The Geometry of Aggregative Games

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Abstract

We study aggregative games in which strategy sets are convex intervals of the real line and payoffs depend only on a player’s own strategy and the sum of all players’ strategies. We give sufficient conditions on each player’s payoff function to ensure the existence of a unique Nash equilibrium in pure strategies, emphasizing the geometric nature of these conditions. These conditions are almost best possible in the sense that the requirements on one player can be slightly weakened, but any further weakening may lead to multiple equilibria. We discuss the application of these conditions to several examples, chosen to illustrate various aspects of their use. When payoffs are sufficiently smooth, these conditions can be tested using derivatives of the marginal payoff and we illustrate these tests in the applications introduced earlier.

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1 Introduction

Very little can be said about the properties of a completely general \( n \)-player noncooperative game. For this reason, researchers have imposed additional structure on such games in order to obtain stronger statements about their equilibrium properties. The burgeoning literature on supermodular games ([4][29], [41], [42]), and recent explorations of potential games ([30]), are notable examples. However, many interesting and important economic applications remain beyond the scope of these games. Even the simplest of contests, as well as surplus and costsharing games and Cournot models with constant elasticity demand functions, fail to possess the monotonicity property that defines both sub- and supermodular games. The scope of potential games, too, is limited. For example, Theorem 4.5 of Monderer and Shapley [30] shows that for a Cournot oligopoly game to be a potential game the industry demand function must be linear. This paper explores the implications of imposing an aggregative – more generally, an additively separable – restriction on the form of interdependence between players. The resulting family of games includes the examples cited above, since the applicability of our approach does not depend on monotonicity. Nor, in the Cournot example, does it require any special restrictions on the form of the demand function. Moreover, if a sub- or supermodular game also has an aggregative structure, our approach augments the analytical tricks developed in existing treatments and can extend the scope of analysis.

Many commonly studied simultaneous-move games have a similar structure in which each player’s payoff is a function of her own strategy and the sum of the strategies of all players. Selten [34] called such games ‘aggregative’. Applications include Cournot oligopoly, private provision of public goods, search models, cost and surplus sharing games – of which open access resource games are special cases – and Tullock rent-seeking contests with linear technology. Further applications can be found, via a transformation of the strategy space, in models of competition with differentiated products (Spence, [36], [37], Dixit and Stiglitz [20], Blanchard and Kiyotaki [3]) and in rent-seeking contests with nonlinear technology (Tullock [40], Szidarovszky and Yakowitz [39]).

Referring to such games, Shubik [35] said: “Games with the above property clearly have much more structure than a game selected at random. How this structure influences the equilibrium points has not yet been explored in depth.” A number of authors have studied existence of pure strategic equilibria in aggregative games in the context of specific applications such as Cournot oligopoly (for example McManus [27], [28] and Novshek [33]). Such authors sometimes use methods applicable to a wider range of aggrega-
tive games. Indeed, Kukushkin’s proof of the existence of an equilibrium of an aggregative game when best replies are non-increasing [26] uses a modification of Novshek’s approach to Cournot oligopoly. Dubey et al [22] also establish existence under assumptions of strategic complementarity or substitution, as these terms were defined by Bulow, Geanokoplos and Klemperer [4]. However, they use a somewhat different approach (pseudo-potential functions). Furthermore, although the family of aggregative games permits nonmonotonic best responses, all the above references confine attention to games with monotonic best responses.

In this paper, we focus on uniqueness as well as existence. A unique equilibrium may increase the predictive power (and thus the falsifiability) of a model. It also avoids equilibrium selection issues and relieves the modeller of the task of explaining how players overcome coordination problems. Conditions for existence and uniqueness of several aggregative games may be found in the literature. Most intensively studied are the Cournot oligopoly game ( Szidarovszky and Okuguchi [38], Kolstad and Mathiesen [25] ) and the public goods contribution games (Andreoni [1], Bergstrom, Blume and Varian [2] and Cornes, Hartley and Sandler [7]). Recently, Watts [43] (see also Cornes and Hartley [8]) has established such conditions for cost and surplus sharing game and Szidarovszky and Yakowitz [39] have proved existence and uniqueness in risk-neutral rent-seeking contests. Most of these authors use distinct approaches to establish their results, and yet the fact that all these games are aggregative, together with general results on existence, prompts the question of whether there is a general technique for investigating those situations when such games are well-behaved. Indeed, the aim of this paper is to develop such techniques and apply them to the games mentioned as well as several others. We also examine comparative statics. More specifically, we introduce assumptions on the payoffs of a player such that, if the payoffs of all players satisfy these conditions, the game will have a unique equilibrium. We stress again that these conditions permit nonmonotonic best responses. We argue that they are the best possible on individual payoffs in the sense that, if they are not satisfied, games involving such a player may exhibit multiple equilibria, even if the payoffs of all other players do satisfy the conditions.

The approach adopted by Novshek and generalized by Kukushkin identifies equilibria as fixed points of the sum of correspondences from the aggregate to the strategy space (“backwards reaction correspondence”), one for each player\(^1\). If each player’s correspondence is single-valued, continuous,

\(^1\)The backwards reaction correspondence was first suggested by Selten [34], and exploited by him to establish existence of equilibrium in an oligopoly model.
decreasing where positive and has large enough supremum, the game will have a unique equilibrium. Conditions under which this holds have been derived for several applications and more generally by Corchon ([5], [6]), who shows that sufficient conditions for existence of a unique equilibrium in an aggregative game are payoffs that are concave in own strategy and satisfy a condition close to and implied by subadditivity, together with compact, convex strategy sets. Such Nash equilibria also have other desirable properties. However, such conditions may be overly restrictive in applications. For example, in Cournot oligopoly, they rule out iso-elastic demand functions and they are not satisfied in open access resource games with standard assumptions on preferences. Nor do they apply to rent-seeking contests. In all these games, best responses as a function of the aggregate strategy of a player’s rivals initially rise and subsequently fall as the aggregate increases from zero. In Section 3, we describe a weaker set of conditions which may be applied to all the above games. These conditions include or generalize all the existence and uniqueness results described above. Although our conditions are less restrictive than Corchon, we are nevertheless able to obtain comparative statics on the behavior of the aggregate and payoffs. For example, we can unambiguously sign the effect on payoffs of adding new players. All these authors use the “backward reaction function” of Novshek [33] and Selten. However, uniqueness requires that the aggregate backward reaction function be decreasing or at least has slope less than unity. Our modification is to divide players’ reaction functions by the aggregate strategy to obtain a “share function”. Consistency requires the aggregate share function to equal one in equilibrium and, if such functions are decreasing, the equilibrium will be unique. Interestingly, the assumptions usually invoked in the applications cited above to games with nonmonotonic best response functions typically imply fulfillment of precisely this condition. No additional bounds on responses need to be introduced in order to ensure existence or uniqueness of Nash equilibrium.

The layout of the paper is as follows. In Section 2, we formally define aggregative games and describe notation. In Section 3, we describe our geometrical conditions (regularity) for ensuring existence and uniqueness of Nash equilibria. We also introduce share functions and prove that regularity implies the existence of a continuous share function that is decreasing where positive. Section 4 extends the analysis to comparative statics of payoffs. In Section 5, we apply the methodology to establish existence and uniqueness
(and comparative statics and competitive limits, where appropriate) for three applications. The sufficient conditions in Section 3 are on the payoffs of individual players and, in Section 6, we investigate their necessity. Firstly, we show how regularity can be slightly weakened for one player without losing existence, uniqueness and comparative statics results of equilibria. However, no further weakening of these conditions is possible. In all our analyses, the only smoothness condition we have imposed is continuity. However, regularity can be tested more conveniently when payoffs are twice differentiable in the interior of the payoff space. Sufficient conditions for regularity when payoffs are differentiable are established in Section 7, and applied to the three examples introduced in Section 5. Finally, Section 8 offers conclusions and discusses several extensions of the methodology.

2 Aggregative games

We consider the simultaneous-move game \( G = (I, \{S_i\}_{i \in I}, \{\pi_i\}_{i \in I}) \), in which each of the finite set of players \( I \) has a strategy set \( S_i = [0, w_i] \) for some \( w_i > 0 \). (In some applications, the natural strategy set may be \( \mathbb{R}_+ \). However, if strategies \( x_i > w_i \) are dominated\(^4\), the theory to be described is still applicable.) Denote \( \prod_{j \in I} S_j \) by \( S \) and \( \prod_{j \in I \setminus \{i\}} S_j \) by \( S_{-i} \). We write \( x_i \in S_i \) for Player \( i \)'s strategy and \( X \) for \( \sum_{i \in I} x_i \). If \( x \in S \) is a strategy profile, \( \pi_i : S \rightarrow \mathbb{R} \) denotes the payoff function of Player \( i \). Henceforth, we assume, without explicit statement, that \( \pi_i \) is continuous except possibly at \( x = 0 \). (The exceptional treatment of the origin is useful in some applications\(^5\).)

We call such a game aggregative if, for each \( i \in I \), there is a function \( v_i : S_i \rightarrow \mathbb{R} \), where

\[
S_i = \{(x_i, X) : 0 \leq x_i \leq \max \{w_i, X\}\},
\]

such that

\[
\pi_i(x) = v_i(x_i, X) \text{ for all } x \in S \text{ satisfying } \sum_{i \in I} x_i = X. \tag{1}
\]

\(^4\)An example is a Cournot oligopoly in which average cost is positive and non-decreasing and price approaches or is equal to zero for large output; if \( w_i \) is the breakeven level of output, \( x_i > w_i \) is dominated by \( x_i = 0 \).

\(^5\)For example, in a rent-seeking game, the sum of payoffs of all players is equal to the rent minus the aggregate expenditure on rent-seeking, provided at least one player’s expenditure is positive. If all expenditures are zero, so are all payoffs. Hence, the sum of payoffs must be discontinuous at the origin and therefore the payoff of at least one player must also have this property.
Since feasibility dictates that \( X \leq \sum_{i \in I} w_i \), we could have imposed (1) only for such \( X \). However, we do not restrict the definition in this way, since our focus is on conditions on \( v_i \) ensuring a unique Nash equilibrium and well-behaved comparative statics for any set of competitors with payoffs also satisfying these conditions. Not restricting \( X \) also permits the study of limiting equilibria as the number of players becomes large. With slight notational abuse, we shall write the aggregative game as \( G = (I, w, \{v_i\}_{i \in I}) \), where \( w = \{w_i\}_{i \in I} \).

