Unemployment Risks and Optimal Retirement in an Incomplete Market

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Abstract

We develop a new approach for solving the optimal retirement problem for an individual with an unhedgeable income risk. The income risk stems from a forced unemployment event, which occurs as an exponentially-distributed random shock. The optimal retirement problem is to determine the individual’s optimal consumption and investment behaviors and optimal retirement time simultaneously. We introduce a new convex-duality approach for reformulating the original retirement problem and provide an iterative numerical method to solve it. Reasonably calibrated parameters say that our model can give an explanation for lower consumption and risky investment behaviors of individuals and a relatively higher stock holdings for the poor. We also analyze the sensitivity of an individual’s optimal behaviors in changing her wealth level, investment opportunity, and the magnitude of preference for post-retirement leisure. Finally, we find our model explains a procyclical pattern of the number of unemployed job leavers.
1 Introduction

Starting from pioneering works by Merton (1969, 1971), intertemporal models of optimal consumption and portfolio choice have evolved into some interesting generalizations. Among them, Bodie et al. (1992), Heaton and Lucas (1997), Koo (1998), Viceira (2001), Farhi and Panageas (2007) and others explored life-cycle models, in which non-tradable labor income is incorporated into the problem of optimal consumption and portfolio choice.

Bodie et al. (1992) examine the impact of labor flexibility on an individual’s optimal behavior including consumption and risky investment under the assumption that the labor supply is certain over her life cycle. Heaton and Lucas (1997) and Koo (1998) investigate how uncertain income stream can affect an individual’s optimal consumption and investment behavior. Further, Viceira (2001) illustrates individual’s optimal behaviors when she faces an exogenous shock of retirement, assuming that exogenous retirement shocks and mortality risks arrive in a random way, but with constant probabilities. Farhi and Panageas (2007) investigate an individual’s optimal consumption, investment and retirement behaviors simultaneously under the assumption that the individual’s income rate is certain while working. All these papers permit only one of the following two assumptions, not both: risky labor income or endogenous (or voluntary) retirement opportunity.

Considering both income risks and endogenous retirement opportunity in the classical life-cycle models is a complicated job. Only a few researchers such as Liu and Neis (2003), Bodie et al. (2004), Dybvig and Liu (2009), and Jang et al. (2012) have successfully resolved the optimal voluntary retirement problems with income risks, but they consider a complete market in which income risks can be hedged away by purchasing and selling some financial instruments traded in that market. This paper deals with both income risks and endogenous retirement in an incomplete market. We assume individuals cannot eliminate income risks because it stems from exogenously forced unemployment events, and investigate the individual’s optimal consumption, portfolio choice, and retirement behaviors in this serious situation.

Investigating the impact of income risks from unemployment events on an individual’s optimal behaviors has been an important task for economists. Caroll (1992) shows that an unemployment event could have a major impact on an individual’s current consumption and saving behaviors. Cocco et al. (2005) show that even a 0.05% probability of being unemployed in any given year has a large effect on an individual’s portfolio choice, particularly early in
life. Further, Wachter and Yogo (2010) stress the fact that unemployment risks could be significant for the poor in the sense that the optimal amount of risky investment increases at a sufficiently low wealth level. Also, Lynch and Tan (2011) show that the amount invested in the risky assets could be relatively lower in a persistent unemployment state and assert that the unemployment state pays only 5% of permanent labor income.\(^1\) This paper might give a significant impetus to clarify the effect of unhedgeable unemployment risks on individual’s optimal behaviors in an incomplete market.

From a technical standpoint, solving financial problems constructed in an incomplete market, such as pricing derivatives and choosing investor’s optimal consumption and investment strategies, is extremely complex due to the non-uniqueness of the equivalent martingale measure, and obtaining a closed-form solution of the problems is hardly possible. Just a little literature, such as Kim and Omberg (1996), Chacko and Viceira (2005), Sangvinatsos and Wachter (2005), and Liu (2007), have provided closed-form solutions of financial problems defined in an incomplete market.

It is well-known that there are two approaches for solving financial problems derived from an incomplete market: the dynamic programming approach (DPA) and the martingale approach (MA). Koo (1998) and Henderson (2005) exploit the DPA to solve optimal consumption and investment problems incorporating labor income risks. They derive a non-linear partial differential equation and solve it. He and Pearson (1991), Svensson and Werner (1993), Teplá (2000), and Keppo et al. (2007) apply the MA to solve financial problems in the presence of income risks in an incomplete market.

Most existing literature concerning income risks, no matter they are hedgeable or unhedgeable, models the income stream as a standard geometric Brownian motion, which permits use of the MA. For instance, Karatzas et al. (1991) and He and Pearson (1991) suggest choosing the minimum local equivalent martingale measure with respect to unhedgable risks represented by a geometric Brownian motion (GBM). Further, Duffie et al. (1997) demonstrate viscosity solutions for hedging problems in incomplete markets in which a stochastic income generated by a GBM cannot be hedged by using a traded risky asset. However, we assume the income risks originally stem from forced unemployment events and such exogenous un-

\(^1\)Cocco et al. (2005) define unemployment state as the state of zero income, however, Gakidis (1998) shows that even unemployed individuals may have other sources of income (e.g., income coming from unemployment benefits and social welfare).
employment events occur following an exponential distribution with positive intensity. This kind of model is new in the optimal retirement literature, and we develop a new approach for solving the optimal retirement problem with the income risks by using the DPA and a convex-dual function of our objective function. Specifically, we derive a differential equation for the convex-dual function and develop an iterative numerical method to solve it. As far as we know, this is the first paper to develop a method for solving the optimal retirement problem with a down-jump event of income which is modeled not using any Brownian motion.

With reasonably calibrated parameters we obtain some interesting features concerning individual’s optimal behaviors and income risks in the incomplete market. In our model, we find

• income risks stemming from forced unemployment events might significantly lower individual’s consumption, investment in risky assets and voluntary retirement wealth level,

• income risks stemming from forced unemployment events might be an explanation for the findings of Cocco et al. (2005), Benzoni et al. (2007), and Lynch and Tan (2011), in that stock holdings in cash-on-hand can increase at a sufficiently low wealth level,

• certainty equivalent wealth gain, the maximum wealth that an individual is willing to give up in exchange for the market without unemployment risks, decreases as wealth and/or investment opportunity grow(s), and

• certainty equivalent wealth gain has a bigger value for an individual with a higher preference of leisure after retirement.

The first finding about voluntary retirement wealth level could be an explanation of a procyclical pattern of the number of unemployed job leavers who have voluntarily left their current jobs; the proportion of job leavers increases during economic recessions and decreases during economic expansions.\(^2\) Our result says that soaring income risks due to forced unemployment events during economic recessions induce people who have slightly smaller wealth than the voluntary retirement wealth level planned during the past economic expansion to enter early retirement.

Our paper is organized as follows: In section 2 we establish a financial market with forced unemployment events and formulate our problem in the market. We also introduce a new convex-dual approach and an iterative numerical method to solve the problem. In section 3.2 we display some analytical results including optimal consumption and investment strategies of an individual, and section 4 shows numerical implications of our model in a normal market. Section 5 analyzes the relationship between the number of job leavers and business cycles and Section 6 concludes the paper.

2 The Model

2.1 The Financial Market

Following the conventional models, we assume an individual can trade two assets in the financial market: a bond (or a risk-free asset) and a stock (or a risky asset). The bond price $B_t$ evolves by the relationship

$$dB_t = rB_t dt,$$

where the positive constant value $r$ is considered to be a risk-free interest rate. On the other hand, the stock price $S_t$ follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu > 0$ is the expected rate of the stock return, $\sigma > 0$ is the stock volatility, and $W_t$ is a standard Brownian motion defined on a suitable probability space.

We assume the individual is in the workforce at the beginning and wants to retire voluntarily someday in the future. She is exposed to forced (or involuntary) unemployment risks such that she loses her job by compulsion whenever an exogenous unemployment shock arrives before the voluntary retirement date. We assume the forced unemployment event occurs following an exponential distribution with intensity $\delta$, namely; for some time $t \geq 0$

$$\text{probability of } \{\tau_U \leq t \} = 1 - e^{-\delta t},$$

where $\tau_U$ is the forced unemployment time. During working status she gets an income of $I_1 > 0$ per unit time, and $I_2 > 0$ ($I_1 > I_2$) per unit time after retirement. The assumption of
a positive income after retirement reflects the fact that most countries provide unemployment allowances and other public welfare services for retired people.\(^3\)

Elmendorf and Kimball (2000) and Gormley et al. (2010) emphasize the important role of insurance against large and negative wealth shocks such as unemployment risks in an individual’s optimal policies. However, private insurance markets for hedging labor income risks are not competitive relative to other insurance markets (Cocco et al. 2005). In this sense, we are assuming that there is no financial vehicle for eliminating or diminishing the forced unemployment risks, thus, the financial market is incomplete. The forced unemployment time considered in this paper is not an optimal stopping time given by the set of information of stock price movements but a random time, resulting in market incompleteness.

