Characterization of Bidding Behavior in Multi-Unit Auctions and Applications

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Abstract

This paper provides first order conditions for a very general class of single or multi-unit auctions. Under mild conditions we characterize bidding-behavior and generalize previous standard results in the literature. As an application we obtain sufficient conditions for truth-telling, monotonic best reply strategies and identification results for multi-unit auctions.

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1 Introduction

Many experimental and empirical work suggest that the participants of auctions do (or at least may) not follow their equilibrium strategies. Although there is a considerable debate about this point, it highlights the assumption that equilibrium behavior might be too strong. An alternative approach is to assume only that the players follow rationalizable strategies, instead of equilibrium strategies. Pursuing this idea, Battigalli and Siniscalchi (2003) show that some empirical and experimental findings can be explained. Nevertheless, they still assume what Harsanyi (1967-8) calls consistency of beliefs, that is, the subjective probability that players attribute to the distribution of signals of the opponents is just a conditional distribution and the conditional distribution of all players comes from the same prior distribution. This is almost always assumed in game theory and one may think that nothing can be said without this basic assumption. To the contrary, we show that if we adhere to the even weaker assumption that bidders are rational, we can still characterize their strategic behavior.

Our model encompasses a very general class of seal-bid auction models. We allow for interdependent values, asymmetric valuations, any attitude towards risk, non-monotonic valuations, non-separable transfers, dependent signals of any dimension (universal type spaces), unitary or multiple unit demands auctions with just sellers or buyers or both. Under general conditions we prove what we call the basic principle of bidding. This formalizes in a general setup an intuition that auction theorists already have. That is “a rational bidder bids in order to equalize the marginal benefit of bidding (the utility that she obtains in case of winning) to the marginal cost of bidding”.

1For a survey of experimental works, see Kagel (1995) and for the empirical literature on auction data, see Laffont (1997).
2This is also called common prior assumption.
In one sense, it is obvious that in smooth optimization problems, at the optimum the marginal benefit (derivative of the objective function) equals to the marginal cost (shadow price) of the constraints. Nevertheless, this is not exactly the case for auctions, where the marginal cost does not come from a constraint. This is also different from the classic firm’s problem: marginal revenue equals to marginal cost, because we do not need to assume separability of revenues and costs. In auctions, the marginal costs and benefits come from another source. The basic trade-off that a bidder faces is that a higher bid, although it increases the probability of winning, it may also decreases the payoff in case of winning. Using the Leibiniz rule, to differentiate an integral that depends on the variable both in the region of integration and in the integrand, we obtain two terms. These two terms can be interpreted as marginal benefit and marginal cost.

Although the prove is reminiscent of Leibiniz rule in differential calculus, we rely on the differential theory of measures. When we introduce additional assumptions, i.e., continuously differentiability of payoffs with respect to bids, we provide first order conditions that generalize those obtained by Milgrom and Weber (1982) for first- and second-price auctions, Krishna and Morgan (1997) for the all-pay auction and war of attrition, and Williams (1991) for buyers’-bids double auctions. When one introduces the additional hypotheses of risk neutrality, symmetry and monotonicity of the utility function, the characterization provided reduces to the ones on those papers. In addition, we provide first order conditions for the multi-unit discriminatory, uniform and Vickrey auction.

The payoff characterization lemma, which is the main result of this paper, and which is valid in the most general setting, interdependent values, asymmetric valuations, any attitude towards risk, non-monotonic valuations or separability of transfers, dependent signals of any dimension and unitary or multiple unit demands, opens the way to a general approach to equilibrium existence for
general auction models like in Araujo and De Castro (2005). It can also provide insights for empirical and experimental studies, since every bid (even the initial or the apparently inconsistent ones in a repeated game) bears valuable information about the players’ beliefs and the first order conditions is a first step towards characterizing rational behavior in general auctions. Also, as the recent literature on econometric identification of auction models has pointed out, characterizing best reply bidding strategies allows for identification in many standard auction formats. Along this approach, our result provides ground for general econometric identification of multidimensional auction models. We present some of these results in subsection 4.3. We also use our results to give a simple characterization of truth-telling in multi-unit auctions. After our results, the proof that Vickrey auction is truth-telling is immediate.

The paper is organized as follows. Section 2 presents the model and notation. Section 3 contains the main results and some examples of direct applications. Section 4 uses the results to obtain a proof of the truth-telling property of Vickrey auctions. We also prove a monotonic-best reply result that generalizes for multiunits auctions a result of Araujo and de Castro (2007), which is the key result for their proof of equilibrium existence in single-object auctions. In this section we also report results on inidentification in multiunit auctions, some of which are new, to the best of our knowledge. Section 5 is the appendix, contains some proofs.

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3See Athey and Haile (2005) for a survey on the main issues regarding the econometric identification of auction models.
2 The Model

2.1 Players and Information

There are \( N \) strategic players, we denote by \( \mathcal{N} = \{1,...N\} \) the set of strategic players. Player \( i \in \mathcal{N} \) receives a signal (i.e., private information), \( t_i \in T_i \) where \( T_i \) is the information set of player \( i \). We denote by \( t = (t_1, t_2, ..., t_N) = (t_i, t_{-i}) \) the vector of all players’ information, where \( t_{-i} = (t_1, t_{i-1}, t_{i+1}, ..., t_N) \), as usual.

Let \( \mathcal{F}_i \) be a \( \sigma \)-field of subsets of \( T_i \) and define \( T \equiv \prod_{i=1}^{N} T_i \) and the product \( \sigma \)-field over \( T \), \( \mathcal{F} = \prod_{i=1}^{N} \mathcal{F}_i \). We assume that there is a probability space \((T, \mathcal{F}, \tau)\). Define the probability spaces, \((T_i, \mathcal{F}_i, \tau_i)\), where each \( \tau_i \) is the marginal probability of \( \tau \) (i.e., \( \tau_i(A) = \tau\{t \in T : t_i \in A\}\)). For notational simplicity we assume that \( \tau \) is the same for every agent but, nothing that follows depends on that. That is, our results do not require the common prior assumption. If \( g \) is a function of \( t_{-i} \), we denote the expectation of \( g \) with respect to \( \tau_{-i} \), given \( t_i \), by \( E[g|t_i] \).

Notice that individual signals may be dependent and of arbitrary dimension. We allow for the existence of an uninformed and non-strategic player, named 0. This is the seller in traditional auctions. For double auctions, there is no such player. We denote by \( \mathcal{N}_0 = \{0,1,...N\} \) the set of all players (strategic and non-strategic). Also, \( \mathcal{N}_{0-i} \) denotes the set of strategic and non-strategic players except for agent \( i \), and similarly for \( \mathcal{N}_{-i} \).

2.2 Objects and Bidding

There are \( K \) identical indivisible objects. Each player \( i \in \mathcal{N}_0 \) comes to the auction with \( e_i \in \{0,1,2,...\} \) units of the same object, and \( \sum_{i=0}^{N} e_i = K \). After receiving its signal, a strategic player submits a sealed proposal, that is, a bid.

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4Our model is inspired in auction games, although it can encompass a general class of discontinuous games. For convenience and easy understanding, we will use the terminology of auction theory, such as “bidding functions” and “bids” for strategies and actions, respectively.
(or offer) that is a vector of real numbers, $b_i \in B \subseteq \mathbb{R}^K$ where $B$ denotes the set of valid bids, that is, $B = \{ b \in \mathbb{R}^K : b_k \geq b_{k+1} \text{ for } k = 1, \ldots, K \} \cap [\underline{b}, \overline{b}]$, $b_{i,k}$ is the maximum value that bidder $i$ is willing to pay for the $k$'th unit, given that he is receiving $k - 1$ units; and $[\underline{b}, \overline{b}]$ denotes a $K$ dimensional rectangle that bounds the set of all bids. Since bids are non-increasing we are implicitly assuming that there are no complementarity among objects. Bids are in units of account (i.e., dollars). The non-strategic player 0 also places a bid $b_0 \in B$, meaning that there is a reserve price for each unit. For instance, in a one-object auction ($K = 1$) where all players are buyers, if $\max_{j=1,\ldots,N} b_{j,1} < b_{0,1}$, this means that none of the bidders are willing to pay the reserve price. The difference is that $b_0$ is known for everyone at the time the auction takes place, while $b_j$, $j \neq 0$, is not known for bidder $i \neq j$, $i \in \mathcal{N}$. We denote by $b$ the vector of all players’ bids, $b \in \mathbb{R}^{(N+1)K}$.

### 2.3 Allocation and Payoffs

The “auction house” computes the bids and determines how many units each player receives. If player $i$ wins a $k$'th unit, his payoff is increased by $u_{i,k}(t, b)$, where $u_{i,k} : T \times \mathbb{R}^{(N+1)K} \rightarrow \mathbb{R}$. Thus, if player $i \in \mathcal{N}$ ends the auction with exactly $m_i \in \{0, 1, \ldots, K\}$ units, his payoff is $\sum_{k=0}^{m_i} u_{i,k}(t, b)$. In the examples we shall restrict to separable transfers so, for later reference, for each player $i$ and unit $k$, let $v_{i,k} : T \rightarrow \mathbb{R}$ be a function such that $v_{i,k}(t)$ represents the (marginal) value, in units of account, of the $k$'th unit for player $i$ when the vector of signals is $t \in T$.

