Abstract

We analyze the problem of a regulator that sets both capital and liquidity requirements to maximize social welfare in a framework in which a bank decides its level of solvency risk facing a risk-return trade-off. Capital requirements reduce risk-shifting through a "skin-in-the-game" channel, but substituting deposits for capital is socially costly. Liquidity requirements mitigate short-term withdrawal risk, but aggravate risk-shifting because they reduce the bank’s returns. We find that liquidity and capital requirements complement each other when the cost of capital or the return on loans is high and offset each other otherwise, so that regulators should set liquidity and capital requirements jointly taking into consideration capital and liquidity feedback loops.

Keywords: Liquidity, Capital, Basel III, Solvency risk, Liquidity risk

JEL Codes: G01, G20, G21, G28, E58

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dmacedo@emp.uc3m.es, sergio.vicente@uc3m.es.
"By way of motivation, note that before the financial crisis, we had a highly developed regime of capital regulation for banks—albeit one that looks inadequate in retrospect—but we did not have formal regulatory standards for their liquidity."


"In the first instance, at least, this was a liquidity crisis. Its fast-moving dynamic was very different from that of the savings and loan crisis or the Latin American debt crisis of the 1980s. The phenomenon of runs instead recalled a more distant banking crisis—that of the 1930s. Despite this defining characteristic of the last crisis, measures to regulate liquidity have by-and-large lagged other regulatory reforms [...]".


1 Introduction

Before the last financial crisis, most of the regulatory efforts were concentrated on capital requirements. Although concerns regarding illiquidity risk have been pervasive in the regulatory debate dating back to at least Bagehot (1873) celebrated essay, the emphasis on capital had relegated liquidity regulation to the background. However, during the 2007 early 'liquidity phase' of the financial crisis, several banking institutions experienced liquidity-driven difficulties despite their adequate capital levels (Gorton (2010), Brunnermeier (2009), Banerjee and Mio (2015)). By September, Northern Rock suffered a bank run after seeking (and obtaining) liquidity support from the Bank of England. As response to the 2007-2009 financial crisis, the Basel Committee for Banking Supervision introduced the Liquidity Coverage (LCR) and the Net Stable Funding (NSFR) ratios within Basel III to reinforce the resilience of banks to illiquidity risk.\(^1\)

In general, capital and liquidity regulation have been analyzed in isolation. Classical treatments of bank capital regulation have emphasized the role of capital as a means of mitigating excessive risk-taking by banks (e.g., Furlong and Keeley (1989), Rochet (1992), Hellmann et al. (2000), Repullo (2004)). On the other hand, liquidity regulation has typically been thought of as a way of dealing

\(^1\)The LCR requires that banks hold a buffer of liquid assets so as to cover cash outflows during a 30-day window under liquidity distress. The NSFR, somehow redundantly, complements the former by requiring a long-term stable funding structure that protect banks against maturity mismatches. The British FSA issued the "Turner Review" in 2009, which included a set of guidances to regulate banks’ liquidity in a similar spirit to the Basel Liquidity Coverage Ratio (see Turner et al. (2009)).
with issues of maturity mismatch or refinancing risk (e.g., Farhi et al. (2009), Freixas et al. (2011), Perotti and Suarez (2011), Calomiris et al. (2015)). And it has not been until recently that the regulation of capital and liquidity has been addressed together (e.g., Vives (2014), De Nicoló et al. (2014), Walther (2015)).

In this paper we analyze joint liquidity and capital regulation in a bank risk-taking framework. Our model features a bank that is partially funded with deposits and protected by limited liability. Moreover, there is a deposit insurance scheme that insulates depositors against potential bank failures. A consequence of these three features is that the bank does not internalize the losses that its potential failure inflicts on the deposit insurance scheme, leading to excessive risk-shifting. This market failure can be partially mitigated by a social welfare maximizer banking regulator, who can employ two regulatory tools: a capital and a liquidity requirement. The bank takes its capital structure as given (through capital requirements) and chooses its (unobservable) insolvency risk profile facing a standard risk-return trade-off. Capital requirements increase the bank’s skin in the game by reducing its deposit liabilities, therefore mitigating the bank’s incentives to risk-shift. Additionally, the bank is subject to an uncertain (exogenous) level of early deposit withdrawals, which generates illiquidity risk. Liquidity requirements reduce the bank’s exposure to liquidity crises. We show that the effectiveness of capital (respectively, liquidity) requirements depends on the bank’s level of liquidity (respectively, capital). Hence, optimal liquidity and capital requirements need be determined jointly.

We assume that substituting deposits for equity entails a social cost, so that capital requirements

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2 Although their focus is on the provision of liquidity aid by a Lender of Last Resort, Rochet and Vives (2004) address the convenience of establishing a liquidity and a capital ratio in combination with LoLR interventions.

3 Since deposits are insured, the rates of return on deposits do not reflect the riskiness of the bank’s portfolio.

4 In our framework, capital is used to mitigate the bank’s risk-shifting incentives. If the bank’s risk profile were verifiable, the first-best policy intervention would consist of a zero capital requirement.

5 We want to emphasize that we are taking a rather extreme vision of solvency and liquidity risks as independently generated phenomena. Arguably, one may expect that the risk of facing a large amount of early withdrawals be positively correlated with the risk profile of the bank, at least if withdrawing at an early stage may be influenced to a certain extent by the bank’s fundamentals and not only by depositors’ idiosyncratic motives. For instance, one may reasonably expect that liquidity shocks may be originated by signals of solvency problems. Moreover, a situation of liquidity distress may well lead to solvency troubles. However, we want to abstract from potential feedback loops between the generation of liquidity and solvency risk in the basic model to highlight that capital and liquidity requirements feed back into each other even if liquidity and solvency risks are generated independently. Nevertheless, in an extension we conduct an exploration of a model in which early withdrawals are relatively more likely the higher the level of solvency risk. In this setup, reducing solvency risk has a double effect and, consequently, capital requirements are larger than in the base model. However, the nature of the feedback effects of capital and liquidity remains.
weigh the marginal return of capital–through its impact on reducing insolvency risk–and its social cost. The effect of liquidity on capital requirements is that the share of liquid assets held by the bank affects the marginal return of capital. For instance, if the bank’s liquidity is low, the likelihood that the bank survives a liquidity crisis is low. Hence, the expected return of reducing insolvency risk is low as well, because the bank may nevertheless fail due to liquidity driven troubles. Consequently, the marginal return of capital is low when liquidity is low. As liquidity increases, so does the probability of surviving a liquidity crisis and therefore the return of reducing insolvency risk. Hence, liquidity requirements are a complementary tool to capital regulation when liquidity requirements are small, for the marginal return of capital increases with liquidity when the liquidity level is low. However, liquidity holdings reduce the social loss in the event of a bank failure, because liquidity holdings constitute a cash-in-hand asset that can be used to reduce the deposit insurance toll in that event. When liquidity levels are sufficiently high, the positive effect of reducing illiquidity risk is outweighed by the reduction of the social loss in the event of a bank failure. Hence, liquidity requirements constitute an offsetting tool to capital requirement when these are high.

On the other hand, the rationale for liquidity regulation in this paper is to reduce illiquidity risk beyond the bank’s will. In a laissez-faire economy, the bank would choose a certain level of liquidity to balance out the positive effect of reducing illiquidity risk with the opportunity cost of liquidity, which is given by the foregone investments in more profitable assets. The regulator faces the same liquidity trade-off as the bank in nature, but liquidity is more valuable for the regulator than for the bank for two reasons. First, a bank failure is more costly for society than for the bank, because the bank does not internalize the loss that its failure inflicts on the deposit insurance fund. Second, liquid assets are valuable in the event of the bank failure. This is a source of value of liquidity that the bank neglects, because the bank does not take into account the value of its assets in the event of a failure. Hence, the opportunity cost of liquidity in terms of foregone investment opportunities is smaller for society than for the bank. As a result, the regulator establishes liquidity requirements in excess of the level that the bank would choose in the absence of regulation. An immediate implication of binding liquidity requirements is that they harm the bank’s expected profits. Hence,

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6 Previous research has emphasized that substituting deposits for equity is socially costly. For instance, Van den Heuvel (2008) and Gorton and Winton (2014) state that capital requirements are socially costly because substituting capital for deposits entails eliminating a valuable source of liquidity for consumers. Myers and Majluf (1984) argue that issuing new equity could be costly when old shareholders and managers have access to information that new investors do not have. Bolton and Freixas (2006) points out that the presence of asymmetric information makes equity capital more costly than other sources of bank funding. Capital may also be costlier than deposits due to its relative scarcity (Martinez-Miera and Suarez (2014)) or because it carries a higher return than deposits in segmented markets (Allen et al. (2015)).
liquidity requirements harm the bank’s incentives to reduce insolvency risk.

The effectiveness of liquidity requirements does in turn depend on the level of bank’s equity capital. On the one hand, increasing capital reduces the negative effect of a binding liquidity requirement on the bank’s choice of insolvency risk. The reason is that the wedge between the bank’s and society’s objective function narrows down as capital increase—recall that this wedge is driven by the fact that the bank is partially funded with non-risk-priced deposits and is insulated by limited liability. Hence, by reducing the negative effect of liquidity on the bank’s incentives to curtail insolvency risk, raising capital leads to an increase of the marginal social return of liquidity requirements. On the other hand, an increase in capital leads to a reduction of insolvency risk. Hence, as capital increases, the value of the bank’s upside payoff carries a higher weight in the regulator’s objective function, because the solvent state is more likely to occur. The higher likelihood that the bank’s loan investments turn out successfully imply that the value of surviving a liquidity crisis is higher, but also that the opportunity cost of holding liquidity instead of granting loans increases as well. For a sufficiently high liquidity level, so that the bank is resilient enough to liquidity shocks, the effect of the increased opportunity cost of holding liquidity dominates. Hence, capital requirements are a complementary tool to liquidity requirements when they are low, while they constitute an offsetting tool to liquidity requirements when they are high.

Summing up, we have two regions for liquidity and capital. In the ‘complementary tools’ region, increasing the level of one of the regulatory variables leads to an increase of the effectiveness of the other. The optimal capital and liquidity requirements fall in that region when the cost of capital and the opportunity cost of liquidity are large, so that the level of optimal capital and liquidity are small. In the ‘offsetting tools’ region, on the contrary, increasing the level of one regulatory tool offsets the effectiveness of the other tool. The optimal capital and liquidity requirement fall in that region when the cost of capital and the opportunity cost of liquidity are small, so that the optimal requirements are large.

There are several important ingredients that we abstract from in the main analysis in order to keep the analysis tractable. We extend the analysis to allow for asset liquidation at (exogenous) fire sales prices in order to overcome a liquidity shock exceeding the bank’s cash reserves. Although the main feedback effects between liquidity and capital prevail, namely that capital and liquidity are complementary or offsetting tools depending on the level of both variables, there are some changes that are worth noting. First, the magnitude of optimal liquidity requirements is unambiguously smaller than in the absence of asset liquidation. The reason is that the bank is now able to overcome larger liquidity shocks and therefore the marginal return of liquidity requirements through reducing illiquidity risk diminishes. It is worth remarking that, although liquidity requirements are reduced,
illiquidity risk unambiguously reduces the combined effect of lower liquidity and asset liquidation allows the bank to overcome higher liquidity shocks. On the other hand, capital requirements are unambiguously larger. Since illiquidity risk is smaller, the value of reducing insolvency risk—and therefore, of capital requirements—is larger.

Finally, we consider a modified version of our model in which a Lender of Last Resort (LoLR) comes to the rescue of the bank in case it faces a shortage of liquid funds to meet early withdrawals. We model the LoLR as an agent that is committed to provide liquidity assistance until a certain (exogenous) level at a given rate of return.\(^7\) The introduction of an LoLR reduces—or, if the LoLR is unconstrained, eliminates—illiquidity risk, leading an unambiguous reduction of liquidity requirements as compared to our benchmark. The reduction of illiquidity risk makes the reduction of insolvency risk more valuable. Hence, capital requirements should be higher under an LoLR. How effective capital requirements are do also depend on the rate at which the bank borrows from the LoLR, for this rate negatively affects the bank’s payoff and therefore its incentives to reduce insolvency risk. Hence, since the magnitude of the LoLR’s subsidy to the bank constitutes a mere transfer among agents in the economy that does not affect social welfare, the bank should borrow from the LoLR at the minimum possible rate.

The rest of the paper is organized as follows. In the next section we review related research. Section 3 contains the description of the basic model. We analyze the bank’s asset choices and the main effects of both liquidity and capital on the bank’s incentives to reduce insolvency risk in Section 4. The main analysis is contained in Section 5. We conduct several extensions in Section 7, where we analyze the implications of enriching the baseline model. We conclude in Section 8.

2 Related Literature

Several papers analyze the role of equity capital as a way to reduce excessive bank risk-taking. Furlong and Keeley (1989) shows that capital enhances the incentives of banks to reduce risk-shifting through increasing shareholders’ losses in case of default. Hellmann et al. (2000) notes that although capital may reduce risk-shifting by putting banks’ equity capital at risk, they may also induce higher risk-taking because they erode banks’ charter value. Their analysis calls for a combination of capital requirements and caps on deposit rates to avoid the negative effect of deleveraging on franchise values. Repullo (2004) reexamines the relationship between capital requirements and risk-taking

\(^7\)For instance, a LoLR may derive utility from bailing out an otherwise failing bank but only, as in Repullo (2005), as long as the liquidity shortfall lies below a certain threshold related to the level of solvency risk. Moreover, the LoLR may additionally face limitation of funds or institutional constraints of some sort that may prevent the LoLR to rescue the bank beyond a certain level of liquidity needs.
behavior in a setup in which banks imperfectly compete for depositors. This paper shows that capital requirements reduce bank’s risk-taking because the extra cost of capital born by regulated banks is passed onto depositors through reduced deposit rates, so that capital requirements do not erode banks’ charter values. Our paper draws from these insights to address the effect of capital on risk-taking. In our model, capital reduces the bank’s reliance on deposits, therefore increasing the bank’s ‘skin in the game’. As a consequence, capital requirements reduce risk-shifting incentives. The main novelty of our analysis is that we combine a liquidity requirement with a capital requirement. We find that the effectiveness of capital requirement depends on the level of liquidity requirements.

There is a handful of papers looking into the joint regulation of capital and liquidity requirements. In an early treatment focusing on the Lender of Last Resort (LoLR), Rochet and Vives (2004) analyze the optimal LoLR intervention policy in a model in which investors may face a coordination failure to roll-over credit to a solvent but illiquid bank. In this model, solvency and liquidity requirements could prevent the coordination failure, but they may entail a large cost in terms of foregone returns, which the LoLR intervention can mitigate. In our model, we show that the LoLR policy should be jointly established with liquidity and capital requirements. In our model, the role of the LoLR is to alleviate (or eliminate) illiquidity risk faced by a solvent bank. Calomiris et al. (2015) argues that liquidity, as a verifiable and riskless asset, can be used as a commitment device to engage in efficient risk management. The paper shows that in the presence of idiosyncratic liquidity shocks, a liquidity insurance scheme with liquidity requirements is optimal, as it diversifies away individual shocks while avoiding a common pool resource problem. Moreover, since liquidity enhances risk management, liquidity requirements lead to book equity being a more precise indicator of equity market values, thus identifying a channel by which liquidity enhances the value of equity requirements. In our paper, in contrast, liquidity leads to an increase in insolvency risk because it erodes the bank upside payoff by foregoing more profitable investments. As noted by König et al. (2015), which studies the effect of liquidity regulation on banks’ overall risk, liquidity requirements harm the bank’s upside payoff by reducing the share of assets invested in loans. De Nicoló et al. (2014) find out that while (mild) capital requirements lead to a higher volume of lending and a lower default probability than in the case of unregulated banks, these positive effects disappear when combined with liquidity requirements. In our model, in contrast, we find that liquidity requirements complement capital requirements by increasing the likelihood of surviving a liquidity crisis and, consequently, increasing the value of capital a tool to reduce insolvency risk. Vives (2014) analyze a global game in which investors’ decisions to roll-over debt are strategic complements and solvency refers to the financial intermediary fundamentals to cope with coordinated actions. The analysis relates liquidity—to deal with liquidity issues—and solvency ratios—to cope
with insolvency—to the level of transparency, as agents’ information is a main driver of coordinated actions. The paper results calls for a joint determination of prudential and disclosure policies. In a recent treatment, Walther (2015) analyzes the problem of a set of banks that have incentives to induce excessive systemic risk through leverage. While the liquidity ratios in Basel III can be used to reduce the advent of fire sales, capital requirements can be used to avoid individual bank failures. In our model, in contrast, although capital and liquidity requirements are set to reduce solvency and illiquidity risk, respectively, capital and liquidity feed back into each other because the effectiveness of one instrument depends on the level of the alternative tool.

3 The basic setup

In this section, we describe the basic aspects of our model. Some of the assumptions are discussed further below. We consider a four-period economy, \( t \in \{-1, 0, 1, 2\} \), where all agents are risk-neutral and there is a zero discount factor. In this economy there is a social welfare maximizer regulator that sets both liquidity and capital requirements \((l^*, k^*)\) at \( t = -1 \). The capital structure of the bank at time \( t = 0 \) consists of an amount \( d \) of common deposits, which are available upon demand both at \( t = 1 \) and at \( t = 2 \); an amount \( b \) of long-term deposits, which are cashable at \( t = 2 \) only; and an amount \( k \) of equity capital, which is required to either meet or exceed the capital requirement, that is, \( k \geq k^* \). We normalize the bank funds to 1, so that we have that \( d + b + k = 1 \). Consequently, we interpret the capital requirement as a fraction of the bank’s assets. We fix the amount of deposits \( d \) exogenously and let the regulator set a minimum capital requirement \( k^* \in [0, 1 - d] \).

We assume that common deposits constitute a substantial share of the bank’s funds. In particular, we assume that \( d \geq d^* \), for some \( d^* > 0 \), whose value we define precisely on Definition 6 in Appendix A. For simplicity, we assume that all deposits are completely insured by a deposit insurance scheme, so that we can normalize (excess) interest rates on common deposits to 0. Long-term deposits, on the contrary, carry an (excess) interest rate \( r \geq 0 \) which, for simplicity, we take as exogenous.\(^8\) Moreover, we assume that equity capital is costlier than deposits, both from a social perspective and to the bank. In particular, we assume that each unit of equity capital carry a(n excess) shadow cost of \( \rho \), with \( \rho > r \). For ease of exposition, we assume that the cost of raising a unit of equity capital to the bank is the same as the social shadow cost.\(^9\) We assume that the

\(^8\)Although we assume that long-term deposits are insured, depositors may require a higher return than on common deposits because common deposits are available upon demand, so that they constitute a source of liquidity for consumers. Consumers may forgo some liquidity in exchange for an extra return of \( r \).

\(^9\)Nonetheless, this assumption does not play any role in our analysis, as banks will not be willing to hold equity capital in excess of capital requirements, that is, capital will effectively not be a choice variable for the bank.
The bank is protected by limited liability, signifying that the bank is not required to meet its deposit obligations in the event that its loan does not perform.

At time \( t = 0 \), the bank decides which fraction \( l \) of its resources to store as a liquid asset (i.e. cash) and which amount \( 1 - l \) to invest in a profitable project (i.e. a loan). The loan is used to finance a long term project, which delivers its returns at \( t = 2 \). If successful, the loan returns \( M \) units per unit of investment, which we refer to as the project profitability (or profitability, for short). The investment returns 0 if it fails. The probability of a success \( \theta \) is an endogenous choice for the bank. Following Dell’Ariccia and Marquez (2006) and Allen et al. (2011), we let the bank choose an unobservable solvency level \( \theta \in [0,1] \), which determines the probability of success of its lending portfolio.\(^{10}\) Choosing a solvency level \( \theta \) carries a cost \( c \theta^2 \), for some \( c > 0. \)\(^{11}\) This assumption gives rise to a standard risk-return trade-off, as higher risk (i.e. lower \( \theta \)) leads to a higher return. We refer to insolvency risk as to the value \( 1 - \theta \).