2.2 The Retirement Problem

The retirement problem explored in this paper is a variation of the problem explored by Farhi and Panageas (2007), but it allows an individual to be exposed to forced unemployment risks. The individual wants to find the maximal score of her expected utility for consumption \(c(t)\):

\[
\max_{(c, \pi, \tau)} E \left[ \int_0^{\tau \wedge \tau_U} e^{-\beta t} U_1(c(t)) \, dt + e^{-\delta (\tau \wedge \tau_U)} \int_{\tau \wedge \tau_U}^\infty e^{-\beta(t-\tau \wedge \tau_U)} U_1(Kc(t)) \, dt \right],
\]

where

\[
U_1(c) = \ln c,
\]

\(\pi\) is the dollar amount invested in the stock, \(\tau\) is optimal voluntary retirement time, \(\beta > 0\) is the individual’s subjective discount rate, and \(K > 1\) denotes preferences for not working (or leisure).\(^4\)

The wealth process \(X(t)\) of the individual is given by

\[
dX(t) = \begin{cases} 
(rX(t) - c(t) + I_1) \, dt + X(t)\pi(t)\sigma(dW(t) + \theta \, dt), & \text{for } 0 \leq t < \tau \wedge \tau_U, \\
(rX(t) - c(t) + I_2) \, dt + X(t)\pi(t)\sigma(dW(t) + \theta \, dt), & \text{for } t \geq \tau \wedge \tau_U,
\end{cases}
\]

with \(X(0) = x\), where \(\theta\) represents the Sharpe ratio \((\mu - r)/\sigma\). By utilizing the conditional expectation of \(\tau_U\), we can rewrite the individual’s objective function (or the value function) as

\[
\phi(x) \equiv \max_{(c, \pi, \tau)} E \left[ \int_0^\tau e^{-(\beta+\delta)t} \left\{ U_1(c(t)) + \delta U_2(X(t)) \right\} \, dt + e^{-(\beta+\delta)\tau} U_2(X(\tau)) \right], 
\]

\(^3\)If we take the definition of unemployment in Lynch and Tan (2011), \(I_2\) is about 5% of \(I_1\).

\(^4\)A utility of this kind is shown in Dybvig and Liu (2009).
where
\[
U_2(z) = \frac{1}{\beta} \left[ \ln \left( \beta \left( z + \frac{I_2}{r} \right) \right) \right] + \frac{1}{\beta} \left( r + \frac{\theta^2}{2} - \beta(1 - \ln K) \right), \quad \text{for } z > 0.
\]

In fact, \(U_2(z)\) is the value function of the classical Merton’s problem with infinite investment horizon under the condition that the investor has a logarithmic utility and an income stream \(I_2\) forever.

### 2.3 Problem Reformulation

#### 2.3.1 A new convex-duality approach

We utilize the conventional dynamic programming approach to resolve our incomplete market problem. The variational inequality of (1) becomes
\[
(\beta + \delta)\phi(x) - (rx + I_1)\phi'(x) + \frac{\theta^2}{2} \phi''(x) + 1 + \ln \phi'(x) \geq \delta U_2(x),
\]
\[
\phi(x) \geq U_2(x).
\]

From the inequality we establish a problem with one free boundary, more specifically, we want to solve the following equation:
\[
\begin{cases}
(\beta + \delta)\phi(x) - (rx + I_1)\phi'(x) + \frac{\theta^2}{2} \phi''(x) + 1 + \ln \phi'(x) = \delta U_2(x), & 0 < x < \hat{x}, \\
\phi(x) = U_2(x), & x \geq \hat{x}, \\
\phi(\hat{x}) = U_2(\hat{x}), \\
\phi'(\hat{x}) = \frac{1}{\beta} \frac{1}{\hat{x} + \frac{I_2}{r}}.
\end{cases}
\]

In our model, it is possible for an individual to retire with a negative wealth at the forced unemployment date and we permit this case. This makes the problem difficult to be well-defined; hence, we extend our problem into the problem defined in the region of \((-I_1/r, \infty)\) containing some negative values of wealth, by assuming that the first derivative of the post-retirement value function \(U_2\) is continuous at zero wealth level. The assumption does not
violate the increasing and concave properties of utility functions.\(^5\) Now, set
\[
\lambda(x) \equiv \phi'(x) \quad \text{and} \quad \hat{\lambda} \equiv \frac{1}{\beta \hat{x} + \frac{I_2}{r}}.
\]
Then,
\[
\lambda(x)(\theta^2 + \beta + \delta - r) + \frac{\lambda'(x)}{\lambda(x)} - \lambda'(x)(rx + I_1) - \frac{1}{2} \theta^2 \lambda(x)^2 \frac{\lambda''(x)}{\lambda(x)}^2 = \begin{cases} 
\delta \frac{1}{\beta x + \frac{I_2}{r}}, & \text{if } x > 0, \\
\delta \frac{r}{\beta I_2}, & \text{if } -\frac{I_1}{r} < x \leq 0.
\end{cases}
\]
(4)

We introduce a modification of the conventional convex-duality approach by Karatzas and Shreve (1998) to solve our incomplete market problem. At first, we define a decreasing and convex function \(G(\cdot)\) by
\[
G(\lambda(x)) \equiv x + \frac{I_1}{r}.
\]
(5)

Hence, we get
\[
G'(\lambda(x)) \lambda'(x) = 1 \quad \text{and} \quad G''(\lambda(x)) \lambda'(x)^2 + G'(\lambda(x)) \lambda''(x) = 0.
\]
For convenience, let \(G(\lambda(x)) = G\) and \(\lambda(x) = \lambda\), then, (4) becomes
\[
\lambda(\theta^2 + \beta + \delta - r) - xG' + \frac{1}{\lambda G''} - \frac{rG}{G''} + \frac{1}{2} \theta^2 \lambda^2 \frac{G'''}{G'} = \begin{cases} 
\delta \frac{1}{G - \frac{I_1}{r} + \frac{I_2}{r}}, & \text{if } G > \frac{I_1}{r} \\
\delta \frac{r}{\beta I_2}, & \text{if } G < \frac{I_1}{r}.
\end{cases}
\]
(6)

From (5) and (6), we get
\[
-\frac{1}{2} \theta^2 \lambda^2 \frac{G'''}{G''} - \lambda G'(\lambda)(\theta^2 + \beta + \delta - r) + rG(\lambda) + \frac{\delta}{\beta} \frac{G''(\lambda)}{G'(\lambda)} = \frac{1}{\lambda},
\]
\[
G(\hat{\lambda}) = \frac{1}{\beta \hat{\lambda}} + \frac{I_1 - I_2}{r}, \quad \lambda > \hat{\lambda},
\]
(7)

where \(\left(G(\lambda) - \frac{I_1}{r}\right)^+ = \max\{G(\lambda) - \frac{I_1}{r}, 0\}\). We add one more constraint,
\[
G(\infty) = 0,
\]
(8)

\(^5\)It guarantees the boundedness of the first derivative of the value function everywhere. Roughly speaking, the investor with such preference is risk-averse for positive values of wealth and risk-neutral for negative values of wealth. Piecewise connected utility functions of this kind are seen in lots of articles in economics. For example, Venter (1983) introduces a utility function which is constant for negative values of wealth, and also says it is designed to reflect bankruptcy laws.
which implies the individual’s marginal utility λ goes to infinity as initial wealth x goes down to \(-\frac{I_1}{\beta}\).

Technically, for the case of \(\delta = 0\), without any forced unemployment risks, the problem formulated by equations in (7) and (8) has an analytic solution, whereas the problem for the case of \(\delta > 0\) is unlikely to have. We develop an iterative numerical method to solve it in the subsequent section.

### 2.3.2 The iterative method

First, we define \(\alpha_\delta > 0\) and \(\alpha_\delta^* < 0\) as the two roots of

\[
F(\alpha; \delta) \equiv -\frac{1}{2}\theta^2(\alpha - 1) + \alpha(\beta + \delta) + \beta = 0.
\]

We conjecture the general solution of (7) as

\[
G(\lambda) = \frac{1}{\lambda(\beta + \delta)} + A(\lambda)\lambda^{-\alpha_\delta} + A^*(\lambda)\lambda^{-\alpha_\delta^*}
\]

subject to

\[
A'(\lambda)\lambda^{-\alpha_\delta} + (A^*(\lambda))'\lambda^{-\alpha_\delta^*} = 0.
\]

Putting the relationship in (9) into (7), we get

\[
G(\lambda) = \frac{1}{\lambda(\beta + \delta)} + B(\lambda)\lambda^{-\alpha_\delta} + \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\left[(\alpha_\delta - 1)\lambda^{-\alpha_\delta} \int_\lambda^\lambda \mu^{\alpha_\delta - 2} \ln \beta \left\{\left(G(\mu) - \frac{I_1}{r}\right)^+ + \frac{I_2}{r}\right\} d\mu\right]
\]

\[
+ (\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^*} \int_\lambda^\lambda \mu^{\alpha_\delta^* - 2} \ln \beta \left\{\left(G(\mu) - \frac{I_1}{r}\right)^+ + \frac{I_2}{r}\right\} d\mu,
\]

and

\[
\frac{1}{\beta\lambda} + \frac{I_1 - I_2}{r} = \frac{1}{\lambda(\beta + \delta)} + B(\hat{\lambda})\hat{\lambda}^{-\alpha_\delta} + \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\int_\lambda^\lambda \mu^{\alpha_\delta - 2} \ln \beta \left\{\left(G(\mu) - \frac{I_1}{r}\right)^+ + \frac{I_2}{r}\right\} d\mu,
\]

for

\[
B(\hat{\lambda}) = A(\hat{\lambda}) + \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\hat{\lambda}^{\alpha_\delta - 1} \ln \frac{1}{\hat{\lambda}}.
\]