To simplify the exposition, it is convenient to focus on non-null (\( x \neq 0 \)) equilibria. Note that there cannot be a null equilibrium if, for any \( i \in I \), there is \( x \in (0, w_i] \), for which \( v_i(x, x) > v_i(0, 0) \). (In a Cournot oligopoly, the condition says that at least one firm can make positive monopoly profits.) Any equilibrium must satisfy \( X > 0 \).

3 Existence and Uniqueness

In this section, we investigate existence and uniqueness of non-null equilibria in pure strategies. We introduce two assumptions: the aggregate crossing condition [ACC] and radial crossing condition [RCC]. To describe and exploit these, a little notation and a preliminary lemma are needed.

When \( \arg \max_{x_i \in S_i} \pi_i(x) \) is a convex set for all \( x_{-i} \in S_{-i} \), we shall say that Player \( i \) has convex best responses. In an aggregative game, best responses depend only on \( X_i = \sum_{j \in I \setminus \{i\}} f_j \) and it is convenient to write \( B_i(X_{-i}) \) for the set of best responses.

**Condition 3.1 (CBR)** \( B_i(X_{-i}) \) is a convex set.

The continuity properties of \( v_i \) imply that \( B_i \) has closed graph except possibly at the origin\(^6\). It is also useful to observe that the graph of \( B_i \) either crosses or lies entirely to the right (in the \((x_i, X_{-i})\)-plane) of any non-horizontal line. This connectedness property is set out in the following lemma\(^7\).

**Lemma 3.1** Assume CBR for all \( X_{-i} \geq 0 \). Suppose that \( x_{i}^0 \in B_i(X_{-i}^0) \) and \( X_{-i}^0 \leq \alpha + \beta x_{i}^0 \), where \( \alpha \) and \( \beta \) are real numbers. Then there exists \( X_{-i}' \geq X_{-i}^0 \) and \( x_{i}' \in B_i(X_{-i}') \) such that \( X_{-i}' = \alpha + \beta x_{i}' \).

\(^6\)Recall that payoffs need not be continuous at the origin.

\(^7\)In fact \( B_i \) is connected in the conventional sense but this is more complicated to prove and not needed in the sequel.
The proof of this, and all other formal propositions, is given in the appendix. For Player $i$ and any $X > 0$, it is useful to focus on the set of strategies $x_i$ that the player can choose in a Nash equilibrium in which the value of the aggregate is $X$. Each such $x_i$ must be a best response to $X_i = X - x_i$. Hence, the graph of the correspondence that maps $X$ into the set of strategies consistent with equilibrium $X > 0$ is

$$L_i = \left\{ (x_i, X) \in \tilde{S}_i' : x_i \in B_i(X - x_i) \right\},$$

where $\tilde{S}_i' = \tilde{S}_i \setminus \{0\}$. Note that $L_i$ is the image of the graph of $B_i$ under the linear transformation $(x_i, X_i) \mapsto (x_i, x_i + X_i)$, which leads to the following corollary of Lemma 3.1.

**Corollary 3.1** Suppose that $(x_i^0, X^0) \in L_i$ and $X^0 \leq \alpha + \beta x_i^0$, where $\alpha$ and $\beta$ are real numbers. Then there exists $(x_i', X') \in L_i$ such that $X' = \alpha + \beta x_i'$ and $X' - x_i' \geq X - x_i$.

Our remaining conditions may now be stated as follows.

**Condition 3.2 (ACC)** Player $i$’s best responses satisfy the aggregate crossing condition at $X$ if there is at most one $x_i$ satisfying $(x_i, X) \in L_i$.

**Condition 3.3 (RCC)** Player $i$’s best responses satisfy the radial crossing condition at $\sigma$ if there is at most one value of $X$ satisfying $(\sigma X, X) \in L_i$.

Geometrically, these conditions can be visualized graphically with $X$ on the horizontal and $x_i$ on the vertical axis. Then Conditions ACC and RCC state that $L_i$ meets a vertical line at $X$ and a ray through the origin with slope $\sigma$ at most once. Figure 1, Panel (a), shows a situation in which all three conditions are satisfied. In panel (b), best responses are not everywhere convex. Panels (c) and (d) depict violations of Conditions ACC and RCC respectively. In both of these panels, there is also a value of $X_{-i}$ for which the set of responses is an interval. Our next lemma demonstrates that the appearance of this feature alongside violations of one or the other of the crossing conditions is no coincidence. It is convenient to write it using the following terminology.

**Definition 3.4** Player $i$ is regular if best responses satisfy

1. CBR for all $X_{-i} \geq 0$,
2. ACC for all $X > 0$,
3. RCC for all $\sigma \in (0, 1]$.

**Lemma 3.2** If Player $i$ is regular, $B_i$ is singleton-valued for $X_{-i} > 0$.

The case $X_{-i} = 0$ is complicated by the possibility of discontinuity at the origin. The radial crossing condition with $\sigma = 1$ implies that $\arg \max v_i (X; X)$ is either a singleton or empty. Hence, there are two possible cases: (i) $v_i (X, X)$ is maximized at $X_i > 0$, or (ii) $v_i (X, X)$ has no maximum in $X > 0$. Note that in Case (i), $(X_i, X_j) \in L_i$ and we shall refer to $X_i$ as the *participation value* of Player $i$. In a Cournot oligopoly, $X_i$ is the monopoly output of firm $i$. In Case (ii), it is convenient to set $X_i = 0$.

Under the assumptions of Lemma 3.2, we can define a best response function, which we write $b_i (X_{-i})$: $B_i (X_{-i}) = \{ b_i (X_{-i}) \}$. Since it has a closed graph, $b_i$ is a continuous function. It follows that, if $L_i$ crosses the line $X = X^0$, it crosses $X = X'$ for all $X' > X^0$. For, if we define the continuous function

$$
\psi_i (X_{-i}) = b_i (X_{-i}) + X_{-i},
$$

(3)
there exists some $X^0_i \leq X^0$ for which $\psi_i (X^0_i) = X^0 < X'$. Since $\psi_i (X') > X'$, the intermediate value theorem implies that there is $X'_i$ satisfying $\psi_i (X'_i) = X'$, which is equivalent to our claim. The aggregate crossing condition implies that $L_i$ crosses $X = X^0$ exactly once for each $X^0$ in a semi-infinite interval. This allows us to define a function $r_i$ on this interval, by taking $(r_i(X^0), X^0)$ to be the crossing point. We call $r_i$ the replacement function of Player $i$. Note that this function has closed graph $(L_i)$ and is therefore continuous. For our purposes, it is more convenient to use the share function defined as $s_i (X) = r_i (X) / X$. The radial crossing condition implies that, for any $\sigma \in [0, 1]$, there is at most one value of $X$ satisfying $s_i (X) = \sigma$. Since $L_i \subset \tilde{S}_i$, we must also have $s_i (X) \leq \omega_i / X$ and we can conclude that $s_i$ is strictly decreasing where positive. In Case (i), the domain of both $r_i$ and $s_i$ is $[X_i, \infty)$. (If $\omega_i$ were defined for $X < X_i$, we would have $s_i (X) > 1$, which is impossible.) In Case (ii), the domain of $r_i$ and $s_i$ is $(0, \infty)$ and we write

$$\overline{\sigma}_i = \sup_{X > 0} s_i (X) = \lim_{X \to 0^+} s_i (X)$$

for the least upper bound of the share function. The following result summarizes and extends these observations. The proofs of the remaining assertions are straightforward and are omitted.

**Proposition 3.1** Regularity is a necessary and sufficient condition for the existence of a share function for Player $i$, which is strictly decreasing where positive and has domain $[X_i, \infty)$ or $\mathbb{R}_{++}$. The former case occurs if and only if $i$ has positive participation value $X_i$ in which case $s_i (X_i) = 1$ and $s_i (X) < 1$ for all $X > X_i$. In either case, either (a) there is $X_i > 0$ such that $s_i (X) = 0$ if and only if $X \geq X_i$, or (b) $s_i (X) \to 0$ as $X \to \infty$.

Figure 2 shows the four possible shapes of the graph of the share function. The distinction between the cases (a) and (b) rests on whether $L_i$ meets the $x = 0$ axis. If so, $X_i$ is the greatest lower bound of the intersection of $L_i$ and this axis. Furthermore, $L_i$ coincides with this axis for $X \geq X_i$, otherwise continuity would imply a contradiction of the radial crossing condition (for small enough $\sigma > 0$). We shall refer to $X_i$ as the dropout value of Player $i$. In a Cournot oligopoly, $X_i$ is the competitive output level of the $i$’th firm. That is, the output at which price falls to the marginal cost of Player $i$ at the origin. In case (b), it is convenient to set $X_i = +\infty$, so the dropout value is always defined.

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8This is our name for the “backwards reaction function”. It is intended to capture the idea that the value of the replacement function at $X$ is the output level that, if subtracted from $X$, will be exactly replaced by the player, maintaining the aggregate level $X$. 

9
Remark 3.5 A detailed examination of the arguments leading to the proposition shows that the boundedness of the strategy set of Player $i$ is not required for all conclusions in the proposition. In particular, if the strategy set is $\mathbb{R}_+$ and best responses are unique (or possibly empty if $X_{-i} = 0$) then all the conclusions in the proposition except (b) remain valid. Indeed, it is straightforward to see that a share function that does not meet the axis must be either strictly increasing or strictly decreasing. In the latter case, an additional assumption is needed to establish that the share function vanishes asymptotically.

Share functions allow us to compute equilibria because of the following result, easily proved by chasing definitions.

Lemma 3.3 Suppose that all players have share functions. Then $\hat{x}$ is a non-null Nash equilibrium if and only if $\hat{X}$ lies in the domain of all $s_i$ and $\hat{x}_i = \hat{X} s_i \left( \hat{X} \right)$ for all $i \in I$, where $\hat{X} = \sum_{i \in I} \hat{x}_i$. 

10
This lemma implies that there is an equilibrium with aggregate value \( \hat{X} \) if and only if the aggregate share function \( s_t(X) = \sum_{i \in I} s_i(X) \) satisfies \( s_t(\hat{X}) = 1 \). Note that the domain of the aggregate share function is \( X \geq \bar{X} = \max_{i \in I} X_i \), where the maximum is over players with finite participation value, if any, and is \( \mathbb{R}^{++} \), otherwise. Under the conclusions of Proposition 3.1, the aggregate share function is continuous and approaches zero as \( X \to \infty \). If at least one player has positive participation value, \( s_t(\max_{i \in I} X_i) \geq 1 \) and there is a unique \( X \) satisfying \( s_t(X) = 1 \). If no player has a positive participation value, there is a unique equilibrium if and only if the aggregate share function exceeds 1 for small enough \( X \). This gives the following existence and uniqueness result.