We model illiquidity risk assuming that some depositors have liquidity needs at \( t = 1 \). As in Diamond and Dybvig (1983), we assume that a fraction \( \beta \) withdraws its deposits at \( t = 1 \), but we add aggregate uncertainty in the actual fraction \( \beta \) of early withdrawals. We assume that \( \beta \) is drawn from a common knowledge random variable with support \([0,d]\) and an associated c.d.f. \( F(\cdot) \) and a well-defined density \( f(\cdot) \). We assume that this distribution is log-concave and that the density function is continuous and positive everywhere in the interior of its support. For any given level of liquidity \( l \) held by the bank, illiquidity risk is given by \( 1 - F(l) \). For simplicity, we assume that the bank will be liquidated if the deposit withdrawals exceeds its cash reserves, that is, if \( \beta > l \). We relax this assumption in Sections 6 and 7.1, where we allow for a Lender of Last Resort intervention to rescue an solvent but illiquid bank and early partial asset liquidation at discounted prices, respectively.

Finally, we make technical assumptions, Assumptions 1-4, which we outline and discuss in Appendix A.

\section{Discussion of assumptions}

Some of the ingredients of the model deserve further explanation. On the one hand, we assume that the bank is protected by limited liability. The immediate implication of this assumption is that the bank does not internalize the losses that its default inflicts on society. Limited liability

\(^{10}\)For instance, a bank may invest resources into developing a credit score model to improve the quality of its creditworthiness appraisals.

\(^{11}\)See also Besanko and Kanatas (1996) and Dewatripont and Tirole (1994) for analogous arguments of costly actions that enhance banks’ expected returns operating through a reduction of the probability of a project failure.
induces a wedge between the bank’s objective function and social welfare, so that there is room for welfare-improving regulation.

Also, we assume that deposits are insured, so that they carry either a zero interest rate (common deposits) or an interest rate of \( r \) (long-term deposits). The results that we obtain do not strictly depend on the assumption that deposits are insured, but on the underlying assumption that deposit rates are independent of the bank’s choice of risk \( \theta \) and the regulator’s choices of liquidity and capital requirements \( (l^*, k^*) \). If, on the contrary, deposits were not insured and the choice of risk were observable, deposit interest rates would be proportional to the inverse of the bank’s choice of risk \( 1/\theta \). In this case, the bank would internalize the effect of its choice of insolvency risk on depositors and the bank would act as if not insulated by limited liability.

We assume that some fraction of the bank’s liabilities is levied in the form of long-term deposits. In the absence of long-term deposits, any increase in the capital requirement would necessarily require either a reduction of the amount of common deposits (since, in that case, we would have that \( d + k = 1 \)) or an enlargement of the bank’s balance sheet at \( t = 0 \). Hence, capital requirements would interfere with either the common deposit base—the one that is subject to early withdrawals—or with the bank’s operating scale, which would also affect the proportion of common deposits over the bank’s balance sheet at \( t = 0 \). In order to clean out the effect of capital requirements without interfering with the common deposit base or the asset size, we assume that the bank can always raise an amount \( b \) of long-term deposits so that \( b + k = 1 - d \). Assumption 1 in Appendix A on the minimum value for the social cost of capital ensures that the optimal capital requirement never exceeds \( 1 - d \).

We assume that equity capital is raised from outside shareholders. Essentially, as in Bolton and Freixas (2000) and Repullo (2013), we assume that the bank must allocate a fraction \( \alpha \) of its ownership profits to reward its outside equityholders in exchange for their provision \( k \) of outside equity capital at an exogenously given competitive rate \( \rho \), so that:

\[
\alpha \cdot \Pi_B (l, \theta, k) = (1 + \rho) \cdot k,
\]

where \( \Pi_B (l, \theta, k) \) stands for the bank’s profits and is defined precisely below, in equation (1) on Section 4.

Regarding the early withdrawal phase, we assume that the amount of early withdrawals is a

\[12\] Nonetheless, the results that we obtain generalize to a model in which capital requirements lead to an expansion of the bank’s balance sheet at \( t = 0 \), insofar as deposits constitute a certain (considerable) share of the assets—if capital were too large a fraction of the bank’s funds, the bank would choose to hedge against any potential early withdrawal, so that we would not have liquidity risk. An extended version of this paper with an analysis of this case is available from the authors upon request.
random variable with an exogenously given distribution function. Unlike in Diamond and Dybvig (1983), we need aggregate uncertainty in the amount of early withdrawals in order to induce illiquidity risk. If the actual amount were known to the bank, it would store exactly as much liquidity as that amount so as to meet its early demand for deposits. The log-concavity assumption implies that the reverse hazard rate $g(l)$, which is given by the ratio $g(l) = \frac{f(l)}{F(l)}$ for all $l \in (0, d)$, is strictly decreasing and has an asymptote at $l = 0$, i.e., $\lim_{l \to 0} g(l) = +\infty$ (Bagnoli and Bergstrom (2005)), a property which we will draw from in our main result. This assumption provides an Inada-type condition for liquidity, since any increase in liquidity at $l = 0$ yields an infinite return (from 0 to positive profits). Moreover, the log-concavity assumption on the distribution of liquidity shocks implies that there are decreasing returns to liquidity since, as we show below, the marginal return of liquidity will be decreasing. The family of log-concave distributions with compact support includes the uniform distribution and all beta distributions with shape parameters at least as large as one, which encompass a large variety of shapes.

Moreover, we assume that the distribution function of early withdrawals is independent of the (solvency) risk profile adopted by the bank. Arguably, one may expect that the amount of early withdrawals be positively correlated with the risk profile of the bank, at least if withdrawing at an early stage may be influenced to a certain extent by the bank’s fundamentals and not only by depositors’ idiosyncratic motives. While we acknowledge that this correlation may be very relevant, the exercise that we carry out here assumes that illiquidity risk is originated independently of insolvency risk, because we want to highlight that liquidity and insolvency risk are interrelated through the bank’s decisions even when originated independently. Nonetheless, in Section 7.2, we relax this assumption assuming that, for any choice of solvency $\theta$, the distribution of early withdrawals $F(\cdot; \theta)$ is such that if $\theta_1 < \theta_2$, then $F(\cdot; \theta_1)$ first-order stochastically dominates $F(\cdot; \theta_2)$. Hence, higher levels of insolvency risk induce a distribution of "larger" early withdrawals in a first-order stochastic sense.

Finally, notice that our framework does not include any negative externality from a banking failure. Including an externality in the form of, for instance, a positive shadow cost associated to the bank liquidation or deposit insurance provision would reinforce our case for regulation, but would not add further insight.

4 Bank’s asset choices

Before addressing the optimal regulation of capital and liquidity, we first analyze the problem of a bank that takes both its capital structure and the share of resources that must be stored as liquid
assets as exogenously given and decides how much to invest in reducing the risk associated to its loan portfolio. This section allows us to assess the roles of liquidity and capital on the bank’s incentives to reduce risk.

We focus on three aspects. First, in Section 4.1, we find the liquidity level that maximizes the bank’s profits for any predetermined level of capital. This choice will serve as a benchmark to assess how the optimal liquidity requirement differs from the value that maximizes the bank’s profits. Second, in Section 4.2, we analyze the interaction between liquidity and insolvency risk. Finally, in Sections 4.3 and 4.4 we assess the role of capital in shaping the bank’s choice of risk.

The bank’s problem consists of choosing a fraction of cash \( l_B \) and a screening level \( \theta_B \) so as to solve

\[
\max_{\theta \in [0,1]} \quad \Pi_B (\theta, l, k) \\
\text{s.t.} \quad \Pi_B (\theta, l, k) \geq 0
\]

where the bank’s objective function is given by:

\[
\Pi_B (\theta, l, k) \equiv \int_0^\theta \left[ \frac{M \cdot (1 - l) + l - \beta - (1 - \beta + b \cdot r - k)}{\text{Loan Value}} \cdot \text{Upside payoff} \right] \cdot f (\beta) \, d\beta - \frac{c}{2} \cdot \theta^2 - (1 + \rho) \cdot k. \tag{1}
\]

The upper bound of the integral is given by the fact that the bank can only survive the early withdrawal phase if it holds enough cash so as to meet the demand for deposits at \( t = 1 \), that is, as long as \( l \geq \beta \). Moreover, the bank can only make positive profits if its investment is successful, which occurs with probability \( \theta \). Upon successful completion of the project, the bank will obtain an amount \( M \cdot (1 - l) \) from its loan investment, as well as an amount of cash \( l - \beta \). Its deposit liabilities amount to \( 1 - \beta + b \cdot r - k \).\(^{13}\) Finally, we include the constraint that the bank makes non-negative profits.\(^{14}\)

\(^{13}\)Notice that the amount \( \beta \) of early withdrawals does not affect bank’s profits, as long as the bank holds enough cash at time \( t = 1 \) so as to survive the early withdrawal phase. We show that early withdrawals indeed do affect banks’ profits in Section 7.1, where we allow for asset liquidation to cope with excessive withdrawals. Nonetheless, the main insights about the effect of liquidity on solvency risk generalizes to the framework in which asset liquidation is possible.

\(^{14}\)Below, we solve the problem of a regulator that maximizes social welfare. Hence, the non-negative-profit constraint for the bank will be harder to meet. In Appendix A we impose conditions on \( M, c \) and \( \rho \) so that the constraint does not bind in that problem. Consequently, it will not bind for this problem either. We henceforth ignore this constraint.
In order to rewrite the bank’s profit maximizing problem, we define the bank’s upside payoff as:

\[
\pi(l, k) \equiv \underbrace{M \cdot (1 - l) + l}_{\text{Loan Value}} - \underbrace{(1 + b \cdot r - k)}_{\text{Deposit Liabilities}}. \tag{2}
\]

This expression stands for the bank’s payoff conditional on a successful project completion once the cost of equity and the cost of reducing risk have been deducted. Observe that, for any level of equity capital \(k\), \(\pi(l, \cdot)\) is a strictly decreasing function of cash holdings, reflecting the fact that each unit of cash holdings carries an opportunity cost in terms of foregone loan opportunities. Also, for any liquidity level \(l\), \(\pi(\cdot, k)\) is a strictly increasing function of capital. While capital is costlier than deposits and, consequently, any additional unit of capital hurts the bank’s profits, equity capital increases the bank’s upside payoff. This feature reflects a central element of our analysis, namely that capital increases the bank’s skin-in-the-game by reducing its deposit liabilities.\(^{15}\)

Substituting out equation (2) into equation (1), we can write the bank’s objective function as:

\[
\Pi_B(\theta, l, k) = \theta \cdot F(l) \cdot \pi(l, k) - \frac{c}{2} \cdot \theta^2 - (1 + \rho) \cdot k. \tag{3}
\]

### 4.1 The bank’s profit-maximizing liquidity level

We first analyze the profit-maximizing liquidity level for a predetermined level of capital \(k\), which will serve as a benchmark to assess optimal liquidity requirements. We write \(l_B(k)\) for the bank’s profit maximizing liquidity level to highlight its dependence on the level of equity capital \(k\). The first order condition for an interior bank’s profit-maximizing level of liquidity is given by:

\[
[l] \quad g(l_B(k)) \cdot \pi(l_B(k), k) - \underbrace{(M - 1)}_{\text{Mg. Value of Surviving Early Withdrawals \ Opportunity Cost of Liquidity}} = 0, \tag{4}
\]

where \(g(l_B(k)) \equiv \frac{f(l_B(k))}{F(l_B(k))}\) stands for the reverse hazard rate of the distribution of early withdrawals, evaluated at the optimum \(l_B(k)\).

Observe from equation (3) that the bank’s choice of liquidity \(l_B(k)\) is given by the liquidity level that maximizes \(F(l) \cdot \pi(l, k)\). The marginal effect of liquidity on bank’s profits, which is given by equation (4), can be split into two components pushing in opposite directions. On the one hand, liquidity increases the probability of surviving the early withdrawal phase. On the other hand, there is an opportunity cost of liquidity: the bank foregoes a loan return of \(M\) in exchange for 1 unit of stored cash. Hence, \(F(l) \cdot \pi(l, k)\) is hump-shaped and \(l_B(k)\) is its maximizer. The

\(^{15}\)Observe that the role of capital in this model is simply to reduce the amount of deposit liabilities. Hence, capital serves the purpose of increasing the bank’s skin-in-the-game. As we shall see below, capital enhances the bank’s incentives to reduce its risk level.
assumption that the distribution of early withdrawals is log-concave ensures that \( g(\cdot) \) is strictly decreasing with an asymptote at \( l = 0 \) (Bagnoli and Bergstrom (2005)): when liquidity is very small, the likelihood of surviving the early withdrawal phase is very small and, consequently, the marginal value of holding liquidity exceeds the opportunity cost of liquidity in terms of foregone loan investments. Finally, observe that the profit-maximizing liquidity level \( l_B(k) \) is independent of the bank’s choice of insolvency risk. The reason for this is that the bank chooses its optimal liquidity level conditioning on the event that the project is successful, as it obtains no rents at the interim period \( t = 1 \) or in case the project fails.

We have argued above that equity capital increases the bank’s upside payoff. Hence, the marginal value surviving early withdrawals increases with equity capital. On the contrary, the opportunity cost of liquidity is independent of capital. Hence, the optimal level of liquidity increases with capital. We summarize this discussion in the following instrumental result, which is straightforward to derive.

**Lemma 1 (Bank’s profit-maximizing liquidity level)** For any level of capital \( k \), we have that:

(i) There exists a unique level of liquidity \( l_B(k) \) that maximizes the bank’s profits, which is given by the liquidity level that maximizes the (hump-shaped) expected bank’s upside payoff \( F(l) \cdot \pi(l,k) \).

(ii) The bank’s profit-maximizing liquidity value \( l_B(k) \) is strictly increasing in capital.

Figure 4.1 depicts the effect of capital on the bank’s choice of liquidity, which is represented on the horizontal axis. The solid and dashed decreasing lines correspond to the marginal value of surviving the early withdrawal phase \( g(l) \cdot \pi(l,k) \) for two different levels of equity capital \( k_2 > k_1 \), respectively. The marginal value \( g(l) \cdot \pi(l,k) \) of surviving the early withdrawal phase corresponds to the product of two positive strictly decreasing functions and is therefore strictly decreasing. The marginal survival value for \( k_2 \) consists of an upward shift of the marginal survival value for \( k_1 \), reflecting the fact that \( \pi(l,k_2) > \pi(l,k_1) \). The marginal value of liquidity \( M - 1 \) is independent of both liquidity and capital. It is depicted as a horizontal solid line. The profit-maximizing liquidity level \( l_B(k) \) is given by the intersection between \( g(l) \cdot \pi(l,k) \) and \( M - 1 \). Hence, we have that \( l_B(k_2) > l_B(k_1) \).
4.2 The effect of liquidity on insolvency risk

We analyze now the interaction between liquidity and the bank’s choice of insolvency risk. The first order condition for an interior solution to the bank’s insolvency risk choice $\theta_B$ is given by:

$$B(l;k) = \frac{1}{c} F(l) \cdot \pi(l,k), \quad (5)$$

where $B(l;k)$ stands for the solvency level chosen by a bank with liquidity $l$ and equity capital $k$.

Observe that the marginal return of reducing insolvency risk is given by the bank’s upside payoff $F(l) \cdot \pi(l,k)$, weighted by the factor $c$. Intuitively, the bank obtains an amount $\pi(l,k)$ if it ends up being solvent. But a precondition for the bank to be solvent is that it survives the early withdrawal phase, whose likelihood is given by $F(l)$. Consequently, the bank chooses a higher level of solvency the higher its upside payoff. Since $l_B(k)$ maximizes the bank’s upside payoff $F(l) \cdot \pi(l,k)$, it follows that $\theta_B(l,k)$ is also maximized at $l_B(k)$. The following result is immediate in the light of Statement (i) on Lemma 1.

**Proposition 1 (Liquidity and insolvency risk)** For any given level of equity capital $k$, the bank’s choice of solvency $\theta_B(l,k)$ is a hump-shaped function of its liquidity holdings $l$, and is maximized at the bank’s profit-maximizing liquidity level $l_B(k)$.

Figure 4.2 illustrates Proposition 1. The dashed line corresponds to the bank’s upside payoff $F(l) \cdot \pi(l,k)$ for a given level of capital $k$. As argued in Lemma 1, this function is hump-shaped.

$^{16}$In order to ensure that $\theta_B(\cdot,\cdot)$ is interior, so that the regulation problem is interesting, we make an assumption on the range of $c$ (see Appendix A).
in liquidity. The solid line represents the optimal solvency level, which is a scaled (down, by $c$) transformation of the bank’s upside payoff.

We shall see below that liquidity requirements exceed the bank’s profit maximizing level, which is the level that a bank would choose in the absence of liquidity regulation. Hence, liquidity requirements will induce a higher level of insolvency risk. The following corollary highlights the fact that an excessive value of liquidity harms solvency.$^{17}$

**Corollary 1 (Binding liquidity requirements increase insolvency risk)** Any level of bank’s liquidity in excess of the bank’s profit-maximizing liquidity level $l_B(k)$ induces a higher bank’s choice of insolvency risk than in the absence of liquidity regulation. Formally, $\frac{\partial \theta_B(l, k)}{\partial l} < 0$ for any $l > l_B(k)$.

### 4.3 The effect of equity capital on insolvency risk

We now turn into the analysis of the effect of capital on the bank’s choice of solvency. We first state the result, which follows immediately from differentiating equation (5), and then provide intuition for it.

**Proposition 2 (Equity capital and insolvency risk)** For any level of liquidity $l$, solvency $\theta_B(l, k)$ is a strictly increasing function of equity capital $k$.

$^{17}$This result is reminiscent of the result found by König et al. (2015), who shows that liquidity may harm insolvency risk.
As argued above, equity capital increases the bank’s upside payoff \( \pi(l, k) \). Hence, the bank’s marginal return of increasing its level of solvency, which is given by \( F(l) \cdot \pi(l, k) \), is larger the higher the amount of equity capital. As a consequence, an elevation of equity capital leads to an increase in the solvency level.

### 4.4 Capital and the effect of liquidity on insolvency risk

We have seen above (Corollary 1) that liquidity in excess of the bank’s profit-maximizing level of liquidity \( l_B(k) \) leads to a reduction of solvency, because it harms the bank’s profits. We shall see below that liquidity requirements will be set in excess of the bank’s profit-maximizing liquidity level. Hence, optimal liquidity requirements will harm solvency. The following result states that capital mitigates this effect. This result plays an essential role in the determination of the joint optimal capital and liquidity requirements.

**Proposition 3 (Impact of equity capital on effect of liquidity on solvency)** Equity capital reduces the (negative) effect of excessive liquidity on solvency. That is, for any \( l > l_B(k) \), \( \left| \frac{\partial \theta_B(l, k)}{\partial l} \right| \) is strictly decreasing in equity capital \( k \).

In order to provide intuition for this result, consider the effect of liquidity on solvency, which is proportional to the first order condition of the bank’s problem with respect to liquidity, as stated in equation (4):

\[
\frac{\partial \theta_B(l, k)}{\partial l} \propto \frac{g(l) \cdot \pi(l, k)}{M} - \frac{(M - 1)}{\text{Opportunity Cost of Liquidity}} < 0 \text{ for } l > l_B(k).
\]

Observe that equity capital increases the marginal value of surviving early withdrawals without affecting the opportunity cost of liquidity. Hence, increasing capital makes the (negative) effect of excess liquidity on solvency less negative.

Figure 4.4 depicts the bank’s choice of solvency as a function of its liquidity holdings for two levels of capital (the upper curve for a higher level of capital than the lower curve, i.e., \( k_2 > k_1 \)). The picture illustrates the three effects of equity capital on illiquidity and insolvency risk stated above. First, an increase in equity capital shifts the bank’s profit-maximizing level of liquidity to the right (Lemma 1, Statement (ii)), that is, \( l_B(k_2) > l_B(k_1) \). Second, an increase in capital leads to an upward shift of the solvency curve (Proposition 2), that is, \( \theta_B(l, k_2) > \theta_B(l, k_1) \) for any \( l \). Finally, the upper solvency curve is flatter than the lower solvency curve to the right of the profit-maximizing level of capital (Proposition 3), that is, \( \left| \frac{\partial \theta_B(l, k_2)}{\partial l} \right| < \left| \frac{\partial \theta_B(l, k_1)}{\partial l} \right| \) for all \( l > l(k_1) \).
5 Regulation of liquidity and capital

We now proceed to the analysis of the optimal regulation of capital and liquidity. In particular, we analyze the problem of a regulator that sets capital and liquidity requirements at $t = -1$ so as to maximize social welfare. More precisely, the regulator sets up a minimum liquidity and capital requirement pair ($l^*, k^*$) so as to solve the following problem:

$$\max_{l \in [0,d], k \in [0,1-d]} \Pi_R (l, k)$$

$$s.t. \quad b + k = 1 - d$$

$$\Pi_B (\theta_B (l,k), l,k) \geq 0$$

where the regulator’s objective function is given by:

$$\Pi_R (l,k) \equiv \int_0^l \theta_B (l,k) \cdot M \cdot (1-l) \cdot f(\beta) d\beta + l - (1+b \cdot r - k) - \frac{c}{2} \cdot \theta_B^2 (l,k) - (1+\rho) \cdot k. \quad (6)$$

First, observe that the level of solvency is not verifiable by the regulator and is therefore chosen by the bank. Hence, the regulator must take the bank’s self-enforcing reaction to liquidity and capital requirements $\theta_B (l,k)$ as given. The first constraint follows from our assumption that the regulator cannot change the scale of the bank’s resources, which we have normalized to 1. Hence, $b$ is determined by the choice of $k$. The second constraint is the participation constraint for the bank, which has to obtain non-negative profits. Assumption 2, laid out in Appendix A, guarantees that the bank has non-negative profits. Hence, we can prescind of the bank’s participation constraint,
which we omit henceforth.