Moreover, the relationship of \(\phi(\hat{x}) = U_2(\hat{x})\) in (3) implies

\[
\frac{\theta^2}{\beta^2(\beta + \delta)}(1 - \alpha_\delta) + \ln K
\]

\[
= \frac{\delta}{\beta} \ln \beta + \hat{\lambda}(I_1 - I_2)\left(1 + \frac{\alpha_\delta \theta^2}{2r}\right) - \frac{\delta(\alpha_\delta^* - 1)}{\beta} \lambda^{-\alpha_\delta^* + 1} \int_\lambda^\lambda \mu^{\alpha_\delta^* - 2} \ln \beta \left\{\left(G(\mu) - \frac{I_1}{r}\right)^+ + \frac{I_2}{r}\right\} d\mu.
\]
The derivation of the relationship in (12) is in Appendix. Further, we prove the uniqueness and existence of the solution of (10) and the strictly decreasing property of $G(\cdot)$. We also verify the solution obtained from the new convex dual approach is a solution of the variational inequality of (2).

Now we state an iterative numerical method to find the solution $G(\cdot)$ of (7).

**The iterative procedure:**

Step 0. Notice that, if $\delta = 0$, we can easily get $B(\hat{\lambda})$ from (11) and, thus, obtain $G(\lambda)$ from (10). Putting this $G(\lambda)$ into (12), we get $\hat{\lambda}$. Suppose $\delta \neq 0$, but has a sufficiently small value. We exploit $G(\lambda)$ and $\hat{\lambda}$ for the case where $\delta = 0$ as the initial values of our iteration method.

Step 1. Since we have an initial $\hat{\lambda}$ and $G(\lambda)$, we can get $B(\hat{\lambda})$ from (11).

Step 2. Update $G(\lambda)$ by using equation (10).

Step 3. Putting the updated $G(\lambda)$ into (12), we can obtain a new $\hat{\lambda}$.

Step 4. Repeat steps 1, 2 and 3 until $\hat{\lambda}$ converges.

3 Analytical Results

3.1 Lower and Upper Bounds for Critical Wealth Level

Even though it is hard to have a closed form of the threshold level $\hat{\lambda}$ in the variational inequality (2), we can derive analytical lower and upper bounds.

To simplify notation, set $L_\delta$ to be the left hand side of (12) and $\psi_\delta(\hat{\lambda})$ to be its right hand

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6This condition of $\delta$ is necessary in order to guarantee the uniqueness and existence of the solution of (10) and to verify the fact that the solution obtained from the new convex dual approach is a solution of the variational inequality of (2). We display the possible range of $\delta$ in Theorem 7.1, 7.3 and 7.4 in Appendix.
side, that is, we let
\[
L_\delta \equiv \frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \ln K,
\]

\[
\psi_\delta(\hat{\lambda}) \equiv \frac{\delta}{\beta} \ln \beta \hat{\lambda} + \hat{\lambda}(I_1 - I_2)(1 + \frac{\alpha_\delta \theta^2}{2r})
- \frac{\delta(\alpha_\delta^* - 1)}{\beta} \hat{\lambda}^{\alpha_\delta^* + 1} \int_{\alpha_\delta^*}^{\infty} \mu^{\alpha_\delta^* - 2} \ln \left\{ \left( G(\mu) - \frac{I_1}{r} \right)^+ + \frac{I_2}{r} \right\} d\mu.
\]

Moreover, we define two functions
\[
\phi_\delta(\hat{\lambda}) \equiv \frac{\delta}{\beta} \ln \beta \hat{\lambda} + \hat{\lambda}(I_1 - I_2)(1 + \frac{\alpha_\delta \theta^2}{2r})
\]

and
\[
\bar{\phi}_\delta(\hat{\lambda}) \equiv \frac{\delta}{\beta} \ln \beta \hat{\lambda} + \hat{\lambda}(I_1 - I_2)(1 + \frac{\alpha_\delta \theta^2}{2r}) + \frac{\delta}{\beta} \ln \left( \max \left\{ \frac{1}{\beta \hat{\lambda}}, \frac{I_2}{r} \right\} \right),
\]

which can be lower and upper bounds of \(\psi_\delta(\hat{\lambda})\) respectively.

**Proposition 3.1** If \(I_2 \geq r\),

\[
\lambda_\delta^0 \leq \hat{\lambda} \leq \lambda_\delta^1,
\]

where \(\lambda_\delta^0\) and \(\lambda_\delta^1\) are found by

\[
\bar{\phi}_\delta(\lambda_\delta^0) = L_\delta \quad \text{and} \quad \phi_\delta(\lambda_\delta^1) = L_\delta.
\]

**Proof.** It is obvious that

\[
\psi_\delta(\hat{\lambda}) \geq \phi_\delta(\hat{\lambda}).
\]

Since \(\left( G(\mu) - \frac{I_1}{r} \right)^+ + \frac{I_2}{r} \leq \max \left\{ \frac{1}{\beta \hat{\lambda}}, \frac{I_2}{r} \right\}\), under the assumption of \(I_2 \geq r\),

\[
\psi_\delta(\hat{\lambda}) \leq \frac{\delta}{\beta} \ln \beta \hat{\lambda} + \hat{\lambda}(I_1 - I_2)(1 + \frac{\alpha_\delta \theta^2}{2r}) + \frac{\delta}{\beta} \ln \left( \max \left\{ \frac{1}{\beta \hat{\lambda}}, \frac{I_2}{r} \right\} \right) = \bar{\phi}_\delta(\hat{\lambda}).
\]

Since both \(\bar{\phi}_\delta(\hat{\lambda})\) and \(\phi_\delta(\hat{\lambda})\) are monotonically-increasing and continuous functions with

\[
\bar{\phi}_\delta(0) = \phi_\delta(0) = -\infty, \quad \text{and} \quad \bar{\phi}_\delta(+\infty) = \phi_\delta(+\infty) = +\infty,
\]

\(\lambda_\delta^0\) and \(\lambda_\delta^1\) are lower and upper bounds for \(\hat{\lambda}\). **Q.E.D.**
The upper and lower bounds for the critical wealth level $\hat{x}$, at which the individual enters voluntary retirement, also can be written as

$$G(\lambda^0_3) - \frac{I_1}{r} \leq \hat{x} \leq G(\lambda^0_3) - \frac{I_1}{r},$$

if we use the result in the theorem and the definition of function $G(\cdot)$ in (5). Notice that $\hat{\phi}_3(\lambda^0_3)$ becomes $\bar{\phi}_3(\lambda^0_3)$, or equivalently, the upper and lower boundaries in the proposition are identical, where intensity $\delta$ is zero. For this case, the boundaries become the corresponding critical wealth level described in Farhi and Panageas (2007), in which individuals are not exposed to any forced unemployment risk.

### 3.2 Optimal Consumption and Investment Strategies

We find the optimal consumption and investment strategies by exploiting the terms of $\hat{\lambda}$, $B(\cdot)$ and $G(\cdot)$, which can be obtained by the iterative numerical method in the previous section.

**Theorem 3.1** The optimal consumption $c$ and risky portfolio $\pi$ are given as

$$c(t) = (\beta + \delta)\left(x + \frac{I_1}{r}\right) - (\beta + \delta)B(\hat{\lambda})\lambda^*(x)^{-\alpha_3}$$

$$- \frac{2\delta(\beta + \delta)}{\theta^2(\alpha_3 - \alpha_3^*)\beta} \left[\alpha_3 (\alpha_3 - 1)\lambda^*(x)^{-\alpha_3} \int_{\hat{\lambda}}^{\lambda^*(x)} \mu^\alpha \ln \beta \left\{ \left( G(\mu) - \frac{I_1}{r} \right) + \frac{I_2}{r} \right\} d\mu \right] + (\alpha_3^* - 1)\lambda^*(x)^{-\alpha_3^*} \int_{\lambda^*(x)}^{\infty} \mu^{\alpha_3^*-2} \ln \beta \left\{ \left( G(\mu) - \frac{I_1}{r} \right) + \frac{I_2}{r} \right\} d\mu,$$

and the solution of

$$\pi(t) = \frac{\theta}{\sigma \lambda^*(x)(\beta + \delta)} + \frac{\theta \alpha_3 B(\hat{\lambda})\lambda^*(x)^{-\alpha_3}}{\ln \beta (x + \frac{I_2}{r})}$$