**Theorem 3.6** Suppose all players in the aggregative game \( G = (I, w, \{v_i\}_{i \in I}) \) are regular. If no player has a positive participation value, suppose further that

\[
\sum_{i \in I} \sigma_i > 1.
\]

Then, \( G \) has a unique non-null Nash equilibrium.

If no player has a positive participation value, and (5) is invalid, \( G \) has no such equilibrium.

Figure 3 shows the graphs of share functions in a 3-player game. The thick line is the graph of the aggregate share function, obtained by summing the individual share functions vertically. Note that equilibrium \( \hat{X} \) exceeds \( X \), which explains the terminology ‘participation value’. Furthermore, Player \( i \) is active (\( \hat{x}_i > 0 \)) if and only if \( \hat{X} < X_i \), which explains the terminology ‘dropout value’. In the figure, Player 1 is inactive at the equilibrium.

Submodularity implies, but is weaker than, regularity for all players, at least when ACC holds. Indeed, suppose that best responses are decreasing in the sense that

\[
x \in B_i(X_{-i}), x' \in B_i(X'_{-i}), X'_{-i} > X_{-i} \implies x' \leq x,
\]

with strict inequality if \( x > 0 \). Then, it is immediate that, for any \( \sigma \) satisfying \( 0 < \sigma < 1 \),

\[
\sigma X \in B_i((1-\sigma)X)
\]

can be satisfied by at most one value of \( X \). This says that RCC holds for \( \sigma \in (0,1) \). Suppose that, in addition, ACC is satisfied for all \( X > 0 \) and, if there is a best response to \( X_{-i} = 0 \), it is unique. Then player \( i \) is regular.

However, RCC is a weaker condition than submodularity. Indeed, in the sequel, we discuss a class of supermodular games in which all players are reg-
ular. More generally, regularity does not entail monotonic best responses. For example, in a Cournot oligopoly with isoelastic demand and constant (positive) marginal costs, all players are regular, yet best responses are initially increasing but eventually decreasing. Nevertheless, we shall show that submodularity and supermodularity in addition to regularity can sometimes yield stronger comparative statics than regularity alone as it allows us to sign the slope of the replacement function.

**Proposition 3.2** Suppose all players in the aggregative game \( G = (I, w, \{v_i\}_{i \in I}) \) are regular and the game is submodular [supermodular]. Then the replacement function \( r_i \) is strictly decreasing [increasing], where positive.

**Proof.** See Appendix □

The previous proposition applies to individual players; if some players had increasing and others decreasing best responses, individual replacement functions would inherit these properties. However, such mixed games appear to be uncommon in practice. We shall exploit this proposition in Section 4, which deals with comparative statics.
4 Comparative Statics

In this section, we discuss comparative statics, noting that such analyses are much more intricate in a strategic than in a price-taking environment. The key result is the following.

**Lemma 4.1** Suppose Player $i$ is regular, has share function $s_i$ and $v_i(x_i, X)$ is strictly increasing in $X$ for all $x_i > 0$. If $X^2 > X^1 > 0$ and $X^1 \geq X_i$ (participation value), then

$$v_i(X^1s_i(X^1), X^1) \leq v_i(X^2s_i(X^2), X^2)$$  \hspace{1cm} (6)

and the inequality (6) is strict if $X^1 < X_i$.

If $v_i(x_i, X)$ is strictly decreasing in $X$ for all $x_i > 0$, the same results hold but with the inequality (6) reversed.

This lemma can be applied to show that adding extra players to a game does not reduce $X$ and makes existing players (weakly) worse or better off according as $v_i$ is decreasing or increasing in $X$. If one of the additional players is active (chooses a positive strategy in equilibrium), currently active players are strictly worse (or better) off.

**Theorem 4.1** Let $G^k = (I^k, w^k, \{v_i^k\}_{i \in I^k})$ for $k = 1, 2$ and suppose that $I^1 \subset I^2$ and $w_1^i = w_2^i, v_1^i = v_2^i$ for $i \in I^1$. Suppose further that all players in $I^2$ are regular and $v_i(x_i, X)$ is strictly increasing [decreasing] in $X$ for all $x_i > 0$ and $i \in I^1$. If $G^1$ has a (unique) non-null Nash equilibrium $\hat{x}^1$, there is an equilibrium $\hat{x}^2$ of $G^2$. Supposing $\hat{x}_i^2 > 0$ for some $i \in I^2 \setminus I^1$ and writing $\hat{X}^k = \sum_{j \in I^k} \hat{x}_j^k$,

1. $\hat{X}^2 > \hat{X}^1$,

2. inactive players in $G^1$ are inactive in $G^2$,

3. active players in $G^1$ are better [worse] off in $G^2$ than in $G^1$,

4. if the game has decreasing {increasing} best responses, $\hat{x}_i^2 \leq \{>\} \hat{x}_i^1$ for all $i \in I^1$. In the former case, the inequality is strict if $\hat{x}_i^1 > 0$.

Note that regularity and monotonicity of $v_i(x_i, X)$ are not sufficient to allow us to sign individual responses. However, as Part 3 of the theorem shows, this does not prevent us signing changes in payoffs.

Part 1 of the theorem can be illustrated graphically. Suppose that there are initially two players. Figure 4 shows the graphs of their share functions,
\( s_1(X) \) and \( s_2(X) \). The associated aggregate share function, graphed by the thick continuous line, takes the value 1 at the Nash equilibrium, \( X = \hat{X}^1 \). Now a third player, whose share function is \( s_3(X) \), enters the game. The aggregate share function of the new game is graphed by the thick dashed line, and equilibrium now occurs at \( X = \hat{X}^2 \).

In a Cournot game, decreasing demand implies that profits \( (\nu_i) \) strictly decrease with aggregate output \( (X) \), for a given level of firm output. The theorem shows that entry increases output and has an adverse effect on incumbent firms. That a condition such as regularity is needed for such a conclusion was shown by McManus [27], [28].

As a second application of the lemma, we consider the effect of an idiosyncratic change in payoffs of a single player \( i \in I \). This yields two aggregative games, \( G^1 \) and \( G^2 \), where \( G^k = \left( I^k, w^k, \{v^k_i\}_{i \in I^k} \right) \) and \( w^1_i = w^2_i, v^1_i = v^2_i \) for all \( i \in I \setminus \{i\} \). The next result gives conditions on the change of payoffs for player \( i \) entailing an increase in equilibrium aggregate.

**Theorem 4.2** Suppose that (i) all players in \( I \) are regular in \( G^1 \) and \( G^2 \); (ii) \( v_i(x_i, X) \) is strictly increasing [decreasing] in \( X \) for all \( x_i > 0 \) and all
\( i \in I \setminus \{i\}; \) (iii) \( X_i^2 \geq X_i^1 \) and \( s_i^2 (X) > s_i^1 (X) \) for all \( X \) in the domain of \( s_i^2 \).

If \( G^1 \) has a (unique) Nash equilibrium non-null \( \tilde{x}^1 \), there is an equilibrium \( \tilde{x}^2 \) of \( G^2 \). Supposing \( \tilde{x}_i^2 > \tilde{x}_i^1 \) and writing \( \tilde{X}^k = \sum_{j \in I^k} \tilde{x}_j^k \),

1. \( \tilde{X}^2 > \tilde{X}^1 \),

2. players inactive in \( G^1 \) are inactive in \( G^2 \),

3. players other than \( i \), active in \( G^1 \), are better [worse] off in \( G^2 \) than in \( G^1 \).

If the game has decreasing [increasing] best responses,

4. \( \tilde{x}_i^2 \leq \{ \geq \} \tilde{x}_i^1 \) with strict inequality if \( \tilde{x}_i^1 > 0 \), for \( i \in I \setminus \{i\} \),

5. \( \tilde{x}_i^2 > \tilde{x}_i^1 \).

The condition that the share function of player \( i \) in \( G^1 \) exceeds that in \( G^2 \) is equivalent to the requirement that the best responses of Player \( i \) are higher in \( G^2 \) than \( G^1 \).

Theorem 4.2 applies only to a change in payoffs of a single player. Obviously, the theorem may be applied cumulatively to changes in the payoffs of a proper subset of players. In some applications, we may wish to analyze a change in all payoffs. For example, an increase in costs in an input market or imposition of a tax may lead to an increase in average and marginal costs for all firms. In general, consider a change in all payoffs in a game in which all players are regular (in both games) and, for all \( i \in I \), we have \( (x^k, X) \in I^k \) for \( k = 1, 2 \) implies that \( x^1 \leq x^2 \) and that this inequality is strict if \( x^2 > 0 \). Repeated application of Part 1 of the theorem shows us that equilibrium \( X \) increases\(^9\). In general, we are unable to sign changes in individual strategies except in the case of increasing best responses, where all strategies increase.

### 5 Applications

In this section, we introduce three applications chosen to illustrate the application of the aggregate and radial crossing conditions. In each case, we give conditions for regularity and thus for the existence of a unique equilibrium. We also briefly discuss comparative statics where appropriate.

\(^9\)Condition (ii) is only needed for signing the change in payoffs of players whose payoffs do not change. There are no such players in this example.
5.1 Search games

We consider a version of the “coconut economy” search game introduced by Diamond [19] which omits production and is also discussed by Milgrom and Roberts [29] and Dixon and Somma [21]. Each player $i$ in the set of players $I$ exerts effort $x_i$ in searching for trading partners. Search incurs a benefit which is proportional both to own effort and to the aggregate effort exerted by the other players as well as a cost described by a cost function $c_i$. The payoff function take the form:

$$\pi_i(x) = \theta x_i \left( \lambda_i + \sum_{j \neq i} x_j \right) - c_i(x_i),$$

where $\theta > 0$ is a parameter scaling the overall return to search and $\lambda_i \geq 0$ represents a payoff from search effort that does not result in meeting a trading partner$^{10}$.

**Example 5.1 (Search)** Player $i$’s strategy set$^{11}$ is $[0, w_i]$ and payoff is:

$$v_i(x_i, X) = \theta x_i (\lambda_i + X - x_i) - c_i(x_i).$$

Much of the interest in such games lies in their multiple equilibria and the consequent coordination problems. If $\lambda_i = 0$ for all $i$, $\hat{x} = 0$ is an equilibrium. Here, we focus on unique non-null equilibria, which, if at least one $\lambda_i$ is positive, will be the unique equilibrium. It is readily checked that this game is supermodular, which guarantees existence of an equilibrium as well as monotone comparative statics. We include the game here to illustrate the fact that supermodularity is not inconsistent with regularity and to show that existence of multiple non-null equilibria places a limit on how fast costs can increase. Indeed, we show that if the marginal cost at $x$ increases faster than $x$ there will be a unique non-null equilibrium. Specifically, we impose the following condition on Player $i$.

