Rolling the Skewed Die to Meet Aspirations

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February 26, 2018

Abstract

Skewness is pervasive across financial instruments and investment decisions, and the literature has documented that many investors seek idiosyncratic skewness in the portfolios. In response, there are some theoretical models that study implications of the preference for skewness, but using utility functions where the preference for right skewness is hard-wired. In this paper we study the demand for skewness –both right and left skewness– using a utility function with microeconomic and evolutionary foundations in the spirit of Friedman and Savage (1948). We consider a parsimonious set of securities that allow the agent to select the exact optimal level of right or left skewness. Our analysis yields a rich set of results broadly consistent with empirical observations.

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1 Introduction

Skewness is pervasive among financial securities—options, growth stocks...— and other types of investments—private equity and VC. Furthermore, skewness-seeking is a first order reason of investment decisions, and it can explain some of the most challenging empirical puzzles contemplated in the literature—for example, the value puzzle, Zhang (2013). However, standard utility functions—in particular, CRRA utility functions—cannot explain the demand for skewness we observe in practice. While the current literature, especially during the last ten years, has explored lottery characteristics—i.e., skewness—of many securities, it has not tackled the reasons that drive individual investors to demand skewness. Within the scarce literature that tries to provide a foundation for the demand of skewness is the work on aspirational utility (Diecidue and van den Ven 2008, in the spirit of Friedman and Savage 1948). Their utility includes a jump that represents the discontinuity in utility derived from crossing a certain threshold of wealth.

In this paper we analyze the demand for skewness that results from an utility function in the spirit of Diecidue and van den Ven (2008) but derived from microeconomic foundations. In particular, we consider an economic agent who cares both about consumption and status, as independent goods that provide utility. The consumption good is divisible and contributes to total utility in the same way as in the standard CRRA case. However, status depends on a non-divisible good—to simplify we assume that the only utility provided by this good is through the status recognition; our conclusions do not depend on this result. Status-seeking is related to a number of preferences popular in the financial economics literature as relative wealth concerns—or external habit formation, Campbell and Cochrane (1999)—and habit formation (Sundaresan 1989 and Constantinides 1990). Rayo and Becker (2007) show that this type of utility provides an evolutionary edge. More recently, Roussanov (2010) proposes a utility model that includes status-seeking and derives some investment implication.

The resulting utility reminds of the framework first established by Friedman and Savage (1948). They are motivated by the observation that some investors simultaneously buy in-
surance and lotteries which, they argue, cannot be explained by standard utility models. However, in their analysis they only consider the notion of “volatility,” the second moment of the distribution. Unlike the previously cited works, we are interested in the third moment, the skewness. This clarifies the first part of the puzzle raised by Friedman and Savage (1948), since lotteries amount to taking long positions in right skewness and buying insurance implies a short position in left skewness. Therefore, they are consistent decisions for an investor with strong preference for right skewness. Yet, the second part of the Friedman and Savage (1948) argument, namely the shortcomings of standard utility models, is still relevant, since in general, standard utility models cannot explain positions in skewness—for example, although CRRA utility in principle implies a preference for right skewness, the optimal policy of an investor with CRRA utility is to sell short a lottery because its negative expected return and high variance dominate the effect of positive skewness.

Interest on the demand for skewness and its effect on equilibrium prices is not new. Kraus and Litzenberger (1976) already explore its implications, assuming a utility function that puts a larger weight on the third moment. More recently, Harvey and Siddique (2000) explore implications of the relation among higher moments—in particular, co-skewness—in the cross-section of stocks. Mitton and Vorkink (2007) show that many investors have a preference for skewness in their portfolios and strategically choose securities that are avoided by investors who prefer a diversified portfolio. Kumar (2009) studies stocks with lottery-like properties and shows they are chosen by people who also buy lotteries. Boyer and Vorkink (2014) study lottery properties among stock options.

In this paper we are interested in the drivers of the demand for skewness resulting from a utility function with microeconomic and evolutionary foundations. The class of aspirational utility functions we consider is similar to a CRRA function, but with a reference point R, and a positive jump in utility if the agent can spend more than R. Next we introduce a parsimonious set of securities possibly skewed that allow investors to choose the exact level of skewness—right or left—optimal for their position in the utility function. From this setting, through a concavification of the utility function we can derive the four seasons of the
demand for skewness. In particular, we show analytically that the relative position of the reference point R (i.e., how far away the aspiration is), with respect to the agent’s current consumption level \( C_0 \) is a critical factor in determining the demand for skewness. If the agents’ aspiration is only marginally higher than the current endowment, they choose to sell skewness. This is because the proximity of the aspiration point encourages the agents to select a security that lands on the aspiration level with relatively high chance. Such a security - high chance of small gain - is negatively skewed. On the other hand, by exactly symmetric arguments, as the aspiration level moves further away from the current wealth, agents choose to buy right-skewed securities. This is in sharp contrast to the standard mean-variance analysis where agents shun any form of gambling with zero or negative expected returns.

We also explore how endogenous demand for skewness changes with respect to parameters. Predictably, the size of the jump is a main factor. If attaining the aspiration leads to a big jump in utility, agents choose less (right-)skewed securities. Intuitively, a big jump implies greater importance of aspiration: the agent is then forced to demand securities that can help get to the aspiration level with higher probability, albeit at the expense of a lower level of consumption if the gamble fails. Such securities contain low or even negative skewness. Analogous results are presented for the level of risk aversion and initial wealth of the agent.

In the binomial setting we introduce, agents’ choice of skewness mechanically fixes the level of volatility. It is impossible to separate the choice of volatility from the choice of skewness. To investigate the role of volatility in the aspirational setting, we introduce tri-nominal securities (that embed binomial securities as a special case). This allows us to separate volatility from skewness. This yields a somewhat surprising result, but in line with the Friedman and Savage (1948) analysis: the aspirational agents do not necessarily choose to minimize volatility as they would in the standard mean-variance setting. Instead, agents choose just the right amount of volatility to propel themselves to their aspirations. In the aspirational setting, volatility can be desirable insofar as it helps them attain their aspirations. In similar vein, extensions are also made to consider a broader range of securities to investigate the
consequence of variations in the first moment.

We also consider multiple (two) reference points. In the two-reference point case, we find that agents can either choose to mind both reference points, mind only one reference point, or mind neither. If agents choose to mind both reference points, the level of skewness they demand is determined entirely by the position of the reference points relative to their current consumption level ($C_0$). In everyday parlance, the agent gets ‘trapped in’ between a neighbor to whom they want to catch up to, and a neighbor they do not want to fall behind. The agent’s demand for skewness is determined by the relative strength of these considerations.

2 Setup

In this section, we introduce and motivate ‘aspirational utility’: a standard utility function augmented by elements of ‘goal’ and its ‘attainment’ (Diecidue and Van de Ven, 2007). The ‘goal’ in this setting is represented as a position in the agent’s wealth or consumption level, the satisfaction of achieving this aspiration is expressed by a discontinuity or ‘jump’ in utility level.

2.1 A Motivation: Local, Bulky Status Goods

To see how aspirational utility can arise from a natural setting, we consider the notion of ‘status’ in the spirit of Roussanov (2010). In Roussanov’s model, agents care not only about standard consumption (as represented by power utility over consumption) but also about their wealth level relative to the average wealth level of the economy, a feature which represents agent’s ‘status concerns’. While we believe incorporating status is a meaningful endeavor, we make two observations that further enhances the realism of this feature. First, status goods are often bulky, indivisible purchases. (Consider luxury cars, or mansions for example.) Second, unlike the setting of Roussanov where the benchmark of status is uniform across agents of all wealth types (the average wealth level of the economy), it is reasonable to assume that status goods are wealth dependent: for example, jewelry for the poor, large
house for the rich.

We will consider an example that embodies status concerns in the spirit of Roussanov’s model (2010), with the two enrichments described above. Let $W^i$ denote the wealth level of an agent $i$. Suppose that for agents of wealth level $W^i \in (0, \$1 \text{ million})$ the relevant status good is ownership of a small house that costs $0.5 \text{ million}$, and for agents of wealth level $W^i \in ($1 \text{ million}, \$2 \text{ million})$ the relevant status good is a luxury house with private pool that costs $1.5 \text{ million}$ to acquire. This reflects the locality and indivisibility of status goods; that status goods are often bulky purchases that are endemic to the peer groups within wealth brackets. Let $S^i$ denote the status good consumed by agent $i$, and let $c^i$ denote the standard consumption good (bread and butter) consumed by agent $i$. The total consumption of agent $i$ ($C^i$) consists of $c^i$ and $S^i$, (that is, $C^i = c^i + S^i \leq W^i$.) Given standard (e.g. power utility) utility over $c^i$, and the local, indivisible nature of $S^i$, an agent optimally chooses the amount $(c^i, S^i)$. The left-hand panel of Figure 1 gives a graphical description of $U(C^i)$ when the optimal choice $(c^i, S^i)$ is made. The derivation can be made rigorous, but the graph is intuitive. The jumps are inheritance from the local, indivisible nature of status goods. They occur at around $0.7 \text{ (A)}$ and $1.7 \text{ (B)}$ million, slightly above the cost of owning the status goods ($0.5 \text{ and } 1.5 \text{ million}$), representing the fact that agents would buy status goods only after they surpass their ‘subsistence level’. The marginal utility of $C$ is high following the jump, since the agents were forced to be thrifty in order to purchase the local status good. Once the marginal utility drops, agents ponder another jump in status (point B), etc.

The key element of this utility is the ‘jump’. The right-hand panel of Figure 1 depicts a different (single) jump, associated to the corresponding (single) jump at point A of the left-hand panel. This discontinuous jump described in the right-hand panel is essentially the ‘aspirational utility’ proposed by Diecidue and Van de Ven (2007). Although the formal shape is different, one can use a well-known concavification argument to show that they yield identical (expected) utility maximization, and hence to the extent of microeconomic analysis, they are indifferent. In short, local, indivisible status concerns lead to aspirational
utility.

2.2 A Formal Representation of Aspirational Utility

Consider an agent who maximizes expected utility (EU), with initial endowment $C_0$. However, this agent differs from the typical EU-maximizing agent in that (i) he has an aspiration $R$ ($R > C_0$), and (ii) he discounts payoffs that fall below $R$, so that his utility is $\delta u(C)$ when $C < R$, and $u(C)$ when $C \geq (R)$ ($0 < \delta < 1$). Otherwise, we maintain the conventional assumptions on the behavior of $u(\cdot)$: $u'(\cdot) > 0$ and $u''(\cdot) < 0$.

More formally, we write the agent’s utility function as:

$$U(c) = \delta u(c)\mathbb{1}_{(0,R)}(c) + u(c)\mathbb{1}_{[R,\infty)}(c)$$  \hspace{1cm} (1)
2.3 Binomial Martingales

To understand how the agent’s EU-maximization is performed under this setting, we introduce the following set of securities. Let $L(p)$ be a binomial martingale (i.e., fair game with two outcomes) with $p \in (0, 1)$. That is, $L(p)$ is a gamble which costs $\pi$ to purchase, and pays $M$ with probability $p$.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
<th>Value</th>
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<tbody>
<tr>
<td>Fail</td>
<td>$1-p$</td>
<td>$-\pi$</td>
</tr>
<tr>
<td>Success</td>
<td>$p$</td>
<td>$M-\pi$</td>
</tr>
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To ensure that $L(p)$ is a martingale, we require: $0 = p(M-\pi) + (1-p)(-\pi)$. This pins down the amount $M$ as a function of price of lottery $\pi$ and $p$; $M(\pi, p)$.

Similarly, let $L^*(p)$ be an extended binomial martingale with $p \in (-\infty, 1)$. That is, it is a gamble which pays $C_S$ with “probability” $p^*$ and $C_F$ with “probability” $1-p^*$, such that:

$$0 = p(M-\pi) + (1-p)(-\pi), p \in (-\infty, 1)$$

Note that the only difference is the domain of $p$. That we allow $p < 0$ means that at this stage, $p$ and $1-p$ should be interpreted as ‘weights’, rather than probabilities. The reason for extending the domain is to ensure that the structure of the problem remains consistent and transparent, even when we consider the optimization problem in more general settings, as will be shown later. The ‘real-world’ interpretation of $p < 0$ will be provided in Theorem 2.