$$+ \frac{2\delta}{\sigma \theta (\alpha_3 - \alpha_3^*) \beta} \left[ \alpha_3 (\alpha_3 - 1)\lambda^*(x)^{-\alpha_3} \int_{\hat{\lambda}}^{\lambda^*(x)} \mu^\alpha \ln \beta \left\{ \left( G(\mu) - \frac{I_1}{r} \right) + \frac{I_2}{r} \right\} d\mu \right] + (\alpha_3^* - 1)\lambda^*(x)^{-\alpha_3^*} \int_{\lambda^*(x)}^{\infty} \mu^{\alpha_3^*-2} \ln \beta \left\{ \left( G(\mu) - \frac{I_1}{r} \right) + \frac{I_2}{r} \right\} d\mu,$$

where $\lambda^*(x)$ is a decreasing function with respect to wealth $x$ and the solution of

$$x + \frac{I_1}{r} = \frac{1}{\lambda^*(x)(\beta + \delta)} + B(\hat{\lambda})\lambda^*(x)^{-\alpha_3}$$

$$+ \frac{2\delta}{\theta^2 (\alpha_3 - \alpha_3^*) \beta} \left[ \alpha_3 (\alpha_3 - 1)\lambda^*(x)^{-\alpha_3} \int_{\hat{\lambda}}^{\lambda^*(x)} \mu^\alpha \ln \beta \left\{ \left( G(\mu) - \frac{I_1}{r} \right) + \frac{I_2}{r} \right\} d\mu \right] + (\alpha_3^* - 1)\lambda^*(x)^{-\alpha_3^*} \int_{\lambda^*(x)}^{\infty} \mu^{\alpha_3^*-2} \ln \beta \left\{ \left( G(\mu) - \frac{I_1}{r} \right) + \frac{I_2}{r} \right\} d\mu.$$
and
\[ B(\hat{\lambda}) = \hat{\lambda}^{\alpha_4} \left[ \frac{\delta}{\beta(\beta + \delta)^2} \frac{1}{\hat{\lambda}} + \frac{I_1 - I_2}{r} - \frac{2\delta}{\beta^2(\alpha_4 - \alpha_5)\beta} \left\{ (\alpha_5^* - 1)\hat{\lambda}^{-\alpha_5^*} \int_{\hat{\lambda}}^{\infty} \beta^{-1}\int_{\mu}^{\infty} \beta^{-1} \right\} \right]. \]

**Proof.** \((5)\) and the first-order conditions with respect to \(c(t)\) and \(\pi(t)\), which were used in deriving the variational inequality \((2)\), yield the optimal consumption \(c(t)\) and risky portfolio proportion \(\pi(t)\). Q.E.D.

The optimal consumption of the classical Merton’s (1969, 1971) problem in the presence of the income \(I_1\) is represented as
\[ c^M(t) = \beta \left( x + \frac{I_1}{r} \right), \]
so the marginal propensity to consume (MPC) out of wealth, \(\frac{\partial c(t)}{\partial x}\), is constant. However, in our model without any forced unemployment risk, namely \(\delta = 0\), the first and second terms of the right hand side of \((17)\), which spring up due to the voluntary retirement event, yield the optimal consumption
\[ c^{VR}(t) = \beta \left( x + \frac{I_1}{r} \right) - \beta B(\hat{\lambda})\lambda^*(x)^{-\alpha_4}. \]
This implies the MPC out of wealth should be positive and the optimal consumption is a concave function with respect to wealth. Since \(B(\hat{\lambda}) > 0\) for \(\delta = 0\), the individual with voluntary retirement option consumes relatively less than the one in the classical Merton problem because she is likely to accumulate wealth by cutting down her consumption to enter retirement early. Moreover, since \(\lambda^*\) is a decreasing function with respect to wealth level the effect of the voluntary retirement option on her consumption behavior becomes significant as her wealth increases. This might be an explanation of the finding by Farhi and Panageas (2007) and Dybvig and Liu (2009), that individuals could dramatically decrease their consumption near the critical wealth level.

The third term of the right hand side of \((17)\) consists of two parts which are closely associated with the impact of income risks stemming from forced unemployment events. Notice that the first (integral) part gives a negative effect to the individual’s consumption and the second (integral) part affects it conversely. As an individual’s wealth approaches the critical wealth
level \( \hat{x} \), \( \lambda^*(x) \) gets closer to \( \hat{\lambda} \), and subsequently, in the limit case the first part disappears and the second part has a fixed value. Hence, if an individual’s wealth reaches near the critical wealth level, the third term of the right hand side in (17) could make a positive impact on the individual’s consumption and offset the negative impact of the second term of the right hand side in (17), which stands for the voluntary retirement option value. So it might be possible for people facing forced unemployment risks to consume more than individuals not exposed to them, near retirement time.

When it comes to the optimal risky investment, the classical Merton’s (1969, 1971) strategy, \( \pi^M \), must be

\[
\pi^M(t) = \frac{\theta}{\sigma} \left( x + \frac{I_1}{r} \right).
\]

However, using the first and second terms of the right hand side of (18) we can get the optimal risky portfolio, \( \pi^{VR} \), for the case where there exists no forced unemployment risk, i.e., \( \delta = 0 \):

\[
\pi^{VR}(t) = \frac{\theta}{\sigma} \left( x + \frac{I_1}{r} \right) + \frac{\theta}{\sigma} \alpha_0 B(\hat{\lambda}) \lambda^*(x)^{-\alpha_0}.
\] (19)

Notice that the second term of (19) is associated with voluntary retirement and is a positive and increasing function with respect to initial wealth \( x \). An individual permitted voluntary retirement is willing to take more risk than one considered in the classical Merton’s set-up. Also it seems that the risky investment, \( \pi^{VR} \), increases according to the growth of an individual’s wealth level. Intuitively, people who have slightly smaller wealth than the critical wealth level give their attention to retire voluntarily as soon as possible, so they tend to invest more in risky assets even though they may end up losing relatively large amounts of money. Similar observations were reported by Farhi and Panageas (2007) and Dybvig and Liu (2009).

The third and fourth terms of the right hand side of (18) are closely associated with involuntary unemployment. Notice that the third term is negative, so it decreases an individual’s stockholding, while the forth term consisting of two integral parts urges stock investment.

We can get an upper bound of \( \pi \) where individual’s wealth goes up to a sufficiently close level to the critical wealth level and a lower bound where individual’s wealth goes down to zero:

**Corollary 3.1** For \( I_2 \leq \frac{r}{\beta \hat{\lambda}} \)

\[
\lim_{x \uparrow \hat{x}} \pi(t) \leq \frac{\theta}{\sigma} \frac{1}{\lambda (\beta + \delta)} + \frac{\theta}{\sigma} \alpha_0 B(\hat{\lambda}) \hat{\lambda}^{-\alpha_0} - \frac{2\delta \alpha_0}{\sigma \theta (\alpha_0 - \alpha_0^*) \beta \hat{\lambda}} \ln \frac{1}{\hat{\lambda}}.
\]
Moreover,
\[
\lim_{\lambda \to 0} \pi(t) \geq \frac{\theta}{\sigma} \lambda^*(x)(\beta + \delta) + \frac{\theta}{\sigma} \beta \lambda^*(x) \lambda^*(-\alpha) - \frac{2\delta \alpha \beta}{\sigma(\alpha \beta - \alpha^\lambda)} \ln \left( \frac{\beta I_2}{r} \right) \lambda^*(x) \lambda^*(-\alpha) \lambda^* \lambda^\alpha - 1.
\]

**Proof.** We first utilize inequality \((G(\mu) - I_1/r)^+ + I_2/r \leq \max \left\{ \frac{1}{\beta \lambda}, \frac{I_2}{r} \right\}\), and take the limit of \(\lambda^* \downarrow \hat{\lambda}\) to derive the last term of the right hand side of the first inequality. On the other hand, the last term of the right hand side of the second inequality is derived if we use \(G(\mu) \geq 0\) for all \(\mu\) and take the limit of \(x \downarrow 0\). Q.E.D.

### 4 Numerical Implications

In this section, we investigate the behaviors of critical wealth level, optimal consumption, and optimal risky investment in economically reasonable circumstances. We exploit the iterative procedure in Section 2.3.2.

#### 4.1 Baseline Parameters

The baseline parameters are fixed to \(r = 3.71\%\), which is the annual rate of return from rolling over 1-month T-bills during the time period of 1926-2009.\(^7\) and we assume \(\beta\) has the same value as \(r\). We utilize \(\mu = 11.23\%\) and \(\sigma = 19.54\%\), which are the return and standard deviation of the world’s large stocks during the time period of 1926-2009.\(^8\) We set \(K = 3\) following the assumption used by Dybvig and Liu (2009), and \(I_1 = 1\) and \(I_2 = 0.05\) following the results of Lynch and Tan (2011), who conclude the unemployment state pays only 5% of permanent labor income.

#### 4.2 Critical Wealth Level

Table 1 shows critical wealth level \(\hat{x}\)’s for various parameter values, such as forced unemployment intensity \(\delta\), expected rate of stock \(\mu\), stock volatility \(\sigma\), and leisure \(K\). It is obvious that an individual would be better off entering voluntary retirement wherever her wealth level is not less than \(\hat{x}\), so that she can enjoy more leisure after retirement. Table 1 shows that such

\(^7\)Source: Bureau of Labor Statistics.
\(^8\)Source: Datastream.
critical wealth level \( \hat{x} \) decreases as forced unemployment intensity \( \delta \) increases. Intuitively, individuals with a higher \( \delta \) enter the voluntary retirement stage earlier even though the wealth level at retirement is not relatively high, because they willingly submit to such utility losses in exchange for avoiding utility losses stemming from forced unemployment risks.