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5If the model does not specify a reserve price it is usual to assume $b_{\text{min}} = 0$.

6Unknown reserve prices can be modeled as the bid of a strategic bidder.

7We consider the dependence on $b$ instead of $b_i$ because we want to include in our results auctions where the payoff depends on bids of the opponents, such as the second-price auction, for instance. Also, this allows the study of “exotic” auctions, i.e., auctions where the payment is an arbitrary function of all bids.

8More precisely, his payoff would be $u_{i,0}(t, b) + \sum_{k=1}^{m_i} u_{i,k}(t, b) \delta_{i,k}$ where $\delta_{i,k} = 0$ if $k \leq e_i$ and $\delta_{i,k} = 1$ if $k > e_i$. To simplify notation we just write, without loss of generality: $\sum_{k=0}^{m_i} u_{i,k}(t, b)$.
If $m_i < e_i$, the player has sold $e_i - m_i$ units in the auction and if $m_i > e_i$, the player has bought $m_i - e_i$ units in the auction. No negotiation was made if $m_i = e_i$.

Given $b_{-i}$, let $s_i = (s_{i,1}, s_{i,2}, ..., s_{i,K})$, with $s_{i,1} \leq s_{i,2} \leq ... \leq s_{i,K}$, denote the (inverse) residual supply curve facing bidder $i$. That is, $s_{i,K}$ is the highest of the bids by players $j \neq i$, $s_{i,K-1}$ is the second highest and so on. Thus, for getting (for sure) at least one unit, bidder $i$’s highest bid must be above $s_{i,1}$, that is, $b_{i,1} > s_{i,1}$. For bidder $i$ earning at least two units, it is necessary $b_{i,2} > s_{i,2}$ and so on. Figure 1 illustrates this.

![Bid (b) and supply (s) curves for bidder i. In the situation displayed, bidder i receives three units, because b_{i,3} > s_{i,3} but b_{i,4} < s_{i,4}.](image)

In order to decide who wins an object, we will assume that the auction house uses an allocation (or tie-breaking) rule.

**Definition 1** An allocation rule is any function $a : \mathbb{R}^{(N+1)K} \rightarrow [0, 1]^{(N+1)K}$ such that:

1. If $b_{i,k} < s_{i,k}$ then $a_{i,k}(b) = 0$.
2. If $b_{i,k} > s_{i,k}$ then $a_{i,k}(b) = 1$. 
3. If for some $k$, $a_{i,k}(b) = 1$ then for all $k' \leq k$, $a_{i,k'}(b) = 1$.

4. $\sum_{i=0}^{N} \sum_{k=1}^{K} a_{i,k}(b) = K$.

The interpretation is the following. If $a_{i,k}(b) = 1$ then player $i$ wins at least $k$ objects. If $a_{i,k}(b) = 0$ then player $i$ wins at most $k - 1$ objects. Formally, the first condition says that if player $i$’s $k$-th bid is lower than the $K - k + 1$ highest competing bid he will not be awarded the $k$-th object. The second condition says that if player $i$ bids higher for unit $k$ than the $K - k + 1$ highest competing bids then he will win at least $k$ objects. The third says that if he wins at least $k$ objects then he must also win at least $1, \ldots, k - 1$ objects. The fourth says that at most $K$ units are allocated among the $N$ agents.

Observe that in the definition of allocation rules, there is freedom to define the rule only when $b_{i,k} = s_{i,k}$, provided the other conditions are satisfied. Thus, it is sufficient to define the rule for ties.

This setting is very general and applies to a broad class of discontinuous games, as we exemplify below.

Allocation Rules

**Example 1 (Nominal Allocation Rule)** Let us suppose that the bidders are numbered following a given order (say, the lexicographic order for their names). We can define that, in the case of a tie, the bidder with the least number, among those that are tying, gets the object. It is easy to see that this rule satisfies all conditions in definition.

Another example of allocation rule is the standard one, that splits randomly the objects.

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9Obviously, the utility function is specified only for bidders, that is, for $i \neq 0$. 

8
Example 2 (Standard Allocation Rule) In the case of a tie, the objects involved in the tie are randomly divided among the tying bidders. Formally: if \( b_{i,k} = s_{i,k} \) then \( a_{i,k}(b) = p/q \) where \( p \) is the number of the objects to be allocated in the tie, that is, \( p = K - \# \left\{ \left( j, \tilde{k} \right) : \text{such that } b_{j,\tilde{k}} > b_{i,k} \right\} \) and \( q \) is the number of tying bids, that is, \( q = \# \left\{ \left( j, \tilde{k} \right) : \text{such that } b_{j,\tilde{k}} = b_{i,k} \right\} \).

Auctions

Example 3 (Single unit auctions). \( u_{i,1}(t, b) = U_i(v_{i,1}(t) - b_{i,1}) \) and \( u_{i,0}(t, b) = 0 \) corresponds to a first-price auction with risk aversion or risk loving\(^\text{10}\). If \( U_i(x) = x \), we have risk neutrality. If \( u_{i,1}(t, b) = v_{i,1}(t) \) and \( u_{i,0}(t, b) = -b_{i,1} \) we have the all-pay auction. If \( u_{i,1}(t, b) = v_{i,1}(t) - s_{i,1} \) and \( u_{i,0}(t, b) = 0 \) we have the second-price auction. If \( u_{i,1}(t, b) = v_{i,1}(t) - s_{i,2} \) and \( u_{i,0}(t, b) = 0 \) we have the third-price auction. If \( u_{i,1}(t, b) = v_{i,1}(t) + b_i - s_{i,1} \) and \( u_{i,0}(t, b) = -b_i \) we have the war of attrition. We can have also combinations of these games. For example, \( u_{i,1}(t, b) = v_{i,1}(t) - \alpha b_i - (1 - \alpha)s_{i,1} \) and \( u_{i,0}(t, b) = 0 \), with \( \alpha \in (0, 1) \), gives a combination of the first- and second-price auctions.

Example 4 (Multi-unit auction with unitary demand). It is also useful to consider \( K \)-unit auctions with unitary demand, among \( N \) buyers, \( 1 < K < N \). In this case, \( b_{j,k} < b_{0,1} \), for all \( j = 1, \ldots, N \) and \( k = 2, \ldots, K \). Then, a pay-your-bid auction is given by \( u_{i,1}(t, b) = v_{i,1}(t) - b_{i,1} \) and \( u_{i,0}(t, b) = 0 \). If it is a uniform price with the price determined by the highest looser’s bid, \( u_{i,1}(t, b) = v_{i,1}(t) - s_{i,K} \) and \( u_{i,0}(t, b) = 0 \). If it is a uniform price with the price determined by the lowest winner’s bid, \( u_{i,0}(t, b) = 0, u_i(t, b) = v_{i,1}(t) - \max \{b_{i,1}, s_{i,K}\} \).

Example 5 (Multi-unit auctions with multi-unit demand). \( u_{i,1}(t, b) = v_{i,1}(t) - b_{i,1}, \ldots, u_{i,K}(t, b) = v_{i,K}(t) - b_{i,K} \) and \( u_{i,0}(t, b) = 0 \) corresponds to a multiple unit auction with discriminatory price. If \( u_{i,1}(t, b) = v_{i,1}(t) - p(b), \ldots, \)

\(^{10}\)If we put \( u_i(t, b) = U_i(v_i(t) - b_{i,1}) \) we can have any attitude towards risk.
\( u_{i,K}(t, b) = v_{i,K}(t) - p(b) \) and \( u_{i,0}(t, b) = 0 \) it correspond to a uniform multiple unit auction. There are two different uniform price auctions: \( p(b) \) can be the lowest winner’s bid (as in some actual treasury bills auctions) or \( p(b) \) can be the highest looser’s bid (as described by Krishna 2002). If \( u_{i,1}(t, b) = v_{i,1}(t) - s_{i,1}, \ldots, u_{i,k}(t, b) = v_{i,k}(t) - s_{i,k}, u_{i,K}(t, b) = v_{i,K}(t) - s_{i,K} \) and \( u_{i,0}(t, b) = 0 \) we have Vickrey auction.

**Strategies and Order Statistics**

The strategy of a bidder \( i \in \mathcal{N} \) is a bidding function \( b_i : T_i \to B \). We will use bold type for bidding functions. Notice that we do not specify a strategy for the non-strategic player, \( i = 0 \). We will restrict to integrable strategies, that is, we assume that the vector of strategies is \( b = (b_i)_{i \in \mathcal{N}} \) \( \in \prod_{i \in \mathcal{N}} \mathbb{L}^1(T_i, B) \).