Using the set of securities $L^*(p)$, the agent can construct a consumption scheme by exercising freedom over two dimensions: (i) which security ($L^*(p)$; choice is over $p$) she wishes to purchase and (ii) how much of that security she wishes to purchase. Let $N$ be the number
of security she purchases. Let $C_S$ be the consumption she enjoys if each unit of her security pays $M - \pi$. Let $C_F$ be the consumption she gets if each unit of her security pays $-\pi$. Without loss of generality, we fix $\pi = 1$ going forward, since the nominal amount she invests in the bet can be adjusted by $N$.

\[
C_S = C_0 + N(M - 1)
\]

\[
\begin{aligned}
&\text{Success} \\
&\quad \text{p} \\
C_0 &\quad \text{Fail} \\
&\quad 1-p \\
C_F = C_0 - N
\end{aligned}
\]

Note here that because $L^*(p)$ is a martingale, the structure of his consumption scheme is also a martingale (can easily be verified algebraically by eliminating $N$ in equations (5), (6) and adding in (7) below), namely:

\[
C_0 = pC_S + (1-p)C_F.
\]  

(3)

Under this setup, we now move on to the utility maximization problem of the agent.

### 2.4 Utility Maximization

The single jump EU-maximization problem (with the extended $L^*(p)$) is:

\[
\max_{p \in (-\infty, 1)} \max_{N \in [0, \infty)} (1 - p)U(C_F) + pU(C_S)
\]

(4)

such that:

\[
C_F = C_0 - N
\]  

(5)
\[ C_S = C_0 + N(M - 1). \]  

\[ 0 = p(M - 1) + (1 - p)(-1) \]

The following lemma allows us to reduce a dimension.

**Lemma 1.** Consider the single jump EU-maximization problem described in equations (4) - (7), with \( C_0 \): initial wealth and \( R \): reference point (\( C_0 < R \)).

Consider an associated reduced form of the EU maximization problem:

\[
\max_{p \in (-\infty, 1)} (1 - p)U(C_F) + pU(R) \tag{8}
\]

such that \( C_F \) satisfies

\[ C_0 = pR + (1 - p)C_F, \]

and let \( p^* \) be the maximand (solution) to this reduced problem.

Assume that the following holds.

\[
U'(C^*_F) := U'(\frac{C_0 - p^*R}{1 - p^*}) := \delta u'(\frac{C_0 - p^*R}{1 - p^*}) > u'(R) := U'(R) \tag{9}
\]

Then any optimal consumption scheme in the original problem satisfies \( C_S = R \).

Assumption (9) of Lemma 1 is not automatically satisfied unless \( \delta = 1 \) (then, since \( C_F < R \), and marginal utility decreases in \( C \), automatic), yet is still innocuous. It only requires that the marginal utility at \( C_F \) is higher than the marginal utility at \( R \). This is intuitive; once the agent attains the reference (aspiration) point, the hankering dissipates and marginal utility goes down, almost by definition.
This Lemma is useful because for each and every p, we can forget about the argument N. Any N which induces a $C_S$ higher than R is irrelevant for the purpose of the optimization, and thus, N is determined automatically. Hence, using Lemma 1, we can reduce the setup described in equations (4)-(7) to the following setup:

$$\max_{p \in (-\infty, 1)} (1 - p)U(C_F) + pU(R)$$

such that $C_F$ satisfies

$$C_0 = pR + (1 - p)C_F$$

From now on, we stick to this setup, which means we will only look at the parameter space $(\delta, R; C_0) \subset \mathbb{R}^3$ that satisfies assumption (9).

Before laying down a set of results that characterizes the behavior of solutions to the problem, we introduce an ancillary lemma that relates $p$ with the level of skewness embodied in security $L^*(p)$. The definition of skewness we use is ‘Pearson’s moment coefficient of skewness’.

$$S(p) = \mathbb{E} \left[ \left( \frac{X - \mu}{\sigma} \right)^3 \right]$$

**Lemma 2.** Let $S : p(0, 1) \mapsto \mathbb{R}$ denote ‘Pearson’s moment coefficient of skewness’ of the consumption scheme induced by $L(p)$. Then:

(i) $S(p)$ is monotonically decreasing in $p$

(ii) $S(p) \uparrow \infty$ as $p \downarrow 0$

(iii) $S(\frac{1}{2}) = 0$

(iv) $S(p) \downarrow -\infty$ as $p \uparrow 1$
This Lemma describes the relationship between $p$ and skewness implied by $p$. In particular, it shows that skewness is monotone in $p$ and hence, the $p^*$ chosen as the solution to the EU-maximization problem (10) uniquely determines the level of skewness that the agent’s choice implies. If $p^* = \frac{1}{2}$, the agent is demanding a symmetric security. If $p^* < \frac{1}{2}$, the agent is demanding a positively-skewed security; a security that has ‘lottery-like’ feature. If $p^* > \frac{1}{2}$, the agent is demanding a negatively-skewed security; e.g., a security that delivers modestly positive returns most of the time, but very negative returns in rare, but unfortunate states. Note that the returns have to be ‘very negative’, because of the martingale assumption, and increasingly so as $p^*$ approaches 1.

3 Single Jump Optimization

Now, we go back to the single jump EU-maximization problem. We specialize $u(\cdot)$ to be the power utility function with $\gamma > 1$. For technical comfort (and realism), we assume $1 \leq C_0(< R)$. Namely, first, it is natural to assume that $C_0$ is lower than R. Also, requiring $1 \leq C_0$ is to ensure the utility value is positive on that domain, so that that multiplying by $\delta$ is indeed a discount. This does not harm generality, since we can always translate the utility function to be positive. Also, to make the problem non-trivial, we assume $0 < \delta < 1$. Theorems 1 and 2 characterize the solution to the EU-maximization problem, $p^*$.

3.1 The ‘Four Seasons of Gambling’

**Theorem 1.** Let $u(c)$ be the power utility function and let $L^*(p)$ be an extended binomial martingale with $p$: “probability” of success ($p \in (-\infty, 1)$). Fix $C_0$: initial wealth, and consider the reduced single jump EU-maximization problem (10) with $R$: reference point ($C_0 < R$). Let $p^*$ be the “probability” that maximizes the expected utility. Then:

(i) $\frac{\partial p^*}{\partial (\frac{C_0}{R})} > 0$
(ii) as \( \left( \frac{C_0}{R} \right) \uparrow 1 \) (i.e., as \( C_0 \uparrow R \)), \( p^* \uparrow 1 \)

(iii) \( \exists \left( \frac{C_0}{R} \right)^* \in (0,1) \) such that \( p^* \leq 0 \)

Theorem 1 describes how \( p^* \) - the optimal security demanded by the agent with aspirational preference - changes with the position of his aspiration (R) relative to his current consumption (\( C_0 \)).

(i) asserts that as R moves further out, away from \( C_0 \), the agent demands more positively skewed security. In everyday parlance, this means that as her aspiration becomes ‘unrealistically high’, the agent starts to demand more and more ‘lottery-like’ securities to meet the aspiration. When \( C_0 \) is far away from R, attempting such a big jump in consumption (\( R - C_0 \)) with high chance comes at the cost of disastrously low consumption if the attempt fails. (Recall that all securities in the choice set are fair gambles.) Thus, the agent rationally avoids the abysmally low utility levels in bad states, through the purchase of positively skewed security (with low \( p^* \)). Moreover, the positive sign on the derivative asserts that this relationship is monotonic. That is, as agents’ aspirations move far from (close to) \( C_0 \), the optimal choice of skewness will increase (decrease) monotonically.

(ii) describes the opposite situation. As R becomes indistinguishably close to \( C_0 \), the agent prefers a negatively-skewed security; that which takes her to the aspiration (R) with high chance at the expense of a low-chance event of a disaster. While it is true that the martingale assumption requires that \( C_F \) be low (because \( p^* \) is close to 1), the proximity of \( C_0 \) to R ensures that the associated \( C_F \) is not unacceptably low. In other words, the proximity of the aspiration ensures that the agent can avoid exposing herself to a big downside risk, even while she is taking negatively skewed bets.

(iii) describes what happens when the aspiration is too far, or equivalently, when the agent is ‘too poor’. (iii) asserts that at some point, the aspiration will be so remote that it is optimal to choose \( p^* \leq 0 \). At this point, however, we are interpreting \( p^* \) as ‘optimal weights’, because \( p^* \) has no real world analogue. The full meaning of (iii) will be revealed in Theorem 2.
Taken together, (i), (ii), and (iii) show that as $R$ gets pushed away from $C_0$, the agent’s demand runs through the entire gamut of securities, starting from the most negatively skewed to the most positively skewed. This happens for any fixed $\delta \in (0, 1)$.

**Theorem 2.** Assume (as is in the real world) that only $L(p)$ with $p \in (0, 1)$ is available. When $p^* \leq 0$, the agent chooses $C_0$ over any $L(p)$ with $p \in (0, 1)$.

Theorem 2 and Theorem 1 together tell us what happens as $R$ increases relative to $C_0$. At first, as $C_0$ stands close to $R$, the agents demand negatively skewed securities for the reasons explained above. As $R$ increases, the agents start demanding more symmetric securities (e.g., stocks), then positively skewed ‘lottery-like’ securities. As we move the $R$ away from $C_0$ further, agents eventually reach a certain threshold where they stay with $C_0$, thereby stop demanding risky securities altogether. We may call this the ‘four seasons of gambling’ as depicted in Figure 1.

![Figure 2: The Four Seasons of Gambling](image)

The horizontal axis represents the $\delta \in [0, 1]$, and the vertical axis represents $\frac{R}{C_0} \in [1, 3]$. Fix
any \( \delta \), and start from \( \frac{R}{C_0} = 1 \). Theorems 1 and 2 says that when \( \frac{R}{C_0} \approx 0 \), agents start fresh by buying negative skewness (Spring, light blue). Then, as \( \frac{R}{C_0} \) increases, they start to buy symmetric securities (Summer, yellow), then positively skewed securities (Autumn, brown), and ultimately, they stop (Winter, navy). Although the picture is truncated at \( \frac{R}{C_0} = 3 \), if we were to elongate the picture, we would indeed see that winter hits agents for all \( \delta \).

It is worth mentioning the role of volatility in the optimization problem. From equation (20), we know that volatility in increasing in \( p^* \), and inversely related to skewness. This means that when \( R \) is very close to \( C_0 \), agents choose a negatively skewed security, even though it embodies a very high volatility. Intuitively, this can be interpreted as a situation where the aspiration point is tantalizingly close to the current income \( C_0 \), and the agent will choose to attain it (and hence enjoy the utility that hikes over the kink) with very high probability, even at the expense of the commensurately low \( C_F \) and the large volatility this entails. On the other hand, when \( R \) moves farther away from \( C_0 \) agents see it as ‘too remote’. They grow increasingly more interested in limiting the downside risk, while hitting the aspiration point becomes a lottery event. As the downside risk is reduced, it also increasingly evaporates their hopes of hitting the aspiration point. These forces diminish the volatility embedded in the optimal \( L(p^*) \), until ultimately, agents are unwilling to accept any volatility whatsoever \( (p^* \leq 0) \) and the four seasons of gambling is over. In short, the most stark deviation from the mean-variance framework (recall that in this martingale setting, agents do not tolerate any volatility in the security) is when the aspiration point is very close to the current consumption level \( C_0 \), in the sense that agents are willing to buy securities with huge volatilities embodied in them in order to obtain \( R - C_0 \). Ultimately, such deviation dissipates as \( R \) moves far away from \( C_0 \).

### 3.2 Comparative Statics

We are now ready to harvest a few comparative statics results that illustrates the choice of skewness by aspirational agents.
3.2.1 Size of Jump

Theorem 3. \( \frac{dp^*}{d\delta} < 0 \)

Theorem 3 explains what happens as \( \delta \) increases. Recall that \( \delta \) is how much the agent keeps when the consumption falls short of the reference point \( R \). As \( \delta \) increases, agents take the aspirations less seriously, and can hence afford to attain it with lower probability (like lottery). In other words, when \( \delta \) is high making it less important to attain \( R \), agents demand securities with higher skewness which limits the downward risk (i.e., low \( C_F \)) at the expense of lower chance of attaining \( R \). However, as \( \delta \) decreases, agents must take the kink more seriously, forcing them to give up more of their \( C_F \) in bad state to ensure that they reach \( R \) with higher probability. The agents thus demand increasingly negative-skewed securities. Figure 1 also illustrates this.

3.2.2 Initial Endowment

Theorem 4. Fix \( \delta, \gamma, \) and \( \xi := \frac{C_0}{R} \). Then, \( \frac{dp^*}{dC_0}|_{\delta, \gamma, \xi} > 0 \)

Theorem 4 is also very intuitive. As agents become wealthier, they need to worry less and less about the disastrous states where marginal utility is extremely high. Hence, they can afford to take more and more downside risk, thereby demanding less skewed securities. This coincides with the empirical findings (e.g., Kumar 2009) that report more active engagement in lottery-like bets among consumers with lower income.