**EA** The cost function $c_i$ is continuous, differentiable for positive argument and $c_i'(x)/x$ is a positive, strictly increasing function of $x > 0$.

If, for example, $c_i = kx^\alpha$, where $k > 0$, then **EA** is satisfied if and only if $\alpha > 2$.

$^{10}$Perhaps from finding coconuts lying on the ground.

$^{11}$In the original game, strategy sets were unbounded.
Proposition 5.1 If $EA$ holds for Player $i$ in the Search game, Example 5.1, then $i$ is regular. The participation value $X_i$ is positive if and only if $\lambda_i > 0$. If $\lambda_i = 0$,

$$\bar{\sigma}_i = \left[1 + \lim_{x \to 0^+} \frac{c'_i(x)}{\theta x}\right]^{-1}. \quad (7)$$

Together with Theorem 3.6, the preceding proposition shows that, if $EA$ holds for all players, then there is a unique non-null equilibrium, except possibly if $\lambda_i = 0$ for all $i$. In the latter case, we also require $\sum_{i \in I} \bar{\sigma}_i > 1$, where $\bar{\sigma}_i$ satisfies (7).

We also note that $v_i$ is strictly increasing in $X$ for all $x_i > 0$ so, by Theorem 4.1, additional searchers lead to increased search effort by existing searchers and an improvement in their payoffs.

5.2 Public good contribution games

Our next application is the classic problem of voluntary subscription to the provision of a public good. Cornes et al. [7] provide a recent discussion of this model. A set $I$ of consumers has to decide non-cooperatively what quantity of a public good to provide. Consumer $i \in I$ chooses how much, $x_i$, of her income $m_i$ to devote to a public good. Preferences are represented by an ordinal utility function $u_i(y_i;X)$ where $y_i$ is expenditure on private consumption and $X$ is total expenditure on the public good.

Example 5.2 (Pure Public Goods) Player $i$’s strategy set is $[0,m_i]$ and payoff is utility:

$$\pi_i(x) = v_i(x_i,X) = u_i(m_i - x_i,X) \text{ when } X > 0$$

and $\pi_i(0) = v_i(0,0) = 0$.

The following is a generalization of a well-known condition.

PA Player $i \in I$ has continuous, strictly increasing preferences and the equal-price income expansion paths is upwards sloping.

PA is most readily exploited in terms of the set of (absolute values of) the marginal rate of substitution which we denote by $MRS_i(y,X)$. [That is, the set of slopes of supporting lines (with $X$ on the horizontal axis) to the upper preference set at $(y,X)$] In particular, if $1 \in MRS_i(y,X)$ and $\delta' \in MRS_i(y',X')$, where $y, X > 0$, we require $\delta' \leq 1$ if $X' \geq X, y' \leq y$ and $\delta' \geq 1$ if $X' \leq X, y' \geq y$, with strict inequality in both cases if $(y',X') \neq (y,X)$. This requirement is implied by, but weaker than, normality of both goods.
Proposition 5.2 If PA holds for a player in the game Public Good Contribution game, Example 5.2, then that player is regular.

Since payoffs are continuous, existence of a unique equilibrium is assured. Further, PA implies that $v_i$ is a strictly increasing function of $X$, for given $x_i$ and that best responses are decreasing. Theorem 4.1 shows that contributions are offset by a reduction in contributions by current players, but not enough to reduce total public good provision. Consequently, current players, even non-contributors, are made better off. These results reflect the standard notions of free and easy riding discussed in Cornes and Sandler [15] for example.

5.3 Contests

Our final application concerns contests for a biddable rent with risk averse contestants. The corresponding game played by risk neutral contestants is strategically equivalent to a Cournot oligopoly model with unit elastic demand, provided production functions are strictly increasing (Vives [42]). This equivalence does not extend to risk averse contestants, but the game is still aggregative and can be analyzed by using the methods described above\(^{12}\).

Formally, suppose Player $i \in I$ spends $y_i \geq 0$ on seeking an indivisible rent $R$ which can be won by only one player. The probability that $i$ wins the rent is given by the contest success function:

$$p_i(y) = \frac{f_i(y_i)}{\sum_{j \in I} f_j(y_j)},$$

where $f_i$ is a strictly increasing function. We assume that Player $i$ is risk averse or risk neutral and has preferences over lotteries described by a von Neumann-Morgenstern utility function $u_i$. In this example, it is useful to transform the state space by writing $x_i = f_i(y_i)$. Since $f_i$ is strictly increasing, it has an inverse function which we denote $g_i$.

Example 5.3 (Rent Seeking) Player $i$’s strategy set is $[0, f_i(R)]$ and payoff is expected utility:

$$\pi_i(x, X) = \frac{x_i}{X} u_i(R - g_i(x_i)) + \left[ \frac{X - x_i}{X} \right] u_i(-g_i(x_i)) \text{ when } X > 0$$

and $\pi_i(0) = v_i(0, 0) = u_i(0)$.

Consider the following condition.

**RA** (i) Player $i$ is either risk averse with constant absolute risk aversion or risk neutral;

(ii) $f_i$ is continuous\(^{13}\), concave, differentiable in $x > 0$ and satisfies $f'_i(x) > 0$ and $f_i(0) = 0$.

Note that the second part of the condition implies that $g_i(0) = 0, g'_i(x) > 0$ for $x > 0$ and that $g_i$ is convex. The first part of the condition requires that $u_i(z) = 1 - \exp\{-\alpha_i z\}$ with $\alpha_i > 0$ or $u_i(z) = z$. Existence and uniqueness in the case when all players are risk neutral was established by Szidarovzsky and Okuguchi\([39]\).

**Proposition 5.3** If **RA** holds for Player $i$ in the Rent-seeking Game, Example 5.3, then that player is regular. Player $i$ has a finite dropout value if and only if $g'_i = \inf_{x>0} g'_i(x) > 0$, in which case the dropout value satisfies $\overline{X}_i = \beta_i / g'_i$, where

$$\beta_i = \frac{1 - \exp\{-\alpha_i R\}}{\alpha_i}$$

if $\alpha_i > 0$ and $\beta_i = R$ if $\alpha_i = 0$.

For $X > 0$, the share function $s_i$ satisfies

$$1 - \frac{1 - \alpha_i \beta_i}{1 - \alpha_i \beta_i \sigma_i(x)} - \alpha_i X g'_i \left[ X s_i (X) \right] = 0.$$ 

It follows that $\sigma_i = \lim_{X \to 0^+} s_i(X) = 1$ and, therefore, from Theorem 3.6 that the game has a unique equilibrium, provided there are two or more players.

Furthermore, $v_i$ can be written in the form

$$u_i (-g_i(x_i)) + \frac{x_i}{X} \left[ u_i (R - g_i(x_i) - u_i (-g_i(x_i)) \right]. \quad (8)$$

Since $u_i$ is strictly increasing, $v_i(x, X)$ is strictly decreasing in $X$ for $x > 0$. It follows from Theorem 4.1 that additional contestants make existing active contestants worse off. Note that we cannot, in general, sign the changes in individual expenditure, since we do not have monotonic best responses, \(^{13}\)Continuity for $y_i > 0$ is a consequence of concavity.
Indeed, we cannot even conclude that aggregate expenditure: \( \sum_{i \in I} y_i \) increases. Whilst aggregate \( X \) certainly does increase, there is typically no simple mapping, let alone a monotonic function, from \( X \) to aggregate expenditure, except when \( f_i \) is linear and identical for all \( i \).

Finally, note that Player \( i \) has a finite dropout value if and only if \( g_i > 0 \) and, since \( g_i \) is the inverse function of \( f_i \), this holds if and only if

\[
\int f_i = \sup_{y > 0} f_i^\prime (y) < \infty. \tag{9}
\]

For example, with the transformation function \( f_i (y) = c_i y^r \) introduced by Tullock, the dropout value is finite if \( r = 1 \) but not if \( r < 1 \). Further applications of this approach to contests may be found in [9].

6 Weak regularity

Study of the applications above prompts the question of whether regularity is a necessary, as well as sufficient, condition on individual players for a unique equilibrium. Uniqueness of equilibrium in the knife-edge case of a share function that is decreasing but not always strictly is less straightforward. If a single player has such a share function, equilibrium will still be unique. The share functions of the regular players are strictly decreasing where positive, so the aggregate share function will be strictly decreasing at share value 1 which implies at most one equilibrium. However, if the graphs of two or more players had such horizontal sections, the graph of the aggregate share function could have a horizontal segment with unit share, resulting in multiple equilibrium values of \( X \). In Figure 5, the share function of each of three players has a horizontal stretch at the share value of 1/3. Similarly, it is possible to have vertical sections in the graph of one player (turning the share function into a correspondence) without losing existence and uniqueness. Once again, were two or more players to have a vertical section at the same value of \( X \), multiple equilibria would be possible, though the equilibrium value of \( X \) would still be unique.

In the remainder of this section, we relate these properties of share functions to geometric properties of the set \( L \) (weak regularity) and then examine necessity of these properties for existence and uniqueness of equilibrium. In particular, we show that existence of a unique equilibrium is assured if one player is weakly regular and the rest are regular. Furthermore, no further
weakening of these assumptions is possible, provided such assumptions impose restrictions solely on the payoffs of individual players.

**Definition 6.1** Player $i$ is weakly regular if

1. $B_i (X_{-i})$ is a singleton for all $X_{-i} > 0$ and $B_i (0)$ is either a singleton or empty,

2. the set $\{x : (x, X) \in L_i\}$ is convex for all $X > 0$,

3. the set $\{X : (\sigma X, X) \in L_i\}$ is convex for all $\sigma \in (0, 1]$.

It follows from Lemma 3.2 that a regular player is also weakly regular. For weakly regular players, we can define a convex-valued share correspondence for any $X > 0$ by

$$S_i (X) = \left\{ \frac{x}{X} : (x, X) \in L_i \right\}.$$ (10)

Note that $S_i$ has a closed graph, except possibly at $X = 0$, and therefore Corollary 3.1 holds. It follows that, if $S_i (X^0) \neq \emptyset$ and $X' > X^0$, then $S_i (X') \neq \emptyset$. Indeed, this is essentially the same argument used above to prove that the domain of the replacement function is a semi-infinite interval.