5.1 The role of regulation

For the remainder of the paper it will be useful to write the bank’s (net) deposit liabilities, which play a central role in the design of the optimal regulatory scheme, as follows:

\[
D(l,k) \equiv 1 + b \cdot r - k - l.
\]  

(7)

The bank’s liabilities consist of an amount \(1 - b - k\) of common deposits, as well as an amount \(b \cdot (1 + r)\) of long-term deposits. Hence, \(D(l,k)\) stands for the amount of loan earnings that the bank will have to allocate to paying its deposit liabilities in case it succeeds—observe that the amount of liquid assets \(l\) that the bank uses to meet early withdrawals can also be used at \(t = 2\) and is therefore fully deducted from its deposit liabilities.

There is a natural alternative interpretation for \(D(l,k)\) as the deposit insurance loss in case of a bank failure: \(D(l,k)\) represents the depositors’ claims that the bank cannot meet if it fails. This expression is core in our analysis. Indeed, combining expressions (1) and (6), the former evaluated at the bank’s optimal choice of solvency, we can write the regulator’s objective function as

\[
\Pi_R(l,k) = \Pi_B(l, \theta_B(l,k), k) - [1 - \theta_B(l,k) \cdot F(l)] \cdot D(l,k).
\]  

(8)

Regulation is needed because the bank does not internalize the harm that its potential failure inflicts on the deposit insurance scheme, which is captured by the second term in expression (8). Put differently, since the bank is protected by limited liability, it does not face the downside of a failure. On the contrary, society experiences a welfare loss of \(D(l,k)\) when the bank cannot meet its deposit obligations, an event which occurs with probability \(1 - \theta_B(l,k) \cdot F(l)\).

In a (first-best) world in which the bank’s choice of insolvency risk were verifiable, the regulator would set a solvency requirement in excess of the bank’s laissez-faire choice precisely because the bank does not internalize the social loss following a failure. In order to see that elevating the level of solvency is welfare-enhancing, we can differentiate expression (8) and write the effect of the solvency level on the regulator’s objective function as:

\[
\frac{\partial \Pi_R(l,k_R(l))}{\partial \theta_B} = \frac{\partial \Pi_B(l, \theta_B(l,k), k)}{\partial \theta_B} + F(l) \cdot D(l,k) > 0. 
\]  

(9)

The first term in expression (9) captures the effect of increasing solvency on the bank’s profit. This term is zero, because it corresponds to the bank’s first order condition for an optimal level of
solvency. Loosely speaking, the bank takes care of this portion of the effect. The second term in expression (9) is positive, reflecting the fact that the bank chooses too low a solvency level because it does not internalize the social loss inflicted on the deposit insurance scheme when it fails.

Capital requirements would not play any role in this framework and would be set to zero. However, in a (second-best) world in which the choice of insolvency risk is not verifiable, capital requirements induce a reduction of insolvency risk through increasing the bank’s skin in the game, as seen above (Proposition 2). The purpose of capital regulation, which entails the substitution of a valuable part of the depositors’ base, is therefore to reduce the level of insolvency risk beyond the bank’s will.

Liquidity requirements increase the bank’s resilience against the advent of excessive early withdrawals. In the absence of regulation, the bank would hold the (the profit-maximizing) liquidity level \( l_B(k) \) derived above (Lemma 1 (i)). In trading off illiquidity risk and investment in loans, the bank’s laissez-faire choice \( l_B(k) \) ignores the expected (social) cost of a potential bank failure captured by the second term in expression (8). Liquidity requirements are set taking this additional effect into account. The reason why liquidity and capital must be set jointly is because the effectiveness of one regulatory tool depends on the use of the other.

In what follows, we solve Problem (R). In order to get some insight on the effect of liquidity on the effectiveness of capital, in the following subsection we construct the optimal capital response curve \( k_R(l) \), which establishes a scheme of optimal capital requirements for any given level of liquidity \( l \). In Subsection 5.3 we assess the effect of capital on the effectiveness of liquidity. With this purpose, we construct the optimal liquidity response curve \( l_R(k) \), which maps each level of capital \( k \) to its corresponding optimal liquidity requirement. In Subsection 5.2 we put them together to establish the joint optimal liquidity and capital requirements \( (l^*, k^*) \), which is the unique pair satisfying \( l^* = l_R(k^*) \) and \( k^* = k_R(l^*) \). We discuss the implications of our analysis in Subsection ??, where we argue whether liquidity and capital are complementary or offsetting tools.

### 5.2 Optimal capital response curve

In this section, we construct the optimal capital response curve \( k_R(l) \), which maps each liquidity level \( l \) to an optimal capital requirement. Differentiating the regulator’s objective function with
respect to capital, we have that an interior optimal capital response curve must satisfy: \(^\text{18}\)

$$
\frac{d\Pi_R (l, k_R (l))}{dk} = \frac{\partial \Pi_R (l, k_R (l))}{\partial \theta_B(k)} \cdot \frac{\partial \theta_B (l, k_R (l))}{dk} - \underbrace{(\rho - r)}_{Direct \ Effect:<0} = 0.
$$

On the one hand, there is a (negative) direct effect of capital of imposing capital requirements, which is given by the shadow cost difference \(\rho - r > 0\) of substituting deposits by equity capital. In addition, there is a (positive) indirect effect of capital on social welfare, which follows from the fact that capital increases the level of solvency beyond the inefficiently low level that the bank would set in the absence of a capital requirement.

Recall from Expression (9) that \(\frac{\partial \Pi_R (l, k_R (l))}{\partial \theta_B(k)} > 0\). Combining this observation with Proposition 2, which states that equity capital induces a higher level of solvency, that is, \(\frac{\partial \theta_B (l, k_R (l))}{dk} > 0\), we have that the indirect effect of capital on social welfare is positive.

Liquidity does not interfere with the direct effect, but it does have an impact on the indirect effect. Hence, the optimal capital response depends on the liquidity level. The following proposition characterizes the optimal capital response curve \(k_R (l)\).

**Proposition 4 (Optimal capital response curve)** There exists a cost of capital \(\rho_{\text{max}}\) (whose value is determined on Definition 3 in Appendix A) such that:

(i) (Zero capital requirement when capital too costly) For any \(\rho \geq \rho_{\text{max}}\), the optimal capital response is \(k_R (l) = 0\) for any \(l\).

(ii) (Capital response curve hat-shaped) For any \(\rho < \rho_{\text{max}}\), there exist liquidity levels \(0 < l_1 (\rho) < \hat{l} (\rho) < l_2 (\rho) \leq d\) such that the optimal capital response curve is hump-shaped for \(l \in (l_1 (\rho), l_2 (\rho))\), attaining a maximum at \(\hat{l} (\rho)\), and zero otherwise, that is:

\[
k_R (l) : \begin{cases} 
= 0 & \text{for } l \leq l_1 (\rho) \\
\text{Increases} & \text{for } l_1 (\rho) < l < \hat{l} (\rho) \\
\text{Decreases} & \text{for } \hat{l} (\rho) < l \leq l_2 (\rho) \\
= 0 & \text{for } l > l_2 (\rho)
\end{cases}
\]

(iii) (Capital response independent of \(M\) and "larger" as \(\rho\) decreases) As \(\rho\) decreases, we have that \(l_1 (\rho)\) shifts to the left, \(l_2 (\rho)\) shifts to the right (as long as \(l_2 (\rho) < d\)) and \(k_R (l)\) larger for any \(l \in (l_1 (\rho), l_2 (\rho))\). Moreover, \(k_R (l)\) is invariant in \(M\).

**Proof.** See Appendix C. \[\blacksquare\]

\(^{18}\)Below, we find the conditions under which the optimal capital requirement is not interior.
The intuition behind this result is as follows. For notational simplicity, we write the indirect effect of capital referred to in Expression (10) as follows:

\[ IEK(l, k) \equiv \frac{\partial \Pi_R(l, kR(l))}{\partial \theta_B} \cdot \frac{\partial \theta_B(l, kR(l))}{\partial k} = F^2(l) \cdot D(l, k) \cdot \frac{1 + r}{c}. \]

Capital increases social welfare because it induces the bank to adopt a higher level of solvency which, in turn, increases social welfare. In particular, the optimal capital requirement must be set at the level that equalizes the indirect marginal return of capital \( IEK(l, k) \) to the shadow cost \( \rho - r \) of substituting depositors by equityholders. When the cost of capital is very high (\( \rho \geq \rho_{\text{max}} \)) the indirect marginal return of capital lies below the shadow cost of capital, so that the optimal capital response is zero for all liquidity levels.

Now consider the case in which capital is not too costly, that is, consider a value of the cost of capital such that \( \rho < \rho_{\text{max}} \). First, notice that \( D(l, k) \) is strictly decreasing in capital: substituting depositholders by equityholders reduces the bank’s net deposit liabilities.\(^{19}\) Consequently, \( IEK(l, k) \) is strictly decreasing in capital as well. Therefore, the marginal return of capital is maximized when \( k = 0 \). Now, observe that when liquidity is low, \( IEK(l, 0) \) is small: when the probability of surviving the early withdrawal phase \( F(l) \) is small, the expected return of increasing solvency is small and therefore the return of capital is small as well. On the other extreme, when liquidity is high, the term \( IEK(l, 0) \) is also small: when the value of the deposit insurance loss in case of failure \( D(l, 0) \) is small, the wedge between the bank and the regulator’s objective functions is small as well. Consequently, we have that when liquidity is either low or high (formally, either when \( l \leq l_1(\rho) \) or when \( l \geq l_2(\rho) \)), the optimal capital response is zero, because the marginal social return of the first unit of capital is smaller than the shadow cost of substituting depositors by equityholders.

For intermediate values of liquidity, for which the return of the first unit of capital exceeds its opportunity cost, it is optimal to set a positive capital response. In this case, in the liquidity range \( l \in (l_1(\rho), l_2(\rho)) \), the optimal capital requirement is given by:

\[ IEK(l, k) = \rho - r. \] (11)

The capital response curve is hump-shaped in \( l \in (l_1(\rho), l_2(\rho)) \). Intuitively, \( IEK(l, k) \) is hump-shaped because there are two effects of liquidity in the return of capital pushing in opposite directions. As liquidity increases, the probability of surviving excessive early withdrawals increase. Hence, the probability of reaching the loan maturity phase—which is the time at which increasing solvency matters—increases. However, as liquidity increases, the wedge between the bank and the

\(^{19}\) Notice that we can write \( D(l, k) = D(l, 0) - k \).
regulator’s objective functions shrinks, making regulation less effective. Our assumption that $F(\cdot)$ is log-concave ensures that when liquidity is low the first effect dominates, while when liquidity is large the second one is the predominant one.

So far, we have conducted this analysis fixing the cost of capital $\rho$. As $\rho$ decreases, the incremental cost of capital $\rho - r$ is reduced. A glance at equation (11) reveals that cheaper capital leads to an upward shift of the optimal capital response curve in the range in which is positive: since the right-hand-side diminishes, the capital response must be increased so as to reduce the left-hand-side as well. Moreover, the minimum liquidity level $l_1(\rho)$ for which the marginal return of the first unit of capital exceeds the incremental cost of capital, which satisfies $IEK(l_1(\rho), k) = \frac{(\rho - r)c}{1 + r}$ decreases as well. If $l_2(\rho)$ is interior, then it satisfies the same condition as $l_1(\rho)$, that is, $IEK(l_2(\rho), k) = \frac{(\rho - r)c}{1 + r}$.

Then, a reduction of the cost of capital leads to an increase in the maximum level of liquidity $l_2(\rho)$ for which there is a positive capital requirement. If, on the contrary, we have that $IEK(d, k) = \frac{(\rho - r)c}{1 + r}$, then the capital requirement is positive for the largest possible value of liquidity, that is, $l_2(\rho) = d$. In this case, $l_2(\rho)$ does not change as $\rho$ decreases. Finally, $\hat{l}(\rho)$ is decreasing in $\rho$ because the marginal social return of capital $IEK(l, k)$ is decreasing in capital. Hence, as capital increases, the maximum of the marginal social return of capital decreases.

### 5.3 Optimal liquidity response curve

In this section, we construct the optimal liquidity response curve $l_R(k)$, which maps each capital level $k$ to an optimal liquidity requirement. Differentiating the regulator’s objective function with respect to liquidity, we have that an interior liquidity requirement satisfies:

$$\frac{d\Pi_R(l_R(k), k)}{dl} = \frac{\partial\Pi_R(l_R(k), k)}{\partial l} + \frac{\partial\Pi_R(l_R(k), k)}{\partial \theta_B} \frac{\partial \theta_B}{\partial l} = 0. \quad (12)$$

We can split the effect of liquidity on social welfare into a direct and an indirect effect. The direct effect of liquidity captures the effect of liquidity on social welfare ignoring the effect of liquidity on the bank’s choice of insolvency risk. The indirect effect of liquidity accounts for the (negative) impact of liquidity on the objective function through its influence on insolvency risk. As we shall see, capital influences both effects. Hence, the optimal liquidity requirement depends on the level of capital. The following proposition characterizes the optimal liquidity response curve $l_R(k)$.

---

$^{20}$As we shall see below, the optimal liquidity requirement is always interior.
Proposition 5 (Optimal liquidity response curve) (i) (Liquidity requirements binding) The liquidity requirement is binding for the bank for any level of capital, that is, $l_R(k) > l_B(k) > 0$ for all $k$.

(ii) (Liquidity response either hump-shaped or increasing) There exists $\bar{M} > M$ such that:

(ii.1) If $M < \bar{M}$, the liquidity response $l_R(k)$ is a hump-shaped function of capital, that is, there exists a capital threshold $\hat{k}(M) < 1 - d$ such that:

$$l_R(k) : \begin{cases} \text{Increases} & \text{for } k < \hat{k}(M) \\ \text{Decreases} & \text{for } k > \hat{k}(M) \end{cases}.$$

(ii.2) If $M \geq \bar{M}$, the liquidity response $l_R(k)$ is strictly increasing in capital.

(iii) (Liquidity response independent of $\rho$ and "smaller" as $M$ increases) For any $M' > M$, we have that $l_R(k) \mid_{M'} < l_R(k) \mid_{M}$. Moreover, $l_R(k)$ is invariant in $\rho$.

Proof. See Appendix C. ■

In order to provide some intuition for this result, we first construct the "direct-effect liquidity response" $l_D(k)$. The liquidity scheme $l_D(k)$ corresponds to the liquidity response curve that would be set by a regulator that ignored the (negative) indirect effect of liquidity on solvency level. Hence, for any given value of capital $k$, the "direct-effect liquidity response" curve $l_D(k)$ satisfies:

$$\frac{\partial \Pi_R (l_D(k), k)}{\partial l} = \left( \frac{1}{\text{Cash-in-hand}} - \frac{\theta_B (l_D(k), k) \cdot M \cdot F (l_D(k)) \cdot [g(l_D(k)) (1 - l_D(k)) - 1]}{\text{Opportunity Cost}} \right) = 0. \quad (13)$$

The direct effect of liquidity on the social value is given by two factors. On the one hand, liquidity constitutes a cash-in-hand asset that reduces the bank’s net deposit liabilities $D(l, k)$ regardless of the loan performance. The marginal value of cash-in-hand is therefore 1. On the other hand, storing cash has an opportunity cost in terms of foregone loans. Observe that an "expected loan maximizer bank" (ELVB) would choose a liquidity level $l_{ELVB}$ satisfying $g(l_{ELVB}) (1 - l_{ELVB}) = 1$.\footnote{An "expected loan maximizer bank" (ELVB) would choose $l$ so as to maximize the expected value of its loan portfolio, that is: $l_{ELVB} = \arg \max_{l \in [0, d]} \theta \cdot F(l) \cdot M \cdot (1 - l)$. Observe that the bank that we are modelling maximizes its expected profits which, in addition to loans, include its liabilities, that is: $l_B(k) = \arg \max_{l \in [0, d]} \theta \cdot F(l) \cdot \pi(l, k)$.} Hence, any liquidity level in excess of $l_{ELVB}$ contributes negatively to the expected loan value.

We now argue how capital affects the "direct-effect liquidity response" curve $l_D(k)$. Capital increases the probability of a loan success by the skin-in-the-game effect on the bank’s choice of solvency (first term of Equation (13)). However, it does not affect the cash-in-hand effect (second term of Equation (13)). Hence, capital increases the relative weight of the expected loan value.
versus the cash-in-hand effect: the more resilient the banking system, the higher the opportunity cost of storing liquidity. Hence, as capital increases, \( l_D(k) \) comes closer to \( l_{ELVB} \). Consequently, the "direct-effect liquidity response" curve \( l_D(k) \) is strictly decreasing in capital \( k \).

Nonetheless, for any given \( k \), the "direct-effect liquidity requirement" \( l_D(k) \) lies above the bank’s profit-maximizing liquidity level \( l_B(k) \), that is, \( l_D(k) > l_B(k) \). The reason is that the bank does only value the cash-in-hand asset in the event of a success. Hence, the cash-in-hand factor in the bank’s optimality condition is weighted by \( \mathbb{F}(l_B(k)) < 1 \).

Consider now the indirect effect of liquidity, which we can write as:

\[
IEL(l,k) \equiv \frac{\partial \Pi_R(l,k)}{\partial \theta_B(l,k)} \cdot \frac{\partial \theta_B(l,k)}{\partial l} = [F(l) \cdot D(l,k)] \cdot \frac{\partial \theta_B(l,k)}{\partial l} < 0 \text{ iff } l > l_B(k) .
\]  

From Lemma 1, insolvency risk is minimized at the bank’s profit-maximizing liquidity level \( l_B(k) \). Hence, we have that \( \frac{\partial \theta_B(l_B(k),k)}{\partial l} < 0 \) for any \( l > l_B(k) \) and, in particular, for \( l_D(k) \). Moreover, recall from Equation (9) that an increase in solvency is welfare enhancing. Hence, the indirect effect of liquidity on social welfare is negative, reflecting the fact that liquidity requirements above the bank’s profit-maximizing liquidity level reduce the bank’s loan investments and therefore harm its incentives to reduce insolvency risk. As a consequence, the optimal liquidity requirement must be smaller than the a "direct-effect liquidity requirement" would be, that is, \( l_R(k) < l_D(k) \).

Nonetheless, the indirect effect does not completely offset the direct effect. In order to see why, observe that the indirect effect diminishes as liquidity reduces and it completely vanishes at the bank’s profit-maximizing liquidity level \( l_B(k) \), since \( \frac{\partial \theta_B(l_B(k),k)}{\partial l} = 0 \). Hence, when \( l = l_B(k) \) the indirect effect is zero, while the direct effect is positive. Therefore, we have that the optimal liquidity response does always constitute a binding requirement for the bank, that is:

\[
l_B(k) < l_R(k) < l_D(k) .
\]  

Capital does therefore exert two opposing effects over the optimal liquidity response curve \( l_R(k) \). As capital increases, the negative effect of liquidity on welfare diminishes. The reason is twofold. First, because \( D(l,k) \) gets reduced, so that the effect of the level of solvency on welfare diminishes. Second, because the negative effect of liquidity on solvency also reduces ("slope effect" of Proposition 1 displayed in Figure 4.4). Hence, as capital increases, liquidity requirements must increase. However, as these effects vanish out, an additional effect pushing in the opposite direction emerges. As capital increases, the bank becomes more solvent ("level effect" of Proposition 2 displayed in Figure 4.4). Consequently, the opportunity cost of liquidity in terms of foregone loans increases. If this "opportunity cost" effect gets to dominate, the optimal liquidity response curve \( l_R(k) \) eventually decreases. However, the "opportunity cost" effect becomes strong enough so as to eventually dominate only if the project profitability \( M \) is not too large.
The effect of the project profitability $M$ on the "opportunity cost" effect can be seen in Equation (13). The project profitability $M$ constitutes a weighting factor of the expected loan value versus cash-in-hand—the higher the project profitability, the higher the opportunity cost of liquidity. As the project profitability $M$ increases, liquidity requirements approach the liquidity level $l_{ELVB}$ that an "expected loan maximizer bank" (ELVB) would choose for any level of capital. Hence, the first term on Equation (13) becomes less responsive to an elevation of capital as $M$ increases. When $M$ is sufficiently large the optimal liquidity response function $l_R(k)$ is always increasing in $k$.