Note that Theorem 4 is saying a little more than Theorem 1-(ii), which is effectively saying that

\[
\frac{\partial p^*}{\partial C_0}|_{\delta, \gamma, R} > 0.
\]

The difference here in Theorem 4 is that we are fixing \( \xi := \frac{C_0}{R} \) instead of \( R \). Thus, here we are not decreasing the distance between \( C_0 \) and \( R \) as we increase \( C_0 \). The point here is that \( p^* \) increases even when \( R \) increases along with \( C_0 \), and that the increase of \( p^* \) is purely an endowment effect, rather than effect of \( C_0 \) approaching \( R \) as in Theorem 1-(ii).
3.2.3 Risk Aversion

Theorem 5. Fix $C_0$ and $\delta$. Suppose $R$ is ‘big enough’ to satisfy:

$$
(R/C_0)^{-\gamma} \left[ \left( \frac{R}{C_0} \right) \frac{\log R}{C_0} - 1 \right] + O(\frac{1}{R}) < \delta,
$$

where $O(\frac{1}{R})$ is a positive term which vanishes at the rate of $\frac{1}{R}$. Then,

$$
\frac{\partial p^*}{\partial \gamma}|_{\delta,C_0,R} < 0
$$

We first give some numerical examples to get a feel for how stringent the assumption is. For reasonable parameters values such as $\delta = 0.8$, $\gamma = 2$, $C_0 = 2$, $\frac{\partial p^*}{\partial \gamma} < 0$ will hold whenever $R > 3.491$. For $\delta = 0.8$, $\gamma = 2$, $C_0 = 1.5$, $\frac{\partial p^*}{\partial \gamma} < 0$ will hold whenever $R > 2.227$. For $\delta = 0.8$, $\gamma = 2$, $C_0 = 1$, $\frac{\partial p^*}{\partial \gamma} < 0$ will hold whenever $R > 1.159$. The role of this R-threshold (higher than which will allow full monotonicity) can be interpreted as follows. Suppose the agent has $C_0$ in his hands. Higher $\gamma$ implies that his utility function is more concave. In the power utility setting, this extra concavity is achieved by pulling down the utility of the agent both on positive outcomes ($C > 1$) and negative outcomes ($C < 1$) with $C = 1$ as the anchoring case. (Refer to figure below.) In our setting, the agent with higher $\gamma$ discounts both $u(C_S)$ and $u(C_F)$ more heavily than in the log-utility case. This has consequences on $p^*$. The depressed $u(C_S)$ affects $p^*$ unequivocally; it acts to lower $p^*$. Intuitively, this is because the reduced upside discourages the agent from taking much downside risk (in the form of lower $C_F$) in return, and the agent consequently decreases $p^*$ to ensure that $C_F$ does not fall too low. (Recall that we are envisioning a martingale situation.) Hence, the agent with higher $\gamma$ chooses lower $p^*$. Meanwhile, the effect of the depressed $u(C_F)$ on $p^*$ can go both ways. When $\gamma$ is higher and the downside is even lower, the agent faces a predicament: on the one hand, she wishes to avert the painful depth of the downside by choosing to increase $C_F$. However, this can only be done at the cost of lowered $p^*$, again, because of the martingale assumption. The lowered $p^*$ means that she is undermining her very chance of avoiding the
downside, albeit perhaps a less painful one. Since the agent faces this inevitable trade-off, the effect of depressed $u(C_F)$ is ambivalent. Nevertheless, we still can say this with absolute certainty: as $C_S = R$ grows, so does the magnitude by which $u(C_S)$ gets depressed. (Refer to Figure below.) This means that the unequivocal effect of the depressed upside (i.e., lower $p^*$) will grow bigger and bigger, until at some point it dominates the (ambivalent and hence limited) effect of the depressed downside. This is precisely our assumption in Theorem 5: as $R$ exceeds a certain threshold, the effect of $\gamma$ on $p^*$ becomes monotonic. Modulo the assumption, the broad-strokes conclusion of Theorem 5 is intuitive: agents with higher risk-aversion will choose lower $p^*$ securities to minimize their perceived downside.

![Figure 3: Power utility functions with different $\gamma$](image)

Figure 3: Power utility functions with different $\gamma$
4 Double Jump Optimization

It is realistic to envision a situation where there are multiple aspirations. For example, while the aspirational agent may be eager to accomplish a goal that she has yet to attain, she could equally be concerned that a past accomplishment may be revoked. We model this situation as a two jumps, where \( C_0 \) is in between the two aspirations. \((R_1 < C_0 < R_2)\)

4.1 Setup

We modify the \( U(c) \) in (1) to accommodate two aspirations; let

\[
U(c) = \delta_2 u(c)\mathbb{I}_{[0,R_1]}(c) + \delta_1 u(c)\mathbb{I}_{[R_1,R_2]}(c) + u(c)\mathbb{I}_{[R_2,\infty)}(c)
\]

with \( 0 < \delta_2 < \delta_1 < 1 \).

The double jump EU maximization problem is identical to (4) - (7), except for the modified \( U(c) \). Consequently, we can use Lemma 1 again to reduce dimension just as before. Therefore, for any 5-tuple set of parameters:

\[
(\delta_1, R_1, \delta_2, R_2; C_0)
\]

(with \( 0 < \delta_2 < \delta_1 < 1 \), \( R_1 < C_0 < R_2 \) and assumption (9) of Lemma 1),

we can define the double jump EU maximization problem to be:

\[
\max_p (1 - p)U(C_F) + pU(R_2)
\]

such that \( C_F \) satisfies (4). Since any \((\delta_1, R_1, \delta_2, R_2; C_0)\) (with parameters in suitable range) defines such a problem, we will succinctly denote the EU maximization problem as: \( \mathcal{M}(s) \) for \( s = (\delta_1, R_1, \delta_2, R_2; C_0) \). Figure 4.1 illustrates an aspirational utility with two jumps.

Next, consider \( \tilde{s} = (R_2, \delta_2; C_0) \). We can define a single-jump problem using these three
Figure 4: Aspirational Utility with Two Jumps
parameters. This is nothing more than a single jump problem generated by erasing the intermediate jump. We denote this EU maximization problem as $\mathcal{M}(\tilde{s})$ for $\tilde{s} = (R_2, \delta_2; C_0)$, and call this the single jump problem associated with $\mathcal{M}(s)$.

We now introduce some notations and definitions.

**Definition 1.** Let $S \subset \mathbb{R}^5$ be the set of all $(\delta_1, R_1, \delta_2, R_2; C_0)$ such that $0 < \delta_2 < \delta_1 < 1$, $R_1 < C_0 < R_2$ and assumption (9) holds. Consider $s = (\delta_1, R_1, \delta_2, R_2; C_0) \in S$, and $\tilde{s} = (R_2, \delta_2; C_0)$. Then, $s$ is indistinguishable from $\tilde{s}$ iff $\mathcal{M}(s)$ induces the same solution (maximand $p^*$ and maximized EU) as $\mathcal{M}(\tilde{s})$.

Intuitively, when the parameters are arranged in such a way that one of the two aspirations (in this case, the lower one; $R_1$) does not play any role in the agent’s maximization problem, we may say that such parameter setting $s$ is indistinguishable from its ‘reduced form’ single-jump parameter setting $\tilde{s}$. In this case, it is redundant to think of it as a double-jump problem because it is possible to reduce it to a single-jump problem. This definition allows us to partition $S$ into two parts.

**Definition 2.** $H = \{ s \in S : s \text{ is indistinguishable from } \tilde{s} \} \subset S$

Naturally, $S = H \cup H^c$ and any $s \in S$ belongs to either $H$ or $H^c$. When $s \in H$, the parameters are configured in such a way that it can be reduced to a single-kink problem and in this sense the double kink problem is ‘trivial’. When $s \in H^c$, the parameters do not allow for such a reduction.

### 4.2 Optimal Choice Under Two Jumps

A natural question to ask given this setup would be: when is reduction possible ($s \in H$) and when it is not ($s \in H^c$)? The following Theorem addresses this question.

**Theorem 6.** Fix all 4 elements of $s$ except $R_1$ (i.e., fix the ex-$R_1$ 4-tuple of $s$, and allow $R_1$ to float.). Let $R_1^* = \inf \{ R_1 : (\delta_1, R_1, \delta_2, R_2; C_0) \in H \}$. 


Then, $R_1^*$ acts as a demarcation point; when $R_1 < R_1^*$, $(\delta_1, R_1, \delta_2, R_2; C_0) \in H^c$ and when $R_1 \geq R_1^*$, $(\delta_1, R_1, \delta_2, R_2; C_0) \in H$. Moreover, when $R_1^* > 1$, $R_1^* \uparrow R_2$ as $\delta_1 \uparrow 1$.

This Theorem helps us discern whether the double-jump problem is an authentic double-jump problem, or whether it is reducible to a single-jump problem. The pivot variable is $R_1$, the intermediate location of the jump. The Theorem indicates that there is a threshold value ($R_1^*$), higher than which all problems are reducible, and vice versa. Roughly speaking, the higher the $\delta_1$ and the lower the $R_1$, the more likely it will be authentically double jumped, whence the intermediate reference point actually matters. This is intuitive; higher $\delta_1$ heightens the vertical stature of the intermediate reference point. Similarly, lower $R_1$ increases the horizontal stature of the intermediate reference point, and hence its importance. Overall, when these forces wrinkle the utility function sufficiently, the EU-optimization departs from a single jump problem. Moreover, when $\delta_1$ increases (and approaches 1 monotonically), the intermediate jump becomes significant for monotonically increasing sets of $R_1$ until it is so high that it is significant for any $R_1$ less than $R_2$. This monotonicity, however, holds only on $R_1^* > 1$, and this requirement ensures that we are not looking at a rather degenerate case where $R_2$ and $\delta_2$ are simultaneously so low that the problem is dominated by the near-zero marginal utility induced by $\delta_2 \approx 0$.

The natural order of business now is to describe the behavior of $p^*$ when $s \in H^c$, i.e, when the double jumped problem truly involves two aspirations. This is done in the following theorem.

**Theorem 7.** On any $s \in H^c$, $M(s)$ admits a solution $p^*$ which is characterized by $\rho = \frac{C_0 - R_1}{R_2 - R_1}$, (which represents the position of initial wealth ($C_0$) relative to the two reference points), namely:

(i) $p^*$ is monotonically increasing in $\rho$; $\frac{\partial p^*}{\partial \rho} > 0$.

(ii) $\lim_{\rho \downarrow 0^+} p^* \leq 0$

(iii) $\lim_{\rho \uparrow 1^-} p^* = 1$
Theorem 6 and Theorem 7 together describe how agents react to a double jump problem. Imagine a situation where the agent is sitting on an initial wealth. On the upside, she sees a dream to which he aspires. On the downside, there is a point below which she does not want to drop. (e.g. she does not want to fall below a level where she will be unable to pay the rent, and will be kicked out of the neighborhood.) In some cases, the downside drop may be ignored \((s \in H)\). Theorem 6 tells us that this is the case if the drop itself is not too big (e.g. when \(\delta_1\) is low and \(\delta_1 - \delta_2\) is relatively small) or if the drop is too close to the aspiration point (e.g., when \(R_1 \approx R_2\), so the agent already lumps them together when she makes decisions.) When these are not the case, the agent has to take both jumps seriously \((s \in H^c)\). Theorem 7 tells us what happens when this is the case. When this happens, the agent gets ‘trapped’ in between the aspirations, in the sense that the agent’s demand for security is determined by \(\rho\), the position of her initial wealth relative to the two aspirations. When her initial wealth is close to \(R_1\), the agent, in fear of dropping below \(R_1\), demands more lottery-like securities that minimizes the downside, until at some point, she stops demanding risky securities altogether and just consumes \(C_0\) \((p^* \leq 0)\). On the other hand, as \(C_0\) is safely far from \(R_1\) and close to \(R_2\), the agent begins to demand negatively skewed securities that gets her to \(R_2\) with great certainty, albeit at the risk of (much less likely) slips below \(C_0\).

5 Departure from Fair Gambles: Sub-martingale and Super-martingale

The analysis thus far has limited its scope to martingales (fair-games with zero returns). This allowed undivided focus on the choice of skewness, however it begs the question: what happens when we vary the first (mean returns) and second moments (volatility). We now add variations in the first moment to see how this affects the choice of skewness.

