p \).  

\[\square\]

**Derivation of the perfect predictor in the two country model**

Let the world capital investment be denoted by \( I(k^2, k^2) := w(k^1) + w(k^2) - 2\bar{x}p \).

**The Case** \( w(k^1) \leq w(k^2) < d \)

If \( q^e \geq b(k^i, p), \forall i = 1, 2 \), the budget constraint is binding and irrespective to expected future price the price law is

\[
    p = \frac{w(k^1) + w(k^2)}{2\bar{x}}.
\]

From equation (37), we know there are two possible cases.

\[
p_1 = \Psi(p, k^1, k^2) := \begin{cases}
        2A \bar{x} p^2 + \frac{R(w(k^1)) + R(w(k^2))}{2} p - c & \text{if } q^e < b(k^1, p) \\
        R(I(k^2, k^2))p + \alpha V_e[e] \left( \bar{x} - \frac{w(k^1)}{p} \right) - c & \text{if } b(k^1, p) \leq q^e < b(k^2, p).
    \end{cases}
\]

The case \( w(k^2) \leq w(k^1) < d \) can be obtained analogously.

**The Case** \( d \leq w(k^1) \leq w(k^2) \)

If \( b(k^i, p) \leq q^e, \forall i = 1, 2 \), the budget constraint is binding and irrespective to expected future price the price law is

\[
    p = \frac{w(k^1) + w(k^2)}{2\bar{x}}.
\]

From equation (38), we know there are five possible cases.
\[ p_1 = \Psi(p, k^1, k^2) := \]
\[
\begin{cases}
(1 - \delta)p - c & \text{if } q^e < z(k^1, p) \\
2A\bar{x}p^2 + \frac{R(w(k^1)) + R(w(k^2))}{2} - c & \text{if } z(k^2, p) \leq q^e < b(k^1, p) \\
\frac{\alpha V^\nu[\varepsilon] A(d - w(k^2) + \bar{x}p)p}{\alpha V^\nu[\varepsilon] + Ap^2} + (1 - \delta)p - c & \text{if } q^e < z(k^2, p) \land z(k^1, p) \leq q^e < b(k^1, p) \\
\left( \bar{x} - \frac{w(k^1)}{p} \right) \alpha V^\nu[\varepsilon] + (1 - \delta)p - c & \text{if } b(k^1, p) \leq q^e \land z(k^2, p) \leq q^e < b(k^2, p). \\
\end{cases}
\]

The case \( d \leq w(k^2) \leq w(k^1) \) can be obtained analogously.

**The Case** \( w(k^1) < d \leq w(k^2) \)

If \( b(k^i, p) \leq q^e, \forall i = 1, 2 \), the budget constraint is binding and irrespective to expected future price the price law is
\[
p = \frac{w(k^1) + w(k^2)}{2\bar{x}}.
\]

We do not have to consider the case \( b(k^2, p) \leq q^e < b(k^1, p) \). \( q^e < b(k^1, p) \) can be rewritten as \( q^e < \frac{w(k^1)\alpha V^\nu[\varepsilon]}{p} + 2Ap + (1 - \delta)p \leq b(k^2, p) \) which is a contradiction.

From equation (37) and (38), we know there are four possible cases.

\[ p_1 = \Psi(p, k^1, k^2) := \]
\[
\begin{cases}
\frac{\alpha V^\nu[\varepsilon] A(d - w(k^1) + \bar{x}p)p}{\alpha V^\nu[\varepsilon] + Ap^2} + (1 - \delta)p - c & \text{if } q^e < b(k^1, p) \land q^e < z(k^2, p) \\
2A\bar{x}p^2 + \frac{R(w(k^1)) + R(w(k^2))}{2} - c & \text{if } q^e < b(k^1, p) \land z(k^2, p) \leq q^e \\
\left( \bar{x} - \frac{w(k^1)}{p} \right) \alpha V^\nu[\varepsilon] + (1 - \delta)p - c & \text{if } b(k^1, p) \leq q^e \land q^e < z(k^2, p) \\
R(I(k^1, k^2))p + \alpha V^\nu[\varepsilon] \left( \bar{x} - \frac{w(k^1)}{p} \right) - c & \text{if } b(k^1, p) \leq q^e \land z(k^2, p) \leq q^e < b(k^2, p). \\
\end{cases}
\]

The case \( w(k^2) < d \leq w(k^1) \) can be obtained analogously. \(\square\)
Proof of Proposition 8