On the other hand, the critical wealth level increases as \( \mu \) increases or \( \sigma \) or \( K \) decreases, ceteris paribus. After retirement, an individual in our model faces a tradeoff between utility gains owing to the increase of leisure and utility losses from the significant reduction of income. In a financial market with a higher expected rate of stock return, an individual is willing to postpone her voluntary retirement because a better investment opportunity makes her worry less about forced unemployment risks. Similarly, a lower stock volatility also provides her with more investment opportunity, so she worries less about forced unemployment risks. Evidently, a lower quantity of leisure after retirement hinders an individual from entering voluntary retirement earlier.

### 4.3 Optimal Consumption and Risky Investment

Figure 1 shows the amounts of optimal consumption and investment in the risky stock as a function of initial wealth for several values of forced unemployment intensity \( \delta \). An individual’s consumption \( c \) grows as initial wealth level \( x \) becomes higher, but it falls as her forced unemployment possibility \( \delta \) grows. A higher demand on precautionary savings against a higher forced unemployment risk could be an explanation of the latter observation. (See

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( \mu )</th>
<th>( \sigma )</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1023</td>
<td>0.1123</td>
<td>0.1223</td>
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<td>0</td>
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<td>4</td>
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<td>74.7296</td>
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<td>79.4360</td>
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<td>88.8817</td>
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<td>70.8381</td>
</tr>
<tr>
<td></td>
<td>119.5770</td>
<td>75.3832</td>
<td>59.3462</td>
</tr>
<tr>
<td></td>
<td>58.7642</td>
<td>72.6748</td>
<td>85.3255</td>
</tr>
<tr>
<td></td>
<td>77.4196</td>
<td>72.6748</td>
<td>68.3373</td>
</tr>
<tr>
<td></td>
<td>115.1860</td>
<td>72.6748</td>
<td>56.8938</td>
</tr>
</tbody>
</table>

Table 1: Critical wealth levels \( \hat{x} \) for various parameter values of \( \delta, \mu, \sigma, \) and \( K \). \( \beta = 0.0371, r = 0.0371, \mu = 0.1123, \sigma = 0.1954, K = 3, I_1 = 1, \) and \( I_2 = 0.05 \) are used for default parameter values.
Figure 1: Optimal consumption and risky investment as a function of initial wealth: 
\( \beta = 0.0371, r = 0.0371, \mu = 0.1123, \sigma = 0.1954, K = 3, I_1 = 1, \) and \( I_2 = 0.05 \) are used for parameter values.
Malley and Moutos, 1996; Meng, 2003; Gruber, 1997; Arent et al., 2012). In terms of optimal risky investment, Figure 1 shows an interesting feature. Admittedly, if wealth is not small enough, the optimal risky investment increases as initial wealth increases. However, for poor people this might not be true; up to some wealth level, some of them with a higher forced unemployment possibility lessen their investment in the risky stock even though they have more wealth. This observation implies that forced unemployment risks could be an important explanation for the findings of Benzoni et al. (2007) and Lynch and Tan (2011), in that they find the stock holdings can increase at a sufficiently low wealth level. Moreover, the result seems to be consistent with that of Cocco et al. (2005), who address whether the amount of optimal risky investment can rise due to unemployment risks when wealth level is low enough.

Figure 2 says that optimal consumption to wealth ratio and optimal risky investment to wealth ratio decrease at decreasing rates as retirement time approaches, or equivalently, as \( x \) goes to \( \hat{x} \). The decreasing properties of optimal consumption and risky investment with respect to wealth ratio seem to result from the intense aspirations toward early voluntary retirement under our set-up. Notice also that the decreasing rate becomes bigger as forced unemployment possibility \( \delta \) increases. This fact indicates that our model can reflect the higher intense aspirations toward early voluntary retirement of an individual facing a higher forced unemployment risk.

The second result in Figure 2 is compatible with the observation of Polkovnichenko (2006): an individual with a higher unemployment risks has to invest more in stock by using savings to finance consumption, because she cannot expand income much by choosing labor supply in the unemployment state. Economists have a consensus that the flexibility in labor supply causes higher investment in stock (e.g., Bodie et al., 1992), and our model says that forced unemployment risks might lead to a much higher allocation to stock when the wealth of an individual with a higher unemployment risks is low.

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9Much empirical literature alludes to precautionary savings against background risks, which cannot be controlled by an individual and are independent of endogenous risks. (Kimball, 1992, 1993; Eeckhoudt and Kimball, 1992; Elmendorf and Kimball, 2000; Carroll, 2001; Carroll and Kimball, 2008). Background risks consist of two main factors, income risk and housing effect, and Bodie et al. (1992), Koo (1998), and Heaton and Lucas (2000) emphasize the importance of income risk.

10The latter result is consistent with that of Carroll (1992). According to her, the latter result, meaning the property of increasing consumption at a higher wealth level, is mostly due to individual's impatience.
Figure 2: Optimal consumption to wealth ratio and optimal risky investment to wealth ratio as a function of initial wealth originated by the critical wealth level $\hat{x}$. $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.01$ are used for parameter values.
4.4 Certainty Equivalent Wealth Gain

We define certainty equivalent wealth gain (henceforth, CEWG) to be the maximum wealth level that an individual in our model is willing to give up in exchange for the market without forced unemployment risks. We now investigate how much CEWG is in a normal economic situation and how sensitive CEWG is when $\mu$, $\sigma$, and $K$ change.

**Definition 4.1** $\Delta(x)$ is the certainty equivalent wealth gain at initial wealth level $x$ if it satisfies

$$\tilde{\phi}(x - \Delta(x)) = \phi(x),$$

where $\tilde{\phi}(x)$ is the value function $\phi(x)$ with $\delta = 0$.

Figure 3 represents CEWG as a function of initial wealth level. It decreases as wealth increases, implying that individuals become less sensitive to forced unemployment risks as they
accumulate wealth. Intuitively, an individual confronting forced unemployment risks would be disentangled from it near the critical wealth level, because the possibility of unemployment for the small period is very low and she can retire soon. Thus, CEWG becomes smallest when wealth approaches the critical wealth level. Moreover, the value of CEWG becomes bigger as \( \delta \) increases. A smaller \( \delta \) means a higher critical wealth level, so the individual has more time to prepare for forced unemployment risks; thus CEWG should be smaller.

Table 2 and 3 display the sensitivity of CEWG to the changes of expected rates of stock return \( \mu \) and stock volatility \( \sigma \), respectively. We find the values of CEWG for the extensive range of wealth, \([\hat{x} - 50, \hat{x}]\). It seems that an individual participating in a financial market with a higher expected stock return and/or a lower stock volatility has a lower CEWG. Intuitively, as investment opportunity in the financial market grows, individuals facing forced unemployment risks become less stressful because they are more likely to enter voluntary retirement.

The sensitivity of CEWG for various \( K \), preference for leisure, is illustrated in Figure 4. It seems that CEWG is much bigger for a bigger preference for leisure if an individual’s wealth is far less than her critical wealth level. This implies individuals with a higher preference
Table 3: Certainty equivalent wealth gain to wealth ratios $\Delta(x)/x$ for various $\sigma$ and $\delta$: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $K = 3$, $I_1 = 1$, and $I_2 = 0.05$ are used for parameter values. $\hat{x}$ is the critical wealth level which is different for each $\delta$.

<table>
<thead>
<tr>
<th>$x \backslash \sigma$</th>
<th>$\delta = 0.01$</th>
<th>$\delta = 0.02$</th>
<th>$\delta = 0.03$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{x} - 50$</td>
<td>0.0372 0.0620</td>
<td>0.0784 0.1345</td>
<td>0.1266 0.2214</td>
</tr>
<tr>
<td>$\hat{x} - 40$</td>
<td>0.0239 0.0348</td>
<td>0.0538 0.0721</td>
<td>0.0795 0.1164</td>
</tr>
<tr>
<td>$\hat{x} - 30$</td>
<td>0.0147 0.0215</td>
<td>0.0294 0.0473</td>
<td>0.0435 0.0714</td>
</tr>
<tr>
<td>$\hat{x} - 20$</td>
<td>0.0113 0.0118</td>
<td>0.0224 0.0235</td>
<td>0.0329 0.0355</td>
</tr>
<tr>
<td>$\hat{x} - 15$</td>
<td>0.0067 0.0096</td>
<td>0.0151 0.0193</td>
<td>0.0245 0.0294</td>
</tr>
<tr>
<td>$\hat{x} - 10$</td>
<td>0.0038 0.0082</td>
<td>0.0082 0.0165</td>
<td>0.0129 0.0257</td>
</tr>
<tr>
<td>$\hat{x} - 5$</td>
<td>0.0020 0.0043</td>
<td>0.0047 0.0102</td>
<td>0.0079 0.0171</td>
</tr>
<tr>
<td>$\hat{x}$</td>
<td>0.0007 0.0010</td>
<td>0.0022 0.0028</td>
<td>0.0041 0.0055</td>
</tr>
</tbody>
</table>

Figure 4: Certainty equivalent wealth gains to wealth ratios $\Delta(x)/x$ as a function of initial wealth level: $\delta = 0.03$, $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $I_1 = 1$, and $I_2 = 0.05$ are used for parameter values.
of leisure have more stress if their wealth level is relatively low compared with their critical wealth level. This tendency is mitigated as their wealth gets closer to the critical wealth level.