5.1 Setup

First, we consider sub-martingales. Sub-martingales are stochastic processes with positive drift. Recall that \(C_S = C_0 + N(M - 1)\) and \(C_F = C_0 - N\). To specify the deviation from
martingale, let:

\[ \alpha := p(M - 1) - (1 - p) \geq 0. \]  \hspace{1cm} (14)

Namely, \( \alpha \) is the expected gain from buying one unit of \( L(p) \), which costs 1 to purchase. Thus, the positive sign on \( \alpha \) is what makes this gamble a sub-martingale. When the agent purchases \( N \) units of \( L(p) \), the expected value of consumption is:

\[ \mathbb{E}[C] = pC_S + (1 - p)C_F = C_0 + N\alpha. \]

The second term \( (N\alpha \geq 0) \) represents the ‘better than fair’ component in the consumption scheme. Recall that in the martingale case, \( \mathbb{E}[C] = C_0 \) and \( N \) was automatically pinned down by Lemma 1 and \( \alpha = 0 \).

We first state formally the EU-optimization problem in the sub-martingale case:

\[
\max_{p \in (-\infty, 1), \ N \in [0, \infty)} (1 - p)U(C_F) + pU(C_S) \]

such that:

\[ C_F = C_0 - N \]

\[ C_S = C_0 + N(M - 1). \]

\[ \alpha := p(M - 1) - (1 - p) \geq 0. \]

Note that the analogue of (3) in the sub-martingale case (obtained by eliminating \( N \) from (5), (6) and substituting (14) in) is:

\[
\left( \frac{p}{1 + \alpha} \right) C_S + \left( \frac{(1 - p) + \alpha}{1 + \alpha} \right) C_F = C_0, \]  \hspace{1cm} (16)
This is the ‘budget constraint’ on the consumption scheme, set by the sub-martingale condition (14). Note that for every $C_0$ and $C_S$, this sub-martingale budget constraint implies a $C_F$ higher than that implied by the martingale budget constraint (3). This reflects the fact that $L(p)$ here represents a ‘better than fair’ gamble; namely that $\alpha \geq 0$.

In the sub-martingale case, we cannot automatically invoke Lemma 1 anymore because Assumption (9) takes a different form, as is outlined in Lemma 3 below. We therefore have to solve the full problem, which amounts to maximizing (15) under the budget constraint (16). Doing so yields an interesting departure from the mean-variance analysis. The following two subsections illustrate the two consequences.

5.2 $\alpha > 0$: ‘Winter (keep $C_0$)’ is Replaced by Lottery-Sales

**Theorem 8.** For any $\alpha > 0$, $\exists p^* \in (0, 1), N^* \in (0, \infty)$ such that $EU(p^*, N^*) > U(C_0)$.

The significance of this theorem is that in the sub-martingale case, the default option for the EU-maximizing agent is no longer to ‘do nothing’ (i.e., just sit on $C_0$) as in the Mean-Variance case. By doing nothing, the agent gets $U(C_0)$, which is strictly dominated by $EU(p, N)$ for a suitable choice of $p$ and $N$ which is always available as long as $\alpha > 0$.

This begs the question: how can agents always enjoy an expected utility level higher than $U(C_0)$ even when they are risk-averse? The answer is that when agents are allowed to choose the level of skewness embedded in the securities, they effectively end up selling lottery, whose payoff profile increasingly resembles an arbitrage as $p \uparrow 1$. (Note that here, the term ‘lottery’ is in the colloquial sense; purchase of a lottery entails loss in the mean, typically with high level of positive skewness, and vice versa for sale of a lottery.) To see this, note that the payoff structure of $L(p)$ in the sub-martingale case (obtained by manipulating the constraints in problem (15)) is:
In particular, when $p \approx 1$,

$$\frac{\alpha + (1-p)}{p}$$

whose payoff almost resembles an arbitrage, increasingly so as $p \uparrow 1$. Such a near-arbitrage opportunity may increase $N$ indefinitely, however it is bounded by the constraint $C_F(p^*) > 0$, effectively acting as an upper-bound for $N$. The following is a pictorial illustration of Theorem 8.
5.3 $\alpha > 0$: Lowers Demand for Positive Skewness

Positive $\alpha > 0$ (strict sub-martingale) introduces an additional effect, encapsulated in the following result.

**Theorem 9.** $\frac{\partial p_{SM}^*}{\partial \alpha} > 0$

The result can be understood intuitively in the following way. Higher $p_{SM}^*$ is desirable because it increases the chance of attaining the good outcome $U(C_S(p_{SM}^*))$. The trade-off is, of course, that it pushes $C_F(p_{SM}^*)$ down lower. However, this downside can be mitigated if $\alpha$ is high. Thus, the EU-maximizing agent can now afford to enjoy a higher $p_{SM}^*$, hence $\frac{\partial p_{SM}^*}{\partial \alpha} > 0$.

Lastly, the following Lemma shows how Assumption (9) has to be modified in the sub-martingale case.

**Lemma 3.** Consider the single-kink EU-maximization problem described in (4), with constraints (5), (6) and (14).

Consider an associated reduced form EU maximization problem:
\[
\max_{p \in (-\infty, 1)} (1 - p)U(C_F) + pU(R)
\]
such that \(C_F\) satisfies
\[
\left(\frac{p}{1 + \alpha}\right) R + \left(\frac{(1 - p) + \alpha}{1 + \alpha}\right) C_F = C_0
\] (17)
and let \(p_{SM}^*\) be the maximand (solution) to this reduced problem. Let \(C_F(p_{SM}^*)\) denote the \(C_F\) which satisfies (17) at \(p_{SM}^*\).

Assume that the following holds.
\[
U'(C_F(p_{SM}^*)) > U'(R) \left(1 + \frac{\alpha}{(1 - p_{SM}^*)}\right)
\] (18)

Then any optimal consumption scheme in the original problem satisfies \(C_S = R\).

It may look as though Assumption (18) is harder to satisfy than Assumption (9) because \(\frac{\alpha}{(1 - p_{SM}^*)} \to \infty\) as \(p_{SM}^* \to 1\). However, it turns out that Assumption (18) is almost always satisfied for reasonable values of \(\alpha\), due to Inada Conditions. Intuitively, when \(p_{SM}^*\) is close to 1 and \(C_S \gg R\), this means \(C_F(p_{SM}^*)\) is already very close to 0, if not already 0. By Inada Conditions, this implies very high \(U'(C_F(p_{SM}^*))\), satisfying Assumption (18). Therefore, fortunately, we do not lose much by always assuming \(C_S = R\), as it turns out to be a good approximation for the solution to the full problem. Conclusion: the sub-martingale problem does not allow for Lemma 1, but for practical purposes, we can approach the sub-martingale problem as a reduced-form optimization problem, just as we did in the martingale case.

5.4 \(\alpha < 0\): Super-martingales

By switching the sign of (14) we can extend the result in the previous subsection to super-martingales. Namely when \(\alpha \leq 0\), decreasing \(\alpha\) monotonically reduces \(p^*\) and increases preference for positive skewness. (i.e., \(\frac{\partial p_{SM}^*}{\partial \alpha} > 0\), as before.) However, somewhat differently from the \(\alpha > 0\) case, the replacement of ‘no gambling’ by lottery sales is no longer available in the super-martingale case, simply because negative \(\alpha\) depletes the opportunity for near-arbitrage. One small caveat in the super-martingale case is that the choice of \(L(p)\) in this
case must be limited to those such that \( p < 1 + \alpha \), \((\alpha < 0)\). This is because \( L(p) \) offer an arbitrage opportunities if otherwise. \((\text{Since } C_S = M - 1 = \frac{\alpha + 1 - p}{p} \text{ and } C_F = -1, \text{ we need to ensure that } \alpha + 1 - p > 0, \text{ otherwise arbitrage can be made by shorting } L(p).)\)

An interesting exercise can be done using super-martingales, starting from the following observation. Taylor approximating a CRRA utility function reveals a mechanical sign on the moments, in particular the mean (positive), volatility (negative) and skewness (positive). This conclusion can, in part, justify the sale and purchase of lottery. Namely, if we characterize standard lottery as a security that has very high positive skewness yet minutely negative return, we can justify trading lottery by ascribing it to the demand for skewness, at the expense of loss in mean. Meanwhile, in the KUWJ setting we provided, we show that preference for negative skewness can also be justified \((p^* \approx 1)\). In the CRRA setting, buying negative skewness would have to be corroborated by positive mean since negative skewness reduces utility. However, in the setting we provide, agents can choose to sell skewness and yet accept negative returns. This is described in the figure below. The colors represent the amount of returns that can be taken away from a martingale bet in order to stop the agent from buying \( L(p) \). The red zone represents the situation where the magnitude of mean return that can be taken away (foregoable return) is the highest, and blue zone, the lowest. The horizontal axis represents the jump size which decreases from left to right. The vertical axis represents the distance between \( R \) and \( C_0 \), which increases from high to low. The picture shows that clearly there is some mean that can be traded away against benefits of trading skewed securities, even when the agent is pursuing a negative skewed payoff. In fact, the foregoable return is highest when the agent is trading negatively skewed securities, and lowest when trading positively skewed securities.

Overall, the picture is intuitive. The foregoable return is 0 when \( p^* = 0 \) because under that configuration, the agent can gain nothing by taking on a risky bet. When the agent is close to the aspiration point, and when the size of jump is higher, the foregoable return is higher because there is something to gain from taking on risk to attain the aspiration point. When the agent is too far from the aspiration point, or if the size of the jump is too small, the
agent becomes increasingly unwilling to give up much in return.

![Figure 6: Maximum Tolerable Negative Expected Return](image)

6 Volatility and Skewness: A Role Analysis

Another meaningful departure (from binomial martingale) is to separate volatility from skewness. In the stringent binomial martingale setting, the choice of \( p^* \) simultaneously fixes volatility and skewness of \( L(p^*) \). Because of this entanglement, it was impossible to see clearly what the agent’s preferences for volatility looks like in the aspirational setting. The aim of this section is to explore what the relative roles of volatility and skewness are in the EU-maximization process. To isolate the two effects, our first goal is to design a set of securities that controls one, yet varies the other.
6.1 The New Consumption Scheme: Tri-nomial Martingales

We generate a variant of the binomial martingale introduced before. For the sake of brevity, we suppress the underlying $L(p)$ and start directly with the consumption scheme, keeping in mind there always exists a corresponding $L(p)$.

\[
C_S = C_0 + N(M - 1)
\]

where

\[
p_1 + p_2 + p_3 = 1, \text{ and } \mathbb{E}[C_0] = p_1C_S + p_2C_M + p_3C_F = C_0 \quad (19)
\]

As before, (19) dictates that the trinomial consumption scheme is also a martingale. The key difference is that we assign positive probability mass ($p_2$) on an intermediate branch, $C_M = C_0$. Because consumption level stays at $C_0$ on this node, the expected outcome is to (1) reduce volatility while (2) keeping skewness relatively stable. Indeed:

Figure 6.1 illustrates the movements of volatility and skewness as probability mass on $C_M$ (i.e., $p_2$) varies. As expected, volatility is muted down as $p_2$ increases, whereas skewness is kept relatively stable.

This allows us to address the task of analyzing separately how elements of volatility and skewness are embedded in the agent’s demand for $L(p)$. We begin by asking, “if single-kinked EU-maximizing agents were offered the opportunity to choose securities with same (similar) skewness but lower volatility (by increasing $p_2$), which security would the agent
choose?” For this, define:

**Definition 3.** Let \( \mathcal{T}(\mathcal{P}, \mathcal{C}) \) denote the trinomial consumption scheme depicted above, with \( \mathcal{P} = (p_1, p_2, p_3) \) and \( \mathcal{C} = (C_S, C_0, C_F) \) such that \( p_3 C_F + p_2 C_0 + p_1 C_S = C_0 \). Consider \( \mathcal{B} := \mathcal{C}(\mathcal{P'}, \mathcal{C'}) \) with \( \mathcal{P'} = (p_1, p_3) \) and \( \mathcal{C'} = (C_S, C_F) \). \( \mathcal{B} \) is the *binomial consumption scheme associated to* \( \mathcal{T} \).

In this definition, \( \mathcal{B} \) is simply the binomial analogue of the trinomial consumption scheme when \( p_2 = 0 \), while preserving the martingale property and the positions of \( C_S \) and \( C_F \). It can be interpreted as the most volatile security in the family of trinomial securities with identical consumption positions and similar (by construction) levels of skewness.
6.2 Principle of Maximal Volatility

Theorem 10 and 11 characterize the aspirational agent’s choice over tri-nomials.