For a vector of strategies \( b = (b_i)_{i \in \mathcal{N}} \), let \( b_{-i} \) be the vector of strategies of all strategic players except player \( i \), we denote by \( s_i \), for a fixed \( b_{-i} \), the function \( s_i : T_{-i} \to \mathbb{R}^{KN} \) that orders the \( NK \) vector \((b_0, b_{-i}(t_{-i}))\) from the highest to the lowest bid. Notice that we include the non-strategic bid \( b_0 \).

Given \( b_{-i} \) and \( j, 1 \leq j \leq KN \), define the distribution function, \( F_{s_{i,j}}(\cdot|t_i) \) on \( \mathbb{R} \), by \( F_{s_{i,j}}(\beta|t_i) \equiv \tau_{-i}({\{t_{-i} \in T_{-i} : s_{i,j}(t_{-i}) < \beta}\}|t_i}) \) and let \( f_{s_{i,j}}(b|t_i) \) be its Radon-Nykodim derivative with respect to the Lebesgue measure. We denote by \( F_{s_{i,j}}\perp(b|t_i) \) the singular part of \( F_{s_{i,j}}(b|t_i) \).

In the examples, sometimes we will restrict to monotone strategies. In such cases we will implicitly assume that \( T_i = \mathbb{R} \) and use the following notation. Given \( t \in T \), we define \( t_{(-i)} \) as \( t_{(-i)} = \max_{j \neq i} t_j \).

**Expected Payoff**

In order to simplify notation below, we will write \((\cdot)\) in the place of \((t_i, t_{-i}, b_0, b_i, b_{-i}(t_{-i})), (\beta, \cdot)\) in the place of \((t_i, t_{-i}, b_0, (\beta, b_{-j-i}), b_{-i}(t_{-i}))\) and \((\circ)\)
in the place of \((b_0, b_i, b_{-i} (t_{-i}))\). Thus, if the bid \(b_0\) and the profile of bidding functions \(b_{-i}\) are fixed, the expected payoff of bidder \(i\) of type \(t_i\), when bidding \(b_i\), is:

\[
\Pi_i(t_i, b_0, b_i, b_{-i}) \equiv \int_{T_{-i}} u_{i,0} (\cdot) \tau_{-i}(dt_{-i}|t_i) + \sum_{k=1}^{K} \int_{T_{-i}} a_{i,k} (\cdot) u_{i,k} (\cdot) \tau_{-i}(dt_{-i}|t_i),
\]

which is equivalent to:

\[
\Pi_i(t_i, b_0, b_i, b_{-i}) = \int_{T_{-i}} u_{i,0} (\cdot) \tau_{-i}(dt_{-i}|t_i) + \sum_{k=1}^{K} \int_{T_{-i}} u_{i,k} (\cdot) 1_{[b_{i,k} > s_{i,k}]} \tau_{-i}(dt_{-i}|t_i) + \sum_{k=1}^{K} \int_{T_{-i}} a_{i,k} (\cdot) u_{i,k} (\cdot) 1_{[b_{i,k} = s_{i,k}]} \tau_{-i}(dt_{-i}|t_i).
\]

**Remark 1** There are two important ways in which the third term in the above expression may be omitted. If for all \(k = 1, \ldots, K\), the distribution \(F_{s_{i,k}} (\cdot|t_i)\) has no atoms and therefore, the tie-breaking rule (i.e., allocation rule \(a\)) is not important and, if the auctioneer keeps the objects in case of ties. That is, when \(a_{i,k} (\cdot) = 0\) whenever \(b_{i,k} = s_{i,k}\).

## 3 Bidding Behavior

Our first result is a characterization of the payoff through its derivative with respect to the bid given by an integral expression (i.e., a kind of first fundamental theorem of calculus). For this, we will need the following assumption:
Condition 1 \( u_{i,k} : T \times \mathbb{R}^{K(N+1)} \to \mathbb{R} \), \( k = 0, 1, \ldots, K \) are absolutely continuous on \( b_{i,k} \) and \( \partial_{b_{i,k}} u_{i,k} \) is essentially bounded.\(^{11}\)

Our main result is the following:

**Lemma 1 (Payoff Characterization)** Assume condition 7. Fix \( b_0 \), a \( b_i \) in the interior of \( B \) and profile of bidding functions \( b_{-i} \). Let \( b_i' \) denote the vector obtained from \( b_i \) by substituting the coordinate \( b_{i,j} \) by \( b_{i,j} - 1 \). Then, for all \( j = 1, \ldots, K \) the payoff of bidder \( i \) when bidding \( b_i \) can be expressed as:

\[
\Pi_i(t_i, b_0, b_i, b_{-i}) = \Pi_i(t_i, b_0, b_i', b_{-i})
\]

\[
\quad + \int_{[b_{i,j-1}, b_{i,j})} \partial_{b_{i,j}} \Pi_i(t_i, b_0, (\beta, b_{i,-j}), b_{-i}) d\beta
\]

\[
\quad + \int_{[b_{i,j-1}, b_{i,j})} E[u_{i,j}(\cdot)|t_i, s_{i,k} = \beta] \frac{1}{f_{s_{i,j}}(\beta|t_i)} d\beta
\]

\[
\quad + \sum_{k=1}^{K} E[a_{i,k}(\cdot) u_{i,k}(\cdot) 1[b_{i,k} > s_{i,k}]|t_i]
\]

where \( E[\cdot|t_i] \) is the expectation with respect to the measure \( \tau_i(-|t_i) \), and for almost all \( b_{i,j} \):

\[
\partial_{b_{i,j}} \Pi_i(t_i, b_0, (\beta, b_{i,-j}), b_{-i}) = E[\partial_{b_{i,j}} u_{i,0}(\beta, \cdot)|t_i]
\]

\[
\quad + \sum_{k \neq j} E[\partial_{b_{i,j}} u_{i,k}(\beta, \cdot) 1[b_{i,k} > s_{i,k}]|t_i]
\]

\[
\quad + E[\partial_{b_{i,j}} u_{i,j}(\beta, \cdot) 1[\beta > s_{i,j}]|t_i]
\]

\[
\quad + E[u_{i,j}(\cdot)|t_i, s_{i,j} = \beta] f_{s_{i,j}}(\beta|t_i)
\]

\(^{11}\) Absolute continuity with respect to \( b_{i,k} \) implies that \( \partial_{b_{i,k}} u_{i,k} \) exists almost everywhere (with respect to Lebesgue measure) and

\[
u_{i,k}(b_{i,j}, \cdot) - u_{i,k}(b_{i,j-1}, \cdot) = \int_{[b_{i,j-1}, b_{i,j})} \partial_{b_{i,j}} u_{i,0}(\beta, \cdot) d\beta.
\]

Essentially bounded is used to invoke Lebesgue dominated converge theorem.
Proof. See appendix. ■

The most important part of Lemma 1 is the expression of \( \partial b_{i,j} \Pi_i(t_i, b_0, (\beta, b_{i,-j}), b_{-i}) \). When there is no tie with positive probability at \( b_i \) (i.e, \( F_{s_{i,k}} (-|t_i) \) has no atoms) \( \partial b_{i,j} \Pi_i(t_i, b_0, b_i, b_{-i}) \) is, for almost all \( b_{i,j} \), the partial derivative of \( \Pi_i(t_i, b_0, b_i, b_{-i}) \) (see section 5). It is useful to observe that in the expression of \( \partial b_{i,j} \Pi_i(t_i, b_0, (\beta, b_{i,-j}), b_{-i}) \) above, the first three lines capture only the impact of the changing of \( b_{i,j} = \beta \) in the payoff (payment) of each unit, while the last line captures the impact of such a change in the probability of winning the unit \( j \). Note also the difference in the events in the second and the third line: \([b_{i,k} > s_{i,k}]\) and \([\beta > s_{i,j}]\).