Figure 5.3 depicts the relationship between several liquidity measures for $M < \bar{M}$. On the one hand, we have the bank’s profit maximizing liquidity level $l_B(k)$ which, as stated in Lemma 1 is strictly increasing in capital. The upper line corresponds to the direct-effect optimal liquidity response $l_D(k)$, which corresponds to the optimal liquidity response curve that would be set by a regulator that ignored the indirect effect of liquidity on the solvency level. As argued above, this liquidity measures is strictly decreasing in capital. As stated in Proposition 5, the regulator’s liquidity response $l_R(k)$ is a hump-shaped function of capital with a maximum at some capital level $\hat{k}(M)$. Also, as specified by the set of inequalities 15, $l_R(k)$ lies above $l_B(k)$ (i.e., the liquidity requirement will bind) and below $l_D(k)$ (i.e., the indirect effect of liquidity pushes the optimal amount of liquidity requirements down). The indirect effect of liquidity (through its negative effect on solvency) is simply the difference between the direct effect and the overall effect of liquidity. From 3 we have that the negative effect of liquidity on solvency is mitigated as capital increases. Hence, $l_D(k) - l_R(k)$ is strictly decreasing in capital.
5.4 Joint capital and liquidity regulation

We are left with determining how capital and liquidity should be set together. In this section, we show that there is a unique pair of liquidity and capital requirements \((l^*, k^*)\) such that the liquidity requirement is optimal given the capital requirement, i.e., \(l^* = l_R(k^*)\) and, conversely, the capital requirement is optimal given the liquidity requirement, that is, \(k^* = k_R(l^*)\). Moreover, we address the issue of whether capital and liquidity requirements are complements or substitutes depending on the cost of capital and the bank’s profitability. We state the main result of the paper in the following proposition and provide intuition for this result below. Abusing notation, we let \(l^*(\rho, M)\) and \(k^*(\rho, M)\) stand for the optimal liquidity and capital requirements, respectively, for a given cost of capital \(\rho\) and return to investment \(M\).

Proposition 6 (Optimal joint liquidity and capital requirements) For any given cost of capital \(\rho\) and return to investment (opportunity cost of liquidity) \(M\), we have that:

(i) (Uniqueness) There exists a unique pair of liquidity and capital requirements \((l^*, k^*)\).

(ii) (Comparative statics)
   (a) (Zero capital requirements for high cost of capital) There exists \(\bar{\rho}(M) < \rho_{\text{max}}\) such that for any \(\rho > \bar{\rho}(M)\), the capital requirement is zero.
   (b) (Raising the cost of one factor reduces the requirement of that factor) The optimal liquidity requirement \(l^*(\rho, M)\) is a strictly decreasing function of \(M\). The optimal capital requirement \(k^*(\rho, M)\) is a decreasing function of \(\rho\) (strictly decreasing if and only if \(\rho < \bar{\rho}(M)\)).
   (c) (Capital and liquidity complementary tools for low cost of capital and offsetting tools for high cost of capital) There exists a threshold \(\hat{\rho}(M) < \bar{\rho}(M)\) for the cost of capital such that:
      If \(\rho \in (\bar{\rho}(M), \hat{\rho}(M))\), then capital and liquidity requirements are complementary tools: a raise of the cost of capital \(\rho\) leads to an elevation of the liquidity requirement \(l^*\); and a raise of the opportunity cost of liquidity \(M\) leads to an elevation of the capital requirement \(k^*\).
      If \(\rho \in (\rho_{\text{min}}, \hat{\rho}(M))\), then capital and liquidity requirements are offsetting tools: a raise of the cost of capital \(\rho\) leads to a reduction of the liquidity requirement \(l^*\); and a raise of the opportunity cost of liquidity \(M\) leads to a reduction of the capital requirement \(k^*\).
   (d) (Capital and liquidity are always complements when the return to investment is high) The threshold \(\hat{\rho}(M)\) that determines whether capital and liquidity are complementary or offsetting tools is strictly decreasing on the return to investment \(M\). Moreover, there exists a threshold \(\hat{M}\) such that \(\hat{\rho}(\hat{M}) = \rho_{\text{min}}\). Consequently, for any \(M > \hat{M}\), capital and liquidity are complements for any cost of capital \(\rho < \hat{\rho}(M)\).
   (e) (Zero capital requirements for high return to investment) The capital cost threshold \(\bar{\rho}(M)\) that establishes the maximum cost of capital for which capital requirements are positive decreases as
the return to investment enlarges, that is, \( \bar{\rho}(M) \) is strictly decreasing in \( M \).

**Proof.** See Appendix C. ■

Figure 5.4 may help understand the intuition behind Proposition 6. In these figures, liquidity is depicted in the horizontal axis and the optimal capital response curve is drawn against the horizontal axis (that is, as functions are typically depicted). Capital is represented on the vertical axis. We draw the optimal liquidity response curve as a function that maps a certain capital level to a unique optimal liquidity requirement, that is, we invert the axes and plot the curve as a function of the variable represented on the vertical axis. From the perspective of the horizontal axis, the optimal liquidity response curve is a correspondence, mapping each liquidity level to the (potentially two) capital level(s) for which that particular liquidity level constitute an optimal liquidity response.

Consider first the liquidity requirement curve \( l_R(k)|_{M_{\text{Low}}} \) for a given (low) opportunity cost of liquidity \( M_{\text{Low}} \). This curve intersects the capital requirement curves \( k_R(l)|_{\rho} \), which represent the optimal capital response for several values \( \rho_1 > \rho_2 > \hat{\rho}(M_{\text{Low}}) > \rho_3 \) of the cost of capital. The jointly determined capital and liquidity requirement for a given cost of capital \( \rho \) is given by the intersection of the liquidity requirement curve \( l_R(k)|_{M_{\text{Low}}} \) and the corresponding capital requirement curve \( k_R(l)|_{\rho} \). Notice that an increase in the cost of capital \( \rho \) would shift the capital requirement curve \( k_R(l)|_{\rho} \) down-and-rightwards, as seen in Proposition 4. However, the cost of capital does not affect the liquidity requirement curve \( l_R(k)|_{M_{\text{Low}}} \). On the contrary, an increase in the opportunity cost of liquidity \( M \) leads to a downward (leftward, in the picture) shift of the liquidity requirement curve \( l_R(k)|_{M} \), leaving the capital requirement curve \( k_R(l)|_{\rho} \) unchanged. Consequently, increasing the cost of capital leads to a reduction of a positive capital requirement. Also, an increase in the opportunity cost of liquidity leads to a reduction of the liquidity requirement.
When the cost of capital is large ($\rho_1 > \hat{\rho}$), the curve $l_R (k) |_{M_{low}}$ intersects $k_R (l) |_{\rho_1}$ at its (zero) flat portion. Hence, the capital requirement $k^* (\rho_1, M_{Low})$ is zero. Moreover, an increase of the cost of capital $\rho$ would leave capital requirements unchanged at zero, as the liquidity and capital requirement curves would continue to intersect at the flat segment of the capital response curve. The next pair of capital requirements corresponds to a cost of capital $\rho_2 \in (\hat{\rho} (M_{Low}), \bar{\rho})$, which is smaller than $\rho_1$. The optimal capital and liquidity response curves intersect at a point where both curves are upward sloping. In this range, capital and liquidity are complements: an increase of either the cost of capital $\rho$ or of the opportunity cost of liquidity $M$ would lead to a decrease of both the capital and the liquidity requirement. The cost of capital that determines whether capital and liquidity are complements or substitutes is $\hat{\rho} (M_{Low})$, which corresponds to the cost of capital for which both curves intersect at their respective peaks. The last capital response curve, for $\rho_3 < \hat{\rho} (M_{Low})$, depicts a case in which capital and liquidity are substitutes: any change in the cost of one of the factors leads to opposite movements of the optimal capital and liquidity requirements.

The curve $l_R (M_{High})$ in Figure 5.4.BIS corresponds to the optimal liquidity response curve for a higher return to investment. This curve lies below the curve $l_R (M_{Low})$–to the left, in the picture–representing the fact that the optimal liquidity response is strictly smaller for any level of capital the higher the return to investment, which represents the opportunity cost of liquidity. Moreover, we have depicted this curve for a value of $M$ exceeding $\bar{M}$, which is defined as $\hat{\rho} (\bar{M}) = \rho_{\text{min}}$. In this case, the optimal liquidity response is increasing for all capital levels. Formally, we have that the capital level for which the optimal liquidity response achieves a maximum is beyond the maximum...
possible capital requirement, that is, \( \hat{k}(M_{High}) > 1 - d \). This capital level is never an optimal capital requirement due to our assumption that the minimum cost of capital is at least as large as \( \rho_{\text{min}} \), which is precisely set at the value for which the optimal capital requirement hits the boundary, that is, \( k^*|_{\rho_{\text{min}}} = 1 - d \). When the return to investment is sufficiently large \( (M > \bar{M}) \), capital and liquidity are always complements, as long as the cost of capital is low enough so as to have a positive capital requirement (i.e., \( \rho < \bar{\rho}(M) \)). The reason is that the optimal liquidity response for any given capital level is lower the higher the opportunity cost of liquidity. Therefore, as the opportunity cost of liquidity increases, the optimal liquidity response gets closer to the liquidity level value that maximizes the expected loan value, so that the weight of the first term of expression (13)—which measures the effect of the loan value on social welfare and drives liquidity down as it increases—, gets smaller. Hence, the capital level \( \hat{k}(M) \) for which the optimal liquidity response attains a maximum is increasing in \( M \).

Finally, the return to investment does also affect the range for which the optimal capital requirement is positive. In order to see why, consider a given opportunity cost of liquidity \( M \). The range for which the optimal capital response curve is positive is given by the liquidity segment \((l_1(\rho), l_2(\rho))\), which depends on the cost of capital \( \rho \). The capital requirement is positive if and only if the optimal liquidity response when capital is zero exceeds the minimum liquidity level for which the optimal capital response is positive, that is, if and only if \( l_R(0)|_M > l_1(\rho) \). Otherwise, if \( l_R(0)|_M \leq l_1(\rho) \), we have that the optimal requirements are given by \( l^* = l_R(0)|_M \) and \( k^* = 0 \). Therefore, since \( l_1(\rho) \) decreases in \( \rho \), we have that for each \( M \) there exists \( \bar{\rho}(M) \) such that the capital requirement is positive if and only if \( \rho < \bar{\rho}(M) \).

6 Regulation with a Lender of Last Resort

So far, we have assumed that the bank fails if the withdrawal of deposits at \( t = 1 \) exceeds the amount of liquid reserves that the bank keeps. On the other side of the spectrum, we now analyze the optimal joint regulation of capital and liquidity when, in addition to these regulatory tools, there
exists a Lender of Last Resort (LoLR) that is instructed to rescue a solvent bank facing a liquidity shortage at \( t = 1 \). Hence, there is no liquidity risk in this setup. We shall show that the price of LoLR funds can be used as a complementary tool to soften capital and liquidity requirements. Moreover, we identify the conditions for liquidity regulation even in the absence of liquidity risk.

As in Rochet and Vives (2004), we assume that the LoLR can observe whether the bank is solvent or not if called upon to intervene and will only assist the bank if it is solvent.\(^{22}\) In order to avoid accessory complications, we assume that the LoLR can perfectly observe the amount of early withdrawals at the bank, so that there is perfect information about the amount of funds needed by the bank. We also assume that the LoLR has access to an unlimited amount of funds, but that a unit of funds employed in rescuing a bank carries a shadow cost \( \lambda \in \left[1, \frac{M}{d}\right] \), which represent the opportunity cost of providing public funds. This opportunity cost can be thought of as the return from the next best alternative for the funds or as the market distortions induced from levying resources, such as imposing taxation, printing money or issuing public debt. The lower bound \( \lambda = 1 \) stands for the case in which there is no alternative use of the LoLR funds and these can be costlessly levied by the LoLR. In turn, we set the upper bound of the shadow cost of funds so as to guarantee that it would always be socially optimal to rescue a solvent but illiquid bank regardless of the size of the rescue.\(^{23}\) As we shall see below, the optimal requirements vary drastically with the shadow cost of funds \( \lambda \). This aspect may have important implications in the determination of the optimal regulatory policies when the LoLR access to funds is (or not) restricted by, for instance, a small fiscal capacity or a burgeoning public debt.

The problem of the regulator at \( t = -1 \) consists of setting a triplet \( (l^l_{\text{LoLR}}, k^l_{\text{LoLR}}, Q^l_{\text{LoLR}}) \) of liquidity and capital requirements, as well as a price per unit of funds borrowed by the bank in case it falls short of liquid assets at \( t = 1 \). In the event that the amount of early withdrawals exceeds the liquidity reserves held by the bank, the bank may seek assistance from the LoLR. If the LoLR rescues the bank, which will occur with probability \( \theta \)–so that the bank is solvent–, the bank will survive the early withdrawal phase. In this case, the LoLR will provide the bank with an amount \( \beta - l \) of funds. The bank will have to repay the borrowed funds at a price \( Q \geq 0 \) per unit. We can therefore write the bank’s objective function with an LoLR in terms of the baseline model as

\[^{22}\]The classical LoLR doctrine, first formally laid out by Thornton (1802) and Bagehot (1873), establishes that the LoLR should lend to "illiquid but solvent" banks. For instance, abiding by these principles, the ECB’s Emergency Liquidity Assistance (ELA) program is set to assist a "solvent financial institution, or group of solvent financial institutions, that is facing a temporary liquidity problem". Here, we take this position and assume that the LoLR does only intervene when the bank is solvent.

\[^{23}\]Rescuing an illiquid but solvent bank yields \( M \cdot (1 - l) \) and entails a cost of \( \lambda \cdot (\beta - l) \). Hence, a LoLR intervention is optimal if and only if \( \lambda \leq \frac{M \cdot (1 - l)}{\beta - l} \). Hence, the upper bound \( \frac{M}{d} \) on the shadow cost for the LoLR funds guarantees that rescues are optimal for any value of liquidity and early withdrawals.
follows:

\[ \Pi^{\text{LoLR}}_B(\theta, l, k, Q) = \Pi_B(\theta, l, k) + \theta \cdot \int_l^d \left[ \pi(l, k) - (Q - 1) \cdot (\beta - l) \right] \cdot f(\beta) d\beta. \]

The level of solvency chosen by a bank with a liquidity, capital and price of funds triplet \((l, k, Q)\) can then be rewritten in terms of the baseline model solvency level as:

\[ \Theta^{\text{LoLR}}_B(l, k, Q) = \Theta_B(l, k) + \frac{1}{c} \cdot \int_l^d \left[ \pi(l, k) - (Q - 1) \cdot (\beta - l) \right] \cdot f(\beta) d\beta. \]  

(16)

Moreover, the presence of an LoLR induces a higher level of solvency for any given pair \((l, k)\) of liquidity and capital values. The intuition is straightforward: the LoLR eliminates illiquidity risk, thus increasing the bank’s return of reducing insolvency risk.

Analogously to the baseline model, we can write the wedge between the regulator’s and the bank’s objective functions as:

\[ \Pi^{\text{LoLR}}_B(\theta^{\text{LoLR}}_B(l, k), l, k, Q) - \Pi^{\text{LoLR}}_R(l, k, Q) = \left( 1 - \Theta^{\text{LoLR}}_B(l, k) \right) \cdot D(l, k) \]

\[ + \Theta^{\text{LoLR}}_B(l, k) \cdot \int_l^d (\lambda - Q) \cdot (\beta - l) \cdot f(\beta) d\beta. \]  

(17)

Equation (17) is the counterpart of expression (8) in the presence of the LoLR. The first term stands for the loss inflicted on the deposit insurance scheme in the event of a bank failure. This term appears in expression (8), but the LoLR intervention makes it smaller. On the one hand, because illiquidity risk is eliminated. On the other hand, because insolvency risk is reduced, as shown in expression (16). The second term represents the subsidy to the bank from the LoLR intervention, the difference \(\lambda - Q\) capturing the gap between the shadow cost of funds and the price paid by the bank.

Regulation with a LoLR differs from the baseline model in two aspects. On the one hand, the role of liquidity regulation is no longer reducing illiquidity risk—which is eliminated by the LoLR—but

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\[ \text{Strictly speaking, the bank will seek assistance from the LoLR if and only if its continuation value from resorting to the LoLR is positive. Hence, its objective function should read as follows:} \]

\[ \Pi^{\text{LoLR}}_B(\theta, l, k) = \Pi_B(\theta, l, k) + \max \left\{ 0, \theta \cdot \int_l^d \left[ \pi(l, k) - (Q - 1) \cdot (\beta - l) \right] \cdot f(\beta) d\beta \right\}. \]

However, we show below that the optimal value for the price of LoLR funds is always such that \(Q^{\text{LoLR}}_R \leq \frac{\pi(l, k)}{\beta - l} + 1\), so that the bank will always seek help from the LoLR when running short of liquid assets. We can therefore omit the "max operator" from the bank’s objective function.
diminishing the cost of the LoLR intervention. On the other hand, while the role of capital regulation is equivalent to that in the baseline model, capital requirements can be complemented with the price $Q$ of LoLR funds so as to reduce insolvency risk. Observe that the level of solvency, as displayed in equation (16), is a strictly decreasing function of the price $Q$ charged by the LoLR. By subsidizing the LoLR funds, the regulator can increase the bank’s upside payoff, just as capital does. The following proposition characterizes the shape of the optimal regulatory policies, as well as the implications for insolvency risk. For short, we denote $\theta_B^{\text{LoLR}}(\lambda) \equiv \theta_B^{\text{LoLR}}(l_R^{\text{LoLR}}(\lambda), k_R^{\text{LoLR}}(\lambda), Q_R^{\text{LoLR}}(\lambda))$ the equilibrium level of solvency for a given shadow cost of LoLR funds $\lambda$.

**Proposition 7 (Optimal regulation with a LoLR)** There exist thresholds $\bar{\rho}^{\text{LoLR}}$ and $\bar{M}^{\text{LoLR}}$ for the cost of capital and the project return, respectively, such that, for all $\rho < \bar{\rho}^{\text{LoLR}}$ and $M > \bar{M}^{\text{LoLR}}$, we have that:

(i) (Liquidity requirements) There exists a threshold $\lambda_l$ such that the optimal liquidity requirement is such that:

$$ l_R^{\text{LoLR}}(\lambda) : \begin{cases} = 0 & \text{if } \lambda \leq \lambda_l \\ \text{Strictly increasing in } \lambda & \text{if } \lambda > \lambda_l \end{cases}.$$ 

Moreover, the liquidity requirement is strictly smaller than in the baseline model, that is, $l_R^{\text{LoLR}}(\lambda) < l_R(\lambda)$ for all $\lambda$.

(ii) (Capital requirements and price of LoLR funds) There exists a threshold $\lambda_k$ such that the optimal capital requirement is such that:

$$ k_R^{\text{LoLR}}(\lambda) : \begin{cases} \text{Strictly decreasing in } \lambda & \text{if } \lambda < \lambda_k \\ = 0 & \text{if } \lambda \geq \lambda_k \end{cases}.$$ 

In addition, the optimal price of LoLR funds is such that:

$$ Q^{\text{LoLR}}(\lambda) : \begin{cases} = 0 & \text{if } \lambda \leq \lambda_Q \\ \text{Strictly increasing in } \lambda & \text{if } \lambda > \lambda_Q \end{cases}.$$ 

Moreover, there exists another threshold $\hat{\lambda}_k < \lambda_k$ such that the capital requirement is strictly smaller than in the baseline model, that is, $l_R^{\text{LoLR}}(\lambda) < l_R^*(\lambda)$, if and only if $\lambda < \hat{\lambda}_k$.

(iii) (Insolvency risk) Insolvency risk increases with the shadow cost of LoLR funds, that is, $\theta_B^{\text{LoLR}}(\lambda)$ is strictly decreasing in $\lambda$. Nonetheless, insolvency risk is reduced with a LoLR, that is, $\theta_B^{\text{LoLR}}(\lambda) > \theta_B(l^*, k^*)$ for all $\lambda$.