---

15Since $L_i$ is closed, except possibly at the origin.
16Put $\beta = 0$ in the corollary.
Furthermore, we can show that $S_i$ is decreasing in the sense that there is at most one value of $X$ satisfying $S_i(X) = 1$ and

$$\sigma^1 \in S_i(X^1), \sigma^2 \in S_i(X^2), X^2 > X^1 \implies \sigma^2 \leq \sigma^1.$$ 

Note that, if Player $i$ is weakly regular, either $v_i(X,X)$ is maximized at $X_i > 0$, or (ii) $v_i(X,X)$ has no maximum in $X > 0$. As above, we refer to $X_i$ as the participation value.

**Proposition 6.1** Weak regularity is a necessary and sufficient condition for the existence of a non-empty, convex-valued, decreasing share correspondence for Player $i$ with domain $[X_i, \infty)$ or $\mathbb{R}^+$. The former case occurs if and only if $i$ has positive participation value $X_i$ and, in that case, $S_i(X_i) = \{1\}$ and $\delta < 1$ for all $\delta \in s_i(X)$ with $X > X_i$. In either case, either (a) there is $X_i > 0$ such that $S_i(X) = \{0\}$ if and only if $X \geq X_i$, or (b) $\max S_i(X) \rightarrow 0$ as $X \rightarrow \infty$.

Lemma 3.3 generalizes to a characterization of equilibria in terms of share correspondences.

**Lemma 6.1** There is a Nash equilibrium $\hat{\mathbf{x}} \neq 0$ if and only if $\hat{x}_i / \hat{X} \in S_i(\hat{X})$ for all $i \in I$, where $\hat{X} = \sum_{i \in I} \hat{x}_i$.

It follows that $\hat{X}$ is an equilibrium value of the aggregate if and only if

$$1 \in \sum_{i \in I} S_i(\hat{X}),$$

using conventional set addition. If all but one player is regular, it follows from Proposition 3.1 that the share functions of the regular players are strictly decreasing where positive. Since the share correspondence of the exceptional player is non-decreasing, the aggregate share correspondence is strictly decreasing where positive: $\sigma^1 \in S_i(X^1), \sigma^2 \in S_i(X^2), \sigma_2 > 0$ and $X^2 > X^1$ imply $\sigma^2 < \sigma^1$. Thus, there is at most one equilibrium value of $X$ and, for such a value, the strategies of the regular players are uniquely determined. This implies a single equilibrium. Defining $\bar{\sigma}_i = \sup_{X > 0} \max S_i(X)$, for weakly regular players, we have the following generalization of Theorem 3.6.

**Theorem 6.2** Suppose that all but one players in the aggregative game $\mathcal{G} = (I,w,\{v_i\}_{i \in I})$ are regular and the remaining player is weakly regular. If no
player has a positive participation value, suppose further that $\sum_{i \in I} \sigma_i > 1$. Then, $\mathcal{G}$ has a unique non-null Nash equilibrium.

If no player has a positive participation value, and (5) is invalid, $\mathcal{G}$ has no such equilibrium.

A decreasing aggregate share correspondence at unit share value is clearly necessary for a unique equilibrium. However, this does not imply regularity or even weak regularity of the players; for example, an increase in one player’s share function can be offset by a faster decrease in another’s. However, if we rule out such interactions and impose conditions only on individual payoffs, Theorem 6.2 is best possible in the following sense. If there is at most one equilibrium in all games in which a player plays against regular competitors, then that player is weakly regular. Similarly, if there is at most one equilibrium when an individual plays against competitors all but one of whom are regular and the exceptional player is weakly regular, then that individual is regular. Theorem 7.2 shows that it is enough to consider two-player games to justify these claims.

Proposition 6.2 If every game played by a player against a weakly regular opponent with positive participation value has a unique non-null equilibrium, then that player is regular. Similarly, if every game played by a player against a regular opponent with positive participation value has a unique equilibrium, then that player is weakly regular.

7 Smooth payoffs

Establishing regularity by direct application of the aggregate and radial crossing conditions may require some ingenuity. When payoffs are sufficiently smooth, these conditions can be tested by examining the properties of marginal payoffs. In this section, we describe and justify the relevant inequalities as well as discussing comparative statics, the competitive limit under smoothness assumptions.

Throughout this and the next section, we shall assume that $\pi_i(x)$, the payoff of each player$^{17}$, $i \in I$, is a continuously differentiable function of $x_i \in (0, w_i)$ for all $x_{-i} \in S_{-i}$. For an aggregative game in which $\pi_i = v_i(x_i, X)$, we shall write $\gamma_i(x_i, X)$ for the marginal payoff with respect to own strategy and note that, if $(x, X) \in \text{int} S_i$, where the latter denotes the interior of $S_i$, then

$$\gamma_i(x, X) = \frac{\partial v_i}{\partial x_i} (x, X) + \frac{\partial v_i}{\partial X} (x, X).$$

$^{17}$We also maintain our continuity assumption.
Note that \( L_i \) is a (possibly strict) subset of the set of zeroes of \( \gamma_i \) in \( \text{int} \tilde{S}_i \). We shall further assume that \( \gamma_i \) is a continuously differentiable function of \((x_i, X)\) in \( \text{int} \tilde{S}_i \) and refer to payoffs satisfying these differentiability assumptions as **smooth**.

The conditions we shall study are as follows.

**A1** If \((x, X) \in \text{int} \tilde{S}_i \) and \( \gamma_i (x, X) = 0 \), then

\[
\frac{\partial \gamma_i}{\partial x} (x, X) < 0.
\]

We shall show that this assumption implies ACC at any \( X > 0 \). Similarly, the following assumption implies that RCC holds for any \( \sigma \in (0, 1] \).

**A2** If \((x, X) \in \tilde{S}_i \), \( 0 < x < w_i \) and \( \gamma_i (x, X) = 0 \), then

\[
x \frac{\partial \gamma_i}{\partial x_i} (x, X) + X \frac{\partial \gamma_i}{\partial X} (x, X) < 0.
\]

Note that, when \( x = X \), the partial derivatives above may not exist and it is necessary to interpret this inequality as \( \partial^2 \pi_i / \partial x_i^2 < 0 \).

These two conditions are sufficient for regularity.

**Proposition 7.1** If a player has smooth payoffs satisfying **A1** and **A2**, then that player is regular. Furthermore, the share function \( s_i \) is differentiable except possibly at \( X = X_i \) and \( X = \bar{X}_i \) and, if \( X_i < X < \bar{X}_i \), then \( s_i' (X) < 0 \).

Comparative statics can also exploit smoothness of payoffs. Obviously, a sufficient condition for \( v_i (x_i, X) \) to be strictly increasing [decreasing] in \( X \) is \( \partial v_i / \partial X > [<] 0 \) for \( 0 < x_i < X \). A sufficient condition for supposition (iii) in Theorem 4.2 is that \( \gamma_i^1 (x_i, X) < \gamma_i^2 (x_i, X) \) whenever \( 0 < x_i < X \). This can be proved using the fact that, if \( 0 < x_i \leq X \), then \( L_i \) coincides with the set of zeroes of \( \gamma_i (x, X) \). Suppose, \((x^k, X) \in L_i^k \) for \( k = 1, 2 \). If \( x^2 > 0 \), then

\[
\gamma_i^1 (x^2, X) < \gamma_i^2 (x^2, X) = 0.
\]

We can conclude from **A1** and the continuity of \( \gamma_i \) that \( x^1 < x^2 \). (Recall that \( x^k = 0 \) if and only if \( \gamma_i^k (x, X) \leq 0 \) for all \( x \in (0, w_i) \).) If \( x^2 = 0 \), then \( \gamma_i^2 (x, X) \leq 0 \) for all \( x \in (0, w_i) \) and therefore \( \gamma_i^1 (x, X) < 0 \) for all such \( x \), which implies \( x^1 = 0 \).
Comparative statics results are stronger when the game has decreasing or increasing best responses. With smooth payoffs, interior best responses to $X_{-i}$, satisfy

$$\gamma_i(b_i(X_{-i}), X_{-i} + b_i(X_{-i})) = 0 \quad (12)$$

and the implicit function theorem allows us to deduce that the best response function $b_i(X_{-i})$ is differentiable at $X_{-i}$ provided $\partial \gamma_i/\partial x_i + \partial \gamma_i/\partial X \neq 0$. Furthermore,

$$b'_i(X_{-i}) = \frac{-\partial \gamma_i/\partial X}{\partial \gamma_i/\partial x_i + \partial \gamma_i/\partial X}, \quad (13)$$

where the right hand side is evaluated at $(b_i(X_{-i}), X_{-i} + b_i(X_{-i}))$. Under $A1$ and $A2$, we have seen that the denominator in (13) is strictly negative when (12) holds. Hence, the following condition is sufficient for decreasing best responses.

**A2** If $(x, X) \in \text{int} \tilde{S}_i$ and $\gamma_i(x, X) = 0$, then

$$\frac{\partial \gamma_i}{\partial X}(x, X) < 0. \quad (14)$$

Note that $A1$ and $A2^*$ together imply $A2$, except possibly when $x = X$. The latter case is covered if $A2^*$ also holds for $x = X$.

Similarly, a sufficient condition for increasing best responses is.

**A3** If $(x, X) \in \text{int} \tilde{S}_i$ and $\gamma_i(x, X) = 0$, then

$$\frac{\partial \gamma_i}{\partial X}(x, X) > 0.$$

At first sight this may appear to conflict with $A2$ at least when $x$ is small. Note, however, that the inequalities in $A2$ and $A3$ are required to hold only when $\gamma_i = 0$. That this restriction permits both $A2$ and $A3$ is illustrated in the first application in the following subsection.

The application of these conditions is often simplified when $\gamma_i$ can be factorized:

$$\gamma_i(x_i, X) = \phi_i(x_i, X) \tilde{\gamma}_i(x_i, X) \text{ for all } (x_i, X) \text{ satisfying } 0 < x_i < X,$$

where $\phi_i(x_i, X) > 0$ if $(x, X) \in \tilde{S}_i$ and $0 < x < w_i$. Since $\gamma_i = 0 \Leftrightarrow \tilde{\gamma}_i = 0$, we have

$$\frac{\partial \gamma_i}{\partial x_i} = \phi_i \frac{\partial \tilde{\gamma}_i}{\partial x_i} \text{ and } \frac{\partial \gamma_i}{\partial X} = \phi_i \frac{\partial \tilde{\gamma}_i}{\partial X}.$$
when these derivatives are evaluated where $\tilde{\gamma}_i = 0$. This means $A1$, $A2$, $A2^*$ and $A3$ hold for $\gamma_i$ if and only if they hold for $\tilde{\gamma}_i$. We will use this ‘factorization principle’ in several of the applications evaluated below.