The following corollary characterizes best response bids in an intuitive way. It says that under condition 1 the optimum bid is such that the marginal cost of bidding is equal to the marginal utility from bidding. More formally:

**Corollary 1 (Basic Principle of Bidding)**. Assume condition 1. If \( \Pi_i(\cdot) \) is differentiable in \( b_i \) at a bid profile which is optimal and in the interior of \( B \), that is, at \( b_i \in \arg \max_{b_i \in B} \Pi_i(t_i, b_0, b, b_{-i}) \cap \text{int}B \), and there is no tie with positive probability at \( b_i \) i.e, \( (F_{s_{i,k}} (-|t_i) \) has no atoms), then for all \( j \),

\[
E [u_{i,j} (\cdot)|t_i, s_{i,j} = b_{i,j}] f_{s_{i,j}} (b_{i,j}|t_i)
\]

\[
= E [-\partial b_{i,j} u_{i,0} (b_{i,j}, \cdot)|t_i] + \sum_{k=1}^{K} E [-\partial b_{i,j} u_{i,k} (b_{i,j}, \cdot) 1_{[b_{i,k} > s_{i,k}]|t_i}].
\]

Proof. If \( F_{s_{i,k}} (-|t_i) \) has no atoms then \( F_{s_{i,k}}^{-1} (-|t_i) = 0 \) almost everywhere. Therefore, by the payoff characterization lemma:

\[
\Pi_i(t_i, b_0, b_i, b_{-i}) = \Pi_i(t_i, b_0, b_i', b_{-i}) + \int_{[b_{i,j-1}, b_{i,j})} \partial b_{i,j} \Pi_i(t_i, b_0, (\beta, b_{i,-j}), b_{-i})d\beta.
\]
If $\Pi_i(t_i, b_0, b_i, b_{-i})$ is differentiable at $b_i \in \arg\max_{b \in B} \Pi_i(t_i, b_0, b, b_{-i})$ then
\[
\partial_{b_{i,j}} \Pi_i(t_i, b_0, (b_{i,j}, b_{i,-j}), b_{-i}) = 0.
\]
This concludes the proof. ■

Observe that $E [u_{i,j} (\cdot) | t_i, s_{i,k} = b_{i,j}] f_{s_{i,k}} (b_{i,j} | t_i)$ represents the marginal benefit of raising the bid in unit $j$, that is, the expected utility conditional to the event of a tie exactly for that unit. On the other hand, $E [-\partial_{b_{i,j}} u_{i,k} (b_{i,j}, \cdot) | t_i]$ represents the marginal cost of participation and the last term represents the marginal cost of changing bid $b_{i,j}$ — in our model, it may change the expected payment for each other units. Note that this interpretation does not require separability in the monetary transfer (risk neutrality).

This interpretation is useful for explaining bidding behavior in an intuitive way: the players bid to equalize the marginal benefit to the marginal cost of bidding.

The following corollary will be used later to prove a monotone best-reply result.

**Corollary 2 (Payoff Characterization as a Line Integral)** Assume condition 4 and suppose for all $i$ and $k$, $\partial_{b_{i,k}} \Pi_i$ exists and is continuous in $b_i$. Fix $b_0$, a profile of bidding functions $b_{-i}$, two bids $b_i^1$ and $b_i^2$ in the interior of $B$, and a smooth curve $\alpha : [0, 1] \rightarrow B$ such that $\alpha(0) = b_i^0$ and $\alpha(1) = b_i^1$ then, for all $j = 1,..., K$ the payoff of bidder $i$ when bidding $b_i$ can be expressed as:

\[
\Pi_i(t_i, b_0, b_i, b_{-i}) = \Pi_i(t_i, b_0, b_i^0, b_{-i}) + \int_{[0,1]} \nabla_{b_i} \Pi_i(t_i, b_0, \alpha(s), b_{-i}) \cdot \alpha'(s) ds,
\]

where $\nabla_{b_i} \Pi_i(t_i, b_0, \alpha(s), b_{-i}) = (\partial_{b_{i,j}} \Pi_i(t_i, b_0, \alpha(s), b_{-i}))_{j=1,...,K}$. 
3.1 Examples

The examples below show that corollary 1 is a generalization of the necessary first-order conditions for the first and second-price auctions presented in Milgrom and Weber (1982), for the war of attrition and all-pay auctions presented in Krishna and Morgan (1997). The example on double auctions shows that the Basic Principle of Bidding is concise. Such an example is the application of Corollary 1 for double auctions and it presents a comparison with the equivalent expression obtained by Williams (1991).

Example 6 (First Price - Single Object Auction). When we restrict ourselves to the case of the first-price single object auction with risk neutrality: $K = 1$, $u_{i,0} = 0$ and $u_{i,1}(t,b_0,b) = v_{i,1}(t) - b_i$, then $\partial_{b_i} u_{i,1}(t,b_0,b) = -1$. The condition of corollary 1 becomes:

$$b_i = E[v_{i,1}|t, s_{i,1} = b_i] - \frac{F_{s_{i,1}}(b_i|t_i)}{f_{s_{i,1}}(b_i|t_i)} . \quad (3)$$

This (necessary) first-order condition provides a useful way to determine best-reply bids. Note that this expression admits non-monotonic bidding functions, contrary to Milgrom and Weber’s model. It also encompasses asymmetries in valuations and distribution of types. Assuming affiliation and monotonic utilities, Milgrom and Weber (1982) can restrict themselves to the space of monotone symmetric bidding functions (i.e., $b_i = b$ for all $i \in N$). Thus,

$$s_{i,1}(b_{-i}(t_{-i})) = x \iff \max_{j \neq i} b_j(t^j) = x \iff \max_{j \neq i} t^j = (b)^{-1}(x) ,$$

where, in the last equation $(b)^{-1}$ stands for the inverse (generalized) of $b$. This equation says that conditioning on $s_{i,1} = b_i$ is the same to conditioning on $\max_{j \neq i} t^j = t_i$. Recall that $t^{(-i)} = \max_{j \neq i} t^j$. Then $f_{s_{i,1}}(s|t_i) = \frac{f_{t_i}(s|t_i)}{b(s)}$ and
\( F_{s_{-1},1}(s|t_i) = F_{t(-i)}(s|t_i) \). With this, (3) becomes

\[
\frac{db}{dt}(t_i) = \left( E\left[v_{i,1}|t_i, t^{(-i)} = t_i\right] - b(t_i) \right) \frac{f_{t(-i)}(t_i|t_i)}{F_{t(-i)}(t_i|t_i)}
\]

(4)

whose solution is shown to be an equilibrium under affiliation.

**Example 7 (Second Price - Single Object Auction).** In the second price single object auction, Milgrom and Weber’s model is equivalent to \( K = 1 \), \( u_{i,1}(t, b) = v_i(t) - s_{i,1} \) and \( u_{i,0} = 0 \). Then, \( \partial_b u_{i,1}(t, b) = 0 \) and the condition in corollary \( \square \) reduces to

\[
E_{t(-i)}[v_i - b_i|t_i, s_{i,1} = b_i] f_{s_{-1,1}}(b_i|t_i) = 0 \]

which can be simplified to

\[
b_i = E[v_{i,1}|t_i, s_{i,1} = b_i].
\]

Again, with monotonicity and symmetry assumptions, Milgrom and Weber’s expression for the equilibrium bid function can be obtained:

\[
b(t_i) = E\left[v_{i,1}|t_i, t^{(-i)} = t_i\right] \equiv \bar{v}(t_i, t_i).
\]

**Example 8 (All Pay - Single Object Auction).** Krishna and Morgan (1997) extend the method of Milgrom and Weber (1982) to the cases of war of attrition and all-pay auctions. In the all-pay auction, their model is equivalent to \( u_{i,1}(t, b) = v_i(t) - s_{i,1} \) and \( u_{i,0}(t, b) = -b_i \). Then, \( \partial_b u_{i,1}(t, b) = 0 \) and \( \partial_b u_{i,0}(t, b) = -1 \). So, the condition in corollary \( \square \) reduces to

\[
E[v_{i,1}|t_i, s_{i,1} = b_i] f_{s_{-1,1}}(b_i|t_i) = 1.
\]

Under the same hypothesis of monotonicity and symmetry, they find the following differential equation:

\[
\frac{db}{dt}(t_i) = E\left[v_{i,1}|t_i, t^{(-i)} = t_i\right] \frac{f_{t(-i)}(t_i|t_i)}{F_{t(-i)}(t_i|t_i)},
\]

16
whose solution they show to be an equilibrium under affiliation.

**Example 9** (War of Attrition - Single Object Auction). In the war of attrition, Krishna and Morgan (1997) model is equivalent to \( u_{i,1} (t, b) = v_i (t) + b_i - s_{i,1} \) and \( u_{i,0} (t, b) = -b_i \). Then, \( \partial_{b_i} u_{i,1} (t, b) = 1 \) and \( \partial_{b_i} u_{i,0} (t, b) = -1 \). So, the condition in corollary reduces to

\[
E [v_i | t_i, s_{i,1} = b_i] f_{s_{i,1}} (b_i | t_i) = 1 - F_{s_{i,1}} (b_i | t_i).
\]

Again, with monotonicity and symmetry, they derive the equation

\[
\frac{db}{dt} (t_i) = E [v_i | t_i, t^{(-i)} = t_i] \frac{1 - F_{t^{(-i)}} (t_i | t_i)}{f_{t^{(-i)}} (t_i | t_i)},
\]

and the equilibrium is shown to exist under affiliation.