If \( k^1 = k^2 < d \), the dynamical system reduces to a two dimensional system given by

\[
\begin{align*}
k_1 &= \Phi(p, k) := Ak^2 - \bar{x}p \\
p_1 &= \Psi(p, k) := f'(Ak^2 - \bar{x}p)p + (1 - \delta)p - c.
\end{align*}
\]

If \( k^1, k^2 \geq d \), the dynamical system reduces to a two dimensional system given by

\[
\begin{align*}
k_1 &= \Phi(p, k) := Ad^2 - \bar{x}p \\
p &= \Psi(p, k) := f'(Ak^2 - \bar{x}p)p + (1 - \delta)p - c.
\end{align*}
\]

From proposition 7 we know that there exist the stable steady state \( Ad^2 - \frac{\bar{x}}{\bar{y}} \) if and only if \( c \leq 0 \) and \( \delta(Ad^2 - d) > -c\bar{x} \). \( \square \)

Proof of Proposition 9

Suppose \( k^2 < d < k^1 \). Then the steady states is defined by

\[
\begin{align*}
k^1 &= Ad^2 - p \left( \bar{x} + \frac{pA(d - k^2)}{\alpha \mathcal{V}_\nu[\varepsilon]} \right) \\
k^2 &= A(k^2)^2 - p \left( \bar{x} - \frac{pA(d - k^2)}{\alpha \mathcal{V}_\nu[\varepsilon]} \right) \\
p &= p(A(d - k^2) + 1 - \delta) - c
\end{align*}
\]

From (41),

\[
k^2 = d - \frac{\delta}{A} - \frac{c}{Ap}
\]

From (40) and (41),

\[
k^2 = \frac{1}{2A} \pm \sqrt{\frac{1}{4A^2} + \frac{p}{A} \left( \bar{x} - \frac{\delta p + c}{\alpha \mathcal{V}_\nu[\varepsilon]} \right)}
\]

The figure shows the intersections of the equation (42) and (43) for \( \bar{x} \alpha \mathcal{V}_\nu[\varepsilon] > c > 0, Ad > 1 \).

Using the equation (40) we can rewrite equation (39) as

\[
k^1 = Ad^2 + A(k^2)^2 - k^2 - 2p\bar{x}.
\]
Figure 7: Asymmetric Steady States

\( k^1 > d \) requires \( Ad^2 + A(k^2)^2 - 2px > d + k^2 \). For sufficiently high \( d \) this inequality is satisfied. Suppose the asset demand of the country 2 is positive at above steady states. This means

\[
\bar{x} - \frac{p\delta + c}{\alpha V_x(\bar{x})} > 0 \implies p < \frac{x_0 V_x(\bar{x}) - c}{\delta}.
\]

However, \( p \geq \frac{x_0 V_x(\bar{x}) - c}{\delta} \) at the asymmetric steady states. This is a contradiction. \qed

**Proof of Proposition 10**

Suppose \( k^2 < k^1 \). From Proposition 9, we know \( x^1 > 0 > x^2 \) at the incentive compatible trading asymmetric steady state. The capital stock of both poor and rich country at the asymmetric steady state is given by

\[
\begin{align*}
k^1 &= Ad^2 - px^1 \\
k^2 &= A(k^2)^2 - px^2.
\end{align*}
\]

Then, at the asymmetric steady state, \( 0 < k^2 < k^1 < Ad^2 \). Suppose there exist no asset market. Then the evolution of capital accumulation in an economy is given by

\[
k_1 = w(k) = \begin{cases} 
Ak^2 & \text{if } k < d \\
Ad^2 & \text{if } k \geq d.
\end{cases}
\]

The economy without an asset market has three steady states, 0, 1/A and \( Ad^2 \). The steady state 1/A is unstable since the function \( w(k) \) cuts the 45 degree line from below.
Hence, the capital stock of the poor country would be \( k = 0 \) and that of the rich country \( k = Ad^2 \) at steady state.

\[ \square \]

**Proof of Proposition 11**

The dynamical system in the neighborhood of the steady state where \( k^2 < d < k^1 \) is defined by

\[
\begin{align*}
    k_1^1 &= \Phi(k_1^1, k_2^1, p) = Ad^2 - p \left( \bar{x} - \frac{p((1 - \delta) - \epsilon R^2(k_1^1, k_2^1, p))}{2\alpha V_e[\epsilon]} \right) \\
    k_1^2 &= \Phi(k_2^2, k_1^1, p) = A(k_2^2)^2 - p \left( \bar{x} - \frac{p(R^2(k_1^1, k_2^1, p) - (1 - \delta))}{2\alpha V_e[\epsilon]} \right) \\
    p_1 &= \Psi(k_1^1, k_2^1, p) = \frac{p}{2} (1 - \delta + R^2(k_1^1, k_2^1, p)) - c.
\end{align*}
\]

where \( R^2(k_1^1, k_2^1, p) = \frac{\alpha V_e[\epsilon]}{2A} \left(2A\left(d-A(k_2^2)^2+p\bar{x}+\frac{\epsilon R^2(k_1^1, k_2^1, p)}{2\alpha V_e[\epsilon]}\right)+(1-\delta)\right)\).