**Theorem 10.** Consider \( \mathcal{M}(\delta, R; C_0) \). Any optimal tri-nomial consumption scheme: \( \mathcal{T} \), is dominated by its associated binomial consumption scheme: \( \mathcal{B} \).

Theorem 10 tells us that any consumption scheme which is less volatile than the most volatile consumption scheme is dominated, and not chosen. In other words, somewhat against our intuition, agent would choose the *most* volatile security in the family of securities with same (similar) skewness profiles. The reason for doing so becomes more apparent with the next Theorem.

**Theorem 11.** Any solution to a trinomial optimization problem is an associated \( \mathcal{B} \) with \( C_S = R \).

Theorem 11 tells us that in spirit, Lemma 1 holds in the trinomial martingale setting as well. Namely, the agent takes just enough risk to hit \( R \), and does not engage in a consumption scheme that offers \( C_S > R \). Meanwhile Theorem 10 tells us that the agent does not aim to reduce volatility either, as long as \( C_S = R \) can be attained. Reducing volatility only reduces expected gains from attaining \( R \). As long as the agent knows that (s)he has right skewness and \( N \) to efficiently hit the aspiration point \( R \), (s)he buys the entire portfolio rather than try to reduce volatility by assigning weight on \( C_0 \). After all, volatility is the thrust that propels the agent from \( C_0 \) to \( R \). Reducing volatility would detract from the spoils of attaining \( R \). In the following example, we illustrate the relative role of skewness and volatility.

**Example 1: Loss of Skewness is Prohibitively Costly**

Consider the EU maximization problem \( \mathcal{M}(\delta, R, C_0) \) with \( \delta = 0.8 \), \( R = 5 \), \( C_0 = 2.8 \). When the agent is allowed to choose from all available \( L(p) \), the agent will choose \( p^* = 11.22\% \) with skewness of 2.46 to attain \( \text{‘}o\text{’} \). Now, we consider depleting the agent of the opportunity to control skewness. If the agent is only offered the choice of skewness of 0, (i.e. \( p = \frac{1}{2} \)), the agent would be offered \( ‘x’ \), which (s)he does not choose since this is dominated by \( U(C_0) \).
Namely, $EU(o) > U(C_0) > EU(x)$ so the gambling stops. This illustrates the case where loss of skewness is prohibitively costly and the agent will stay with $U(C_0)$ (no gambling).

Figure 8: Loss of Skewness is Prohibitively Costly
Example 2: Loss of Skewness is Not Prohibitive

Now consider the EU maximization problem $\mathcal{M}(\delta, R, C_0)$ with $\delta = 0.8$, $R = 5$, $C_0 = 4.5$. The only difference from Example 1 is $C_0$. When the agent is allowed to choose from all available $L(p)$, the agent will choose $p^* = 81\%$ with skewness of -1.54 to attain ‘o’. In this case, when the agent loses control over skewness, the agent still chooses ‘x’ over $U(C_0)$. Namely, in this case, $EU(o) > EU(x) > U(C_0)$, so the agent still makes a risky choice.

Figure 9: Loss of Skewness is Acceptable
Examples 1 and 2 illustrate that ‘principle of maximal volatility’ is not confined to optimal choices. Namely, in spirit, Theorems 10 and 11 hold even when we restrict the agent’s choice to limited levels of skewness - in the Examples, restricted to $\frac{1}{2}$. In Example 2 when control over skewness is lost, agent does not choose to reduce volatility. It is easy to show that if the agent was offered a trinomial martingale with skewness = 0, the agent would decline and still choose the associated binomial martingale, very much like in Theorem 10. However, like in Theorem 11, this is only to attain R. Once $C_S = R$, the agent does not increase the size of bet (N) to increase $C_S$.

6.3 A Rule of Thumb Under Limited Choice of Skewness

The Examples also illustrate a ‘rule of thumb’ that agents use when they are offered a tri-nomials with limits on the level of skewness they can choose from. When the agent is offered a tri-nominal security with skewness = 0 (or more generally, with restrictions on the level of skewness she can choose), she processes it in the following steps.

**Step 1**: she asks whether the loss of skewness is prohibitive and chooses $C_0$ if it is.

**Step 2**: if the loss of skewness is not prohibitive, she chooses the security with the highest volatility; the associated binomial.

In summary, skewness is welfare improving because it allows the agent to tailor her aspiration more precisely. Restrictions on this freedom is welfare reducing (Example 2), and in some instances, prohibitively so (Example 1). Volatility on the other hand, can work both ways. Volatility is helpful insofar as it takes the agent to R. This is why agents choose the associated binomial (‘principle of maximal volatility’.) However, this is only up to R, and excess volatility over and above R is avoided (Theorem 11.)
7 Conclusions

In this paper we study the demand for skewness—both right and left skewness—using a utility function with microeconomic and evolutionary foundations. We assume that economic agents care both about consumption—divisible good—and status—achieved through the purchase of non-divisible goods. Our resulting utility is in the spirit of Friedman and Savage (1948), however, their analysis focuses on the second moment of the distribution that explains uncertainty. We consider a parsimonious set of securities that allow the agent to select the exact optimal level of right or left skewness. Our analysis yields a rich set of results broadly consistent with empirical observations.
References


Appendix

Proof of Lemma 1:

**Step 1)** Show that any consumption scheme whose $C_S$ exceeds $R$ is dominated by a consumption scheme whose $C_S$ equals $R$.

Let $N_0$ be the $N$ such that $C_S = R$ on equation (6). (It can easily be shown that) EU is continuously concave in $N$ on $N \geq N_0$. It then suffices to show that $\frac{\partial EU}{\partial N} \bigg|_{N_0} < 0$ in the original problem. Namely, if $\frac{\partial EU}{\partial N} \bigg|_{N_0} < 0$, then by continuous concavity of EU, $\frac{\partial EU}{\partial N} \bigg|_{N} < 0$ for all $N \geq N_0$, and (again, by concave continuity of EU) any consumption scheme with $N > N_0$ is dominated by a consumption scheme with $N_0$

To prove this sufficient condition ($\frac{\partial EU}{\partial N} \bigg|_{N_0} < 0$), first, write down the $EU \bigg|_{N \geq N_0}$ under equations (5) - (7):

$$EU \bigg|_{N \geq N_0} = (1-p)U(C_F) + pU(C_S)$$
$$= (1-p)U(C_0 - N) + pU(C_0 + N(M - 1))$$
$$= (1-p)U(C_0 - N) + pU(C_0 + N \frac{1-p}{p})$$

Then differentiate $EU \bigg|_{N \geq N_0}$ with respect to $N$ at $N_0$, $p^*$:

$$\frac{\partial EU}{\partial N} \bigg|_{N_0,p^*} = -(1-p^*)U'(C_0 - N) + p \frac{1-p^*}{p} U'(C_0 + N \frac{1-p}{p}) \bigg|_{N_0}$$
$$= (1-p^*)(U'(C_S) - U'(C_F)) \bigg|_{N_0}$$
$$= (1-p^*)(U'(R) - U'(C_F))$$
$$< 0,$$

where the last inequality follows from the domain of $p$, and assumption (9).
Step 2) Show that any consumption scheme whose \( C_S \) falls below \( R \) is dominated by a consumption scheme whose \( C_S \) equals \( R \).

When \( C_S < R \), \( U(c) = \delta u(c) \). Then by our standard assumptions on \( u(c) \) and Jensen’s Inequality, we know that \( p = 0 \) (no trade) dominates all other consumption scheme, which is by definition, (weakly) dominated by \( p^* \) because \( p=0 \) is in the choice set.

Proof of Lemma 2: Since \( p \in (0, 1) \), \( C_F = \frac{C_0 - pR}{1 - p} \), and \( C_F - C_0 = \frac{p(C_0 - R)}{(1 - p)} \). Also by Lemma 1, \( C_S = R \). Some calculations yield:

\[
\sigma^2 = \frac{p}{(1 - p)}(R - C_0)^2, \tag{20}
\]

and similarly,

\[
\mathbb{E}[(C - C_0)^3] = \frac{p(1 - 2p)}{(1 - p)^2}(R - C_0)^3. \tag{21}
\]

\[ \therefore S(p) = \frac{1 - 2p}{\sqrt{p(1 - p)}} \quad \text{and} \quad S'(p) = \frac{-1}{2p(1 - p)^{3/2}} < 0. \tag{21} \]

Proof of Theorem 1: The optimization problem characterized by (10) is specialized to power utility:

\[
\max_p EU(p) \tag{22}
\]

where

\[
EU = \delta(1 - p) \frac{C_F^{1-\gamma} - 1}{1 - \gamma} + p \frac{R^{1-\gamma} - 1}{1 - \gamma} \tag{23}
\]

and

\[
C_0 = pR + (1 - p)C_F. \tag{24}
\]

Let \( \xi := \frac{C_0}{R} \).

The associated First Order Condition \( \frac{\partial EU}{\partial p} \) gives:
\[ F(p; \cdot) := \frac{\delta}{1 - \gamma} \left( \frac{\xi - p}{1 - p} \right)^{-\gamma} R^{1 - \gamma} \left( \frac{\gamma - 1 + p - \xi \gamma}{1 - p} \right) + \frac{R^{1 - \gamma} + \delta - 1}{1 - \gamma} \] 

(25)

\[ = 0 \]  

(26)

Checking the Second Order Condition,

\[ \frac{\partial F(p; \cdot)}{\partial p} = \frac{\delta}{1 - \gamma} R^{1 - \gamma} \left( \frac{\xi - p}{1 - p} \right)^{-\gamma - 1} \gamma^2 (1 - \xi^2) \frac{(1 - \xi)^2}{(1 - p)^3} < 0 \]  

(27)

Hence, the EU-optimization problem amounts to finding the \( p^* \) which satisfies

\[ F(p^*; \cdot) = 0 \]  

(28)

Proof of part (i):

By Implicit Function Theorem,

\[ \frac{\partial p^*}{\partial \xi} = -\frac{\partial F}{\partial p} \]  

(29)

Partial differentiation yields:

\[ \frac{\partial F}{\partial \xi} = -\frac{\delta}{(1 - \gamma)(1 - p)} R^{1 - \gamma} \gamma (\xi - p) \frac{\gamma - 1}{1 - p} \frac{(\gamma - 1)(1 - \xi)}{1 - p} \]

and

\[ \frac{\partial F}{\partial p^*} = -\frac{\delta}{(1 - \gamma)(1 - p)} R^{1 - \gamma} \gamma (\xi - p) \frac{\gamma - 1}{1 - p} \frac{(1 - \gamma)(1 - \xi)^2}{(1 - p)^2} \]
Hence,
\[
\frac{\partial p^*}{\partial \xi} = \frac{1 - p^*}{1 - \xi} > 0
\]
(30)

Proof of part (ii)

Suppose not. This means that on equation (25), \( p^* \) does not converge to 1 as \( \xi \uparrow 1 \). This in turn allows us to conclude that:

\[
F(p^*; \cdot) \longrightarrow \frac{\delta}{1 - \gamma} R^{1 - \gamma} + \frac{R^{1 - \gamma} + \delta - 1}{1 - \gamma},
\]
as \( \xi \uparrow 1 \)
(31)

Since \( F(p^*; \cdot) = 0 \), so is the limit. (Equals zero.) This implies:

\[
\frac{\delta}{1 - \gamma} R^{1 - \gamma} + \frac{R^{1 - \gamma} + \delta - 1}{1 - \gamma} = 0.
\]
(32)

Rearranging, this becomes: \( R^{1 - \gamma} = 1 \), a contradiction under our assumptions \( R > 1 \) and \( \gamma > 1 \).

Proof of part (iii)

Let
\[
g(\xi) : = F(0; \cdot)
= \frac{\delta}{1 - \gamma} \xi^{1 - \gamma} R^{1 - \gamma} (\gamma - 1 - \xi \gamma) + \frac{R^{1 - \gamma} + \delta - 1}{1 - \gamma}.
\]

To show that \( \exists (\frac{C_0}{R})^* \in (0, 1) \) such that \( p^* = 0 \), we need to show that \( g(\cdot) \) has a root in \((0, 1)\).
First, note that
\[ g(1) = \frac{1 - \delta}{1 - \gamma} (R^{1-\gamma} - 1) > 0 \]

Then, by Intermediate Value Theorem (IVT), it suffices to show that:

\[ \exists C_{\delta,R} \in (0,1), \text{ such that } g(C) < 0. \ (*) \]

Proof of (*)

Rearranging \( g(\xi) \), we get:
\[
 g(\xi) = -\delta \xi^{-\gamma} R^{1-\gamma} (1 - \xi \frac{\gamma}{\gamma - 1}) + \frac{R^{1-\gamma} + \delta - 1}{1 - \gamma} \]

but,
\[
 \xi^{-\gamma} (1 - \xi \frac{\gamma}{1 - \gamma}) > \xi^{-\gamma} \to \infty 
\]
as
\[
 \xi \downarrow 0^+ 
\]
Plugging this back into (22), this implies
\[
 g(\xi) \to -\infty 
\]
as
\[
 \xi \downarrow 0^+ 
\]

Hence, for any \( N \in \mathbb{R} \), \( \exists C \in (0,1) \) such that \( g(C) < -N \), which proves (*). \( \Box \)

Proof of Theorem 2: Recall that
(i) \( \frac{\partial F(p_{\xi})}{\partial p} < 0 \) (from SOC), and
(ii) \( C_F - C_0 = \frac{p^*}{1-p^*} (C_0 - R) \).