5 Economic Downturns and Unemployment Risks

In 2000, “Issues in Labor Statistics” published by the US Bureau of Labor Statistics questions whether unemployed job leavers who have voluntarily quit their jobs is a meaningful gauge of confidence in the job market or not. Some analysts regard the increased number of job leavers as a meaningful gauge of rising confidence in the job market, and one possible explanation of their insistence is that it is good for workers to quit their job voluntarily if they have good prospects for a successful job search.

However, “Issues in Labor Statistics” asserts that the unemployed job leavers show a procyclical pattern; the proportion of job leavers increases during economic recessions and decreases during economic expansions, and, thus, the use of unemployed job leavers as a gauge of confidence in the job market could be problematic. Figure 5 shows this phenomenon intimately, and our model might give a solid explanation for it. Notice that critical wealth level of an individual in our model decreases as forced unemployment risk increases, implying a higher forced unemployment risk could compel individuals to retire at a lower wealth level. Generally, forced unemployment risks are relatively high during economic recessions and relatively low during economic expansions, so we might conclude that soaring forced unemployment risks during economic recessions induce myopic individuals who have slightly smaller wealth than their critical wealth level planned during the past economic expansion to enter early retirement.

More specifically, our model can show the behaviors of myopic individuals under different market conditions. Table 4 shows critical wealth levels for various $\delta$ and $I_1$ under two different market conditions, say, ‘economic expansions’ and ‘economic recessions’. Ang and Bekaert (2002) show that the expected stock return and stock volatility in the US market was $\mu = 0.1394$ and $\sigma = 0.2600$ during the economic expansions and $\mu = 0.1394$ and $\sigma = 0.1313$ during the economic recessions, and we utilize the same parameters for Table 4.\textsuperscript{11}

In the table, myopic individuals who draw up their retirement plan solely reflecting the current financial market conditions seem to retire at a much lower wealth level during the

\textsuperscript{11}These parameters are also used in Jang et al. (2007).
Figure 5: **Unemployed job leavers as a percent of the labor force.** (Source: Bureau of Labor Statistics) The two shaded regions stand for US economic recessions quoted by NBER.

Economic recessions than during the economic expansions, even though the force unemployment possibility and their income rate do not change. For instance, an individual with $I_1 = 1$ and $\delta = 0.02$ makes a plan to retire at the wealth level of 213.2846 during the economic expansions, however, the critical wealth level gets much smaller to 77.4343 during the economic recessions.

Admittedly, a smaller income rate $I_1$ or a bigger forced unemployment possibility $\delta$ during economic recession periods is most likely to consolidate the reduction of critical wealth level.\(^{12}\) Since an individual with an asset portfolio and labor income is exposed to a bigger income risk stemming from forced unemployment events and worse financial market conditions during economic recessions, she inevitably lowers her voluntary retirement wealth level in order to get a relatively higher utility gain (mostly obtained from more leisure time) after retirement. For example, according to Table 4 a myopic individual with $\delta = 0.01$ and $I_1 = 1$ has the

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\(^{12}\)Lynch and Tan (2011) find that the first and second moments of labor income growth depend on business cycles, and Emery et al. (2010) find that in the global crisis the default volume of corporate issuers skyrocketed, for example, the default rate of Moody’s global speculative-graded bonds was 13.0% in 2009, which is about three times bigger than 4.4% in the previous year.
### Table 4: Critical wealth level \( \hat{x} \) for various \( \delta \) and \( I_1 \): \( \beta = 0.0371, r = 0.0371, K = 3, \mu = 0.1394 \) (0.1394), \( \sigma = 0.2600 \) (0.1313) are used for the parameter values under Economic Recessions (Economic Expansions, respectively). \( I_2 \) is fixed as 5% of \( I_1 \).

<table>
<thead>
<tr>
<th>( I_1 ) ( \delta )</th>
<th>Economic Recessions</th>
<th>Economic Expansions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.6</td>
<td>140.4878, 130.2984, 123.4213, 118.6584</td>
<td>373.8310, 354.5471, 338.7705, 325.6472</td>
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<tr>
<td>1.4</td>
<td>122.9268, 114.0716, 108.1111, 103.9982</td>
<td>327.1021, 310.5686, 297.0392, 285.7827</td>
</tr>
<tr>
<td>1.2</td>
<td>105.3658, 97.8359, 92.7833, 89.3122</td>
<td>280.3733, 266.5380, 255.2158, 245.7945</td>
</tr>
<tr>
<td>1</td>
<td>87.8049, 81.5893, 77.4343, 74.5959</td>
<td>233.6444, 222.4478, 213.2846, 205.6607</td>
</tr>
<tr>
<td>0.8</td>
<td>70.2439, 65.3295, 62.0606, 59.8431</td>
<td>186.9155, 178.2852, 171.2241, 165.3506</td>
</tr>
<tr>
<td>0.6</td>
<td>52.6829, 49.0535, 46.6554, 45.0444</td>
<td>140.1866, 134.0316, 128.9996, 124.8166</td>
</tr>
<tr>
<td>0.4</td>
<td>35.1219, 32.7555, 31.2075, 30.1831</td>
<td>93.4578, 89.6541, 86.5518, 83.9794</td>
</tr>
</tbody>
</table>

### 6 Conclusion

We developed a new approach for solving the optimal retirement problem for an individual with an unhedgeable income risk. The income risk stems from a forced unemployment event, which occurs as an exponentially-distributed random shock. The optimal retirement problem is to determine an individual’s optimal consumption and investment behaviors and optimal retirement time simultaneously. Our approach for solving the problem originated from the combination of the DPA and the convex-duality approach, but we introduced a slightly different convex-dual function of the individual’s value function from the conventional ones, and we also provided an efficient iterative numerical method.

Reasonably calibrated parameters show that our model can give an explanation of lower consumption and risky investment behaviors of individuals and relatively higher stock holdings for the poor. Exploiting the concept of CEWG, we glanced at an individual’s optimal
behaviors in changing her wealth level, investment opportunity, and the magnitude of preference of post-retirement leisure.

Finally, we find our model gives an explanation of a procyclical pattern of the number of unemployed job leavers. It could provide an evidence that soaring forced unemployment risks during economic recessions induces people who have slightly smaller wealth than the critical wealth level planned during the past economic expansion to enter early retirement.

7 Appendix

7.1 The Derivation of Equation (12)

We will show equation (12) can be obtained from the value-matching condition at $\lambda = \hat{\lambda}$, or equivalently, at $x = \hat{x}$.

First, rearranging equation (3) we get an equality concerning $\phi(x)$:

$$(\beta + \delta)\phi(x) = (rx + I_1)\lambda(x) - \frac{\theta^2}{2}\lambda^2(x)G'(\lambda(x)) - (1 + \ln \lambda(x)) + \delta U_2(x). \quad (20)$$

If we let

$$H(\lambda) \equiv \frac{1}{\beta + \delta} \left( rG(\lambda)\lambda - \frac{\theta^2}{2}\lambda^2 G'(\lambda) - (1 + \ln \lambda) \right. \left. + \frac{\delta}{\beta} \left( \ln \begin{pmatrix} G(\lambda) - I_1 \end{pmatrix}^+ + \frac{I_2}{r} \right) + \frac{1}{\beta} \left( r + \frac{\theta^2}{2} - \beta(1 - \ln K) \right) \right), \quad (21)$$

then

$$\phi(x) = H(\lambda(x)). \quad (22)$$

Equations (7) and (21) yield

$$H'(\lambda) = \lambda G''(\lambda),$$

so

$$\phi'(x) = H'(\lambda(x))\lambda'(x) = \frac{H'(\lambda(x))}{G'(\lambda(x))} = \lambda(x).$$

Therefore, we can say that $\phi(x)$ is a solution of equation (3) subject to a boundary condition of

$$\phi'({\hat{x}}) = \lambda({\hat{x}}) = \frac{1}{\beta \hat{x} + \frac{I_2}{r}}.$$
Using the condition of \( \phi(\hat{x}) = U_2(\hat{x}) \) in (3), we obtain the value of \( H \) at \( \lambda = \hat{\lambda} \):

\[
H(\hat{\lambda}) = \frac{1}{\beta} \left[ \ln \frac{1}{\hat{\lambda}} + \frac{1}{\beta} \left( r + \frac{\theta^2}{2} - \beta(1 - \ln K) \right) \right].
\]  

(23)