**Proof.** See Appendix. □

The intuition for this result is as follows. Liquidity requirements play the role of reducing the expected cost of the LoLR intervention. However, there is an opportunity cost of liquidity in terms
of foregone loan investments. When the shadow cost of LoLR funds is smaller than the expected 
cost of the LoLR intervention, the optimal liquidity policy is to set a zero requirement. As the 
shadow cost of LoLR funds increase, so does the expected cost of the LoLR intervention, which 
leads to increasing liquidity requirements from $\lambda_t$ onwards.

In order to grasp the intuition behind the joint determination of the optimal capital requirement 
and price of LoLR funds, it is instructive to rearrange the first order condition of the regulator’s 
problem as follows:

$$
\frac{D(l, k)}{\text{Deposit Insurance Loss}} - \int_0^d (\lambda - Q) \cdot (\beta - l) \cdot f(\beta) d\beta = (\rho - r) \cdot c. 
$$

(18)

Observe that the optimal regulatory policy prescribes that the left-hand side of expression (18), 
which represents the social return of reducing insolvency risk, be constant. This expression is 
additively separable in two terms. The first term, which corresponds to the deposit insurance loss 
following from a bank failure, is strictly decreasing in capital $k$, since capital reduces the bank’s 
deposit base. The second term corresponds to the social subsidy that the bank perceives from the 
LoLR assistance. While the social cost of funds is $\lambda$, the bank just faces a price of $Q$. Hence, $\lambda - Q$ 
stands for the wedge between the social and the private cost of LoLR funds. Raising $Q$ reduces the 
subsidy and does therefore reduce the misalignment of social and private goals. Hence, the subsidy 
term increases in $Q$.

The price of LoLR funds is a double-edged sword. On the one hand, the larger the subsidy 
(i.e., the smaller the price $Q$ of LoLR funds), the higher the incentives for the bank to reduce 
insolvency risk. This is due to the fact that the LoLR intervention—and therefore the perception 
of the subsidy—is more likely the more solvent the bank. Hence, the regulator can set a low price 
$Q$ of LoLR funds in combination with capital requirements to help incentives to reduce insolvency 
risk. However, when $\lambda$ is large, too cheap a price $Q$ for LoLR funds may overincentivize the bank 
to choose too low a level of insolvency risk, which would be funded with socially costly LoLR funds.

Let us now move to the analysis of the jointly optimal capital requirements and price of LoLR 
funds policy. For small values of $\lambda$, which represent situations in which the LoLR potential 
intervention is cheap, reducing insolvency risk has a large impact on social welfare. Positive capital 
requirements should be used in this case so as to induce the bank to reduce insolvency risk. In 
addition, capital requirements should be optimally set in combination with the highest possible 
subsidy (i.e., $Q = 0$) so as to further help incentives to reduce insolvency risk. Observe that if 
$Q > 0$ capital requirements should be larger than if $Q = 0$, so that welfare would be reduced. As 
the shadow cost of funds $\lambda$ increase, the social return of reducing insolvency risk reduces, since a
potential LoLR intervention becomes costlier. Hence, capital requirements decrease as $\lambda$ increase. For $\lambda = \lambda_k$, subsidized LoLR funds are sufficient to provide the right incentives for the bank to choose the second-best level of insolvency risk. Hence, capital requirements, which are socially costly, should be set to zero for $\lambda \geq \lambda_k$. As $\lambda$ grows beyond $\lambda_k$, the regulator must increase the price of LoLR funds to avoid the bank to choose an inefficiently low level of insolvency risk at the expense of too cheap LoLR funds. The optimal regulatory policy does therefore prescribe an increasing level of the price $Q$ of LoLR funds as $\lambda$ increases.

Finally, regarding the equilibrium level of insolvency risk, observe that the LoLR intervention increases the social return of the bank being solvent, since liquidity risk is eliminated. Hence, insolvency risk decreases with the LoLR with respect to the baseline model. Nonetheless, the larger the shadow cost of LoLR funds, the lower the expected social welfare, since a potential LoLR intervention becomes costlier. Consequently, the optimal (second-best) level of insolvency risk decreases with the social cost of LoLR funds $\lambda$.

### 6.1 Liquidity risk with a Lender of Last Resort

So far, we have analyzed two polar cases in which the bank fails if withdrawals exceed the bank’s liquidity reserves (baseline model) or there is no illiquidity risk because the LoLR rescues the bank when it is solvent but illiquid (model with an LoLR). In this subsection we analyze the case in which the LoLR described above will intervene to rescue a bank with some common knowledge probability $\alpha \in [0, 1]$. The baseline model corresponds to the case in which $\alpha = 0$, whereas the model with a frictionless LoLR analyzed above corresponds to $\alpha = 1$. This hybrid model allows us to study regulation in a setup in which liquidity reserves play two roles. On the one hand, as in the baseline model, liquidity allows the bank to survive liquidity shocks without having to resort to an LoLR that may potentially not act. On the other hand, as in the model with the frictionless LoLR, liquidity serves as a buffer that reduces the shadow cost of the LoLR intervention.

\[25\] As noted by Fischer (1999), it would be hard for an LoLR to establish in advance whether and under which circumstances it will intervene if called upon to act. As a recent example, in June 2016 the ECB’s governing council reinstated the waiver that allowed Greek sovereign bonds to be accepted as collateral for Greek banks to borrow ECB cash. Previously, Greek banks had only been assisted through the restrictive ELA program for emergency assistance. Moreover, some Central Banks have followed a policy of constructive ambiguity (e.g., Corrigan et al. (1990) and George (1994)), whereby the potential intervention of the LoLR is explicitly left uncertain. Freixas et al. (1999) and He (2000) argue that the LoLR constructive ambiguity policy is an effective tool to mitigate banks’ moral hazard.
In this setup, the regulator’s problem is given by:

$$\Pi^\alpha_R(l, k, Q) \equiv \theta^\alpha_B(l, k) \left(F^\alpha (l) M (1 - l) - \alpha \int_l^d (\lambda - Q) (\beta - l) f(\beta) d\beta\right) - D(l, k) - c \frac{\theta^2}{2} - (1 + \rho) k. \quad (19)$$

Observe that expression (19) is analogous to that of the regulator’s problem with a LoLR, except for two new terms. The probability $$F^\alpha (l)$$ with which the bank survives the early withdrawal phase when holding an amount $$l$$ of liquid reserves is now given by the combination of the LoLR intervention—which rescues an illiquid bank with probability $$\alpha$$—and the bank’s use of its liquidity own reserves in case the LoLR does not intervene, that is:

$$F^\alpha (l) \equiv \alpha + (1 - \alpha) F(l).$$

Also, the level of solvency $$\theta^\alpha_B(l, k, Q)$$ chosen by the bank with capital, liquidity and price of LoLR triplet $$(l, k, Q)$$ in this setup is analogous to expression (16), except that the bank will only survive a liquidity shock with probability $$\alpha$$. It is therefore given by:

$$\theta^\alpha_B(l, k) = \theta_B(l, k) + \frac{\alpha}{c} \int_l^d [\pi(l, k) - (Q - 1) (\beta - l)] f(\beta) d\beta.$$

The following proposition, which describes the nature of equilibria for any pair $$(\alpha, \lambda)$$, relates the findings in the baseline model with those found in the model with a LoLR.

**Proposition 8 (Optimal regulation with an LoLR)** There exist thresholds $$M^{\text{LoLR}}$$, as defined in Proposition 7, and $$\bar{\rho}(M)$$, as defined in Proposition 6 for the the project return and the cost of capital, respectively, such that, for all and $$M > M^{\text{LoLR}}$$ and $$\rho < \bar{\rho}(M)$$, we have a threshold $$\bar{\alpha}(\lambda)$$ for the probability of an LoLR intervention such that:

(i) If $$\alpha < \bar{\alpha}(\lambda)$$, the optimal regulatory policy entails positive liquidity and capital requirements which are determined as the intersection of optimal liquidity and capital response functions, analogously as in the baseline model.

(ii) If $$\alpha \geq \bar{\alpha}(\lambda)$$, either the optimal liquidity requirement is zero or the optimal capital requirement is zero, analogously as in the model with a frictionless LoLR.

**Proof.** See Appendix.

Figure ??? illustrates this proposition. The function $$\lambda_k(\alpha)$$ separates the space in two areas. For any given $$\alpha$$, capital requirements are zero if $$\lambda \geq \lambda_k(\alpha)$$. Analogously as in the model with the frictionless LoLR analyzed above, for values $$\lambda > \lambda_k(\alpha)$$ subsidized LoLR funds are sufficient for the bank to choose the appropriate level of insolvency risk. Hence, capital requirements are zero. The function $$\lambda_k(\alpha)$$ is strictly decreasing in $$\alpha$$: Larger values of $$\alpha$$ are associated to a higher probability of getting the subsidy. In the baseline model, the absence of an LoLR leads to positive capital requirements.
7 Extensions

So far we have analyzed a simplified model in which several important ingredients are absent. In this section, we enrich the basic model and assess the validity of the predictions of the basic setup.

7.1 Asset liquidation

7.1.1 The bank’s problem with asset liquidation

In the basic model the bank fails if the demand for deposits at the interim stage exceeds the bank’s liquidity reserves. In this section we allow for asset liquidation at a discounted price if early withdrawals go beyond the bank’s liquidity level, so that the bank can survive larger liquidity shocks. In this setup, the bank’s problem reads:

\[
\Pi^{AL}_B (\theta, l, k) \equiv \Pi_B (\theta, l, k) + \int_l^{l+\gamma} \theta \cdot \left[ M \cdot \left( 1 - l - \frac{\beta - l}{\psi} \right) - \frac{\text{Leftover Investment}}{\text{Deposit Liabilities}} \right] \cdot f(\beta) \, d\beta.
\] (20)

The first term corresponds to the bank’s expected profit in the event that early withdrawals fall below the bank’s liquidity reserves, which is equivalent to the bank’s problem of the basic model defined in equation (B). The second term stands for the bank’s expected profit in the event that the bank has to liquidate assets in order to overcome a large liquidity shock. For any withdrawal level \( \beta > l \) in excess of the bank’s liquidity reserves, the bank needs to liquidate an amount \( \frac{\beta - l}{\psi} \) so as to obtain an amount \( \beta - l \) of liquid assets, where \( \psi < 1 \) represents the (fire sales) price per unit of loan.\(^{26}\) Hence, after a partial liquidation occurs, the bank is left with an amount \( 1 - l - \frac{\beta - l}{\psi} \) of loans, yielding a return of \( M \) per unit if successful. The bank’s deposit liabilities after satisfying an early withdrawal of \( \beta \) amount to \( D(\beta, k) \), as defined in equation (7). The upper bound of the integral is given by the maximum amount of extractions that the bank can bear, where \( \gamma \) represents the maximum amount of cash that the bank can obtain by liquidating assets and continue operating with a nonnegative continuation expected payoff.\(^{27}\)

\(^{26}\)Clearly, if \( \psi \geq 1 \) the bank could obtain a higher amount of liquidity by liquidating assets in the interim period than by storing cash, in which case the optimal amount of liquidity reserves would be zero.

\(^{27}\)Sensu stricto, \( \gamma \) should be given by the solution to the following system of equations: \( M \cdot \left( 1 - l - \frac{\beta - l}{\psi} \right) - D(\beta, k) = 0 \) and \( \gamma = \beta - l \). However, for tractability, we treat \( \gamma \) as an exogenous variable and impose (sufficient) conditions for the value of \( \gamma \) so that the continuation value of the bank is nonnegative in the case in which the advent of early withdrawals is given by \( \gamma \). Alternatively, one could argue that, even if the bank could put up with an amount of early withdrawals larger than \( \gamma \) by liquidating at a (constant, independent of the liquidating volume) rate of \( \psi \) per loan,
Differentiating equation (20) with respect to \( \theta \), it follows that:

\[
\theta_B^{AL} (l, k) = \theta_B (l, k) + \frac{1}{c} \int_{l}^{l+\gamma} \left[ M \cdot \left( 1 - l - \frac{\beta - l}{\psi} \right) - D(\beta, k) \right] \cdot f(\beta) d\beta.
\]  \tag{21}

Observe that the second term in the previous expression is positive given that asset liquidation only occurs if it enhances the bank’s continuation payoff. Consequently, insolvency risk is strictly smaller than in the absence of asset liquidation: the possibility of liquidating assets to survive larger liquidity shocks makes it more valuable to reduce insolvency risk.

Binding liquidity requirements harms insolvency risk, as the following Proposition states.

**Proposition 9 (Liquidity and solvency risk with asset liquidation)**  (i) The level of solvency \( \theta_B^{AL} (l, k) \) is a hump-shaped function of liquidity which, for any given level of equity capital \( k \), is maximized at the bank’s profit-maximizing liquidity level \( l_B^{AL} (k) \). Consequently, any liquidity requirement \( l > l_B^{AL} (k) \) in excess of the bank’s profit maximizing level induces a lower level of solvency.

(ii) The bank’s profit-maximizing liquidity level with asset liquidation is strictly smaller than in the basic model but the corresponding overall illiquidity risk, which includes the range of asset liquidation, is smaller. Formally:

\[
l_B^{AL} (k) < l_B (k) < l_B^{AL} (k) + \gamma.
\]

**Proof.** See Appendix. \( \blacksquare \)

This proposition is the counterpart of Proposition 1 when we allow for asset liquidation. The reason why the profit-maximizing liquidity level does also maximize solvency is the same as in the basic model: the returns of reducing insolvency risk are given by the expected upside payoff. Therefore, the profit-maximizing liquidity level—which does also maximize the upside payoff—is the value that maximizes the return of reducing insolvency risk.

Proposition 9 also shows that the bank’s incentives to keep liquid assets is reduced. This is due to two effects pushing in the same direction. On the one hand, the possibility of liquidating assets increases the likelihood of surviving the early withdrawal phase, thus increasing the opportunity cost of holding liquidity instead of investing in loans. On the other hand, the value of liquidity in terms of increasing the likelihood of surviving the early withdrawal phase is smaller with asset liquidation, because the possibility of liquidating assets expands the range of liquidity shocks that the bank can meet.\(^{28}\) Finally, the fact that overall illiquidity risk is smaller follows immediately exceeding the amount \( \gamma \) would lead to a disorderly process of liquidation that would make the bank’s continuation non-viable.

\(^{28}\) Notice that the marginal contribution of liquidity to increasing the likelihood of surviving the early withdrawal phase is strictly decreasing in liquidity, because \( g(\cdot) \) is strictly decreasing, which implies that \( g(l + \gamma) < g(l) \).
from the fact that the marginal value of liquidity is reduced when asset liquidation is possible, which implies that the likelihood of surviving the early withdrawal phase must be larger.

7.1.2 Capital and liquidity regulation with asset liquidation

We can write the wedge between the bank and the regulator’s problem as:

\[ \Pi^B_AL (\theta^B_AL (l, k), l, k) - \Pi^R_AL (l, k) = (1 - \theta^B_AL (l, k)) \cdot \left[ \begin{array}{c}
\text{No liquidation} \\
\text{Partial liquidation}
\end{array} \right] \\
\begin{array}{c}
F(l) \cdot D(l, k) + \int_l^{l+\gamma} D(\beta, k) \cdot f(\beta) d\beta, \\
(1 - F(l + \gamma)) \cdot [D(l, k) - \gamma].
\end{array}
\]

The payoff discrepancy between the regulator and the bank occur when the bank fails, since the bank is protected by limited liability. There are three potential sources for the bank failure. First, the bank may overcome a liquidity shock and then become insolvent. In this case the deposit insurance fund liabilities amount to \( D(l, k) = 1 + b \cdot r - l - k \). Second, the bank may undergo a partial liquidation at the interim stage and then become insolvent. In these situations, the social toll following the bank failure is given by \( D(\beta, k) = 1 + b \cdot r - \beta - k \), as part of the deposit liabilities are paid back at the early withdrawal phase making use of the liquidation proceeds. Finally, if the bank fails as the result of a liquidity crisis, the deposit insurance fund uses the bank full liquidation value to reduce its financial obligations. Hence, beyond expanding the size of early withdrawals that the bank can survive, asset liquidation reduce the social toll following a bank failure.

We can now proceed analogously as in the basic model and find the optimal capital and liquidity responses, respectively. As compared to the basic model, we can highlight the following features:

(i) For any given level of the bank’s liquidity holdings, capital requirements have a higher impact in reducing insolvency risk, that is, \( \frac{\partial \theta^B_AL (l, k)}{\partial k} > \frac{\partial \theta^B_AL (l, k)}{\partial k} \).

Since the likelihood of surviving a liquidity shock is higher with the possibility of incurring in asset liquidation, raising capital has a higher impact on the bank’s expected upside payoff. As a consequence, the returns of reducing insolvency risk are higher.

(ii) If the capital response function in the basic model entails a positive capital level, then the capital response function with asset liquidation attains a higher value. Formally, if \( k_R (l) > 0 \), then we have that \( k^AL_R (l) > k_R (l) \).
There are two effects that push in the direction of raising the capital response as compared to the baseline model. On the one hand, as argued in the previous statement, capital is more effective in reducing insolvency risk—the ‘substitution effect’ is larger than the ‘income effect’. On the other hand, the social value of reducing insolvency risk is higher than in the basic model, because the likelihood of surviving the early withdrawal phase is higher and therefore the likelihood of reaching the stage in which being solvent matters.

(iii) The liquidity response function with asset liquidation is strictly lower than in the basic model, that is, $l^A_L (k) < l_R (k)$.

There are two effects making the liquidity response function smaller than in the baseline model. One of them have already been outlined in the bank’s problem: the effectiveness of liquidity in reducing illiquidity risk is diminished. On the other hand, the possibility of liquidating assets raises the opportunity cost of liquidity. As in the case of the bank, one reason for this is the larger likelihood of surviving a liquidity shock and therefore an elevation of the expected value of investing in loans. Additionally, the value of liquidity as a cash-in-hand asset to reduce the social toll in case of failure is reduced, because the proceeds of liquidation can now serve that purpose.

We end this section by stating existence and uniqueness of liquidity and capital requirements.

Proposition 10 (Liquidity and capital requirements with asset liquidation) For any given cost of capital $\rho$ and of loan returns $M$, there exists a unique pair $(l^*_A, k^*_A)$ of liquidity and capital requirements given by $l^*_A = l^A_L (k^*_A)$ and $k^*_A = k^A_L (l^*_A)$.

7.2 Solvency-driven liquidity shocks

So far, we have assumed that illiquidity and insolvency risk are generated independently and yet shown that capital and liquidity should be jointly determined. Our assumption, although extreme in nature, makes sense if, for instance, a large fraction of withdrawals is due to purely idiosyncratic consumption motives. However, it seems reasonable to assume that liquidity risk is influenced by the bank’s asset quality, as in Jacklin and Bhattacharya (1988), where banks’ fundamentals may trigger an information-based bank run. In this section, we show that our main predictions in terms of optimal regulation are qualitatively robust to the introduction of information-based runs.

In particular, consider the baseline model outlined in Section 3, in which the early withdrawals function with solvency-driven shocks $F_T (\cdot)$ is related to the level of insolvency risk as follows:

$$P [\beta \leq l] = F_T (l, \theta, T) = \begin{cases} \theta^T \cdot F (l) & \text{if } l < d \\ 1 & \text{if } l = d \end{cases}$$
where $F(\cdot)$ is the c.d.f of early withdrawals in the baseline model and $T \in (0, \tilde{T})$ for some $\tilde{T} < 1$. The c.d.f $F^T(\cdot)$ can be interpreted as follows. In the baseline model, for any given level of liquidity $l$ held by the bank, $F(l)$ stands for the probability that the amount of early withdrawals is at most as high as $l$. The c.d.f $F^T(\cdot)$ simply weights this probability by a factor $\theta^T$ and places a mass of $1 - \theta^T$ on the (run-triggered) event of all depositors withdrawing early. Observe that for any $0 < 0'$ it follows that $F^T(l, \theta', T) < F^T(l, \theta'', T)$. Hence, the level of solvency $\theta$ induces a first-order stochastic dominance ordering in the family $\{F^T(l, \theta, T)\}_{\theta \in [0,1]}$ whereby the larger the solvency level $\theta$ the "less likely"—in a first-order stochastic dominance sense—early withdrawals. For instance, an insolvent bank (one with $\theta = 0$) would experience a run for certain. On the other side of the spectrum, an absolutely solvent bank (i.e., a bank with $\theta = 1$) would face the smooth early withdrawals c.d.f function $F(\cdot)$ described in the baseline model.