Clearly, from (i), the Expected Utility is maximized at \( p^*(\leq 0) \). Note that from (ii), choosing \( p^*=0 \) is equivalent to choosing \( C_0 \), namely not choosing any gamble. This is certainly an
available option for the agent. Hence, we want to show that the agent chooses \( p=0 \) over \( p \in (0,1) \). To do this, we need to show

\[
EU(p^*) \geq EU(0) > EU(p)
\]

where

\[
p^* \leq 0 < p
\]

By Mean Value Theorem (MVT), \( \exists c \in (0,p) \) such that

\[
EU(p) - EU(0) = EU'(c)(p - 0) < 0
\]

This follows from the fact that \( EU'(c) < 0 \), which can easily be shown by applying MVT again.

**Proof of Theorem 3:** Using Implicit Function Theorem, \( \frac{dp^*}{d\delta} = -\frac{\partial F}{\partial p^*} \).

Note that

\[
F(p^*; \cdot) = 0 \iff \frac{\partial EU}{\partial p} = 0
\]

\[
\iff \frac{\partial(1 - p)\delta C_{1-\gamma}^{1-\gamma-1}}{\partial p} + p R^{1-\gamma -1}_{1-\gamma} = 0
\]

\[
\iff \delta \frac{\partial(1 - p)C_{1-\gamma}^{1-\gamma-1}}{\partial p} + \frac{R^{1-\gamma -1}}{1 - \gamma} = 0
\]

Hence,

\[
\frac{\partial F}{\partial \delta} = \frac{\partial(1 - p)C_{1-\gamma}^{1-\gamma-1}}{\partial p}
\]

\[
= - \frac{R^{1-\gamma -1}1}{1 - \gamma} \frac{1}{\delta} < 0.
\]

Also, from before,

\[
\frac{\partial F}{\partial p^*} = - \frac{\delta}{(1 - \gamma)(1 - p)} R^{1-\gamma -1} \frac{(\xi - p)1 - \gamma -1}{(1 - p)^2} < 0
\]
It thus follows that $\frac{dp^*}{ds} < 0$. □

**Proof of Theorem 4**: Using Implicit Function Theorem, $\frac{\partial p^*}{\partial C_0}|_{\delta, \gamma, \xi} = -\frac{\partial F}{\partial p^*} \frac{\partial F}{\partial C_0}$.

Note that

$$\frac{\partial F}{\partial C_0} = \gamma \delta R^{-\gamma}(1-\xi)^{\gamma-1} > 0.$$  

Also, from before,

$$\frac{\partial F}{\partial p^*} = -\frac{\delta}{(1-\gamma)(1-p)} R^{1-\gamma}(1-\gamma)(1-\xi)^2 < 0$$

It thus follows that $\frac{dp^*}{dc_0}|_{\delta, \gamma, \xi} > 0$. □

**Proof of Theorem 5**: A direct algebraic proof is not amenable. We first suggest a sufficient condition (actually, an equivalence condition) and then use this to prove the theorem.

**Claim 1.** For a given $\gamma$ (and of course, under the fixed $C_0$ and $\delta$ as assumed in the statement of the theorem), let $R_0^*(\gamma)$ denote the $R$ which leads to $p^* = 0$. (Recall from Theorem 1, we know $\exists R_0^*(\gamma)$.) It suffices to show that $\frac{\partial R^*_0(\gamma)}{\partial \gamma} < 0$

**Proof.** Let $\gamma_i < \gamma_j$, and denote the optimal solutions pertaining to $\gamma_i$ and $\gamma_j$ (as functions of $\xi$) as $p_{\gamma_i}^*(\xi)$ and $p_{\gamma_j}^*(\xi)$. Note that $p_{\gamma_i}^*(\xi)$ is well-defined as a function of $\xi$ because we have fixed $C_0$. In fact, in this setting we can treat $p_{\gamma_i}^*(\xi)$ as a continuously differentiable function, as a direct consequence of the Implicit Function Theorem. Note also that by (30), $p_{\gamma_i}^*(\xi)$ and $p_{\gamma_j}^*(\xi)$ can never intersect. To intersect at, say, point $\xi_0$, there must exist a neighborhood of $\xi_0$ upon which $|\frac{\partial p_{\gamma_i}^*(\xi)}{\partial \xi}|$ always exceeds $|\frac{\partial p_{\gamma_j}^*(\xi)}{\partial \xi}|$. However, (30) prohibits this (i.e. plug in $\xi_0$ into (30), and for any $\xi > \xi_0$, $|\frac{\partial p_{\gamma_i}^*(\xi)}{\partial \xi}| < |\frac{\partial p_{\gamma_j}^*(\xi)}{\partial \xi}|$ whenever $p_{\gamma_i}^*(\xi) > p_{\gamma_j}^*(\xi)$ and vice versa for $\xi < \xi_0$ ), asserting our claim that $p_{\gamma_i}^*(\xi)$ and $p_{\gamma_j}^*(\xi)$ can never intersect.

Next, suppose that $\frac{\partial R_0^*(\gamma)}{\partial \gamma} < 0$ holds. Since $\gamma_i < \gamma_j$, this implies $R_0^*(\gamma_i) > R_0^*(\gamma_j)$. Using what we know about $p_{\gamma}^*(\xi)$ from Theorem 1, we can deduce that
\[ p^*_{\gamma_i}(\frac{C_0}{R_0(\gamma_j)}) > 0 = p^*_{\gamma_j}(\frac{C_0}{R_0(\gamma_j)}) , \]

where the inequality follows from combining Theorem 1-(i) (\( p^* \) is monotonically increasing in \( \xi \) and approaches 1 from the left) and the fact that \( 0 = p^*_{\gamma_i}(\frac{C_0}{R_0(\gamma_i)}) \), by definition. Similarly, the equality follows from definition of \( R^*_0(\gamma) \). But since we established that \( p^*_{\gamma_i}(\xi) \) and \( p^*_{\gamma_j}(\xi) \) can never intersect, this inequality at \( \xi = \frac{C_0}{R_0(\gamma_j)} \) must in fact hold uniformly in all \( \xi \), namely, \( p^*_{\gamma_i}(\xi) > p^*_{\gamma_j}(\xi) \) whenever \( \gamma_i < \gamma_j \). Therefore, \( \frac{dp^*}{d\gamma} < 0 \), as desired.

\[ \text{Claim 2. } \frac{\partial R^*_0(\gamma)}{\partial \gamma} < 0. \]

**Proof.** We first specialize (25) by insisting \( p^* = 0 \), as per the definition of \( R^*_0 \):

\[ G(\cdot) = F(0; \cdot) = -\delta C_0^{-\gamma} R - \frac{1}{1-\gamma} C_0^{1-\gamma \gamma} + \frac{R^{1-\gamma} + \delta - 1}{1 - \gamma} = 0 \]

By Implicit Function Theorem,

\[ \frac{\partial R}{\partial \gamma} = -\frac{\frac{\partial G}{\partial \gamma}}{\frac{\partial G}{\partial R}} = \left( \frac{R}{C_0} \right)^{-\gamma} \left[ \frac{(R/C_0) \log R + R}{(R/C_0)^{-1}} \right] + O(\frac{1}{R}) - \delta = 0 \]

where \( O(\frac{1}{R}) := \frac{(1-\delta) C_0^{-1} \log C_0}{R^2 C_0} > 0 \), a positive quantity that converges to 0 at the rate of \( \frac{1}{R} \).

Under the current assumptions, \( R - 1 > C_0 - 1 > 0 \), and \( 1 - \gamma < 0 \), thus \( A = \frac{(R/C_0)^{-1}}{C_0} < 0 \). To sign \( B \), note that \( \frac{(R/C_0) \log R}{(R/C_0)^{-1}} > 1 \) for all \( R > 1 \), and \( O(\frac{1}{R}) > 0 \), so if \( \frac{(R/C_0)^{-1} \log R}{(R/C_0)^{-1}} + O(\frac{1}{R}) < \delta \), this implies \( \frac{(R/C_0)^{-\gamma}}{B^2} < \delta \), and \( B > 0 \). Therefore, under the given assumption, \( \frac{\partial R}{\partial \gamma} < 0 \).

Combining Claim 1 and Claim 2, we arrive at the desired conclusion.

**Proof of Theorem 6:** Note that by definition, \( R_1 < R^*_1 \) automatically implies \( (\delta_1, R_1, \delta_2, R_2; C_0) \in H^c \). It remains to prove that when \( R_1 \geq R^*_1, (\delta_1, R_1, \delta_2, R_2; C_0) \in H \). The proof is constructed as follows. First, (Lemmas 4 - 5) we state some miscellaneous facts which we
will use along the way. Second, we introduce a ‘discriminant’ that will help us tell $H$ and $H^c$ apart, and derive some of its properties (Lemmas 6-10). Third, we will use these to draw conclusions on how parameters should behave to be in either $H$ or $H^c$.

**Lemma 4.** Let $C_F(p^*)$ denote the $C_F$ when the agent’s optimal solution is implemented; i.e., that which satisfies (3) at $p^*$. Then, $\frac{\partial C_F(p^*)}{\partial \delta} > 0$

**Proof.** $C_F(p^*) = \frac{C_0 - p^* R}{1 - p^*}$. By chain rule, $\frac{\partial C_F}{\partial \delta} = \frac{\partial C_F}{\partial p} \cdot \frac{\partial p^*}{\partial \delta} = \frac{C_0 - R}{(1 - p^*)^2} \cdot \frac{\partial p^*}{\partial \delta} > 0$, as product of two negatives.

**Lemma 5.** When $C_F > C_F(p^*)$, $\frac{\partial EU}{\partial C_F} < 0$

**Proof.** First, note that $\frac{\partial C_F}{\partial p} = \frac{C_0 - R}{(1 - p)^2} < 0$, hence $C_F$ is a bijection. It follows that $C_F(p) > C_F(p^*) \iff p < p^*$, and from proof of theorem 1, we know that $\frac{\partial EU}{\partial p} = \frac{\partial EU}{\partial C_F} \cdot \frac{\partial C_F}{\partial p} > 0$ on $p < p^*$.

$\therefore \frac{\partial EU}{\partial C_F} < 0$ as desired.

**Lemma 6.** Consider $s \in S$ and its associated $\bar{s}$. $l_{\delta_2}(c)$ is the tangent line to $\delta_2 \cdot u(c)$ on $\mathcal{M}(\bar{s})$ which passes through the point $(R_2, u(R_2))$. Moreover, the point of tangency is unique.
Figure 10: \( l_{\delta_2}(c) \) is tangent to \( U(c) \) at \( C_F \)

Proof. That it passes through \((R_2, u(R_2))\) is clear. To prove tangency, consider the single kinked EU maximization problem and the FOC condition.

\[
F(p; \cdot) = \frac{\partial U}{\partial p} = u(R_2) - \delta_2 u(C_F) + \frac{1}{1 - p^*} \delta_2 u'(C_F) \cdot (C_0 - R_2) = 0
\]

\[
\iff \delta_2 u'(C_F) = \frac{(1 - p^*)(u(R_2) - \delta_2 u(C_F))}{R_2 - C_0} = \frac{(u(R_2) - \delta_2 u(C_F))}{R_2 - C_F}.
\]

Since RHS = \( \frac{(u(R_2) - \delta_2 u(C_F))}{R_2 - C_F} \) is the slope of \( l_{\delta_2}(c) \) and \( u \) is concave, \( l_{\delta_2}(c) \) is tangent to \( \delta_2 u(c) \) and the point of tangency is \( c = C_F \). Finally, uniqueness of tangency follows from concavity of \( u(c) \).

\[\square\]

Lemma 7. Let \( u(c) \) be the power utility function. Then, \( \exists \hat{c} \in \mathbb{R}^+ \) where \( g'(\hat{c}) = 0 \). Moreover, \( g(\hat{c}) \) is a local and global (hence unique) maximum \( \iff g'(\hat{c}) = 0 \).