### 7.1 Applications

In this section, we apply these conditions to the problems covered in Section 5.

#### 7.1.1 Search games

In the search game discussed in Subsection 5.1, suppose that $c_i$ is convex and twice continuously differentiable for positive arguments. Then,

$$\gamma_i(x, X) = \theta (\lambda + X - x) - c'_i(x).$$

Assume that, for all $x \in (0, w_i)$,

$$0 < c'_i(x) < xc''_i(x).$$

Then,

$$\frac{\partial \gamma_i}{\partial x}(x, X) = -\theta - c''_i(x) < 0,$$

so $A1$ holds. If $\gamma_i(x, X) = 0$, we also have

$$x \frac{\partial \gamma_i}{\partial x_i}(x, X) + X \frac{\partial \gamma_i}{\partial X}(x, X) = \theta (X - x) - xc''_i(x)$$

$$= c'_i(x) - xc''_i(x)$$

$$< 0$$

for $(x, X) \in \tilde{S}_i$, $0 < x < w_i$. Thus $A2$ holds.

Finally, $\partial \gamma_i / \partial X = 1 > 0$, so $A3$ holds, which shows that best responses are increasing; the game is supermodular. Since an increase in $\theta$ increases $\gamma_i$, condition (iii) of Theorem 4.2 applies and we can deduce by sequential application of the theorem, that the search intensity of all players with equilibrium in $(0, w_i)$ increases.
7.1.2 Public good games

In the public good contribution games discussed in Subsection 5.2, suppose that \( u_i \) is twice continuously differentiable for positive arguments. Then,

\[
\gamma_i(x, X) = -\frac{\partial u_i}{\partial q}(m-x, X) + \frac{\partial u_i}{\partial X}(m-x, X).
\]

If \( \frac{\partial u_i}{\partial X} > 0 \) for all positive arguments, we can apply the factorization principle with \( \phi_i = \frac{\partial u_i}{\partial X} \) to divide by \( \phi_i \), which gives

\[
\tilde{\gamma}_i(x, X) = 1 - MRS_i(m-x, X),
\]

where \( MRS_i = \left[ \frac{\partial u_i}{\partial q} / \frac{\partial u_i}{\partial X} \right] \). Now suppose further that

\[
\frac{\partial MRS_i}{\partial q} < 0, \quad \frac{\partial MRS_i}{\partial X} > 0
\]

for positive arguments. Then \( A1 \) and \( A2^* \) follow immediately from (15) and (16). Hence, under these assumptions, players in a public good contribution game are regular and have decreasing best responses.

7.1.3 Contests

In the contests discussed in Subsection 5.3, suppose that \( f_i \) is twice continuously differentiable for positive arguments and \( f'_i > 0, f''_i < 0 \) for all positive arguments. Then a calculation shows that

\[
\gamma_i(x, X) = \exp\{\alpha_ig_i(x)\}\left(\frac{X - \beta_i x}{X}\right)\tilde{\gamma}_i(x, X),
\]

where \( \alpha_i \) is the (constant) coefficient of risk aversion, \( g_i \) is the inverse function of \( f_i \) and

\[
\tilde{\gamma}_i(x, X) = \frac{1}{X} - \frac{1 - \beta_i}{X - \beta_i x} - \alpha_ig'_i(x), \\
\beta_i = 1 - \exp\{-\alpha_iR\} < 1.
\]

Since \( X - \beta_i x > 0 \), we may apply the factorisation principle. If \( 0 < x < X \),

\[
\frac{\partial \tilde{\gamma}_i(x, X)}{\partial x} = -\frac{\beta_i(1 - \beta_i)}{(X - \beta_i x)^2} - \alpha_ig''_i(x) < 0,
\]

where we have used the fact that \( g''_i > 0 \), a consequence of our assumptions
on $f_i'$. This verifies A1 and
\[
x \frac{\partial \gamma_i (x, X)}{\partial x} + X \frac{\partial \gamma_i (x, X)}{\partial X} = -\frac{\beta_i (X - x)}{X (X - \beta_i x)} - \alpha_i x y_i'' (x) < 0
\]
verifies A2.

Hence, under our assumptions on $f_i$, contestants with constant absolute risk aversion are regular.

8 Conclusions

We have explored sufficient conditions on payoffs in aggregative games which ensure a unique (non-null) equilibrium and benign comparative statics, and have illustrated the application of these results to several classes of aggregative games. These conditions are almost the weakest possible requirements on individual payoffs. We have also demonstrated how these conditions can be tested when payoffs are sufficiently smooth. The main tool in our approach is the share function. In fact share functions and correspondences have wider applicability than our use of them in this paper would suggest. For example, [10] studies rent dissipation in a sequential game with entry costs and [11] examines efficient rules for sharing the surplus of a joint production game. In both cases, share functions are the essential analytical tool for deriving the results. In some cases, share functions have to be replaced with correspondences. For example, an application to the analysis of (multiple) equilibria of Tullock rent-seeking contests where the “production function” exhibits increasing returns to scale is given in [13].

Another potential extension is to games in which payoffs depend on the strategies of rivals through some (common) function other than the sum of all strategies. In some cases, a transformation of strategy spaces and payoffs can restore aggregativity. An application is given above in Subsection 5.3 for the case of contests in which the production function $f_i$ is non-linear. Indeed, in the case of a smooth game, it can be shown that for replacement and share functions to exist, such a transformation must be possible. However, where there are kinks in payoffs (as in weakest-link problems, where payoffs depend on own strategy and the minimum of all strategies) there may be no share function. Nevertheless, share correspondences may still be used to analyze such games and, indeed, completely characterize the set of equilibria in both weakest-link and best-shot games. More general aggregation functions are also considered by Dubey et al [22], who use (pseudo-)potential functions to conduct their analysis. However, their analysis is explicitly restricted to
games with strategic substitutes.

Finally, in some games payoffs depend on more than one aggregative function, but it may still be possible to adapt the methods used above. In particular, by isolating aggregative sub-games and then imposing consistency in the overall game, existence, uniqueness and comparative statics can be studied. Hartley and Dickson [23] apply this approach to obtain a number of novel results in market games with a single product and Cornes et al [14] consider games in which groups contribute to “local” public goods that also contribute to a global public good entering the payoffs of all players.

9 Appendix

Proof of Lemma 3.1 Since the set of \((x, X_{-i})\) satisfying \(x_i \in [0, w_i]\) and \(X_{-i} \leq \alpha + \beta x_i\) is bounded we can define \(X^U_i\) to be the least upper bound of \(X_{-i}\) subject to \(x_i \in B_i(X_{-i})\) and \(X_{-i} \leq \alpha + \beta x_i\). Since \(x_i \in B_i(X_{-i})\) implies that \(0 \leq x_i \leq w_i\), there is a sequence \(\{(x^n_i, X^n_{-i})\}\) such that \(X^n_{-i} \to X'_{-i}\), as \(n \to \infty\) and \(\{x^n_i\}\) is convergent, to \(x^U_i\), say. By continuity, \(x^U_i \in B_i(X'_{-i})\) and \(X'_{-i} \leq \alpha + \beta x^U_i\). For any \(X_{-i} > X'_{-i}\), there is \(x_i \in [0, w_i]\) such that \(x_i \in B_i(X_{-i})\) and, by definition of \(X'_{-i}\), we have \(X_{-i} > \alpha + \beta x_i\). It follows by a similar continuity and compactness argument that there is an \(x^L_i\) such that \(x^L_i \in B_i(X'_{-i})\) and \(X'_{-i} > \alpha + \beta x^L_i\). For any \(X_{-i} > X'_{-i}\), there is \(x_i \in [0, w_i]\) such that \(x_i \in B_i(X_{-i})\) and, by definition of \(X'_{-i}\), we have \(X_{-i} > \alpha + \beta x_i\). It follows by a similar continuity and compactness argument that there is an \(x^L_i\) such that \(x^L_i \in B_i(X'_{-i})\) and \(X'_{-i} > \alpha + \beta x^L_i\). If \(x'_i\) is chosen to satisfy \(X'_{-i} = \alpha + \beta x'_i\), then \(x'_i\) is a convex combination of \(x^L_i\) and \(x^U_i\) and, by convexity of best responses, \(x'_i \in B_i(X'_{-i})\). The inequality \(X'_{-i} \geq X^0_{-i}\) is immediate from the construction of \(X'_{-i}\).

Proof of Lemma 3.2 It is useful to view \(B_i(X_{-i})\) as the set of maximizers of \(v_i\) on a line of unit slope through \((0, X_{-i})\); formally,

\[
B_i(X_{-i}) = \arg \max_{x \in S_i} v_i(x, x + X_{-i}).
\]

The lemma is proved by fixing \(X_{-i} > 0\) and deriving a contradiction from the supposition that

\[
B_i(X_{-i}) = [x^*, x^{**}],
\]

where \(0 \leq x^* < x^{**} \leq w_i\).

To achieve this, it proves convenient to define \(X^* = x^* + X_{-i}, X^{**} = x^{**} + X_{-i}, \sigma^* = x^*/X^*, \sigma^{**} = x^{**}/X^{**}\) and note that \(\sigma^* < \sigma^{**}\). We now
consider the line through \((\sigma^*X^**, X^**)\) with unit slope: \(x = \phi(X) = X - (1 - \sigma^*)X^**\). Note that maximizers of \(v_i\) on this line take the form

\[ B_\phi = \{(\phi(X), X) : \phi(X) \in B_i((1 - \sigma^*)X^**)\} \]

and observe that \(X \in [X^*, X^{**}]\) implies \((\phi(X), X) \notin B_\phi\) because of ACC. Similarly, if

\[ X^{**} \leq X \leq \frac{1 - \sigma^*}{1 - \sigma^{**}} X^{**}, \]

then \(\phi(X) \in [\sigma^*, \sigma^{**}]\), which implies \((\phi(X), X) \notin B_\phi\) because of RCC.