The parameter $T$ stands for the elasticity of the c.d.f function $F^T(\cdot)$ with respect to the solvency level $\theta$. Accordingly, the higher $T$, the more sensitive $F^T(\cdot)$ to the solvency level $\theta$. Clearly, $T = 0$ corresponds to the baseline model. The upper threshold $\tilde{T}$ for the elasticity parameter is set so as to avoid the following pathological behavior. In the baseline model, increasing the bank’s liquidity reserves leads to an unambiguous reduction of illiquidity risk. In this setup, in addition to this effect of liquidity on illiquidity risk, increasing the bank’s liquidity reserves beyond the bank’s profit-maximizing liquidity level leads to a reduction of the equilibrium choice of solvency $\theta$, which increases the likelihood of a bank run. The upper threshold $\tilde{T}$ ensures that increasing liquidity, regardless of any optimality consideration, unambiguously reduces illiquidity risk in equilibrium.

Before closing the description of the model, let us carry out the following thought experiment. Consider an exogenously given pair of liquidity and capital $(l, k)$. Suppose that we increase $T$ and we ask whether the solvency level $\theta(l, k)$ that the bank would choose as a function of the pair $(l, k)$ increases or decreases. On the one hand, increasing $T$ makes the effect of solvency on illiquidity risk more important. Hence, the bank would be willing to increase its level of solvency. However, increasing $T$ does also reduce the bank’s expected profits for any given pair $(l, k)$ of liquidity and capital, because illiquidity risk is higher. This effect would push the bank’s choice of solvency in the opposite direction. In order to clean out the effect of solvency-driven bank runs through the effect of solvency on liquidity exclusively, we can "level the playing field" by adding a fictitious transfer $H(T)$ to be added to the bank’s upside payoff, so that the bank’s expected profit in equilibrium does not change as we change $T$. Specifically, we could compute the bank’s solvency function $\theta^T(l, k)$, as well as the optimal liquidity and capital requirements $(l^T, k^T)$ for any given $T$ and compute $H(T)$.
so that:

\[
\Pi_B (\theta (l^*, k^*), l^*, k^*) - \Pi_B (\theta^T (l^T, k^T), l^T, k^T) = \theta (l^*, k^*) \cdot F (l^*) \cdot \pi (l^*, k^*) - \theta^T (l^T, k^T) \cdot F (l^T) \cdot [\pi (l^T, k^T) + H (T)] = 0. \tag{24}
\]

Clearly, \( H (T) \) is strictly increasing, capturing the negative effect of increasing runs on the bank’s expected profits. The following proposition establishes the pattern of optimal liquidity and capital responses as \( T \) varies.

**Proposition 11 (Liquidity and capital with solvency-driven liquidity shocks)** (i) Let \( H (T) \) be as defined in equation (23). Then, if a liquid and solvent bank is compensated with a transfer \( H (T) \), the optimal liquidity response function \( l^T_R (k) \) is strictly decreasing in \( T \) and the optimal capital response function \( k^T_R (l) \) is strictly increasing in \( T \). Moreover, the bank’s choice of solvency \( \theta^T (l, k) \) is strictly increasing in \( T \).

(ii) Let \( H (T) = 0 \). Then, the optimal liquidity response function \( l^T_R (k) \) is strictly decreasing in \( T \). Additionally, there exist thresholds \( \underline{l} \) and \( \overline{l} \), to the left and right of the bank’s profit maximizing level of liquidity, respectively, such that both the optimal capital response function \( k^T_R (l) \) and the bank’s choice of solvency \( \theta^T (l, k) \) are strictly increasing in \( T \) if and only if \( l \in (\underline{l}, \overline{l}) \).

(iii) The triplet \( (\theta^T (l, k), l^T_R (k), k^T_R (k)) \) converges to \( (\theta (l, k), l_R (k), k_R (k)) \) as \( T \) goes to zero.

Consider first the case in which the negative effect of solvency-driven shocks on the bank’s expected profits is eliminated through the fictitious transfer \( H (T) \). On the one hand, increasing the solvency level has a positive effect on illiquidity risk. Therefore, the bank’s choice of solvency is larger as \( T \) increases. Moreover, the social return of capital increases. As a consequence, the optimal capital response is also larger as \( T \) increases. On the other hand, binding liquidity requirements have a negative effect on the choice of solvency. Hence, the negative effect of liquidity on solvency has a larger impact as solvency becomes more important, that is, as \( T \) increases. Consequently, the optimal liquidity response is smaller as \( T \) increases.

Additionally, solvency-driven shocks harm the bank’s expected profit overall. As a consequence, the bank’s choice of solvency, which is higher the larger the bank’s upside payoff, may decreases as \( T \) increases. For the same reason, the optimal capital response may also be decreasing in \( T \). In order to see why, consider a small value of liquidity. When liquidity is small, illiquidity risk is large regardless of the choice of solvency. Consequently, the effect of solvency on illiquidity risk will also be small. In this case, the negative effect of solvency-driven liquidity shocks on the bank’s profits will dominate, so that the bank’s choice of solvency will be smaller as \( T \) increases. Nonetheless, as liquidity approaches the bank’s profit maximizing value of liquidity (and hence its expected profits
increase), the negative effect on profits is dominated by the positive effect of solvency on illiquidity risk. Consequently, for liquidity values around the bank’s profit maximizing one, the choice of solvency increase as $T$ increases.

Our baseline model can be interpreted as the limiting model as $T$ tends to zero. While the optimal liquidity and capital requirements quantitatively change as $T$ varies, the qualitative results of both models are analogous.

8 Conclusion

This paper addresses optimal joint regulation of equity capital and liquidity from a microprudential perspective. We find that liquidity and capital should be jointly regulated, for the level of either regulatory tools affects the effectiveness of the other regulatory tool. More particularly, we find that liquidity complements capital when liquidity is low, because raising liquidity increases the likelihood of surviving a liquidity crisis and therefore the return of capital in reducing insolvency risk. Capital does also complement liquidity when capital is low, because raising capital reduces the negative effect of liquidity on solvency, so that the return of liquidity increases. In the region where this occurs, capital and liquidity are complementary regulatory tools. On the contrary, when liquidity is high, further liquidity raises lead to a diminished return of capital, because a high liquidity value reduces the loss inflicted on the deposit insurance scheme in case of a bank failure and therefore the social value of reducing insolvency risk. Moreover, when the level of capital is high, increasing capital further offsets the return of liquidity. The reason is that when capital is high, insolvency risk is low and therefore the opportunity cost of liquidity—in terms of foregone opportunities to invest in loans—is high as well. In this region, capital and liquidity offset each other as regulatory tools.

Whether optimal capital and liquidity requirements fall in the region where capital and liquidity are complementary tools or offset each other depends on the cost of capital and on the opportunity cost of liquidity. When the cost of capital and the return on loans—which determines the opportunity cost of liquidity—are large, capital and liquidity requirements are low. Hence high cost of capital and liquidity make this regulatory tools complementary. Any exogenous decrease in the price of one of the regulatory tools should therefore lead to an elevation of both requirements.

The results of this paper have an important regulatory implication. Although capital and liquidity requirements should be designed to cope with issues of different nature, namely solvency and illiquidity risk, and even if these are originated independently, regulation should take into account the feedback loops between these two regulatory instruments. This fact suggests that capital and liquidity committees should work together and take into account the cross effects of
either regulatory tool on the effectiveness of the other.
References


A Technical assumptions

We start off by defining the level of liquidity \( l_\Phi(k) \) that maximizes the marginal return of capital for any given value of \( k \), which is implicitly defined by:

**Definition 1 (Capital return maximizing level of liquidity)**

\[
g(l_\Phi(k)) \cdot D(l_\Phi(k), k) \equiv \frac{1}{2}.
\]

Next, we define thresholds for the maximum and minimum values of the social cost of capital.

**Definition 2 (Upper threshold social cost of capital)**

\[
\rho_{\text{max}} \equiv r + F^2(l_\Phi(0)) \cdot D(l_\Phi(0), 0) \cdot \frac{1+r}{c}.
\]

**Definition 3 (Lower threshold social cost of capital)**

\[
\rho_{\text{min}} \equiv r + F^2(l_\Phi(1-d)) \cdot D(l_\Phi(1-d), 1-d) \cdot \frac{1+r}{c}.
\]

In the main text we show that for values \( \rho \geq \rho_{\text{max}} \) the optimal capital response is zero for any given value of liquidity. In addition, in order to limit the number of cases to discuss and avoid a capital requirement in excess of the maximum level of capital that the bank may hold, we assume that the social cost of capital is sufficiently high so as to have an interior solution. Formally:

**Assumption 1**

\[
\rho > \rho_{\text{min}}.
\]

We now define a (lower) threshold for the project profitability as follows:

**Definition 4 (Lower threshold project profitability)**

\[
M \equiv \frac{D(l_\Phi(0), 1-d)}{1-l_\Phi(0)} + \left\{ c \cdot \left[ \frac{\left( F^2(l_\Phi(0)\cdot(1-l_\Phi(0))) \cdot D(l_\Phi(0), 1-d) \right)^2}{2F^2(l_\Phi(0)) \cdot (1-l_\Phi(0))^2} \cdot [(1+\rho_{\text{max}}) \cdot (1-d) + D(l_\Phi(0), 1-d)] \right]^{1/2} \right\}.
\]
The following assumption is a sufficient condition that guarantees that the project profitability is sufficiently high so that the optimal regulatory policy does not entail shutting down the bank.\footnote{This is a strong sufficient condition to guarantee that the social value of the bank is not negative, but that we do otherwise not use to prove our results. Given our choice of $M$, in equilibrium we have that the social value generated by the bank is positive. We could therefore relax this assumption so that the social value produced by the bank is exactly zero. In order to do so, on Definition 4 we could replace the maximum possible equilibrium level of liquidity $l_B(0)$ by the actual equilibrium value of liquidity $l^*$, as well as the maximum level of capital $1 - d$ by the equilibrium value $k^*$. This replacement would reduce the lower threshold for the minimum level of profitability $M$, so that the range of profitability would be enlarged. However, $l^*$ and $k^*$ are determined endogenously in equilibrium, whereas both $l_B(0)$ and $1 - d$ are determined exogenously by the model parametric assumptions. By focusing on these upper thresholds we are able to provide a explicit expression that depends exclusively on the model parameters.}

**Assumption 2**

\[
M \geq M. 
\]

In order to ensure an interior solution for the level solvency we need to impose a condition on the cost $c$ of reducing insolvency risk. Define the (lower) threshold for the cost of reducing insolvency risk as:

**Definition 5 (Lower threshold insolvency risk reduction cost)**

\[
\zeta \equiv F(l_B(1 - d)) \cdot \pi(l_B(1 - d), 1 - d). 
\]

The following assumption guarantees that insolvency risk is not totally eliminated in equilibrium.

**Assumption 3**

\[
c \geq \zeta. 
\]

We also need to impose a condition so as to ensure that the bank is a "proper bank" in the sense that deposits constitute a non-negligible source of funds for the bank. Otherwise, if the bank relies on too large an amount of non-deposit liabilities, a 100% liquidity requirement may be optimal whenever eliminating illiquidity risk entailed a negligible cost in terms of foregone loans because of a small deposit base. In this situation, deposits would simply be stored as liquid assets and would not play affect the bank’s profits. We define the (lower) threshold for the deposit base as:

**Definition 6 (Lower threshold deposits)**
\[ g(d) \cdot (1 - d) \equiv \frac{1}{2(1 + r)}. \]

Notice that log-concavity of \( F(\cdot) \) implies that \( \lim_{l \to 0} g(l) = +\infty \), so that we can make \( g(l) \cdot (1 - l) \) arbitrarily large as \( l \) approaches 0. Moreover, log-concavity requires that \( g(d) < +\infty \), so that \( g(l) \cdot (1 - l) \) can be made arbitrarily close to 0 by choosing values of \( l \) sufficiently close to \( d \). In addition, observe that \( g(l) \cdot (1 - l) \) is a strictly decreasing function of \( l \) for all \( l < d \), since both \( g(l) \) and \( (1 - l) \) are positive strictly decreasing functions of liquidity \( l \). Hence, \( d \) is well-defined, unique and such that \( d < 1 \). The following assumption ensures an interior liquidity requirement, so that liquidity risk is never eliminated in equilibrium.

**Assumption 4**

\[ d \geq d. \]

Finally, we need a condition on the maximum value that the interest rate \( r \) on long-term deposits can achieve, so that deposit liabilities do not explode.

**Definition 7 (Upper threshold long-term deposits interest rate)**

\[ \bar{r} \equiv \frac{1}{1 - d}. \]

We then have that:

**Assumption 5**

\[ r \in [0, \bar{r}]. \]

### B Additional technical assumptions for the asset liquidation case

As the base model, let be \( \tilde{l}_{\Phi_{AL}}(k) \) the level of liquidity that maximizes the indirect effect of capital under asset liquidation for a given value of \( k \), which is implicitly defined as:

\[ g(\gamma + \tilde{l}_{\Phi_{AL}}(k))(D(\tilde{l}_{\Phi_{AL}}(k), k)) = \frac{1}{2}. \]

Then, we set both an upper and a lower bound for capital cost as follows

\[ \rho_{\text{max}} = r + F(\gamma + \tilde{l}_{\Phi_{AL}}(0))^2 \left[ D(\tilde{l}_{\Phi_{AL}}(0), 0) - \frac{\int_{\tilde{l}_{\Phi_{AL}}(0)}^{\gamma + \tilde{l}_{\Phi_{AL}}(0)} (\beta - \tilde{l}_{\Phi_{AL}}(0)) f(\beta) d\beta}{F(\gamma + \tilde{l}_{\Phi_{AL}}(0))} \right] \cdot \frac{1 + r}{c}. \]
and
\[ \rho_{\min} = r + F^2(\gamma + \tilde{l}_\Phi(1 - d)) D \left( \tilde{l}_{\Phi_{AL}}(1 - d), 0 \right) \cdot \frac{1 + r}{c}. \]

The lower bound \( \rho_{\min} \) ensures that capital requirements never exceed \( 1 - d \), while for any \( \rho \geq \rho_{\max} \) the optimal capital response is zero for any \( l \).

According to this, we assume (Assumptions 1-AL):

\[ \rho_{\min} < \rho < \rho_{\max}. \]

Moreover, we impose a minimum cost for the cost of solvency to ensure an interior solvency level denoted by

\[ c_1 = M \cdot F(\gamma + l_{ELV_{AL}}) \cdot (1 - ELV_{AL}), \]

where \( ELV_{AL} \) is implicitly defined as \( g(\gamma + ELV_{AL})(1 - ELV_{AL}) = 1 \).

Moreover, in order to guarantee that \( g(\gamma + l_{R_{AL}}(k))(1 - l_{R_{AL}}(k)) - 1 + \frac{\chi(l)}{c} < 0 \), we define

\[ c_2 = \frac{1 + r(1 - d)}{1 + F(l_{ELV_{AL}})} + g(\gamma + l_{ELV_{AL}}) - 1. \]

Then, we assume (Assumptions 2-AL):

\[ c \geq c = Max[c_1, c_2]. \]

In order to ensure that the bank always has enough asset in place to liquidate at least a fraction \( \frac{\gamma}{\psi} \), we set

\[ \gamma_1 = (d - \tilde{l}_{\Phi_{AL}}(0)) \psi. \]

On the other hand, we implicitly define \( \gamma_2 \)

\[ g(\gamma_2)\pi(0, 0) = M - 1 \]

to ensure that \( \theta_{B_{AL}}(l, k) \) is a hump-shaped function of \( l \).

Then, we define \( \overline{\gamma} \) as

\[ \overline{\gamma} = Min[\gamma_1, \gamma_2], \]

and assume that (Assumptions 3-AL):

\[ \gamma \leq \overline{\gamma}. \]
Finally, we implicitly define $d$ as

$$g(\gamma + \ell_{ELV_{AL}})(d - \ell_{ELV_{AL}}) = \frac{1}{2}.$$ 

Then, we assume (Assumptions 4-AL):

$$d > d.$$ 

## C Omitted Proofs

Before proceeding to the proof of the Propositions in the main body of the text, we show the following instrumental result, which will be useful in the proofs of Propositions 4 and 5.

**Fact 1 (Properties of $\Phi(l, k)$ curve)** Define $\Phi(l, k) \equiv 2g(l) \cdot D(l, k) - 1$, where $g(l) \equiv \frac{f(l)}{F(l)}$ and $D(l, k) \equiv 1 + (1 - d) \cdot r - (1 + r) \cdot k - l$, as defined in Sections 3.1 and 3, respectively. Let $l_\Phi(k)$ be implicitly defined as $\Phi(l_\Phi(k), k) = 0$. Then, the functions $\Phi(l, k)$ and $l_\Phi(k)$ satisfy the following properties:

(i) For all $k \in [0, 1 - d]$, $\lim_{l \to 0} \Phi(l, k) = +\infty$ and $\Phi_d(k, k) < 0$.

(ii) For all $k \in [0, 1 - d]$, $\Phi(l, k)$ is strictly decreasing in $l$.

(iii) For all $l \in (0, d]$, $\Phi(l, k)$ is strictly decreasing in $k$.

(iv) The function $l_\Phi(k)$ is strictly decreasing. Moreover, $l_\Phi(0) \in (0, d)$.

**Proof of Fact 1.**

(i) The first property, $\lim_{l \to 0} \Phi(l, k) = +\infty$, follows from the fact that $F(\cdot)$ is log-concave, so that $\lim_{l \to 0} g(l) = +\infty$. The second property, namely that $\Phi_d(k, k) = 2g(\gamma)(1 + r) \cdot (1 - d - k) - 1 < 0$, follows directly from Assumption 4 on Section A.

(ii) Both $g(l)$ and $D(l, k)$ are positive and strictly decreasing functions, so that the product $g(l) \cdot D(l, k)$ is also positive and strictly decreasing. The result is then immediate.

(iii) This property follows immediately from $\frac{\partial D(l, k)}{\partial l} = -(1 + r) < 0$.

(iv) The locus $l_\Phi(k)$ satisfies $\Phi(l_\Phi(k), k) = 0$. Implicitly differentiating this expression with respect to $k$ yields $\frac{\partial l_\Phi(k)}{\partial k} = -\frac{\partial \Phi(l_\Phi(k), k)}{\partial k} < 0$, the last inequality following from properties (ii) and (iii). We now show that $l_\Phi(0) < d$. From from Property (ii) in Fact 1, we have that $\Phi(l, 0)$ is strictly decreasing in its first argument. Also, $\Phi(l, 0)$ is continuous in its first argument as well. Moreover, $\lim_{l \to 0} \Phi(l, 0) = +\infty$. In addition, it follows from item (i), just proved, that $\Phi(d, 0) < 0$. Hence, it follows that there exists a unique $l_\Phi(0) \in (0, d)$ that solves $\Phi(l_\Phi(k), k) = 0$. ■
Fact 2 (Properties of the regulator’s objective function)  The regulator’s objective function \( \Pi_R (l, k) \) satisfies:

(i) For any given \( l \in [0, d] \), \( \Pi_R (l, k) \) is strictly concave in \( k \) in all of its domain.

(ii) For any given \( k \in [0, ] \), \( \Pi_R (l, k) \) is strictly quasiconcave in \( l \) in the domain \([l_B (k), l_\Phi (k)]\).

Remark 1 For a family of log-concave distribution functions, the function \( \Pi_R (l, k) \) is strictly convex in \( l \) around zero. Hence, \( \Pi_R (l, k) \) is not concave in \( l \) in all of its domain. However, \( \Pi_R (l, k) \) is quasiconcave in \( l \) in all of its domain. We restrict ourselves to showing that \( \Pi_R (l, k) \) is quasiconcave in \( l \) in the domain \([l_B (k), l_\Phi (0)]\) because this weaker statement suffices for our purposes and the proof is (much) less involved.

Proof of Fact 2. 

(i) For any given \( l \in [0, d] \), we have that
\[
\frac{\partial^2 \Pi_R (l, k)}{\partial k^2} = - \frac{F^2 (l) \cdot (1 + r)^2}{c} < 0.
\]

(ii) We prove this property through a succession of steps through which we show that, for any given \( k \), there exists a unique value of liquidity \( l_R (k) \in (l_B (k), l_\Phi (0)) \) such that \( \frac{d\Pi_R (l_R (k), k)}{dl} = 0 \).