**Example 10** (Double Auction). In the analysis of a double auction with private values, risk neutrality, independent types and symmetry among buyers and sellers, Williams (1991) assumes that the payment is determined by the buyer’s bid. So, it is optimum for the seller to bid her value. To analyze the behavior of the buyer \( i \), Williams (1991) reaches the following expression:

\[
\partial_{b_i} \Pi_i (v, \beta) = \left[ nf_1 (\beta) K_{n,m} (b^{-1} (\beta), \beta) \\
+ (m - 1) \frac{f_2 (b^{-1} (\beta))}{b' (\beta)} L_{n,m} (b^{-1} (\beta), \beta) \right] (v - \beta) \]

\[ -M_{n,m} (b^{-1} (\beta), \beta) \]

where \( b \) denotes here the symmetric bidding function followed by all buyers, \( f_1 \) is the common density function of sellers, \( f_2 \) is the common density function of buyers, \( n \) is the number of sellers, \( m \) is the number of buyers and \( M_{n,m} (\cdot, \cdot) \) is
The expression (5) is just a special case of (2). To see this, observe that \( F_{b_i}(\beta) \), the probability that the threshold bid is less or equal to \( \beta \), is given by the probability of the union of following disjoint events: there are \( i \) bids of buyers and \( j \) bids of sellers below or equal to \( \beta \) and \( i + j = m \) (because the \( m \)-th bid determines the threshold between winning and losing). Thus,

\[
F_{s_i,1}(\beta) = \sum_{i+j=m, \atop 0 \leq i \leq m-1, 0 \leq j \leq n} \binom{n}{j} \binom{m-1}{i} F_1(\beta)^i F_2(b^{-1}(\beta))_j (1 - F_1(\beta))^{n-j} (1 - F_2(b^{-1}(\beta)))^{m-1-i},
\]

which is equal to \( M_{n,m}(b^{-1}(\beta), \beta) \) above. Now, it is a matter of length but elementary derivation to confirm that

\[
f_{s_i,1}(\beta) = n f_{1}(\beta) K_{n,m}(b^{-1}(\beta), \beta) + (m-1) \frac{f_2(b^{-1}(\beta))}{b'(\beta)} L_{n,m}(b^{-1}(\beta), \beta),
\]

12To obtain \( K_{n,m}(\cdot, \cdot) \) just substitute \( n-1 \) for \( n \) where it occurs in \( M_{n,m}(\cdot, \cdot) \). To obtain \( L_{n,m}(\cdot, \cdot) \), substitute \( m-2 \) for \( m-1 \) where it occurs in \( M_{n,m}(\cdot, \cdot) \).
which concludes the proof of the claim.

Example 11 (Multiple Object Discriminatory Auction). Let $u_{i,0} = 0$, $u_{i,k}(t, b) = v_{i,k}(t) - b_{i,k}$. Then $\partial b_{i,j} u_{i,k}(t, b) = 0$ if $j \neq k$ and $-1$ if $j = k$. It is easy to show that the condition in corollary[3] reduces to:

$$b_{i,k} = E[v_{i,k}(t) | t_i, s_{i,k} = b_{i,k}] - \frac{F_{s_{i,k}}(b_{i,k} | t_i)}{f_{s_{i,k}}(b_{i,k} | t_i)}.$$  

Assuming monotonic and symmetric bidding functions the first order condition reduces to:

$$\frac{db_k}{dt} (t_i) = (E_{t_i} [v_{i,k}(t) | t_i, s_{i,k} = b_{i,k}] - b_k(t_i)) \frac{F_{t_i} (b_{i,k} | t_i)}{F_{t_i} (b_{i,k} | t_i)}.$$

Example 12 (Multiple Object Vickrey Auction). Let $u_{i,0} = 0$, $u_{i,k}(t, b) = v_{i,k}(t) - s_{i,k}$. Then $\partial b_{i,j} u_{i,k}(t, b) = 0$. Therefore the condition in corollary[7] reduces to:

$$b_{i,k} = E[v_{i,k}(t) | t_i, s_{i,k} = b_{i,k}].$$  

This makes clear that the Vickrey Auction implies revelation of the truth in a general setting.

Example 13 (Uniform price auction) Let $u_{i,0} = 0$, $u_{i,k}(t, b) = v_{i,k}(t) - p$, where $p$ is the payment, which is equal for all units and bidders. There are two common rules for the uniform price auction. One is the highest looser bid, which is the uniform price auction described by Krishna (2002). In this case, the payment is equal to the highest bid among those bids that do not receive the object. A variant is to put the payment equal to the lowest winning bid. We
treat both below. Note that for any $k$,

$$
\partial_{b_{i,j}} u_{i,j} (b_{i,j}, \cdot) = -\partial_{b_{i,j}} p(b) = \begin{cases} 
-1, & \text{if } b_{i,j} \text{ determines the payment} \\
0, & \text{otherwise}
\end{cases}
$$

In the case of the lowest winning bid, $b_{i,j}$ determines the payment in the event $s_{i,j} < b_{i,j} < s_{i,j+1}$. This event contains the event $[b_{i,k} > s_{i,k}]$ if and only if $k \leq j$. Thus, the first order condition becomes:

$$
b_{i,j} = E[v_{i,j} (t) | t_i, s_{i,j} = b_{i,j}] - j \frac{\Pr [s_{i,j+1} > b_{i,j} > s_{i,j}, b_{i,j+1} < s_{i,j+1}]}{f_{s_{i,j}} (b_{i,j}|t_i)}.
$$

In the case of the highest losing bid, $b_{i,j}$ determines the payment if $b_{i,j} < s_{i,j}$, $b_{i,j-1} > s_{i,j-1}$ and $b_{i,j} > s_{i,j-1}$. Similarly,

$$
b_{i,j} = E[v_{i,j} (t) | t_i, s_{i,j} = b_{i,j}] - (j - 1) \frac{\Pr [s_{i,j-1} < b_{i,j} < s_{i,j}, b_{i,j-1} > s_{i,j-1}]}{f_{s_{i,j}} (b_{i,j}|t_i)}.
$$

4 Applications

Here we point out some potential applications and how our main result can be used to give a simple prove of some useful facts about auctions. Some of these results are new.

For all results below, we assume that the strategies $b_{-i}$ of bidder $i$’s opponents are such that the distribution of $s_i$ is absolutely continuous with respect to the Lebesgue measure. Thus, the payoff is given only by the integral of its derivative.

\[^{13}\text{We do not consider situations where two bids may be equal.}\]
4.1 Sufficient Conditions for Truth-telling

It is widely known that second price auctions lead to bidding equal to the truthful expected value by the bidder. This can be easily seen from the first order condition for this auction:

\[ b_i = E[v_i|t_i, s_i = b_i]. \]

The other terms in the first order condition disappear because they are identically zero: \( \partial_{b_{i,1}} u_i (t, b) = 0 \). The reason for that is that the payment does not depend on the own bidder’s bid and it is equal to the bid in the case of a tie. The following result shows that these two conditions are exactly what one needs to obtain truthful bidding.

**Proposition 1** Assume the conditions for Corollary 7. (1) If the bid \( b_{i,j} \) never modifies the payment of any unit, more precisely, \( \partial_{b_{i,j}} u_i (t, b) = 0 \) for all \( k \) and \( (t, b) \), then it is optimal for bidder \( i \) to bid \( b_{i,j} \) such that:

\[ E[u_{i,j} (-)|t_i, s_{i,j} = b_{i,j}] = 0. \]

(2) In addition to the previous condition, assume that \( u_{i,j} (t, b) = v_{i,j} (t) - p_{i,j} (b) \), and that the payment \( p(b) \) is \( b_{i,j} \) in case of a relevant tie at \( b_{i,j} = s_{i,k} \). Then the optimal bid is to bid the expected value of the unit:

\[ b_{i,j} = E[v_{i,j} (t)|t_i, s_{i,j} = b_{i,j}]. \]

**Proof.** (1) It is sufficient to examine the expression of \( \partial_{b_{i,j}} \Pi_i (t_i, b_0, \beta, b_{i,-j}, b_{-i}) \).

(2) Observe that in the conditional event \( s_{i,j} = b_{i,j} \), the payment is \( p(b) = b_{i,j} \).

Some known results are immediate corollaries:
Corollary 3  The first (highest) bid in the uniform price auction (with payment equal to the highest loser bid) is truthful.

Proof. The first bid cannot affect the payment of a winning bidder in a uniform auction, and thus the condition in (1) is satisfied. It is easy to see that the condition in (2) is also satisfied.

Corollary 4  The bids in the Vickrey auction are truthful.

Proof. It is immediate to see that Vickrey auction satisfy the conditions in (1) and (2).

It is useful to illustrate that the condition in (1) is not sufficient to get the result. Consider the following example:

Example 14  (Third price auction) Consider the third-price auction, that is, the auction of a single object with \( n \geq 3 \) bidders, where the bidder with the highest bid wins and pays the third highest bid. Of course, the payment never depends on the winner’s bid, that is, \( \partial_{b_{i,1}} u_{i,1}(t, b) = 0 \). Let us assume risk neutrality and private values, that is, \( u_{i,1}(t, b) = t_i - p(t) \). Then, Proposition 1 (1) implies that:

\[
t_i = E[p(b) | t_i, s_{i,1} = b_{i,1}] < b_{i,1},
\]

because \( s_{i,1} \) is the second highest bid and the expected payment (expected value of the third highest bid) is strictly below it. Thus, the optimal bid in a third price auction is above the own bidder’s value.