It follows that there is \((\phi(X), X) \in B_\phi \subset L_i\) which satisfies either (a) \(X < X^*\), or (b) \(X > (1 - \sigma^*)X^{**}/(1 - \sigma^{**})\). In case (a), we can apply Corollary 3.1 to deduce the existence of \((x', X^*) \in L_i\) such that

\[ X^* - x' \geq X - \phi(X) = (1 - \sigma^*)X^{**} > (1 - \sigma^*)X^*. \]

We conclude that \(x' < \sigma^*X^* = x^*\) and thus that there are two distinct points of \(L_i\) satisfying \(X = X^*\), contradicting aggregate crossing. In case (b),

\[
\phi(X) = X - (1 - \sigma^*)X^{**} \\
> \frac{1 - \sigma^*}{1 - \sigma^{**}} X^{**} - (1 - \sigma^*)X^{**} \\
= \frac{\sigma^{**}}{1 - \sigma^{**}} [X - \phi(X)],
\]

which implies that \(\phi(X) > \sigma^{**}X\). We can apply Corollary 3.1 again to deduce the existence of \((x', X^') \in L_i\) such that \(x' = \sigma^{**}X'\) and

\[ (1 - \sigma^{**})X' = X' - x' \geq X - \phi(X) = (1 - \sigma^*)X^{**}, \]

implying \(X' > X^{**}\). We conclude that there are two distinct points satisfying \(x = \sigma^*X\), giving another contradiction, this time with the radial crossing condition.

**Proof of Proposition 3.2** We use the fact that \(r_i = Xs_i\) satisfies

\[ r_i(X) = b_i \{[1 - s_i(X)] X\}, \]

where \(b_i\) is the best response function. The fact that \(s_i\) is nonincreasing implies that \([1 - s_i(X)] X\) is strictly increasing in \(X\) and therefore
that $r_i$ is strictly decreasing where positive if the game is submodular and strictly increasing if it is supermodular.

**Proof of Lemma 4.1** Since $s_i$ is non-increasing and $s_i(X^1) = 1$ implies $s_i(X^2) < 1$, \[ X^1 [1 - s_i(X^1)] < X^2 [1 - s_i(X^2)] \]

From the definition of share functions we have
\[
 v_i(X^1 s_i(X^1), X^1) = \max_{x \geq 0} v_i(x, X^1 - X^1 s_i(X^1) + x) \\
\leq \max_{x \geq 0} v_i(x, X^2 - X^2 s_i(X^2) + x) \\
= v_i(X^2 s_i(X^2), X^2).
\]

Note that the continuity of $v_i$ implies that $v_i(0,X)$ is non-decreasing in $X$. Indeed, equality can occur only if both maximands are 0 and, in particular, only if $s_i(X^1) = 0$.

The last assertion follows similarly.

**Proof of Theorem 4.1** The existence of an equilibrium of $G^2$ is an immediate consequence of Theorem 3.6. Then,
\[
\sum_{j \in I^2} s_j(\hat{X}^1) \geq \sum_{j \in I^1} s_j(\hat{X}^1) = 1 = \sum_{j \in I^2} s_j(\hat{X}^2).
\]

Since each $s_i$ is non-decreasing, we deduce that $\hat{X}^2 \geq \hat{X}^1$. Equality could only occur if we had $s_j(\hat{X}^2) = 0$ for all $j \in I_2 \setminus I_1$ but this would violate our assumptions and proves Part 1.

If, for some Player $i$, we have $s_i(\hat{X}^1) = 0$, then $s_i(\hat{X}^2) = 0$ by Lemma 3.1, which gives Part 2.

Part 3 follows immediately on application of Lemma 4.1 using the result of Part 1.

Part 4 is an immediate consequence of Proposition 3.2.

**Proof of Theorem 4.2** Regularity implies that all players have share functions, which are the same in both games for all players in $I \setminus \{i\}$. By (iii), $X_i^2 \geq X_i^1$ and, if $X_i^2 \leq X < X_i^2$, then $s_i^1(X) < s_i^2(X)$, implying
\[
\sum_{j \in I \setminus \{i\}} s_j^1(X) + s_i^1(X) < \sum_{j \in I \setminus \{i\}} s_j^2(X) + s_i^2(X).
\]
so that $\hat{X}_2 > \hat{X}_1$. Parts 2, 3 and 4 are proved as in Theorem 4.1. Part 1 implies that the strategy of at least one player must increase in $G^2$. In a submodular game, it follows from Part 4 that this player must be $i$, proving Part 5. When the game is supermodular (has increasing replacement functions), Part 5 follows from:

$$\hat{x}_i^2 = r_i^2(\hat{X}_2) > r_i^1(\hat{X}_2) > r_i^1(\hat{X}_1) = \hat{x}_i^1,$$

where the first inequality follows from $s_i^1 < s_i^2$ and the second from Proposition 3.2.

**Proof of Proposition 5.1** Assumption **EA** implies that $c'_i(x)$ is strictly increasing which implies that $w_i$ is the best response to $X_{-i}$ for $X_{-i} \geq X_i^w$, where

$$X_i^w = \max \left\{ \frac{c'_i(w_i) - \lambda_i}{\theta}, 0 \right\}.$$

The best response to $X_{-i} = 0$ is positive\(^{18}\) (and equal to the participation value) if $\lambda_i > 0$ and is $x_i = 0$ if $\lambda_i = 0$. Since Assumption **EA** also implies that $c'_i(x) \to 0$ as $x \to 0^+$, the (interior) best response $x_i$ to any $X_{-i}$ in the interval $(0, X_i^w)$ satisfies $c'_i(x_i) = \theta (\lambda_i + X_{-i})$, which we can rewrite:

$$X = x_i \left[ 1 + \frac{c'_i(x_i) - \lambda_i}{\theta x_i} \right]. \tag{17}$$

Since $c'_i$ is strictly increasing, we may conclude that best responses are unique. Furthermore, the right hand side of (17) is increasing in $x_i$, which shows that (17) has at most one solution for any $X > 0$. Note also that, if (17) holds for some $x_i \in (0, w_i)$, then $X < w_i + X_i^w$, so $(w_i, X) \notin L_i$. Similarly, if $X \geq w_i + X_i^w$,

$$X \geq w_i + \frac{c'_i(w_i) - \lambda_i}{\theta} > x_i \left[ 1 + \frac{c'_i(x_i) - \lambda_i}{\theta x_i} \right]$$

for any $x_i < w_i$, so (17) cannot hold. These observations establish ACC. Similarly, for $x_i = \sigma X$ and $\sigma \in (0, 1)$,

$$\frac{1}{\sigma} = 1 + \frac{c'_i(\sigma X)}{\theta \sigma X}. \tag{18}$$

\(^{18}\)**EA** implies that $c'_i(x) \to 0$ as $x \to 0$, so marginal payoff approaches $\lambda_i$. 
Proof of Proposition 5.2 We have \((x_i, X) \in L_i\) if and only if
\[
1 \in MRS_i (m_i - x_i, X) \tag{19}
\]
and \(x_i\) is a best response to \(X_{-i}\) if and only if (19) holds with \(X = x_i + X_{-i}\). The discussion above shows that multiple best responses are not possible and also verifies ACC. Note also that \((\sigma X, X) \in L_i\) if and only if \(1 \in MRS_i (m_i - \sigma X, X)\), which, if \(\sigma > 0\), can hold for at most one value of \(X\), verifying RCC.

Proof of Proposition 5.3 When \(0 < x_i < X\), a calculation shows that \((x_i, X) \in L_i\) if and only if it is a zero of the function \(\widetilde{\gamma}_i\), where
\[
\widetilde{\gamma}_i (x_i, X) = \frac{\beta_i (X - x_i)}{X (X - \alpha_i \beta_i x_i)} - g'_i (x_i).
\]
Holding \(X_{-i}\) fixed, the derivative of the first term with respect to \(x_i\) is
\[
\frac{\beta_i (X - x_i)}{X^2 (X - \alpha_i \beta_i x_i)^2} [\alpha_i \beta_i (X + x_i) - 2X] < 0,
\]
since \(x_i < X\) and \(\alpha_i \beta_i \leq 1\). Furthermore, \(g'_i\) is a strictly increasing function and we may conclude that \(\widetilde{\gamma}_i\) is strictly decreasing in \(x_i\). So Player \(i\) has convex best responses.

ACC is a consequence of the fact that \(\widetilde{\gamma}_i\) is a strictly decreasing function of \(x_i\) for \(x_i \in (0, X)\). RCC can be verified by observing that \((\sigma X, X) \in L_i\) if and only if
\[
\frac{\beta_i (1 - \sigma)}{X (1 - \alpha_i \beta_i \sigma)} - g'_i (\sigma X) = 0
\]
and the left hand side is strictly decreasing in \(X\).

To prove the remaining assertions, note that convexity of \(g_i\) implies that
\[
g'_{i} = \lim_{x \to 0^+} g_i (x).
\]
Since \(\widetilde{\gamma}_i\) is a strictly decreasing function of \(x_i\) for given \(X_{-i}\), then \(x_i = 0\) is a best response to \(X_{-i}\) if and only if
\[
\lim_{x \to 0^+} \widetilde{\gamma}_i (x, x + X_{-i}) = \frac{\alpha_i \beta_i}{X_{-i}} - \alpha_i g'_{i} \leq 0.
\]
This holds for some $X_{-i}$ if and only if $g' > 0$ and, in that case, it holds when $X_{-i} \geq \beta_i/\alpha_i g'$. Note that this also establishes the formula for $X_i$.

**Proof of Proposition 6.1** Suppose Player $i$ is weakly regular and let the convex-valued, share correspondence be $S_i(X)$. The proof that the domain of $S_i$ is $[X_i, \infty)$ or $\mathbb{R}_{++}$ is established by a similar argument to that for share functions in Proposition 3.1; we omit the details. By assumption, Player $i$ has at most one best response to $X_{-i}$ and it follows that $S_i(X) = 1$ for at most one value of $X$. This can be established by a similar argument to that in Lemma 3.2. We shall prove that $S_i$ is decreasing by contradiction, so suppose that we had $0 < X^1 < X^2$, $(x^1, X^1), (x^2, X^2) \in L_i$, $\sigma^1 = x^1/X^1$, $\sigma^2 = x^2/X^2$ and $\sigma^1 < \sigma^2$. The best response to $X_{-i} = (1 - \sigma^1)X^2$ is a point on the line of unit slope through $(\sigma^1X^2, X^2)$. Writing $(x^0, X^0)$ for this point, we have either A: $X^0 \leq X^2$ or B: $x^0 > \sigma^1X^0$. In Case A, we can apply Corollary 3.1 to deduce that there exists $(x'_i, X^2) \in L_i$ such that $X^2 - x'_i \geq X^0 - x^0_i = (1 - \sigma^1)X^2$. Hence, $x'_i \leq \sigma^1X^2$ and convexity of the set in Part 2. of the definition of weak regularity implies that

$$\{(x_i, X^2) : \sigma^1X^2 \leq x_i \leq X^2\} \subset L_i.$$  \hfill (20)

Since $(\sigma^1X^2, X^2) \in L_i$ by (20), convexity of the set in Part 3. of the definition implies that,

$$\{ (\sigma^1X, X) : X^1 \leq X \leq X^2 \} \subset L_i.$$  \hfill (21)