The Jacobian matrix of the dynamical system is

\[
J(k_1^1, k_2^1, p) = \begin{pmatrix}
\frac{\partial \Phi^2(\cdot)}{\partial k_1^1} & \frac{\partial \Phi^4(\cdot)}{\partial k_1^1} & \frac{\partial \Phi^1(\cdot)}{\partial p} \\
\frac{\partial \Phi^2(\cdot)}{\partial k_2^1} & \frac{\partial \Phi^4(\cdot)}{\partial k_2^1} & \frac{\partial \Phi^1(\cdot)}{\partial p} \\
\frac{\partial \Phi^4(\cdot)}{\partial k_1^1} & \frac{\partial \Phi^4(\cdot)}{\partial k_2^1} & \frac{\partial \Phi^4(\cdot)}{\partial p}
\end{pmatrix}.
\]

Since the first column of the above matrix has only zero entry, we can consider the sub-matrix

\[
\begin{pmatrix}
\frac{\partial \Phi^2(\cdot)}{\partial k_2^1} & \frac{\partial \Phi^4(\cdot)}{\partial k_2^1} & \frac{\partial R^2(\cdot)}{\partial k_2^1} \\
\frac{\partial \Phi^2(\cdot)}{\partial k_2^1} & \frac{\partial \Phi^4(\cdot)}{\partial k_2^1} & \frac{\partial R^2(\cdot)}{\partial k_2^1} \\
\frac{\partial \Phi^4(\cdot)}{\partial k_2^1} & \frac{\partial \Phi^4(\cdot)}{\partial k_2^1} & \frac{\partial R^2(\cdot)}{\partial p}
\end{pmatrix} = \begin{pmatrix}
2Ak^2 + \frac{(p^2)^2}{2\alpha V_e[\epsilon]} \frac{\partial R^2(\cdot)}{\partial k^2} & \frac{p(R^2(\cdot) - (1 - \delta))}{\alpha V_e[\epsilon]} - \bar{x} + \frac{p^2}{2\alpha V_e[\epsilon]} \frac{\partial R^2(\cdot)}{\partial p} \\
\frac{2\alpha V_e[\epsilon]}{2A} \frac{p\partial R^2(\cdot)}{\partial k^2} & \frac{1}{2} \frac{\partial R^2(\cdot)}{\partial k^2} + \frac{p}{2} \frac{\partial R^2(\cdot)}{\partial p} & \frac{\partial R^2(\cdot)}{\partial p}
\end{pmatrix}.
\]

The determinant and the trace of the above \(2 \times 2\) matrix is

\[
\begin{align*}
\det &= \frac{\alpha V_e[\epsilon]}{2A} \frac{A^2(k_2^2)^2 + (d + 2(k_2^2 + p\bar{x}))A^2\alpha V_e[\epsilon] - A(dp^2 + k^2(-2p^2 + k^2\alpha V_e[\epsilon]))}{(Ap^2 + \alpha V_e[\epsilon])^2} + \frac{2Ak^2\alpha V_e[\epsilon](1 - \delta)}{Ap^2 + \alpha V_e[\epsilon]} \\
\text{tr} &= A\alpha V_e[\epsilon] \left( \frac{A^2(k_2^2)^2 + (d + 2(k_2^2 + p\bar{x}))\alpha V_e[\epsilon]}{(Ap^2 + \alpha V_e[\epsilon])^2} - \frac{A(dp^2 + k^2(-2p^2 + k^2\alpha V_e[\epsilon]))}{(Ap^2 + \alpha V_e[\epsilon])^2} + \frac{(1 - \delta)(\alpha^2 V_e[\epsilon] + 2A\alpha V_e[\epsilon]p^2 + A^2p^2)}{(Ap^2 + \alpha V_e[\epsilon])^2} \right)
\end{align*}
\]
The points (a), (b) and (c) in Figure 8 corresponds to $\delta = (0.625, 0.594719, 0.575)$ in Figure 2. As the value of $\delta$ decreases from 0.625 to 0.575 the determinant crosses 1 at $\delta = 0.594719$ which proofs that the system goes through a Neimark Sacker bifurcation.

Figure 8: Stability triangle

References