Consider an arbitrary \( k \in [0, 1 - d] \) and, for the sake of notational simplicity, rename the first derivative of the bank and the regulator’s objective functions, respectively, as follows:

\[
H_B (l, \theta, k) \equiv \frac{d\Pi_B (l, \theta, k)}{dl},
\]
and

\[
H_R (l, k) \equiv \frac{d\Pi_R (l, k)}{dl}.
\]

Proof.

Step 1 \( H_R (l_B (k), k) > 0 \), where \( l_B (k) \) is the unique value satisfying \( H_B (l_B (k), k) = 0 \).

We can write the regulator’s objective function in terms of the bank’s as follows:

\[
\Pi_R (l, k) = \Pi_B (l, \theta_B (l, k), k) - (1 - \theta_B (l, k) \cdot F (l)) \cdot D (l, k).
\]

We can therefore write the regulator’s first derivative w.r.t. liquidity as:

\[
H_R (l, k) = H_B (l, \theta_B (l, k), k) - \frac{d}{dl} \left[ (1 - \theta_B (l, k) \cdot F (l)) \cdot D (l, k) \right].
\]

From the bank’s FOC w.r.t. to liquidity, we have that \( H_B (l_B (k), \theta_B (l_B (k), k), k) = 0 \). Moreover, from Proposition 1, it follows that \( \frac{\partial}{\partial l} \theta_B (l_B (k), k) = 0 \). Hence, we can write

\[
H_R (l_B (k), k) = 1 + \theta_B (l_B (k), k) \cdot F (l_B (k)) \cdot [g (l_B (k)) \cdot D (l_B (k), k) - 1]
\]
Notice that \( g(l_B(k)) \cdot D(l_B(k), k) \geq 0 \). Hence \( g(l_B(k)) \cdot D(l_B(k), k) - 1 \geq -1 \). Moreover, \( \theta_B(l_B(k), k) \cdot F(l_B(k)) < 1 \) (both \( \theta_B(\cdot, \cdot) \) and \( F(\cdot) \) are smaller or equal to 1, the former always strictly smaller). Hence, we have that \( H_R(l_B(k), k) > 0 \).

**Step 2** \( H_R(l_\Phi(0), k) < 0 \) where, given \( \Phi(l, k) = 2g(l) \cdot D(l, k) - 1 \) as defined in Fact 1, \( l_\Phi(k) \) is the unique value satisfying \( \Phi(l_\Phi(k), k) = 0 \).

Straightforward algebra leads to:

\[
H_R(l_\Phi(0), k) = \frac{F(l_\Phi(0))^2}{c} \cdot \left( M \cdot \pi(l_\Phi(0), k) \cdot [g(l_\Phi(0)) \cdot (1 - l_\Phi(0)) - 1] + D(l_\Phi(0), k) \cdot [g(l_\Phi(0)) \cdot \pi(l_\Phi(0), k) - M + 1] \right) + 1.
\]

Recall that \( \pi(l, k) \equiv M \cdot (1 - l) - D(l, k) \). Abusing notation, we have that \( \frac{\partial \pi(l_\Phi(0), k)}{\partial M} = 1 - l_\Phi(0) \). Recognizing that \( H_R(l_\Phi(0), k) \) depends on \( M \) both directly and through \( \pi(l_\Phi(0), k) \), we can then derive the following expression:

\[
\frac{dH_R(l_\Phi(0), k)}{dM} = 2M \cdot \frac{F(l_\Phi(0))^2}{c} \cdot (1 - l_\Phi(0)) \cdot [g(l_\Phi(0)) \cdot (1 - l_\Phi(0)) - 1].
\]

By construction of \( l_\Phi(0) \), we have that \( g(l_\Phi(0)) \cdot D(l_\Phi(0), k) = 1/2 < 1 \). Hence, it follows that \( g(l_\Phi(0)) \cdot (1 - l_\Phi(0)) - 1 < 0 \). Hence, we have that \( \frac{\partial H_R(l_\Phi(0), k)}{\partial M} < 0 \). Hence, there exists \( \overline{M} \) such that \( H_R(l_\Phi(0), k)|_M \) is strictly smaller than \( \overline{M} \) for all \( M \geq \overline{M} \). Tedious but straightforward algebra shows that \( \overline{M} < \overline{M} \), as defined in Definition 4.

**Step 3** \( H_R(\hat{l}, k) = 0 \) for some \( \hat{l} \in (l_B(k), l_\Phi(0)) \).

It follows directly from Steps 1 and 2 and the continuity of \( H_R(l, k) \) in \( l \).

**Step 4** For any \( \hat{l} \in (l_B(k), l_\Phi(0)) \) such that \( H_R(\hat{l}, k) = 0 \), it follows that \( \frac{\partial H_R(l, k)}{\partial l} < 0 \).

Define the following function:

\[
\chi(l) \equiv M \cdot \pi(l, k) \cdot [g(l) \cdot (1 - l) - 1] + D(l, k) \cdot [g(l) \cdot \pi(l, k) - M + 1].
\]

We can then write the regulator’s first derivative w.r.t. liquidity as follows:

\[
H_R(l, k) = \frac{F(l)^2}{c} \cdot \chi(l) + 1.
\]

By construction of \( \hat{l} \) (Step 3), we have that:

\[
H_R(\hat{l}, k) = \frac{F(\hat{l})^2}{c} \cdot \chi(\hat{l}) + 1 = 0.
\]
Hence, it follows that $\chi (\hat{l}) < 0$.

The first derivative of $H_R (l, k)$ w.r.t. $l$ can be written, for $l > 0$, as:
\[
\frac{\partial H_R (l, k)}{\partial l} = \frac{F(l)^2}{c} \cdot \left( 2g (l) \cdot \chi (l) + \frac{\partial \chi (l)}{\partial l} \right).
\]

Since $\chi (\hat{l}) < 0$, recognizing that $F(l) > 0$ and $g (l) > 0$ for all $l$, it suffices to show that $\frac{\partial \chi (l)}{\partial l} < 0$. We can write:
\[
\frac{\partial \chi (l)}{\partial l} = M^2 \cdot [1 - 2g (l) \cdot (1 - l)] + [M^2 \cdot (1 - l) - D (l, k)] \cdot (1 - l) \cdot \frac{\partial g (l)}{\partial l}
\]
\[
+ [g (l) \cdot (1 - l) - 1] + [D (l, k) \cdot g (l)] \leq
\]
\[
\leq \left[ (M^2 - 1) \cdot (1 - 2g (l) \cdot (1 - l)) \right] + [M^2 \cdot (1 - l) - D (l, k)] \cdot (1 - l) \cdot \frac{\partial g (l)}{\partial l}
\]

By definition of $l_\Phi (0)$, we know that $1 - 2g (l_\Phi (0)) \cdot (1 - l_\Phi (0)) = 0$. Recognizing that $g (l) \cdot (1 - l)$ is strictly decreasing in $l$ (recall that both $1 - l$ and $g (l)$ are strictly decreasing and positive functions, the latter because of log-concavity of the distribution function), it follows that $1 - 2g (\hat{l}) \cdot (1 - \hat{l}) < 0$. Clearly, $M^2 - 1 > 0$. Moreover, since $M > 1$, we have that $M^2 \cdot (1 - l) - D (l, k) > M \cdot (1 - l) - D (l, k) = \pi (l, k) > 0$. From log-concavity of the distribution function, we have that $\frac{\partial g(l)}{\partial l} < 0$ for all $l$. Hence, it follows that $\frac{\partial \chi (l)}{\partial l} < 0$.

**Step 5** There exists a unique $\hat{l} \in (l_B (k), l_\Phi (0))$, which we dub $l_R (k)$, such that $H_R (l_R (k), k) = 0$.

Assume, for the sake of contradiction, that there is more than one $\hat{l} \in (l_B (k), l_\Phi (0))$ such that $H_R (\hat{l}, k) = 0$. Let $\hat{l}_1, \hat{l}_2$, with $\hat{l}_1 < \hat{l}_2$ be the first two values satisfying $H_R (\hat{l}_1, k) = H_R (\hat{l}_2, k) = 0$.

From continuity of $H_R (l, k)$ with respect to $l$, it follows that $\frac{\partial \chi (\hat{l}_2)}{\partial l} > 0$, which contradicts the statement shown in Step 4. ■ ■

**Proof of Proposition 4.** Since, for any given $l$, $\Pi_R (l, k)$ is strictly concave in $k$ in all of its domain, the first order necessary condition for an interior optimal capital requirement is also sufficient. Consider the first order condition of the social welfare objective function with respect to capital:
\[
\frac{d\Pi_R (l, k)}{dk} = \frac{\partial \Pi_R (l, k)}{\partial \theta} \cdot \frac{\partial \theta_B (l, k)}{\partial k} + \frac{\partial \Pi_R (l, k)}{\partial k},
\]
which we can write as follows:
\[
\frac{d\Pi_R (l, k)}{dk} = \frac{1 + r}{c} \cdot F^2 (l) \cdot D (l, k) - (\rho - r).
\]

Take any given set of parameters $c$, $r$ and $\rho$. Fix any $l \in [0, d]$. Then, $k_R (l) > 0$ only if $\frac{d\Pi_R (l, k_R (l))}{dk} = 0$. Hence, if $k_R (l) > 0$ we have that $\Upsilon (l, k_R (l)) = \frac{1}{1+r}$. Moreover, if $\frac{d\Pi_R (l, k)}{dk} < 0$ for all $k \in [0, 1 - d]$, then $k_R (l) = 0$. The proof consists of comparing $\Upsilon (l, k) \equiv F^2 (l) \cdot D (l, k)$ with $\frac{(\rho - r) \cdot c}{1+r}$. In order to do this comparison, we first show that $\Upsilon (l, k)$ satisfies the following properties.
Fact 3  The function $\Upsilon(l, k) \equiv F^2(l) \cdot D(l, k)$ satisfies the following properties.

(a) For any given $k \in [0, 1 - d]$, $\Upsilon(l, k)$ is strictly increasing in $l$ for $l < l_\Phi(k)$ and strictly decreasing in $l$ for $l > l_\Phi(k)$, where $l_\Phi(k)$ was implicitly defined as $\Phi(l_\Phi(k), k) = 0$ in Fact 1.

(b) For any given $l \in (0, d]$, $\Upsilon(l, k)$ is strictly decreasing in $k$.

(c) $\Upsilon(l_\Phi(k), k)$ is strictly decreasing in $k$.

Proof of Fact 3.

(a) Fix $k$, and take the first derivative of $\Upsilon(l, k)$ with respect to $l$ to get $\frac{\partial \Upsilon(l, k)}{\partial l} = F^2(l) \cdot \Phi(l, k)$, where $\Phi(l, k)$ is as defined on Fact 1. Since $F^2(l) > 0$, the sign of $\frac{\partial \Upsilon(l, k)}{\partial l}$ is determined by $\Phi(l, k)$.

From Property (i) in Fact 1 and the definition of $l_\Phi(k)$, we have that $\frac{\partial \Upsilon(l, k)}{\partial l} > 0$ for all $l \in (0, l_\Phi(k))$.

Also, from Property (i) in Fact 1, we have that $\frac{\partial \Upsilon(l, k)}{\partial l} < 0$ for all $l \in (l_\Phi(k), d)$. Hence, the result follows.

(b) This statement follows directly from first differentiating $\Upsilon(l, k)$ with respect to $k$, to obtain $\frac{\partial \Upsilon(l, k)}{\partial k} = -F^2(l) \cdot (1 + r) < 0$.

(c) Take any two $k_1 < k_2$. From (b) it follows that, for any given $l \in (0, d]$, $\Upsilon(l, k_1) > \Upsilon(l, k_2)$.

Also, by definition of $l_\Phi(k)$, we have that $\Upsilon(l_\Phi(k_1), k_1) \geq \Upsilon(l, k_1)$ for all $l \in (0, d]$. Combining these two inequalities, we have that $\Upsilon(l_\Phi(k_1), k_1) > \Upsilon(l, k_2)$ for any given $l \in (0, d]$. In particular, the last inequality holds true for $l = l_\Phi(k_2)$. Hence, we can write $\Upsilon(l_\Phi(k_1), k_1) > \Upsilon(l_\Phi(k_2), k_2)$. ■

Now we show the Proposition statements.

(i) Let $\rho_{\text{max}}$ be the unique value satisfying $\Upsilon(l_\Phi(0), 0) = \frac{(\rho_{\text{max}} - r)c}{1 + r}$. Observe that, for any given $k$, $\Upsilon(l, k)$ is maximized at $l_\Phi(k)$ (Property (a)); also, for any given $l$, $\Upsilon(l, k)$ is maximized at $k = 0$ (Property (b)). Hence, $\Upsilon(l, k)$ is jointly maximized for $(l, k) = (l_\Phi(0), 0)$. Hence, by construction of $\rho_{\text{max}}$, it follows that $\Upsilon(l, k) < \frac{(\rho - r)c}{1 + r}$ for all $(l, k)$, as we wanted to show.

(ii) Now, let $\rho < \rho_{\text{max}}$. We prove this result in a succession of steps.

Step 6  The optimal capital response at $l = 0$ is zero, i.e., $k_R(0) = 0$.

Observe that $\Upsilon(0, k) = 0 < \frac{(\rho - r)c}{1 + r}$ for all $k \in [0, 1 - d]$. Hence, we have that $k_R(0) = 0$.

Step 7  There exists a value $l_1(\rho) \in (0, d)$ such that $\Upsilon(l_1(\rho), 0) = \frac{(\rho - r)c}{1 + r}$. Moreover, $\Upsilon(l, 0) < \frac{(\rho - r)c}{1 + r}$ for $l < l_1(\rho)$ and $\Upsilon(l, 0) > \frac{(\rho - r)c}{1 + r}$ for $l > l_1(\rho)$.

First, recall from the previous statement that $\Upsilon(0, 0) = 0 < \frac{(\rho - r)c}{1 + r}$. Moreover, by construction of $\rho_{\text{max}}$, which is such that $\Upsilon(l_\Phi(0), 0) > \frac{(\rho_{\text{max}} - r)c}{1 + r}$, we have that $\Upsilon(l_\Phi(0), 0) > \frac{(\rho - r)c}{1 + r}$, where $l_\Phi(0) \in (0, d)$ (From Property (iv) in Fact 1). Moreover, it follows from Property (a) in Fact 3, that $\Upsilon(l, k)$ is strictly increasing in $l$ for $l < l_\Phi(0)$. Hence, by continuity of $\Upsilon(l, k)$ in its first argument, it follows that there exists a unique value for liquidity, which we dub $l_1(\rho)$, such that $\Upsilon(l_1(\rho), 0) = \frac{(\rho - r)c}{1 + r}$. It also follows that $\Upsilon(l, 0) < 0$ for $l < l_1(\rho)$ and $\Upsilon(l, 0) > 0$ for $l > l_1(\rho)$. 56
Step 8 The optimal response for \( l \leq l_1 (\rho) \) is zero, i.e., \( k_R(l) = 0 \) for any \( l \leq l_1 (\rho) \).

We have that \( \Upsilon (l, k) \) is strictly decreasing in \( k \) for all \( l > 0 \) (Property (b) in Fact 3). Also, we know from Step 7 that \( \Upsilon (l, 0) < 0 \) for \( l < l_1 (\rho) \). Hence, it follows that \( \Upsilon (l, k) < 0 \) for any \( l < l_1 (\rho) \) and any \( k \in [0, 1 - d] \).

Step 9 There exists a unique liquidity maximizer \( \hat{l} (\rho) \in (l_1 (\rho), d) \) of the capital response function, that is, \( k_R(\hat{l} (\rho)) > k_R(l) \) for all \( l \neq \hat{l} (\rho) \). Moreover, \( \hat{l} (\rho) \) and \( k_R(\hat{l} (\rho)) \) are defined by \( \Phi (\hat{l} (\rho), k_R(\hat{l} (\rho))) = 0 \).

By construction of \( \rho_{\max} \), we have that \( \Upsilon (l_\Phi (0), 0) > \frac{(\rho-r)c}{1+r} \). From Property (c) in Fact 3, it follows that there exists a unique \( k' > 0 \) such that \( \Upsilon (l_\Phi (k'), k') = \frac{(\rho-r)c}{1+r} \). Let \( \hat{l} (\rho) = l_\Phi (k') \). Hence, we have that \( k_R (\hat{l} (\rho)) = k' \). Now, consider \( k'' > k' \). Then, using again Property (c) in Fact 3, it follows that \( \Upsilon (l_\Phi (k''), k'') < \frac{(\rho-r)c}{1+r} \). Now, from Property (iv) in Fact 1, we have that \( l_\Phi (k) \) is strictly decreasing in \( k \). Hence, since \( k'' > k' \), we have that \( l_\Phi (k'') < l_\Phi (k') \). Also, from Property (a) in Fact 3, we know that \( \Upsilon (l, k) \) is strictly increasing in \( l \) for \( l < l_\Phi (k) \). Hence, since \( l_\Phi (k'') < l_\Phi (k') \), it follows that \( \Upsilon (l_\Phi (k''), k'') < \Upsilon (l_\Phi (k'), k'') \). Consequently, \( \Upsilon (l_\Phi (k''), k'') < \frac{(\rho-r)c}{1+r} \). Hence, \( k_R (\hat{l} (\rho)) = k' \) is indeed the maximum capital response, which is achieved at \( \hat{l} (\rho) = l_\Phi (k') \). Finally, from construction of \( l_\Phi (k') \), it follows that \( \Phi (\hat{l} (\rho), k_R(\hat{l} (\rho))) = 0 \).

Step 10 The function \( \Phi (l, k_R(l)) \) is such that \( \Phi (l, k_R(l)) > 0 \) if \( l < \hat{l} (\rho) \).

From Step 9, we know that \( \Phi (\hat{l} (\rho), k_R(\hat{l} (\rho))) = 0 \). Consider \( l < \hat{l} (\rho) \). We also know from Step 9 that \( k_R(\hat{l} (\rho)) > k_R(l) \). From Properties (ii) and (iii) in Fact 1, we know that \( \Phi (l, k) \) is strictly decreasing in \( k \) and in \( l \). Hence, it follows that \( \Phi (l, k_R(l)) > \Phi (\hat{l} (\rho), k_R(\hat{l} (\rho))) = 0 \) for \( l < \hat{l} (\rho) \).

Step 11 For any \( l \in (l_1 (\rho), \hat{l} (\rho)) \), we have that the optimal capital response increases in \( l \), that is, \( k_R (l) \) strictly increasing in \( l \).

From Step 7, we have that \( \Upsilon (l, 0) > \frac{(\rho-r)c}{1+r} \) for \( l > l_1 (\rho) \). Hence, we have an interior solution, that is, \( k_R (l) \) is such that \( \Upsilon (l, k_R(l)) = \frac{(\rho-r)c}{1+r} \). From Property (b) in Fact 3, we know that \( \Upsilon (l, k) \) is strictly decreasing in \( k \). Hence, there exists a unique capital level, which we label \( k_R (l) \), such that \( \Upsilon (l, k_R(l)) = \frac{(\rho-r)c}{1+r} \). Now, we show that \( k_R (l) \) is strictly increasing by showing that \( \frac{\partial k_R (l)}{\partial l} > 0 \). Implicitly differentiating \( \Upsilon (l, k_R(l)) = \frac{(\rho-r)c}{1+r} \) with respect to \( l \) it follows that \( \frac{\partial k_R (l)}{\partial l} > 0 \) if and only \( \frac{\partial \Upsilon (l, k_R(l))}{\partial l} > 0 \), that is, if and only if \( \Phi (l, k_R(l)) > 0 \). Hence, it follows from Step 10 that if \( l < \hat{l} (\rho) \) we have \( \frac{\partial k_R (l)}{\partial l} > 0 \).
Step 12 For any \( l \in (\hat{l}(\rho), l_2(\rho)) \), we have that the optimal capital response decreases in \( l \), that is, \( k_R(l) \) strictly decreasing in \( l \).