The following example is also an useful illustration of the failure of the conditions of Proposition 1.

Example 15  (Discriminatory Highest Others Bid Auction) Consider a multi-unit auction with a standard allocation rule, that is, the highest \( K \) bids win the object. The payment of bidder \( i \) for unit \( j \) is given by \( \max\{b_{l,k} : l \neq i, b_{l,k} \leq b_{i,j}\} \), that is, the payment is the highest defeated bid, not given by bidder \( i \).

\(^{14}\)We thank Robert Marshall for suggesting this example.
himself. It is clear that this payment function is such that the partial derivative with respect to \( b_{i,j} \) is zero almost everywhere, that is, \( \partial b_{i,j} u_{i,k} (t, b) = 0 \). In the event \( s_{i,j} = b_{i,j} \), the payment is \( \max\{b_{l,k} : l \neq i, b_{l,k} \leq b_{i,j}\} = s_{i,j} = b_{i,j} \). Thus, the condition in (2) is satisfied. Thus, it seems that the equilibrium of this auction is truthful. Nevertheless, it is easy to see that the payment for unit \( j \) is above the payment in a standard Vickrey auction. In fact, in the standard Vickrey auction, the payment for unit \( j \) is just \( s_{i,j} \) and it is above this in this auction, but for the last unit. Thus, if the equilibrium for this auction were truthtelling, expected revenue of this auction would be above the Vickrey auction. But this is not possible, because we know (see Krishna and Perry, 1998 or Krishna, 2002), that the Vickrey auction gives the highest revenue among the truthful bidding mechanisms.

The equilibrium of this auction is not truthtelling because condition (1) of Proposition 4 is in fact not satisfied. It requires that own bidder’s bid does not modifies his payments, but it is easy to see that the payment for unit \( j \) increases if the set of bids \( \{b_{l,k} : l \neq i, b_{l,k} \leq b_{i,j}\} \) changes with an increase of \( b_{i,j} \).

4.2 Sufficient Conditions for Increasing Best Reply

Let

\[
V_i (b_i, b_{-i}) = \int \Pi_i (t_i, b_i (t_i), b_{-i}) dt_i
\]

be the ex-ante payoff. We define the interim and the ex-ante best-reply correspondence, respectively, by

\[
\Theta_i (t_i, b_{-i}) = \arg \max_{\beta \in \mathbb{B}} \Pi_i (t_i, \beta, b_{-i}),
\]

There is also another way to see the same problem: condition 1 assumed in Corollary 1 fails to be satisfied. This requires the utility functions to be absolutely continuous, which implies that the utility functions are the integral of their (almost everywhere) derivative. But this is clearly false here (see footnote 11).
and

\[ \Gamma_i (b_{-i}) \equiv \arg \max_{b_i \in \mathcal{L}((0,1),B)} V_i (b_i, b_{-i}). \]

We need the following:

**Definition 2** Given a partial order \( \succeq \) on \( T_i \), we say that a function \( g(t, b) \) is strictly increasing (non-decreasing) in \( t_i \) if \( t_i^2 \succ t_i^1 (t_i^2 \succeq t_i^1) \) implies \( g(t_i^2, t_{-i}, b) > (\geq) g(t_i^1, t_{-i}, b) \) for all \( t_{-i}, b \).

Let \( \geq \) denote the coordinate-wise partial order in \( B \), that is: \( b_{1,i,j} \geq b_{0,i,j} \) for all \( j = 1, \ldots, K \). We write \( b_{1,i} > b_{0,i} \) if \( b_{1,i} \geq b_{0,i} \) and \( b_{1,i} \neq b_{0,i} \).

**Proposition 2** Assume we are under the conditions of Corollary 2. Let \( \succeq \) be a partial order on \( T_i \). For all \( k, j = 1, \ldots, K \), assume that \( u_{i,k} (t, b) \) is absolutely continuous in \( t \) and \( b \), and strictly increasing in \( t_i \); and \( \partial_{b_{i,j}} u_{i,k} (t, b) \) is non-decreasing in \( t_i \) (except, possibly in a set of null measure). Then the following holds:

1. For each \( t_i \), \( \Theta_i (t_i, b_{-i}) \) is non-empty.

2. Consider two types \( t_i^1, t_i^2, t_i^3 \) \( t_i^2 > t_i^1 \) (or \( t_i^2 \succeq t_i^1 \)) and best reply bids for them, that is, \( b_{1,i} \in \Theta_i (t_i^1, b_{-i}) \), \( b_{2,i} \in \Theta_i (t_i^2, b_{-i}) \) and assume that these bids imply different probability of winning, i.e.,

\[ \Pr \left( \{t_{-i} : \exists k \text{ such that } a_{i,k} (b_{1,i}, b_{-i} (t_{-i})) \neq a_{i,k} (b_{2,i}, b_{-i} (t_{-i})) \} \right) > 0. \]

Then \( \sim (b_{1,i} > b_{2,i}) \).

**Proof.** See Appendix. \( \blacksquare \)

\[16\] Given the partial order \( \succeq \), we write \( x \succ y \) if \( x \succeq y \) but \( \sim (y \succeq x) \).
The unidimensional version of the above theorem was used by Araujo and de Castro (2007) to prove equilibrium existence in single unit auctions. The main role of this result in their equilibrium proof is to restrict the set of strategies to a compact set (the set of non-decreasing functions). Restricted to this strategy set, they obtained approximated equilibria of perturbed games, used compactness to obtain a converging subsequence and proved that the limit is equilibrium of the original auction. Maybe the above theorem could be equally useful in obtaining new equilibrium existence results for multiunit auctions, but such results are out of the scope of this paper.

4.3 Identification of Multi-Unit Auctions

There is large literature on structural identification of unitary auctions (see Athey and Haile (2002), (2005)). The case of multiple unit demand auctions has been recently the focus of attention. The problem is of interest in the applied literature because many important markets rely on auction mechanisms to allocate goods or services and these are naturally modeled as markets for the allocation of multiple units. Prominent examples are the markets for treasury auctions and for the demand and supply of electricity (see Hortacsu (2002) for treasury auctions, Hortacsu and Puller (2007) and Wolak (2006) for electricity markets). With very few exceptions (for example McAdams (2007) or Wolak (2006)), most applied work rely on Wilson (1979) share model. One of the salient features of this approach is the use of continuous bid functions that are hardly found in real markets. Therefore, it is of interest to study identification when bids are discrete. The particular institutional settings in which some of these auctions are carried in real markets make the continuous bid assumption unattainable. For example, in England spot electricity market generators make three bids out of their supply function. In the Colombian spot markets, gener-
ators make price offers for generating a fixed (although different among plants) amount of energy per generating unit (see de Castro, Espinosa and Riascos (2007) for an application of our identification results to the Colombian electricity spot market). Below we provide identification results for the two most important multi-unit auctions, the discriminatory auction and the uniform auction. To the extend of our knowledge, the discrete case result for the uniform auction is new (Hortacsu and Puller (2007) report a similar result for continuous bid functions). In all cases, estimation methods assume that observables are generated by a Bayesian-Nash equilibrium.

Recall example 13 where we derived the first order conditions for the multi-unit uniform auction. Consider the case in which agents pay the lowest winning bid.

**Proposition 3** Consider the uniform price auction (see example 13). Then if values are private, the marginal utility of an additional unit is nonparametrically identified from agents bid. Formally,

\[
v_{i,j}(t_i) = b_{i,j} + \frac{\Pr[s_{i,j+1} > b_{i,j} > s_{i,j}, b_{i,j+1} < s_{i,j+1}]}{\int_{s_{i,j}}(b_{i,j}[t_i])}.
\]

**Proof.** To the extend that all agents’ bids are observable and one is able to estimate the second term from the right hand side, identification follows. \( \blacksquare \)

**Remark 2** Equation 6 is analogous to equation (2) in Hortacsu and Puller (2007).

**Remark 3** To estimate the right hand side of equation 6 one can follow the same estimation methods of Hortacsu (2002).

**Proposition 4** Consider the discriminatory multi-unit auction (see example 13). Then if values are private, the marginal utility of additional unit is non-
parametrically identified from agents bid. Formally,

\[ v_{i,j}(t_i) = b_{i,j} + \frac{E_{s_{i,j}}(b_{i,j}|t_i)}{f_{s_{i,j}}(b_{i,j}|t_i)} \]  (7)

**Proof.** The same argument as before. ■

**Remark 4** Equation 7 is analogous to equation (2) in Hortacsu (2002). Notice that Hortacsu (2002) also studies a discrete version in which prices are restricted to lie in on a discrete grid but the divisibility assumption of goods is still assumed.

**Remark 5** To estimate the right hand side of equation 7 one can follow the same estimation methods of Hortacsu (2002).

5 Appendix: Proofs

5.1 Proof of the Payoff Characterization Lemma

The proof follows the demonstration of the Leibiniz rule. The main point is the use of a well known theorem on the derivatives of measures and its integral expression. The theorem we use is in Rudin (1966).