Proof. Clearly, \( u'(c) > 0 \) for all \( c \), hence by lemma 5, \( \frac{(u(R_2) - \delta_2 u(C_F))}{R_2 - C_F} > 0 \). If \( u(c) \) is the power
utility function, then \( u'(c) = c^{-\gamma} \) which tends to \(+\infty\) at \(0^+\) and converges to 0 as \(c \to \infty\). Hence by IVT, \( \exists \hat{c} \in \mathbb{R}^+ \) where \( g'({\hat{c}}) = 0 \). The second part (local and global maximality) is a general property of strictly concave functions.

**Lemma 8.** Consider \( \mathcal{M}(\delta_1, R; C_0) \) and \( \mathcal{M}(\delta_2, R; C_0) \) with \( \delta_1 > \delta_2 \) and define \( l_{\delta_i}(c) \), \( i = 1, 2 \) conformably as before. Then \( l_{\delta_1}(c) > l_{\delta_2}(c), \forall c \in (0, R) \).

**Proof.** By Lemma 3, \( \frac{\partial C_F(\delta)}{\partial \delta} > 0 \), hence \( C_F(\delta_1) > C_F(\delta_2) \). Since \( u'(c) \) is decreasing in \( c \), \( u'(C_F(\delta_1)) < u'(C_F(\delta_2)) \). By Lemma 5, \( \delta_i u'(C_F(\delta_i)) = \frac{u(R)-\delta_i u(C_F(\delta_i))}{R-C_F(\delta_i)} \), \( i = 1, 2 \), and therefore
\[
\frac{u(R)-\delta_1 u(C_F(\delta_1))}{R-C_F(\delta_1)} < \frac{u(R)-\delta_2 u(C_F(\delta_2))}{R-C_F(\delta_2)}.
\]
Note also that \( l_{\delta_1}(R) = l_{\delta_2}(R) \).
\[
\therefore \ l_{\delta_1}(c) > l_{\delta_2}(c), \forall c \in (0, R).
\]

**Lemma 9.** Let \( u(c) \) be the power utility function. Then the following are equivalent. \( g_1 \) and \( g_2 \) are two distinct roots of \( g(c) \) in \((0, \infty) \) \( \iff \ g(\hat{c}) > 0 \), where \( g'(\hat{c}) = 0 \) and \( \hat{c} \) in \((g_1, g_2) \)
\[
\iff \ \max_{c \in (g_1, g_2)} g(c) > 0
\]

**Proof.** We prove the first equivalence. The second equivalence is a direct corollary from lemma 6.

\((\Rightarrow) \) \( \exists 2 \) roots in \((0, \infty) \) implies (by MVT) that \( \exists \hat{c} \) in \((g_1, g_2) \) such that \( g'(\hat{c}) = 0 \), whence by lemma 6 \( g(\hat{c}) \) is a global maximum in \((0, \infty) \). If the global maximum, \( g(\hat{c}) \), \( \leq 0 \), then \( \exists 0 \) or 1 root in \((0, \infty) \), a contradiction.

\((\Leftarrow) \) Suppose \( g(\hat{c}) > 0 \), where \( g'(\hat{c}) = 0 \) and \( \hat{c} \in (0, \infty) \), then by concavity of \( g(c) \), we can pick \( 0 < \underline{c} < \hat{c} < \bar{c} \) such that \( g'(\bar{c}) < 0 < g'(\underline{c}) \). Taylor expanding around \( \underline{c}, \bar{c} \), (and using the fact that \( u(c) \) is a power utility function which tends to \(-\infty\) as \( c \downarrow 0^+ \)), we can show that
\[
\lim_{c \to 0^+} g(c) = \lim_{c \to +\infty} g(c) = -\infty.
\]
\( \therefore \) by IVT, \( \exists \) at least two distinct roots in \((0, \infty) \).

**Lemma 10.** \( g(c) \) has at most two distinct roots in \((0, \infty) \).

**Proof.** Suppose not. Then by lemma 8, there are at least two distinct local maxima\( > 0 \), which contradicts lemma 6, in particular, the uniqueness of local maximum.

With these, we now construct the main body of the proof. By Lemma 9, we know \( g(c) \) has 0, 1, or 2 distinct roots. Let \( S_0 \) denote the subset of \( S \) such that \( g(c) \) has 0 root. Let \( S_1 \) denote the subset of \( S \) such that \( g(c) \) has 1 root. Let \( S_2 \) denote the subset of \( S \) such that
g(c) has 2 roots. Clearly, $S_0, S_1, S_2$ partition $S$; $S_0 \cup S_1 \cup S_2 = S$.

Figure 11: $g(c)$ has 2 roots.

Figure 12: $g(c)$ has no root. (Many of these are ruled out by Lemma 1)

**Claim 3.** $S_0 \subset H$

**Proof.** Given any $s \in S$ and its associated $\bar{s}$, let $p^*$ and $\bar{p}^*$ denote the optimal solution to $\mathcal{M}(s)$ and $\mathcal{M}(\bar{s})$. Let $C_F(p^*)$ and $C_F(\bar{p}^*)$ denote the $C_F$’s when $p^*$ and $\bar{p}^*$ are implemented. Let $EU^*(\mathcal{M}(s))$ and $EU^*(\mathcal{M}(\bar{s}))$ denote the maximized EU when $p^*$ and $\bar{p}^*$ are implemented. For a given $s \in S$, $C_F(p^*)$ can either be higher or lower than $R_1$. We look at the two cases.

Step 1) We first look at $\{s : C_F(p^*) < R_1\}$ and show $\{s : C_F(p^*) < R_1\} \cap S_0 \subset H$.

Here,

$$EU^*(\mathcal{M}(s)) = p^* \cdot U(R_2) + (1 - p^*) \cdot U(C_F(p^*))$$

$$= p^* \cdot u(R_2) + (1 - p^*) \cdot \delta_2 \cdot u(C_F(p^*))$$,
because $C_F(p^*) < R_1$. On the other hand,

$$EU^*(\mathcal{M}(\tilde{s})) = \tilde{p}^* \cdot U(R_2) + (1 - \tilde{p}^*) \cdot U(C_F(\tilde{p}^*))$$

$$= \tilde{p}^* \cdot u(R_2) + (1 - \tilde{p}^*) \cdot \delta_2 \cdot u(C_F(\tilde{p}^*))$$

Since $\tilde{p}^*$ is the unique solution to $\max_{p \in (0, 1)} EU(\mathcal{M}(\tilde{s}))$, we must have $p^* = \tilde{p}^*$, and consequently,

$$C_F(p^*) = C_F(\tilde{p}^*)$$

and

$$EU^*(\mathcal{M}(s)) = EU^*(\mathcal{M}(\tilde{s})), \text{ i.e., } \{s : C_F(p^*) < R_1\} \cap S_0 \subset H.$$  

Step 2) $\{s : C_F(p^*) \geq R_1\} \cap S_0$ is empty; i.e., $C_F(p^*) \geq R_1$ never happens in $S_0$.

Suppose $C_F(p^*) \geq R_1$. Since on $S_0$, $g(c)$ has 0 root, $g(c) < 0$ on the entire domain. Therefore, $l(c) > \delta_1 \cdot u(c)$ and in particular,

$$l(C_F(p^*)) > \delta_1 \cdot u(C_F(p^*)).$$  \hspace{1cm} (42)

Also,

$$l(R_2) = u(R_2)$$  \hspace{1cm} (43)

by definition, so taking linear combinations of the two sides, and recalling that we are assuming to be in $\{s : C_F(p^*) \geq R_1\}$ so that $\delta_1 \cdot u(C_F(p^*)) = U(C_F(p^*))$,

$$(1 - p^*) \cdot l(C_F(p^*)) + p^* \cdot l(R_2) > (1 - p^*) \cdot \delta_1 \cdot u(C_F(p^*)) + p^* \cdot u(R_2) =: EU^*(\mathcal{M}(s)).$$  \hspace{1cm} (44)

Note that $l(\cdot)$ is a linear operator in its argument, and plug in $C_0 = \tilde{p}^* \cdot R_2 + (1 - \tilde{p}^*) \cdot C_F(\tilde{p}^*)$ to get

$$LHS = l(C_0) = \tilde{p}^* \cdot u(R_2) + (1 - \tilde{p}^*) \delta_2 \cdot u(C_F(\tilde{p}^*)) =: EU^*(\mathcal{M}(\tilde{s})).$$  \hspace{1cm} (45)

Thus, $EU^*(\mathcal{M}(s)) < EU^*(\mathcal{M}(\tilde{s}))$, so $p^*$ is never chosen, a contradiction to global optimality of $p^*$. Hence $C_F(p^*) \geq R_1$ never happens in $S_0$.

Claim 4. $S_1 \subset H$
Proof. Pick any \( s \in S_1 \). First, note that \( l_{\delta_1}(c) = l_{\delta_2}(c) \) because \( g(c) \) is concave and unique root defines the tangency, hence the tangent line for on \( C_F(\delta_1) \) and \( C_F(\delta_2) \) is a common line by construction of \( l(c) \). Then, by same logic as in Claim 1, we can show that \( EU^*(\mathcal{M}(s)) = EU^*(\mathcal{M}(\tilde{s})) \). Modulo the (innocuous) assumption that the agent chooses \( \tilde{p}^* = p^* \) when indifferent, \( s \) and \( \tilde{s} \) are indistinguishable, hence \( S_1 \subset H \) as desired. \( \Box \)

Claim 5. \( S_2 = R^+ \cup R^- \) where \( R^+ \subset H \) and \( R^- \subset H^c \)

Proof. Let \( \overline{R} < \overline{R} \) be the two roots of \( g(c) \). Let \( R^+ := \{ s \in S_2 : R_1 \geq \overline{R} \} \) and let \( R^- := \{ s \in S_2 : R_1 < \overline{R} \} \). Clearly, \( S_2 = R^+ \cup R^- \).

Step 1) \( R^+ \subset H \)

It suffices to show that \( U(c) \leq l_{\delta_2}(c) \) on all \( c \in (0, R_2) \). Then, we can use the same argument as in Claim 1 to show that \( EU^*(\mathcal{M}(s)) = EU^*(\mathcal{M}(\tilde{s})) \). (Namely, we partition \( C_F(p^*) \) into \((0, R_1) \) vs \([R_1, R_2) \) and argue that on \((0, R_1) \), \( s \in H \) and the sub-case for \([R_1, R_2) \) is empty.)

When \( c \in (0, R_1) \), \( U(c) \leq l_{\delta_2}(c) \) certainly holds since \( U(c) = \delta_2 \cdot u(c) \cdot 1_{(0,R_1]} \) and \( l_{\delta_2}(c) \) is line tangent to \( \delta_2 \cdot u(c) \cdot 1_{(0,R_1]} \) from above. When \( c \in [R_1, R_2) \), by lemma 6, \( \exists \hat{c} \in (\overline{R}, \overline{R}) \) such that \( g'(\hat{c}) = 0 \) and \( g'(c) < 0 \) everywhere on \( c \in (\hat{c}, \infty) \). \( \therefore g(c) = \delta_1 \cdot u(c) - l_{\delta_2}(c) < 0 \) for all \( c \in (\overline{R}, \infty) \), in particular for all \( c > R_1 \geq \overline{R} \), since we are in \( R^+ \). Hence, \( \delta_1 \cdot u(c) < l_{\delta_2}(c) \) for all \( c \in (R_1, R_2) \) as desired.

Step 2) \( R^- \subset H^c \)

This assertion will be proved in Theorem 7 (Lemma 10).

\( \Box \)

Summing up, we know that \( S_0 \cup S_1 \cup R^+ = H \) and \( R^- = H^c \). Recall that \( R_1^* = \inf \{ R_1 : (\delta_1, R_1, \delta_2, R_2; C_0) \in H \} \). For any \( s \in S_0 \cup S_1 \cup R^+ \), \( R_1^* = 1 \), because for all \( R_1, s \in H \) and \( R_1 \) is defined to satisfy \( R_1 > 1 \). On the other hand by construction of the proof of Claim 3, for any \( s \in R^- = H^c \), \( R_1^* = \overline{R} \). Moreover, for any \( R_1 \geq R_1^* \), the corresponding \( s \) is in \( H \).
Therefore, $R_1^* = \overline{R}$ is the demarcation point between $H$ and $H^c$.