Hence, for small enough $\varepsilon > 0$, we have $(\sigma^1X^2 + \varepsilon, X^2) \in L_i$ by (20) and $(\sigma^1X, X) \in L_i$, where

$$X = X^2 - \frac{\varepsilon}{1 - \sigma^1},$$

by (21). But this means there are two distinct best responses to $X_{-i} = (1 - \sigma^1)X^2 - \varepsilon$, contradicting the assumption of a unique best response. In Case B, we can apply Corollary 3.1 to deduce that there exists $(x'_i, X') \in L_i$ such that $x'_i = \sigma^1X'$ and $X' - x'_i \geq (1 - \sigma^1)X^2$. Hence, $X' \geq X^2$ and Part 3 of the definition of weak regularity implies (21). This shows that $(\sigma^1X^2, X^2) \in L_i$ and, applying convexity again, (20). As we have seen, this contradicts uniqueness of best responses.
The converse result follows from the fact that

\[ L_i = \left\{ (x, X) : \frac{x}{X} \in S_i(X) \right\}. \]

Since \( S_i(X) \) is a convex set for all \( X > 0 \), Part 2 of the definition of weak regularity holds. To justify Part 3, note that

\[ \{X : (\sigma X, X) \in L_i\} = \{X : \sigma \in S_i(X)\}. \]

If we had \( X' < X'' < X''' \) with \( \sigma \in S_i(X') \cap S_i(X'') \), since \( X'' > X' \), \( S_i(X'') \) is non-empty. If \( \sigma'' \in S_i(X'') \), then \( \sigma'' < \sigma \) would conflict with \( S_i \) being decreasing (for \( X'' \) to \( X''' \)). A similar conflict would hold if \( \sigma'' > \sigma \) (for \( X' \) to \( X'' \)). Hence, \( \sigma \in S_i(X'') \), which shows that \( \{X : \sigma \in S_i(X)\} \) is convex, proving Part 3. We prove Part 1 by contradiction, so suppose, to the contrary that best responses were not unique. Specifically, suppose we had \( x_i, x'_i \in B_i(X_{-i}) \) and \( x_i < x'_i \) and let \( X = x_i + X_{-i} \) and \( X' = x'_i + X_{-i} \). Then \( X_{-i} > 0 \) would imply \( x_i/X < x'_i/X' \), contradicting decreasing \( S_i \), since \( x_i/X \in S_i(X) \) and \( x'_i/X' \in S_i(X') \). The same conclusion holds for \( X_{-i} = 0 \) from the supposition that at most one \( X \) satisfies \( S_i(X) = 1 \).

The final assertions can be established by a similar proof to that of Proposition 3.1; we omit the details.

Proof of Proposition 6.2 We will make use of the fact that, given any \( X_i > 0 \) and continuous, share function \( s_i \) defined on \([X_i, \infty)\), which is strictly decreasing where positive and satisfies \( s_i(X_i) = 1 \), there is a regular payoff function for which \( s_i \) is the share function. Indeed, we need only take \( v_i(x, X) \) to be the negative of the distance from \((x, X)\) to the set

\[ L_i = \{(Xs_i(X), X) : X \geq X_i\}. \]

It is readily verified that this \( v_i \) has convex best responses and \( L_i \) satisfies the aggregate and radial crossing conditions for all \( X > 0 \) and \( \sigma \in (0, 1] \), respectively. A similar argument shows that every correspondence satisfying the properties set out in Proposition 6.1, can be realized as the share correspondence of a weakly regular player.

Consider a player, to which we arbitrarily assign the label 1. Equation (10) defines a share correspondence \( S_1 \) for Player 1 and an application of Lemma 3.1 with \( \beta = -1 \) shows that this correspondence is non-empty-valued on a semi-infinite (to the right) interval.

To prove the first assertion of the proposition, we start by showing
that, under the first hypothesis, this correspondence must be strictly decreasing where positive. If, to the contrary, we had \( X'' > X' \) and \( \sigma'' \geq \sigma' \), \( \sigma'' > 0 \), where \( \sigma' \in S_1 (X') \), \( \sigma'' \in S_1 (X'') \), the argument in the first paragraph shows that there is a weakly regular player, 2, say, with share correspondence \( S_2 \) such that \( S_2 (X') = \{ 1 - \sigma' \} \) and \( S_2 (X'') = \{ 1 - \sigma'' \} \). But then Lemma 6.1 leads to the contradiction of two equilibrium values of the aggregate: \( X' \) and \( X'' \). To complete the proof, we show that \( S_1 (X) \) must be single-valued. Suppose, per contra, we had \( \sigma', \sigma'' \in S_1 (X) \), with \( \sigma' < \sigma'' \). Then, there would exist a share function \( s_2 \) with positive participation value such that \( s_2 (X') = 1 \) and \( s_2 (X'') = 0 \). Hence, this game has no equilibrium, contradicting the hypothesis of the proposition. These properties of \( S_1 \) imply that Player 1 is weakly regular.

**Proof of Proposition 7.1** Convexity of best responses follows from quasi-concavity of payoffs in own strategy and the latter follows from the observation that, if \( (x, X) \in \text{int} \bar{S}_i \) and \( \gamma_i (x, X) = 0 \), then \( A1 \) and \( A2 \) imply

\[
\frac{\partial^2 \pi_i}{\partial x_i^2} = X^{-1} \left[ (X - x) \frac{\partial \gamma_i}{\partial x} + x \frac{\partial \gamma_i}{\partial x} + X \frac{\partial \gamma_i}{\partial X} \right] < 0.
\]

This inequality also holds for \( x = X \) by direct application of \( A2 \). This shows that \( \pi_i (x_i, x_{-i}) \) is a continuous function of \( x_i \in [0, w_i] \), has no local minima in \( (0, w_i) \) and is therefore strictly quasi-concave. Note that this implies that, for \( (x, X) \in \bar{S}_i \) with \( 0 < x < w_i \), we have \( (x, X) \in L_i \) if and only if \( \gamma_i (x, X) = 0 \).
We now show that A1 leads to ACC at all $X > 0$. Define

$$
\mu(X) = \inf \{ x \in (0, w_i) : \gamma_i(x, X) < 0 \} = \sup \{ x \in (0, w_i) : \gamma_i(x, X) > 0 \},
$$

where we take the infimum of an empty set to be $w_i$ and the supremum to be 0. The equality of the two definitions is a consequence of the fact that, given $X$, $\gamma_i(x, X)$ changes sign at most once as $x$ increases in $(0, w_i)$ and such a change must be from positive to negative. Note also that $\mu$ is a continuous function on $X > 0$. Indeed, compactness of the range of $\mu$ implies that, if it were discontinuous at $X^0$, there would be a sequence $\{X^n\}$ convergent to $X^0$ on which $\mu(X^n) \rightarrow \mu^+ \neq \mu(X^0)$ as $n \rightarrow \infty$. If, say $\mu^+ > \mu(X^0)$, we would then have $\mu(X^n) > \mu^+ = \left[\mu^+ + \mu(X^0)\right]/2$ for all large enough $n$. Hence, $\gamma_i(\mu^+, X^n) > 0$ for all large enough $n$, which, because of the continuity of $\gamma$ implies that $\gamma_i(\mu^+, X^0) \geq 0$ implying $\mu^+ \leq \mu(X^0)$, a contradiction. A contradiction can be derived similarly if $\mu^+ < \mu(X^0)$. Verification of ACC is completed by showing that $L_i$ is a subset of the graph of $\mu$, for then ACC is immediate. So suppose that $(x', X') \in L_i$ and $X' > 0$. If A: $x' \in (0, w_i)$, we have already noted that $\gamma_i(x', X') = 0$, which is readily seen to imply $x' = \mu(X')$. If B: $x' = 0$ and there is a neighborhood of $X'$ such that $(0, X') \in L_i$ for all $X$ in the neighborhood, strict quasi-concavity of best responses implies that $\gamma_i(x, x + X) < 0$ for all $x \in (0, w_i)$. By considering all $X < X'$ in the neighborhood, we deduce that $\gamma_i(x, X') < 0$ for all small enough $x$ which entails $\mu(X) = 0$ (using the first definition of $\mu$). If C: $x' = 0$ and there exists $X$ arbitrarily close to $X'$ such that $(x, X) \in L_i$ with $x > 0$, we know that $x = \mu(X)$ by (i) and can deduce that $\mu(X') = 0$ from the closedness of $L_i$ and continuity of $\mu$. Finally, if $x' = w_i$, a similar argument (using the second definition of $\mu$) shows that $\mu(X') = w_i$. In all cases, $\mu(X') = x'$ as required.

To complete the proof, we need to show that A2 implies RCC for all $\sigma \in (0, 1]$. This is done by first observing that, for any such $\sigma$ and $X > 0$ with $\gamma_i(\sigma X, X) = 0$, we have

$$
\frac{\partial}{\partial X} \gamma_i(\sigma X, X) = \sigma X \frac{\partial \gamma_i}{\partial \sigma X_i}(\sigma X, X) + X \frac{\partial \gamma_i}{\partial X}(\sigma X, X) < 0,
$$

by A2. This implies that $\gamma_i(\sigma X, X)$ changes sign at most once as $X$ increases from 0 to $w_i/\sigma$ and such a change must be from positive to negative. This observation can be used to modify the proof for ACC.
to show that $A2$ implies $RCC$. We shall omit the details.

Differentiability of the replacement function (and therefore the share function) when $0 < s_i(X) < w_i/X$ follows from applying the implicit function theorem to the first order condition $\gamma_i(Xs_i(X), X) = 0$. To justify this application, we note that $\partial \gamma_i/\partial x_i \neq 0$, by $A1$. Furthermore,

$$s'_i(X) = \left[ x_i \frac{\partial \gamma_i}{\partial x_i} + X \frac{\partial \gamma_i}{\partial X_i} \right] / X^2 \frac{\partial \gamma_i}{\partial x_i},$$

evaluated at $(x_i, X) = (Xs_i(X), X)$. By $A1$ and $A2$, $s'_i(X) < 0$. If $s_i(X) = w_i/X$ we must have $\gamma_i(Xs_i(X), X) \geq 0$. If we had $\gamma_i = 0$, the same argument would hold. If we had $\gamma_i > 0$, then $s_i(X') = w_i/X'$ in a neighborhood of $X$ and $s'_i < 0$ is immediate.

References


[14] Cornes, R. C., R. Hartley and D. Nelson (2005), Groups with intersecting interests, to be presented at PET05, Marseilles, France.