We can get

\[
(\beta + \delta) \frac{1}{\beta} \left[ \ln \frac{1}{\hat{\lambda}} + \frac{1}{\beta} \left( r + \frac{\theta^2}{2} - \beta(1 - \ln K) \right) \right] = r \hat{\lambda} \left( \frac{1}{\beta \hat{\lambda}} + \frac{I_1 - I_2}{r} \right) - \frac{\theta^2}{2} \hat{\lambda}^2 G'(\hat{\lambda})
\]

(24)

if we rearrange the relationship of (21) and rewrite it at the boundary \( \lambda = \hat{\lambda} \). Then, (10) and (11) allow us to obtain

\[
\frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \ln K = \frac{\delta}{\beta} \ln \beta \hat{\lambda} + \hat{\lambda}(I_1 - I_2)(1 + \frac{\alpha_\delta \theta^2}{2r})
\]

\[
- \frac{\delta(\alpha_\delta - 1)}{\beta \hat{\lambda}} \alpha_\delta + 1 \int_\lambda^\infty \mu^{\alpha_\delta - 2} \ln \left\{ \left( G(\mu) - \frac{I_1}{r} \right)^+ + \frac{I_2}{r} \right\} d\mu.
\]

7.2 Theorems Described in Section 2.3.2

**Theorem 7.1 (Uniqueness)** If \( \frac{2\delta}{\theta^2 \beta \lambda} < 1 \), the solution of (10) is unique.

**Proof.** Let \( G_1 \) and \( G_2 \) be the two solutions of (10), then we get

\[
G_1(\lambda) - G_2(\lambda) = \frac{2\delta}{\theta^2 (\alpha_\delta - \alpha_\delta^*) \beta} \left[ \left( \alpha_\delta - 1 \right) \lambda^{-\alpha_\delta} \int_\lambda^\infty \mu^{\alpha_\delta - 2} \left( \ln \beta \left\{ \left( G_1(\mu) - \frac{I_1}{r} \right)^+ + \frac{I_2}{r} \right\} \right) d\mu \right.
\]

\[
- \ln \beta \left\{ \left( G_1(\mu) - \frac{I_1}{r} \right)^+ + \frac{I_2}{r} \right\} d\mu \right] + (\alpha_\delta - 1) \lambda^{-\alpha_\delta} \int_\lambda^\infty \mu^{\alpha_\delta - 2} \left( \ln \beta \left\{ \left( G_1(\mu) - \frac{I_1}{r} \right)^+ + \frac{I_2}{r} \right\} \right) d\mu
\]

- \ln \beta \left\{ \left( G_1(\mu) - \frac{I_1}{r} \right)^+ + \frac{I_2}{r} \right\} d\mu].

However, we know that

\[
\left| \ln \beta \left\{ \left( G_1(\mu) - \frac{I_1}{r} \right)^+ + \frac{I_2}{r} \right\} - \ln \beta \left\{ \left( G_1(\mu) - \frac{I_1}{r} \right)^+ + \frac{I_2}{r} \right\} \right| \leq |G_1(\mu) - G_2(\mu)|,
\]

and this implies

\[
|G_1(\mu) - G_2(\mu)| \leq \frac{2\delta}{\theta^2 \beta \lambda} \sup_\mu |G_1(\mu) - G_2(\mu)|.
\]
The proof is complete. \textbf{Q.E.D.}

The following theorem permits us to take a monotonically-decreasing $G(\lambda)$ under suitable parameter conditions.

\textbf{Theorem 7.2 (Monotonicity)} Suppose that

$$I_2 \geq \frac{1}{\beta + \delta} + \frac{2}{\theta^2(\alpha_\delta - \alpha_\delta^*)} \ln \left( \frac{\beta I_2}{r} \right) > 0,$$

and

$$\frac{2\delta \alpha_\delta}{\theta^2 \beta} \left( \ln \beta + \ln \max \left\{ \frac{1}{\beta \lambda}, \frac{I_2}{r} \right\} \right) < \frac{1}{\beta + \delta}. \quad (25)$$

Then, any solution of (7) satisfies $G'(\lambda) < 0$.

\textbf{Proof.} Any solution of (7) satisfies the integral equation (10). From (11) and the assumption of Theorem 7.2 we deduce

$$B(\hat{\lambda}) \hat{\lambda}^{-\alpha_\delta} = \left[ \frac{\delta}{\beta(\beta + \delta)} \frac{1}{\hat{\lambda}} + \frac{I_1 - I_2}{r} \right]$$

$$\left\{ \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)} \right\} \left( \int_\lambda^\infty \mu^\alpha_\delta^2 \ln \beta \left\{ \left( G(\mu) - \frac{1}{r} \right) + \frac{I_2}{r} \right\} d\mu \right\}$$

$$\geq \frac{\delta}{\beta \lambda} \left[ \frac{1}{\beta + \delta} + \frac{2}{\theta^2(\alpha_\delta - \alpha_\delta^*)} \ln \left( \frac{\beta I_2}{r} \right) \right] + \frac{I_1 - I_2}{r} > 0.$$ 

Since $B(\hat{\lambda}) > 0$,

$$G'(\lambda) = -\frac{1}{\lambda^2(\beta + \delta)} - \alpha_\delta B(\hat{\lambda}) \lambda^{-\alpha_\delta - 1}$$

$$- \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)} \left[ \alpha_\delta(\alpha_\delta - 1) \lambda^{-\alpha_\delta - 1} \int_\lambda^\infty \mu^\alpha_\delta^2 - 1 \ln \beta \left\{ \left( G(\mu) - \frac{1}{r} \right) + \frac{I_2}{r} \right\} d\mu \right]$$

$$+ \alpha_\delta^*(\alpha_\delta^* - 1) \lambda^{-\alpha_\delta^*-1} \int_\lambda^\infty \mu^\alpha_\delta^2 - 2 \ln \beta \left\{ \left( G(\mu) - \frac{1}{r} \right) + \frac{I_2}{r} \right\} d\mu \right\}$$

$$+ \frac{2\delta}{\theta^2 \beta^2} \ln \beta \left\{ \left( G(\lambda) - \frac{1}{r} \right) + \frac{I_2}{r} \right\}, \quad (26)$$

$$\leq -\frac{1}{\lambda^2} \left[ \frac{1}{\beta + \delta} - \frac{2\delta \alpha_\delta}{\theta^2 \beta} \left( \ln \beta + \ln \max \left\{ \frac{1}{\beta \lambda}, \frac{I_2}{r} \right\} \right) \right] - \alpha_\delta B(\hat{\lambda}) \lambda^{-\alpha_\delta - 1}.$$ 

From the assumption of Theorem (7.2), we find the fact of $G'(\lambda) < 0$. \textbf{Q.E.D.}
Theorem 7.3 (Existence) Suppose

\[ I_1 > \frac{\theta^2}{2\beta(\beta + \delta)} (1 - \alpha_\delta) + \ln K - \frac{\delta}{\beta} \ln \frac{\alpha_\delta t_2}{r} + I_2, \]

\[
\frac{\delta}{\beta} \ln \beta + \frac{2\delta}{\theta^2 \beta} (I_1 - I_2) (1 + \frac{\alpha_\delta \theta^2}{2r}) + \frac{\delta}{\beta} \ln \left( \max \left\{ \frac{\theta^2}{2 \beta} \frac{I_2}{r} \right\} \right) < \frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \ln K, \quad \text{and} \\
\frac{\delta}{\beta} \ln \beta + \frac{2\delta}{\alpha_\delta (\beta + \delta)} \left( 1 + \frac{\alpha_\delta \theta^2}{2r} \right) + \frac{\delta}{\beta} \ln \left( \max \left\{ \frac{\theta^2}{2 \beta}, \frac{I_2}{r} \right\} \right) < \frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \ln K.
\]

Then there exists at least one solution \( \hat{\lambda} \) of (12) satisfying \( 0 < \hat{\lambda} < r I_2 \beta \). The corresponding solution \( G(\lambda) \) of (7) is uniquely determined.

Proof. Recall the relationships of

\[
\psi_\delta(\hat{\lambda}) = \frac{\delta}{\beta} \ln \beta \hat{\lambda} + \hat{\lambda} (I_1 - I_2) (1 + \frac{\alpha_\delta \theta^2}{2r}) - \frac{\delta(\alpha_\delta^* - 1)}{\beta} \hat{\lambda}^{-\alpha_\delta + 1} \int_{\lambda}^{\infty} \mu^{\alpha_\delta - 2} \ln \left( \left\{ \frac{G(\mu) - \frac{I_1}{r}}{\frac{I_2}{r}} \right\} + \frac{I_2}{r} \right) d\mu,
\]

\[
L_\delta = \frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \ln K,
\]

\[
\phi_\delta(\hat{\lambda}) = \frac{\delta}{\beta} \ln \beta \hat{\lambda} + \hat{\lambda} (I_1 - I_2) (1 + \frac{\alpha_\delta \theta^2}{2r}),
\]

\[
\bar{\phi}_\delta(\hat{\lambda}) = \frac{\delta}{\beta} \ln \beta \hat{\lambda} + \hat{\lambda} (I_1 - I_2) (1 + \frac{\alpha_\delta \theta^2}{2r}) + \frac{\delta}{\beta} \ln \max \left\{ \frac{1}{\beta \lambda_0}, \frac{I_2}{r} \right\}.
\]