Recall the expression for agents expected payoff:

\[ \Pi_i(t_i, b_0, b_i, b_{-i}) = \int_{T_{-i}} u_{i,0}(\cdot) \tau_{-i}(dt_{-i}|t_i) + \]

\[ \sum_{k=1}^{K} \int_{T_{-i}} u_{i,k}(\cdot) 1_{[b_{i,k}>s_{i,k}]} \tau_{-i}(dt_{-i}|t_i) + \]

\[ \sum_{k=1}^{K} \int_{T_{-i}} a_{i,k}(\cdot) u_{i,k}(\cdot) 1_{[b_{i,k}=s_{i,k}]} \tau_{-i}(dt_{-i}|t_i). \]  (8)

Fix \( j \), and consider each term above separately.
1. The first one has a derivative with respect to \( b_{i,j} \) almost everywhere and is equal to \( E \left[ \partial_{b_{i,j}} u_{i,0} (\cdot) | t_i \right]. \) Also,

\[
E [u_{i,0} (\cdot) | t_i] = \int_{[b_{i,j}, b_{i,j}]} E \left[ \partial_{b_{i,j}} u_{i,0} (\beta, \cdot) | t_i \right] d\beta
\]

2. If the distribution \( F_{s_{i,k}} (-|t_i) \) has no atoms, the third term is equal to zero and its derivative exists and it’s zero.

3. Now consider the second term,

\[
\int_{T_{-i}} u_{i,k} (\cdot) 1_{[b_{i,k} > s_{i,k}]} \tau_{-i} (dt_{-i}|t_i)
\]

There are two cases, \( j \neq k \) and \( j = k. \)

In the first case \( (j \neq k), \) let \( a^n \to (b_{i,j})^+ \) (i.e., \( a^n > b_{i,j} \); the case \( a^n \to (b_{i,k})^- \) is analogous). We have

\[
\int_{T_{-i}} u_{i,k} (t_i, t_{-i}, b_0, (a^n, b_{i,j}), b_{-i} (t_{-i})) 1_{[b_{i,k} > s_{i,k}]} \tau_{-i} (dt_{-i}|t_i)
\]

\[- \int_{T_{-i}} u_{i,k} (\cdot) 1_{[b_{i,k} > s_{i,k}]} \tau_{-i} (dt_{-i}|t_i)
\]

\[= \int_{T_{-i}} (u_{i,k} (t_i, t_{-i}, b_0, (a^n, b_{i,j}), b_{-i} (t_{-i})) - u_{i,k} (\cdot)) 1_{[b_{i,k} > s_{i,k}]} \tau_{-i} (dt_{-i}|t_i)
\]

Since \( u_i \) has bounded derivative with respect to almost all \( b_{i,j}, \)

\[
\lim_{a^n \to (b_{i,j})^+} \frac{u_{i,k} (t_i, t_{-i}, b_0, (a^n, b_{i,j}), b_{-i} (t_{-i})) - u_{i,k} (\cdot)}{a^n - b_{i,j}} = \partial_{b_{i,j}} u_{i,k} (\cdot),
\]

for almost all \( b_i. \) By Lebesgue dominated convergence theorem,

\[
\lim_{a^n \to (b_{i})^+} \int_{T_{-i}} \frac{u_{i,k} (t_i, t_{-i}, b_0, (a^n, b_{i,j}), b_{-i} (t_{-i})) - u_{i,k} (\cdot)}{a^n - b_{i,j}} 1_{[b_{i,k} > s_{i,k}]} \tau_{-i} (dt_{-i}|t_i)
\]

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exists and it is equal to \( E \left[ \partial_{b_{i,j}} u_{i,k} (\cdot) 1_{[b_{i,k}>s_{i,k}]} |t_i \right] \). Also,

\[
E \left[ u_{i,k} (\cdot) 1_{[b_{i,k}>s_{i,k}]} |t_i \right] = \text{initial value} + \int_{[b_{i,j}, \beta_i, j]} E \left[ \partial_{b_{i,j}} u_{i,k} (\beta, \cdot) 1_{[b_{i,k}>s_{i,k}]} |t_i \right].
\]

From now on, we will omit the “initial value” terms, since it is clear that they sum up to \( \Pi_i (t_i, b_0, b_i, b_{i,-i}) \).

In the second case \((j = k)\), let \( a^n \rightarrow (b_{i,k})^+ \) (i.e., \( a^n > b_{i,k} \)); the case \( a^n \rightarrow (b_{i,k})^- \) is analogous. Then,

\[
\int_{T_{i}} u_{i,k} (t_i, t_{-i}, b_0, (a^n, b_{i,-k}), b_{i,-i} (t_{-i})) 1_{[a^n > s_{i,k}]} \tau_{-i} \int (dt_{-i} | t_i)
\]

\[ - \int_{T_{i}} u_{i,k} (\cdot) 1_{[b_{i,k}>s_{i,k}]} \tau_{-i} \int (dt_{-i} | t_i)
\]

\[ = \int_{T_{i}} (u_{i,k} (t_i, t_{-i}, b_0, (a^n, b_{i,-k}), b_{i,-i} (t_{-i}))) - u_{i,k} (\cdot)) 1_{[a^n > s_{i,k}]} \tau_{-i} \int (dt_{-i} | t_i)
\]

\[ + \int_{T_{i}} u_{i,k} (\cdot) (1_{[a^n > s_{i,k}] - 1_{[b_{i,k}>s_{i,k}]}}) \tau_{-i} \int (dt_{-i} | t_i)
\]

\[ = \int_{T_{i}} (u_{i,k} (t_i, t_{-i}, b_0, (a^n, b_{i,-k}), b_{i,-i} (t_{-i}))) - u_{i,k} (\cdot)) 1_{[a^n > s_{i,k}]} \tau_{-i} \int (dt_{-i} | t_i)
\]

\[ + \int_{T_{i}} u_{i,k} (\cdot) 1_{[a^n > s_{i,k} \geq b_{i,k}]} \tau_{-i} \int (dt_{-i} | t_i).
\]

Since \( u_i \) has bounded derivative with respect to almost all \( b_{i,k} \),

\[
\lim_{a^n \rightarrow (b_{i,k})^+} \frac{u_{i,k} (t_i, t_{-i}, b_0, (a^n, b_{i,-k}), b_{i,-i} (t_{-i}))) - u_{i,k} (\cdot)}{a^n - b_{i,k}} = \partial_{b_{i,k}} u_{i,k} (\cdot),
\]
for almost all \( b_{i,k} \). Also, \( 1_{[a^n > s_{i,k}]} \rightarrow 1_{[b_{i,k} > s_{i,k}]} \). These imply that:

\[
\lim_{a^n \rightarrow (b_{i,k})^+} \frac{u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-k}), b_{-i}(t_{-i})) - u_{i,k}(\cdot)}{a^n - b_{i,k}} 1_{[a^n > s_{i,k}]}
\]

for almost all \( b_{i,k} \) and these functions are (almost everywhere) bounded.

Again by Lebesgue dominated convergence theorem, the limit

\[
\lim_{a^n \rightarrow (b_{i,k})^+} \int_{T_{-i}} \frac{(u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-k}), b_{-i}(t_{-i})) - u_{i,k}(\cdot))_1}{a^n - b_{i,k}} 1_{[a^n > s_{i,k}]}|t_i|\tau_{-i}(dt_{-i}|t_i)
\]

exists and it is equal to \( E[\partial_{b_{i,k}}u_{i,k}(\cdot)1_{[b_{i,k} > s_{i,k}]}|t_i] \). Also,

\[
E[u_{i,k}(\cdot)1_{[b_{i,k} > s_{i,k}]}|t_i] = \int_{[b_0, b_{i,k})} E[\partial_{b_{i,k}}u_{i,k}(t_i, t_{-i}, b_0, (\beta, b_{i,-k}), b_{-i}(t_{-i}))1_{[\beta > s_{i,k}]}|t_i]} d\beta
\]

Now, we want to determine:

\[
\lim_{a^n \rightarrow (b_{i,k})^+} \frac{1}{a^n - b_{i,k}} \int_{T_{-i}} u_{i,k}(\cdot) 1_{[a^n > s_{i,k} > b_{i,k}]}|t_i|\tau_{-i}(dt_{-i}|t_i)
\]

For each each \( t_i \in T_i \) and \( b_i \) fixed, define the signed measure \( \rho \) over \( \mathbb{R} \) by

\[
\rho(V; t_i, b_0, b_i, b_{-i}) \equiv \int_{T_{-i}} u_{i,k}(\cdot) 1_{[s_{i,k} \in V]}|t_i|\tau_{-i}(dt_{-i}|t_i).
\]

\[17\] On a \( \sigma \)-field this is synonymous with a countably additive set function.
Then,

\[ \lim_{a^n \to (b_{i,k})} \frac{1}{a^n - b_{i,k}} \int_{T_i} \frac{u_{i,k}(\cdot) \mathbb{1}_{[a^n > s_{i,k} \geq b_{i,k}]}}{\tau_{-i}} (dt_i|t_{i}) \]

\[ = \lim_{a^n \to (b_{i,k})} \frac{1}{a^n - b_{i,k}} (\rho([b_{i,k}, a^n); t_i, b_0, b_i, b_{-i}) \]

\[ = \lim_{a^n \to (b_{i,k})} \rho([b_{i,k}, a^n); t_i, b_0, b_i, b_{-i}) \]

\[ = \frac{\rho([b_{i,k}, a^n); t_i, b_0, b_i, b_{-i})}{m([b_{i,k}, a^n])}, \]

where \( m \) is Lebesgue measure over \( \mathbb{R} \). By Theorem 8.6 of Rudin [1966] this limit exists \( m \)-almost everywhere in \( b_{i,k} \) and we call it \( D\rho(t_i, b_0, b_i, b_{-i}) \).