Finally, suppose we consider the ex-$R_1$ 4-tuple of any $s \in H^c$. Then $R_1^* > 1$. As $\delta_1 \uparrow 1$, the $\overline{R}$ of the associated $g(c)$ monotonically increases to $R_2$, thus proving the claim that $R_1^* \uparrow R_2$ as $\delta_1 \uparrow 1$. □

**Proof of Theorem 7:** Let $s \in R^-$ and let $C_F(p^*)$ be the the $C_F$ associated to $\mathcal{M}(s)$. Let $C_F(\delta_1)$ be the $C_F$ associated to the single-kink utility maximization problem $(\delta_1, R_2; C_0)$. Similarly, let $C_F(\delta_2)$ be the $C_F$ associated to associated single-kink EU problem $\mathcal{M}(\tilde{s})$. Note that $C_F(p^*)$ is in either one of the two intervals: $(0, R_1)$ or $[R_1, R_2)$. We assume $C_F(p^*) \in [R_1, R_2)$ and later verify this. Assuming $C_F(p^*) \in [R_1, R_2)$, consider two sub-cases.

Case 1) $C_F(\delta_1) \leq R_1$

Because we assumed $C_F(p^*) \in [R_1, R_2)$,

$$EU^*(\mathcal{M}(s)) = p^* \cdot U(R_2) + (1 - p^*) \cdot U(C_F(p^*))$$

$$= p^* \cdot u(R_2) + (1 - p^*) \cdot \delta_1 \cdot u(C_F(p^*)).$$

Because we assumed $C_F(\delta_1) \leq R_1$,

$$C_F(\delta_1) \leq R_1 \leq C_F(p^*)$$  \hspace{1cm} (46)

Recall by Lemma 4, $\frac{\partial EU}{\partial C_F} < 0$ on all $C_F(\delta_1) < C_F$, hence the 'best' $C_F(p^*)$ is the smallest $C_F$ that respects $C_F(\delta_1) < R_1 \leq C_F(p^*)$.

$$\therefore C_F(p^*) = R_1.$$  \hspace{1cm} (47)

Case 2) $C_F(\delta_1) > R_1$

In this case, we argue exactly as in the proof of Theorem 6, Claim 1, Step 1 to arrive at:
\[ C_F(p^*) = C_F(\delta_1). \] (48)

Pulling these two cases together,

\[ C_F(p^*) = \max(R_1, C_F(\delta_1)) \] (49)

We now justify our assumption \( C_F(p^*) \in [R_1, R_2) \).

**Claim 6.** \( C_F(p^*) \in [R_1, R_2) \)

**Proof.** Suppose \( C_F(p^*) \in (0, R_1) \). Then, by same logic as Theorem 6, Claim 1, Step 1

\[ C_F(p^*) = C_F(\delta_2) \] (50)

Pulling these together,

\[
C_F(p^*) = \begin{cases} 
C_F(\delta_2), & \text{if } C_F(p^*) \in (0, R_1) \\
\max(R_1, C_F(\delta_1)), & \text{if } C_F(p^*) \in [R_1, R_2)
\end{cases}
\]

Therefore, it suffices to show that agents always choose \( \max(R_1, C_F(\delta_1)) \) over \( C_F(\delta_2) \). We proceed (as before) in two cases.

**Case 1) \( C_F(\delta_1) \leq R_1 \)**

Here, \( \max(R_1, C_F(\delta_1)) = R_1 \), so the task is to compare \( C_F(\delta_2) \) and \( R_1 \). Consider \( g(c) \) and its two roots \( R \) and \( \bar{R} \). By Lemma 6, \( \exists \bar{c} \in (R, \bar{R}) \) such that \( g'(\bar{c}) = 0 \).

\[ \delta_1 \cdot u'(\bar{c}) = l'_{\delta_2}(c) \] (51)

On the other hand, by Lemma 5,

\[ \delta_1 \cdot u'(C_F(\delta_1)) = l'_{\delta_1}(c). \] (52)

Also, (as in the proof of Lemma 7) it is easy to see that
Putting these together,

\[ \delta_1' \cdot u' (C_F(\delta_1)) = l_{\delta_1}'(c) < l_{\delta_2}'(c) = \delta_1' \cdot u' (\tilde{c}). \]

which means \( C_F(\delta_1) > \tilde{c} \). Combine this with the assumptions \( C_F(\delta_1) \leq R_1 \) and \( R_1 < \overline{R} \) (because we are in \( R^- \)), we get:

\[ \tilde{c} < C_F(\delta_1) \leq R_1 < \overline{R} \]

Since \( g(c) \) is concave in \( c \), and \( g'(\tilde{c}) = 0, g'(c) < 0 \) on \((\tilde{c}, R_2)\).

This, (49), and Lemma 8 tell us \( g(c) > 0 \) on \((\tilde{c}, \overline{R})\), in particular, \( g(R_1) > 0 \). Namely,

\[ \delta_1 u(R_1) > l_{\delta_2}(R_1) \]

Finally, since \( C_F(p^*) = R_1 \), \( p^* \) must satisfy

\[ (1 - p^*) C_F(p^*) + p^* R_2 = (1 - p^*) R_1 + p^* R_2 = C_0 \]

We compare

\[ EU(R_1) = (1 - p^*) U(R_1) + p^* U(R_2) \]
\[ = (1 - p^*) \delta_1 u(R_1) + p^* u(R_2) \]

and

\[ EU(C_F(\delta_2)) = l_{\delta_2}(C_0) \]
\[ = l_{\delta_2}((1 - p^*) R_1 + p^* R_2) \]
\[ = (1 - p^*) l_{\delta_2}(R_1) + p^* u(R_2). \]
Using (50), $EU(C_F(p^*)) > EU(C_F(\delta_2))$ as desired.

Case 2) $C_F(\delta_1) > R_1$

Task here is to compare EU under $C_F(\delta_1)$ vs $C_F(\delta_2)$. By Lemma 7, $l_{\delta_1}(c) > l_{\delta_2}(c)$, and in particular for $c = C_0$. Therefore, $EU(C_F(\delta_1)) > EU(C_F(\delta_2))$ as desired.

Therefore, in each of the cases, indeed, $C_F(p^*) \in [R_1, R_2)$ as we assumed.

We can now use (46) without any qualifications. This immediately ties up a loose end we left in Theorem 6 ($R^- \subset H^c$), which we state as a Lemma.

**Lemma 11.** $R^- \subset H^c$.

*Proof.* It suffices to show that $C_F(p^*) \neq C_F(\delta_2)$. Recall that $\delta_2 < \delta_1$, and hence by Lemma 3, $C_F(\delta_2) < C_F(\delta_1)$. $\therefore C_F(\delta_2) < C_F(\delta_1) \leq \max(R_1, C_F(\delta_1)) = C_F(p^*)$, as desired.

We now use (46) to wrap up the proof. Consider the two cases:

Case 1) $C_F(\delta_1) \leq R_1$

By (43), $C_F(p^*) = R_1$. This implies $p^* = \frac{C_0 - R_2}{R_2 - R_1} = \rho$, whence (i), (ii), (iii) follow directly.

Case 2) $C_F(\delta_1) > R_1$

By (43), $C_F(p^*) = C_F(\delta_1)$. By same logic as in Theorem 6, Claim 1, Step 1, we can think of $\mathcal{M}(s)$ as $\mathcal{M}(\delta_1, R_2; C_0)$, which is a setting where Theorem 1 - Theorem 5 apply. (i) follows from Theorem 5 and the chain rule. (ii) and (iii) follow from Theorem 1 and the chain rule.


Proof of Theorem 8: Without loss of generality, suppose that $C < R$ always holds. (Assuming otherwise only strictly increases EU whence the same analysis can be used for the proof.) Consider a hypothetical sub-martingale $L(p)$ with $p = 1$ and $\alpha > 0$. Namely, $L(1)$ is a bet that gives $\alpha$ with probability 1, and gives -1 with probability 0. Then, for any $N^* > 0$, $EU(1, N^*) = U(C_S) = U(C_0 + \alpha \cdot N) > U(C_0)$, with strict inequality. Since EU is continuous in $p$, $\exists p^* \in (0, 1)$ such that $EU(p^*, N^*) > U(C_0)$. \hfill \Box

Proof of Theorem 9: (At this point, an analytic proof does not seem obvious. However, numerically this seems true, so there should be a way to prove this. \hfill \Box

Proof of Lemma 3:

Step 1) Show that any consumption scheme whose $C_S$ exceeds $R$ is dominated by a consumption scheme whose $C_S$ equals $R$.

Let $N_0$ be the $N$ such that $C_S = R$ on equation (6). Then as in the proof of Lemma 1,

$$EU\bigg|_{N \geq N_0} = \left((1 - p)U(C_F) + pU(C_S)\right)$$

$$= \left((1 - p)U(C_0 - N) + pU(C_0 + N(M - 1))\right)$$

Then differentiate $EU\bigg|_{N \geq N_0}$ with respect to $N$ at $N_0$, $p^*$:

$$\frac{\partial EU}{\partial N}\bigg|_{N_0, p^*} = -\left((1 - p^*)U''(C_0 - N) + (M - 1)p^*U''(C_0 + N(M - 1))\right)\bigg|_{N_0}$$

$$= \left((1 - p^*)\left((1 + \frac{\alpha}{1 - p^*})U''(C_S) - U''(C_F)\right)\right)\bigg|_{N_0}$$

$$= \left((1 - p^*)\left((1 + \frac{\alpha}{1 - p^*})U''(R) - U''(C_F)\right)\right)$$

$$< 0,$$
where the last inequality follows from the domain of $p$, and assumption (18).

**Step 2)** Showing that $C_S < R$ never happens is same as in Lemma 1.

**Theorem 10:** We proceed in two steps. In Step 1, we show that the optimal consumption positions are identical to those of the optimized binomial scheme. In Step 2, we show that the probability mass on $C_0$ must be 0 for optimality.

**Step 1:** Consider the standard solution to the EU-maximization problem using *binomial* consumption scheme. Let $C_F^*$ and $C_S^*$ be optimal $C_F$ and $C_S$ in this binomial solution. Let $\mathcal{F}$ with $p = (p_1^*, p_2^*, p_3^*)$ and $C = (C_S^*, C_0^*, C_F^*)$ be the optimal trinomial consumption scheme to $\mathcal{M}(\delta, R; C_0)$. Then $C^* = (C_S^{**}, C_0, C_F^{**})$.

**Proof.** (Given any $p_2, 0 \leq p_2 \leq 1.$) The proof is by comparing the objective functions of the binomial and trinomial optimization. The trinomial optimization problem is:

\[
\max_{p_1, p_2, p_3} p_3 U(C_F) + p_2 U(C_0) + p_1 U(C_S),
\]

subject to

\[
p_3 C_F + p_2 C_0 + p_1 C_S = C_0 \text{ and } p_1 + p_2 + p_3 = 1
\]
or equivalently,

\[
\max_{p_1, p_3} U(C_0) + p_1 (U(C_S) - U(C_0)) + p_3 (U(C_F) - U(C_0)), \quad (58)
\]

subject to

\[
p_1 (C_S - C_0) + p_3 (C_F - C_0) = 0 \text{ and } p_1 + p_3 = 1 - p_2 \quad (59)
\]

Recall that the objective function for the binomial scheme was:

\[
\max_p U(C_0) + p (U(C_S) - U(C_0)) + (1 - p) (U(C_F) - U(C_0)), \quad (60)
\]
subject to
\[ p(C_S - C_0) + (1 - p)(C_F - C_0) = 0 \]  \hspace{1cm} (61)

Scaling (58) - (61) by \( \frac{1}{1 - p_2} \) yields a monotone (affine) transformation of (60) and identical constraint, hence identical solution, up to the C positions. (P* cannot be pinned down yet, because it is determined only up to scaling.)

\[ \square \]

**Step 2:** \( p_2^* = 0 \).

**Proof.** By an argument similar to the proof of Lemma 5-Lemma 7, we know that:

\[ (1 - \alpha)U(C_F) + \alpha U(R) > U(C_0), \] for \( 0 < \alpha < 1 \).

On the other hand, some manipulation on (58) yields

\[ EU = (1 - p_2)\left( \frac{(1 - p_2 - p_1)}{(1 - p_2)} U(C_F) + \frac{p_1}{(1 - p_2)} U(R) \right) + p_2 U(C_0). \]

Hence, to maximize EU, we require \( p_2 = 0 \).

\[ \square \]

**Proof of Theorem 11:** This is really a Corollary of Theorem 10. By Theorem 10, we can reduce the solution space \( \mathcal{T} \) to the space of \( \mathcal{B} \). Then by Lemma 1, the \( C_S \) of the associated \( \mathcal{B} \) must equal \( R \).