From (14), (15) and (16) and the continuity of the function \( \psi_\delta(\hat{\lambda}) \) we conclude that there exists at least one solution of

\[
\psi_\delta(\hat{\lambda}) = L_\delta. \tag{27}
\]

Notice that the following two assumptions of

\[
\frac{2\delta}{\theta^2 \beta \lambda_0^0} < 1 \tag{28}
\]

and

\[
\frac{2\delta \alpha_\delta}{\theta^2 \beta} \left( \ln \beta + \ln \max \left\{ \frac{1}{\beta \lambda_0^0}, \frac{I_2}{r} \right\} \right) < \frac{1}{\beta + \delta} \tag{29}
\]

guarantee the fact that a solution of (27) satisfies (25), and imply the assumption in Theorem 7.1. Also notice that (28), (29) are equivalent to

\[
\lambda_0^0 > \frac{2\delta}{\theta^2 \beta}, \quad \lambda_0^0 > e^{-\frac{\theta^2 \beta}{2 \alpha_\delta (\beta + \delta)}}, \tag{30}
\]

29
respectively. The definition of $\lambda^0_\delta$ and the two inequalities in (30) yield

$$\overline{\phi}_\delta\left(\frac{2\delta}{\theta^2}\right) < L_\delta, \quad \overline{\phi}_\delta\left(e^{-\frac{\theta^2}{2\alpha_\delta(\beta+\delta)}}\right) < L_\delta. \quad (31)$$

The first inequality in (31) can be rewritten as

$$\frac{\delta}{\beta} \ln \frac{2\delta}{\theta^2} + \frac{2\delta}{\theta^2} (I_1 - I_2)\left(1 + \frac{\alpha_\delta \theta^2}{2r}\right) + \frac{\delta}{\beta} \ln \left(\max\left\{\frac{\theta^2}{2\delta}, \frac{I_2}{r}\right\}\right) < \frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \ln K, \quad (32)$$

and the second one also can be restated as

$$\frac{\delta}{\beta} \ln \frac{\theta^2 \beta}{2\delta \alpha_\delta(\beta + \delta)} + e^{-\frac{\theta^2}{2\alpha_\delta(\beta+\delta)}} (I_1 - I_2)\left(1 + \frac{\alpha_\delta \theta^2}{2r}\right) + \frac{\delta}{\beta} \ln \left(\max\left\{\frac{1}{\beta} e^{\frac{\theta^2}{2\alpha_\delta(\beta+\delta)}}, \frac{I_2}{r}\right\}\right) < \frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \ln K. \quad (33)$$

Therefore, we can say that, if (32) and (33) hold (these conditions require a sufficiently small $\delta$), there exists at least one solution of (27). For any solution of (27), the corresponding $G(\lambda)$ is the unique solution of (7).

Moreover, the inequality $\lambda^1_\delta < \frac{r}{\beta I_2}$ is a sufficient condition of the inequality of $\hat{\lambda} < \frac{r}{\beta I_2}$.

It can be rewritten as

$$\overline{\phi}_\delta\left(\frac{r}{\beta I_2}\right) > L_\delta,$$

and it is equivalent to

$$\frac{\delta}{\beta} \ln \frac{r}{I_2} + \frac{r}{\beta I_2} (I_1 - I_2)\left(1 + \frac{\alpha_\delta \theta^2}{2r}\right) > \frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \ln K - \frac{\delta}{\beta} \ln \frac{r}{I_2}.$$

Therefore, we get the first assumption of the theorem

$$I_1 > \frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \ln K - \frac{\delta}{\beta} \ln \frac{r}{I_2} + I_2.$$

Q.E.D.

Now we show that the function $\phi(x)$ defined by (22) solves the variational inequality (2).

**Theorem 7.4**

If

$$\frac{1}{\beta} \frac{r(I_1 - I_2)}{I_2} (1 + \frac{\alpha_\delta \theta^2}{2r}) < \frac{\theta^2}{2} \frac{\delta(1 - \alpha_\delta)}{\beta(\beta + \delta)} + \ln K \quad \text{and}
$$

$$\frac{1}{\beta} e^{\frac{\theta^2}{2(\beta+\delta)}} (I_1 - I_2)\left(1 + \frac{\alpha_\delta \theta^2}{2r}\right) + \frac{\delta I_2}{\beta r} < \ln K$$

are true, the value function $\phi(x)$ defined in (22) is a solution of the variational inequality (2).
Proof. Since
\[ \phi(\hat{x}) = U_2(\hat{x}), \]
for
\[ \psi(x) \equiv \phi(x) - U_2(x) \]
we know \( \psi(\hat{x}) = 0 \). If we show that \( \psi'(x) \leq 0 \) for \( 0 < x \leq \hat{x} \), the second inequality of (2) will follow. It is enough to show that
\[ G(\lambda) < \frac{1}{\beta \lambda} + \frac{I_1 - I_2}{r}, \tag{34} \]
for \( \lambda > \hat{\lambda} \).

Define
\[ \Gamma(\lambda) \equiv G(\lambda) - \frac{1}{\beta \lambda} - \frac{I_1 - I_2}{r}, \]
then, \( \Gamma(\hat{\lambda}) = 0 \) and \( \Gamma(\infty) = -\frac{I_1 - I_2}{r} < 0 \). From the first equality of (7) we can get
\[ -\frac{1}{2} \theta^2 \lambda^2 \Gamma''(\lambda) - \lambda(\theta^2 + \beta + \delta - r) \Gamma'(\lambda) + r \Gamma(\lambda) + \frac{\delta}{\beta} \Gamma'(\lambda) + \frac{r}{\beta \lambda} \lambda^2 = -\frac{\delta}{\beta \lambda} - (I_1 - I_2) + \frac{\delta}{\beta^2} \Gamma(\lambda) + \frac{1}{\beta \lambda} \lambda^2. \tag{35} \]
Since \( \lambda > \hat{\lambda} > \lambda_0^\beta \), the right-hand side of (35) have an upper bound of
\[ -\frac{\delta}{\beta \lambda} \left(1 - \frac{r}{\beta I_2} \lambda_0^\beta \right) < 0 \]
under the condition of
\[ \lambda_0^\beta > \frac{r}{\beta I_2}. \tag{36} \]
Applying the comparison principle (Friedman, 1982) to (35), we obtain \( \Gamma(\lambda) < 0 \), which is equivalent to (34).

Now, we will verify the first inequality in (2). For \( x \geq \hat{x} \), the following equality holds:
\[ (\beta + \delta)\phi(x) - (rx + I_1)\phi'(x) + \frac{\theta^2 \phi'(x)^2}{2 \phi(x)} + 1 + \ln \phi'(x) - \delta U_2(x) \]
\[ = -\frac{I_1 - I_2}{\beta(x + \frac{I_2}{r})} + \ln K, \tag{37} \]
since \( \phi(x) = U_2(x) \). The function \( -\frac{I_1 - I_2}{\beta(x + \frac{I_2}{r})} + \ln K \) is monotonically increasing, so it is sufficient to show
\[ -\frac{I_1 - I_2}{\beta(\hat{x} + \frac{I_2}{r})} + \ln K \geq 0, \]
or equivalently,
\[ \ln K - \hat{\lambda}(I_1 - I_2) \geq 0. \]

Using (12), we get
\[
\ln K - \hat{\lambda}(I_1 - I_2) \geq -\frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \frac{\delta}{\beta} \ln(\beta \hat{\lambda}) + \hat{\lambda}(I_1 - I_2) \frac{\alpha_\delta \theta^2}{2r} \\
\geq -\frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \frac{\delta}{\beta} \ln(\beta \lambda_0) + \lambda_0(I_1 - I_2) \frac{\alpha_\delta \theta^2}{2r} > 0
\]

where
\[
\lambda_0^0 = \frac{1}{\beta} e^{\frac{\theta^2}{\beta} \frac{(1 - \alpha_\delta)}{(\beta + \delta)}}. \quad (38)
\]

We can rewrite the restrictions (36) and (38) as the following inequalities by using the function \( \varphi_\delta(\cdot) \) defined in (15):
\[
\varphi_\delta \left( \frac{r}{\beta I_2^2} \right) < L_\delta, \quad \varphi_\delta \left( \frac{1}{\beta} e^{\frac{\theta^2}{\beta} \frac{(1 - \alpha_\delta)}{(\beta + \delta)}} \right) < L_\delta. \quad (39)
\]

Therefore, the two inequalities in (39) become our assumptions:
\[
\frac{1}{\beta} \frac{r(I_1 - I_2)}{I_2^2} (1 + \frac{\alpha_\delta \theta^2}{2r}) < \frac{\theta^2}{2} \frac{\delta}{\beta(\beta + \delta)} (1 - \alpha_\delta) + \ln K, \quad \text{and}
\]
\[
\frac{1}{\beta} e^{\frac{\theta^2}{\beta} \frac{(1 - \alpha_\delta)}{(\beta + \delta)}} \frac{r(I_1 - I_2)}{I_2^2} (1 + \frac{\alpha_\delta \theta^2}{2r}) + \frac{\delta I_2}{\beta r} < \ln K.
\]

Q.E.D.

References