From construction of \( k_R(l) \), we have that \( \Upsilon(\hat{l}(\rho), k_R(\hat{l}(\rho))) = \frac{(\rho-r)c}{1+r} \). Moreover, since \( \Upsilon(l,k) \) is strictly decreasing in \( k \), it follows that \( \Upsilon(\hat{l}(\rho),0) > \frac{(\rho-r)c}{1+r} \). By continuity of \( \Upsilon(l,k) \) with respect to its first argument, it follows that there exists \( \tilde{l}_2(\rho) > \hat{l}(\rho) \) such that \( \Upsilon(l,0) > \frac{(\rho-r)c}{1+r} \) for all \( l \in (\hat{l}(\rho), \tilde{l}_2(\rho)) \). Hence, for all \( l \in (\hat{l}(\rho), \tilde{l}_2(\rho)) \), we have that \( k_R(l) > 0 \). Hence, for all \( l \in (\hat{l}(\rho), \tilde{l}_2(\rho)) \), \( k_R(l) \) is given by \( \Upsilon(l, k_R(l)) = \frac{(\rho-r)c}{1+r} \). Consider \( l \in (\hat{l}(\rho), \tilde{l}_2(\rho)) \) and let \( k_R(l) \) be such that \( \Upsilon(l, k_R(l)) = \frac{(\rho-r)c}{1+r} \). Recall that \( \Upsilon(\hat{l}(\rho), k_R(\hat{l}(\rho)) = \frac{(\rho-r)c}{1+r} \) for all \( l \in (\hat{l}(\rho), \tilde{l}_2(\rho)) \). Now, since \( \Upsilon(l,k) \) strictly decreasing in \( k \), it follows that \( k_R(l) < k_R(\hat{l}(\rho)) \). Hence, we have shown that \( k_R(l) < k_R(\hat{l}(\rho)) \) for \( l \in (\hat{l}(\rho), \tilde{l}_2(\rho)) \). Now, consider any \( l' \in (\hat{l}(\rho), \tilde{l}_2(\rho)) \) such that \( l' > l \). Then, by the same token, we have that \( k_R(l') < k_R(l) \).

Finally, to complete this statement of the proof, consider the following two cases. First, assume that \( \Upsilon(d,0) \geq \frac{(\rho-r)c}{1+r} \). Then, it follows that \( d = \tilde{l}_2(\rho) \). In this case, we have that \( l_2(\rho) = d \). Now suppose, on the contrary, that \( \Upsilon(d,0) < \frac{(\rho-r)c}{1+r} \). In this case, we have that there exists \( l_2(\rho) < d \) such that \( \Upsilon(l_2(\rho),0) = \frac{(\rho-r)c}{1+r} \). Then, for any \( l \in (l_2(\rho), d) \), we have that \( \Upsilon(l,0) < \frac{(\rho-r)c}{1+r} \), in which case \( k_R(l) = 0 \).

(iii) For the remainder of the proof, let \( \rho' < \rho \). Recall from Step 7 above that \( l_1(\rho) \) is defined as \( \Upsilon(l_1(\rho),0) = \frac{(\rho-r)c}{1+r} \). Then, we have that \( \Upsilon(l_1(\rho),0) > \frac{(\rho'-r)c}{1+r} = \Upsilon(l_1(\rho'),0) \), where the first inequality follows from \( \rho' < \rho \) and the last inequality follows from definition of \( l_1(\rho') \). Since \( \Upsilon(l,0) \) is strictly increasing in \( l \) for any \( l < l_\Phi(0) \) (Property (a) in Fact 3) and \( l_1(\rho) < l_\Phi(0) \), the result follows.

Now, if \( l_2(\rho) = d \), we have argued in the last statement of the proof of item (ii) that \( \Upsilon(d,0) \geq \frac{(\rho-r)c}{1+r} \). Hence, it follows that \( \Upsilon(d,0) > \frac{(\rho'-r)c}{1+r} \). Hence, \( l_2(\rho) = d \). Now suppose that \( l_2(\rho) < d \), in which case \( \Upsilon(l_2(\rho),0) = \frac{(\rho-r)c}{1+r} \). It then follows that \( \Upsilon(l_2(\rho),0) > \frac{(\rho'-r)c}{1+r} \). Since \( \Upsilon(l,0) \) is strictly decreasing in \( l \) for any \( l > l_\Phi(0) \) (Property (a) in Fact 3) and \( l_2(\rho) > l_\Phi(0) \), the result follows.

Finally, consider \( l \in (l_1(\rho), l_2(\rho)) \). Then, we have that \( \Upsilon(l, k_R(l)) = \frac{(\rho-r)c}{1+r} > \frac{(\rho'-r)c}{1+r} \). Hence, since \( \Upsilon(l,k) \) is strictly decreasing in \( k \), it follows that \( k_R(l) \) must increase so as to meet the last inequality. 

**Proof of Proposition 5.**

Since, for any given \( k \), \( \Pi_R(l,k) \) is strictly quasiconcave in \( l \) in the domain \([l_B(k), l_\Phi(0)]\), the first order necessary condition for an interior optimal liquidity requirement is also sufficient.
(i) Any liquidity requirement such that \( l \leq l_B(k) \) is non-binding, so that the bank would choose \( l_B(k) \) if the liquidity requirement is smaller. This requirement is not optimal, since \( \frac{d\Pi_R(l_R(k),k)}{dl} > 0 \) (Step 1 in the Proof of Fact 2). Moreover, we have that there exists a unique value of liquidity \( l_R(k) \in (l_B(k), l(0)) \) such that \( \frac{d\Pi_R(l_R(k),k)}{dl} = 0 \) (Step 5 in the Proof of Fact 2). Hence, the liquidity requirement \( l_R(k) \) is such that \( l_R(k) > l_B(k) \).

(ii) In order to prove this result, we show that there exists a value of capital \( \hat{k}(M) \) such that, for any \( k < \hat{k}(M) \), it follows that \( \frac{\partial R(k)}{\partial k} > 0 \), while for any \( k > \hat{k}(M) \), we have that \( \frac{\partial R(k)}{\partial k} < 0 \). We start by calculating the slope of \( l_R(k) \), which is given by:

\[
\frac{\partial l_R(k)}{\partial k} = - \frac{\partial^2 \Pi_R(l,k)}{\partial k \partial l}.
\]

From the regulator’s problem second order condition with respect to \( l \), we know that \( \frac{\partial^2 \Pi_R(l,k)}{\partial l^2} < 0 \), so that the sign of \( \frac{\partial l_R(k)}{\partial k} \) is given by the second order cross derivative, which in turn is given by:

\[
\frac{\partial^2 \Pi_R(l,k)}{\partial k \partial l} = \frac{(1 + r)F(l)^2}{c} \left[ 2g(l)D(l,k) - 1 \right].
\]

Note that the term in brackets determines the sign of \( \frac{\partial l_R(k)}{\partial k} \), and therefore the sign of \( \frac{\partial l_R(k)}{\partial k} \). Particularly, if the second order cross derivative equals zero, then we have that \( \frac{\partial l_R(k)}{\partial k} = 0 \). Let us now define:

\[
\Phi(l,k) \equiv [2g(l)D(l,k) - 1] = 0, \quad (25)
\]

as the loci of \( k \) and \( l \) such that \( \frac{\partial^2 \Pi_R(l,k)}{\partial l \partial k} = 0 \). Note that the term inside brackets decreases with capital and liquidity. Hence, if the the pair \( (l_R(k),k) \) lies below the curve \( \Phi(l,k) \) evaluated at \( (l_R(k),k) \), then we have that \( 2g(l_R(k))D(l_R(k),k) - 1 > 0 \) and that \( \frac{\partial l_R(k)}{\partial k} > 0 \), as otherwise \( \frac{\partial l_R(k)}{\partial k} < 0 \). Since, for \( k = 0 \), we have that \( \Phi(l_R(0),0) > 0 \), while for \( k = 1 - d \) we have that \( \Phi(\hat{l}_R(1 - d),1 - d) < 0 \), the continuity of \( l_R(k) \) guarantees that there exists at least a value of equity capital \( \hat{k}(M) \) such that \( \Phi \left( l_R(\hat{k}(M)), \hat{k}(M) \right) = 0 \).

We now prove that \( \hat{k}(M) \) is unique. We prove this statement by showing that the curve \( l_R(k) \) never crosses the curve \( \Phi(l,k) \) from above. Hence, \( \frac{\partial l_R(k)}{\partial k} \) cannot change its sign from negative to positive. By the sake of contradiction, assume that there exists a value \( \hat{k}' > \hat{k}(M) \) such that \( l_R(k) \) crosses \( \Phi(l,k) \) at \( (l_R(\hat{k}'), \hat{k}') \) from above. This implies that \( \frac{\partial l_R(k)}{\partial l} < 0 \). However, since \( \Phi(l_R(\hat{k}'), \hat{k}') = 0 \), the slope of \( l_R \) at \( \hat{k}' \) must be \( \frac{\partial l_R(\hat{k}')}{\partial k} = 0 \). Then, there is no value of \( k > \hat{k}(M) \) such that \( \frac{\partial l_R(k)}{\partial k} > 0 \). We thus conclude that \( l_R(k) \) increases for \( k < \hat{k}(M) \) and decreases for \( k > \hat{k}(M) \). Hence, \( l_R(k) \) is a hump-shaped function of capital.
Proof of Proposition 6. (i) (Uniqueness) First, we show that the regulator’s maximization problem satisfies the condition for the Theorem of the Maximum (Berge (1963)) for quasiconcave objective functions over convex-valued sets to apply. Let $M$ take a fixed value. Define the set of possible joint liquidity and capital optimal requirements $\Phi = [0, d] \times [0, 1 - d]$ and the set of all possible values of the cost of capital $\Gamma = [\rho_{\text{min}}, \rho_{\text{max}}]$. Define the real-valued function $\Pi_R : \Phi \times \Gamma \to \mathbb{R}$ as $\Pi_R (l, k, \rho) \equiv \Pi_R (l, k)$, as defined in equation (6), where $\Pi_R (l, k, \rho)$ is simply a relabeling of the regulator’s objective function to let it depend not only on $k$ and $l$, but also on the cost of capital parameter $\rho$. Moreover, define the mapping $\Phi (\rho) : \Gamma \to \Phi$ of possible joint liquidity and capital optimal requirements for each value of the cost of capital $\rho \in \Gamma$. That is, we have that $\Phi (\rho) = [0, d] \times [0, 1 - d]$ for all $\rho \in \Gamma$. Let the pair $(l^* (\rho, M), k^* (\rho, M)) \in \arg \max_{(l,k) \in \Phi (\rho)} \Pi_R (l, k, \rho)$ be the optimal liquidity and capital requirements, for a given fixed $M$, and let $\Pi_R^* (\rho) = \Pi_R (l^* (\rho, M), k^* (\rho, M), \rho)$ be regulator’s value for each $\rho \in \Gamma$. First, we have that $\Pi_R (l, k, \rho)$ is jointly continuous in all its arguments. Moreover, $\Pi_R (l, k, \rho)$ is strictly quasiconcave in $(l, k)$ for each $\rho \in \Gamma$ (we have shown above that $\Pi_R (l, k)$ is strictly concave in $k$ and single-peaked–hence quasiconcave–in $l$; proving that $\Pi_R (l, k)$ is quasiconcave in $(l, k)$ entails computing the bordered Hessian of $\Pi_R (l, k)$, which is tedious but straightforward). Also, $\Phi (\rho)$ is a compact-valued, convex-valued and continuous correspondence for all $\rho \in \Gamma$, because it is an invariant compact and convex set for all $\rho \in \Gamma$. Hence, by the Theorem of the Maximum for quasiconcave functions over convex-valued sets, it follows that $\Pi_R^* (\rho)$ is continuous in $\rho$ and $(l^* (\rho, M), k^* (\rho, M))$ is a (single-valued) continuous function for all $\rho \in \Gamma$, for any given $M$.

(ii.a) (Zero capital requirements for high cost of capital) From Proposition 4, we have that for all $\rho \geq \rho_{\text{max}}$ the optimal capital response curve is zero for all liquidity levels. In this case, we have that the optimal liquidity and capital requirements are given by $l^* = l_R (0)$ and $k^* = 0$.

For any $\rho < \rho_{\text{max}}$, the optimal capital response curve is positive in a range $(l_1 (\rho), l_2 (\rho))$ and zero elsewhere, i.e., $k_R (\rho) > 0$ if and only if $l \in (l_1 (\rho), l_2 (\rho))$. Moreover, we also know that $l_1 (\rho)$ is continuous and decreasing in $\rho$. Hence, $l_1 (\rho)$ attains a maximum at $l_1 (\rho_{\text{max}})$ and a minimum at $l_1 (\rho_{\text{min}})$. Also, from Proposition 5, we know that $l (k) |_M$ is continuous and decreasing in $M$ for all $k$. Hence, $l (0) |_M$ attains a maximum at $l (0) |_M$ for each $k$. Moreover, we know that $\inf_{M \in (M, \infty)} \{ l (0) |_M \} = l_{\text{ELV}}$ and also that $l (0) |_M > l_{\text{ELV}}$ for all $M < \infty$. Let $\tilde{\rho} (M)$ be implicitly defined as $l (0) |_M = l_1 (\tilde{\rho} (M))$. We distinguish two cases. If $l_1 (\rho_{\text{min}}) > l_{\text{ELV}}$, then there exists $\tilde{M} > M$ such that for any $M \geq \tilde{M}$ the optimal liquidity and capital response curves do only intersect in the range in which the optimal capital response curve is flat (case 1). However, if either $l_1 (\rho_{\text{min}}) > l_{\text{ELV}}$ and $M < \tilde{M}$, or if $l_1 (\rho_{\text{min}}) \leq l_{\text{ELV}}$, then the curves do intersect for some $M$ at some point in which the optimal capital response is positive (case 2). In case 1, we have that the
optimal liquidity and capital requirements are given by \( l^* = l_R(0) \) and \( k^* = 0 \). We focus henceforth on case 2.

Let \( \rho \geq \tilde{\rho}(M) \). We show that the optimal liquidity and capital requirements are given by \( l^* = l_R(0) \) and \( k^* = 0 \), respectively, for any \( \rho \geq \tilde{\rho}(M) \). Fix \( M \) and let \( \tilde{\rho} \) be supremum of the set of values of the cost of capital for which the curves intersect at some level for which capital is positive, that is, \( \tilde{\rho} = \sup \{ \rho : k^*(\rho, M) > 0 \} \). If \( \tilde{\rho} \leq \tilde{\rho}(M) \), then we are done. Suppose now that \( \tilde{\rho} > \tilde{\rho}(M) \). Then, by construction of \( \tilde{\rho} \), we have that the curves intersect in the range in which the optimal capital response is zero for \( \rho > \tilde{\rho} \) and for some level of capital bounded away from zero, \( \tilde{k} > 0 \), for \( \rho = \tilde{\rho} \). Hence, it follows that the capital requirement would be \( k^*(\rho, M) = 0 \) for \( \rho > \tilde{\rho} \).

We now show that and \( k^*(\tilde{\rho}, M) = 0 \) as well. Define the sequences \( \rho_n \equiv \tilde{\rho} + \frac{1}{n} \) and \( k^*_n \equiv k^*(\rho_n, M) \) for all \( n \in \mathbb{N} \). On the one hand, we have that \( \lim_{n \to \infty} \rho_n = \tilde{\rho} \). On the other hand, we have that \( k^*_n = 0 \) for all \( n \in \mathbb{N} \), so that it follows that \( \lim_{n \to \infty} k^*_n = 0 \). Therefore, since \( (l^*(\rho, M), k^*(\rho, M)) \) is continuous, it follows that \( k^*(\lim_{n \to \infty} \rho_n, M) = k^*(\tilde{\rho}, M) = 0 \).

From now on, we consider the case \( \rho < \tilde{\rho}(M) \). First, observe that by construction of \( \tilde{\rho}(M) \), we have that \( l_1(\rho) < l(0) |_M \).

(ii.b) (Raising the cost of one factor reduces the requirement of that factor) For any given \( \rho \) and \( M \), we have that \( k_R(l) \rvert_{\rho} \) is strictly decreasing in \( \rho \) for any \( l \in (l_1(\rho), l_2(\rho)) \) and \( l_R(M) \) is strictly decreasing in \( M \). The result follows immediately.

(ii.c) (Capital and liquidity complements for low cost of capital and substitutes for high cost of capital)

First, we fix \( M \) and analyze the impact of changing \( \rho \) on the optimal liquidity requirements. Observe that the optimal liquidity response curve \( l_R(k) \) is increasing in the complementarity region and decreasing in the substitutability region. Moreover, \( l_R(k) \) is independent of \( \rho \). Hence, since \( M \) is fixed, \( l_R(k) \) does not change with changes in \( \rho \). Now, consider the map of optimal capital responses \( \{k_R(l) \rvert_{\rho}\}_{\rho \in (\rho_{\min}, \tilde{\rho}(M))} \). This map consists of a collection of hat-shaped functions that increase in the complementarity region and decrease in the substitutability region. Suppose that, for any given \( \rho \), \( (l^*(\rho, M), k^*(\rho, M)) \) belongs to the complementarity region. Then, an increase in \( \rho \) shifts the optimal capital response curve down. Since the optimal liquidity response function does not change, the optimal requirements are given by moving down along the optimal liquidity response curve, where both curves intersect. Since \( l_R(k) \) is increasing (as \( (l^*(\rho, M), k^*(\rho, M)) \) is in the complementarity region), it follows that the optimal liquidity requirement \( l^*(\rho, M) \) decreases. By the same token, suppose that for any given \( \rho \), \( (l^*(\rho, M), k^*(\rho, M)) \) belongs to the substitutability region. Then, an increase in \( \rho \) also shifts the optimal capital response curve down. Since the optimal liquidity response function does not change, the optimal requirements are given by moving down
along the optimal liquidity response curve, where both curves intersect. Since $l_R (k)$ is decreasing, it follows that the optimal liquidity requirement $l^* (\rho, M)$ increases. Finally, we need to show that there exists a threshold $\hat{\rho} (M)$ such that the optimal liquidity and capital requirements belong to the complementarity region if $\rho > \hat{\rho} (M)$ and to the substitutability region if $\rho < \hat{\rho} (M)$. Observe that for $\hat{\rho} (M)$, the optimal liquidity requirement $l^* (\hat{\rho} (M), M)$ belongs to the complementarity region. As $\rho$ decreases, the optimal liquidity requirement $l^* (\rho, M)$ moves up along the optimal liquidity response curve $l_R (k)$ until the curve reaches its maximum and crosses to the substitutability region, which occurs for a given $\hat{\rho} (M)$. Then, $l^* (\rho, M)$ decreases along the optimal liquidity response curve $l_R (k)$ within the substitutability region for $\rho < \hat{\rho} (M)$.

Now, we fix $\rho < \hat{\rho} (M)$ and analyze the impact of changing $M$ on the optimal capital requirements. Observe that the optimal capital response curve $k_R (l)$ is increasing in the complementarity region and decreasing in the substitutability region. Now, consider the map of optimal liquidity responses $\{l_R (k) \mid M \}_{\rho \in (\hat{M}, \infty)}$. This map consists of a collection of hump-shaped functions that increase in the complementarity region and decrease in the substitutability region. Moreover, $k_R (l)$ is independent of $M$. Hence, since $\rho$ is fixed, $k_R (l)$ does not change with changes in $M$. Suppose that, for any given $M$, $(l^* (\rho, M), k^* (\rho, M))$ belongs to the substitutability region. Then, an increase in $M$ shifts the optimal liquidity response curve down (or to the left, if one looks at Figure 5.4). Since the optimal capital response function does not change, the optimal requirements are given by moving up along the optimal capital response curve, where both curves intersect. Since $k_R (l)$ is decreasing (as $(l^* (\rho, M), k^* (\rho, M))$ is in the substitutability region), it follows that the optimal capital requirement $k^* (\rho, M)$ increases. By the same token, if $(l^* (\rho, M), k^* (\rho, M))$ belongs to the complementarity region, an elevation of $M$ leads to an increase of the optimal capital requirement $k^* (\rho, M)$. Finally, define $\check{M}$ implicitly so that the chosen cost of capital $\rho$ constitutes the threshold that establishes whether $(l^* (\rho, M), k^* (\rho, M))$ are in the complementarity or in the substitutability region, that is, $\rho = \check{\rho} (M)$. Then, for any $M > \check{M}$ it follows that $\rho > \check{\rho} (M)$. In this case, the optimal liquidity and capital requirements $(l^* (\rho, M), k^* (\rho, M))$ belong to the substitutability region. On the contrary, if $M < \check{M}$ then $\rho < \check{\rho} (M)$, so that the optimal requirements $(l^* (\rho, M), k^* (\rho, M))$ belong to the complementarity region.

(ii.d) (Capital and liquidity are always complements when the return to investment is high) It follows directly from Proposition 5, statement (ii.b).

(ii.e) (Zero capital requirements for high return to investment) The cost of capital threshold $\check{\rho} (M)$ that determines whether capital requirements are positive or not is defined as $l (0) \mid_M = l_1 (\check{\rho} (M))$. Since $l_1 (\rho)$ is invariant with $M$ and $l (0) \mid_M$ is strictly decreasing in $M$, the result follows immediately. \blacksquare