Also, \( D\rho \) coincides almost everywhere with the Radon-Nikodym derivative \( \frac{d\rho}{dm}(t_i, b_0, b_i, b_{-i}). \) Therefore,

\[ \rho(V; t_i, b_0, b_i, b_{-i}) = \int_V \frac{\rho([b_{i,k}, a^n); t_i, b_0, b_i, b_{-i})}{m([b_{i,k}, a^n])} \ dm + \rho^{-}(V; t_i, b_0, b_i, b_{-i}) \]

where \( \rho^{-} \) denotes the singular part of \( \rho \), and it has the property

\[ \lim_{a^n \to (b_{i,k})} \rho^{-}(V; t_i, b_0, b_i, b_{-i}) = 0, \]

by the same theorem.

It is easy to see that \( \rho \) is absolutely continuous with respect to the distribution \( F_{s_{i,k}}(\cdot|t_i) \). The Radon-Nikodym Theorem guarantees the existence of the Radon-Nikodym derivative of \( \rho \) with respect to the distribution of \( F_{s_{i,k}}(\cdot|t_i) \), which we denote by \( g \). Therefore, \( g \) is such that

\[ \rho(V; t_i, b_0, b_i, b_{-i}) = \int_V g(\beta) F_{s_{i,k}}(d\beta|t_i) = \int_V g(\beta) f_{s_{i,k}}(\beta|t_i) d\beta \]
and by definition:

\[ \rho(V; t_{i}, b_{0}, b_{i}, b_{-i}) = \int_{\tau_{-i}} u_{i,k} (\cdot) 1_{[s_{i,k} \in V]} \tau_{-i}(dt_{-i}|t_{i}) \]

\[ = E[u_{i,k} (\cdot) 1_{[s_{i,k} \in V]} | t_{i}] \]

\[ = \int_{[b_{0,k}, \infty)} E[u_{i,k} (\cdot) 1_{[s_{i,k} \in V]} | t_{i}, s_{i,k} = \beta] F_{s_{i,k}} (d\beta|t_{i}) \]

\[ = \int_{V} E[u_{i,k} (\cdot) | t_{i}, s_{i,k} = \beta] F_{s_{i,k}} (d\beta|t_{i}) \]

\[ = \int_{V} E[u_{i,k} (\cdot) | t_{i}, s_{i,k} = \beta] f_{s_{i,k}} (\beta|t_{i}) d\beta. \]

Therefore, by the unicity of the Radom Nikodym derivative of \( \rho \) with respect to Lebesgue measure \( m \), we have that:

\[ g(\beta) = E_{\tau_{-i}, t_{i}} [u_{i,k} (\cdot) | s_{i,k} = \beta] f_{s_{i,k}} (\beta|t_{i}), \]

\( m \)-almost everywhere in \( \beta \).

Thus,

\[ \partial_{b_{i,j}} \Pi_{i}(t_{i}, b_{0}, (\beta, b_{-i}, -j), b_{-i}) \]

\[ = E[\partial_{b_{i,j}} u_{i,0} (\beta, \cdot) | t_{i}] \]

\[ + \sum_{k \neq j} E[\partial_{b_{i,j}} u_{i,k} (\beta, \cdot) 1_{[b_{i,k} > s_{i,k}] | t_{i}}] \]

\[ + E[\partial_{b_{i,j}} u_{i,j} (\beta, \cdot) 1_{[\beta > s_{i,k}] | t_{i}}] \]

\[ = E[u_{i,j} (\cdot) | t_{i}, s_{i,j} = \beta] f_{s_{i,k}} (\beta|t_{i}) \]
Finally, by the Lebesgue dominated convergence theorem,

\[
\Pi_i(t_i, b_0, b_i, b_{-i}) = \int_{[b_{i,j-1}, b_{i,j})} \partial_{b_{i,j}} \Pi_i(t_i, b_0, (\beta, b_{i,-j}), b_{-i}) d\beta + \int_{[b_{i,j-1}, b_{i,j})} E [u_{i,j}(\cdot)|t_i, s_{i,k} = \beta] F_{s_{i,j}}^\perp (\beta|t_i)
\]

\[
\sum_{k=1}^K \int_{T_{-i}} a_{i,k}(\cdot) u_{i,k}(\cdot) 1_{[b_{i,k}>s_{i,k}]} \tau_{-i}(dt_{-i}|t_i)
\]

This concludes the proof.

### 5.2 Proof of Proposition 2

**Proof.**

(i) Since \(B\) is compact and \(\Pi_i(t_i, \cdot, b_{-i})\) is continuous if \(F_{b_{-i}}(\cdot)\) is absolutely continuous, the conclusion is immediate.

(ii) We will make use of the expression:

\[
\partial_{b_{i,j}} \Pi_i(t_i, b_0, (\beta, b_{i,-j}), b_{-i}) = E [\partial_{b_{i,j}} u_{i,0}(\cdot)|t_i]
\]

\[
+ \sum_{k \neq j} E [\partial_{b_{i,j}} u_{i,k}(\cdot)|t_i] 1_{[b_{i,k}>s_{i,k}]}|t_i]
\]

\[
+ E [\partial_{b_{i,j}} u_{i,j}(\cdot)|t_i] 1_{[b_{i,j}>s_{i,j}]}|t_i]
\]

\[
+ E [u_{i,j}(\cdot)|t_i, s_{i,j} = \beta] f_{s_{i,j}}(\beta|t_i).\]

Since \(b_1^j > b_2^j\), we can choose a curve \(\alpha : [0, 1] \rightarrow B\), such that \(\alpha(0) = b_2^j\), \(\alpha(1) = b_1^j\) and

\[
\alpha_j'(s) \geq 0, \forall j, s \in [0, 1] \text{ and } \exists j \text{ such that } \alpha_j'(s) > 0, \forall s \in [0, 1]. \quad (10)
\]

By Assumption, \(\partial_{b_{i,j}} u_{i,k}(t_1^j, \cdot) \leq \partial_{b_{i,j}} u_{i,k}(t_2^j, \cdot)\) for all \(k\). Thus,

\[
E [\partial_{b_{i,j}} u_{i,k}(t_1^j, \cdot) 1_{[\alpha_j(s)>s_{i,k}]}] \leq E [\partial_{b_{i,j}} u_{i,k}(t_2^j, \cdot) 1_{[\alpha_j(s)>s_{i,k}]}]. \quad (11)
\]
Since \([0, 1]^{n-1}\) and \(B^n\) are compact and \(u_i\) is (absolutely) continuous, there exists \(\delta > 0\) such that \(u_{i,k} (t_1^1, t_{-i}, b) + 2\delta < u_{i,k} (t_1^2, t_{-i}, b)\) for all \(t_{-i} \in [0, 1]^{n-1}\), all \(b \in B^n\) and all \(k\). For fixed bid \(\beta \in B\) and \(j\), define the functions

\[
g^1(t_{-i}) = u_{i,j} (t_1^1, t_{-i}, \beta, b_{-i} (t_{-i})) , \quad \text{and} \\
g^2(t_{-i}) = u_{i,j} (t_1^2, t_{-i}, \beta, b_{-i} (t_{-i})).
\]

Then, \(g^1(t_{-i}) + 2\delta < g^2(t_{-i})\). By the positivity of conditional expectations\(^\text{18}\)

\[
E[g^2 - g^1 - 2\delta|s_{i,j} = \beta] \geq 0.
\]

Thus, from the independence of types, we conclude that

\[
E[u_{i,j} (t_1^1, \cdot)|t_i, s_{i,j} = \beta] + \delta < E[u_{i,j} (t_1^2, \cdot)|t_i, s_{i,j} = \beta]. \quad (12)
\]

Then, (11), (12), and the expression of \(\partial_{b_i} \Pi_i(t_i, \beta, b_{-i})\) given by the characterization Lemma imply that for almost all \(\beta\),

\[
\nabla_{b_i} \Pi_i(t_2^1, \alpha(s), b_{-i}) > \nabla_{b_i} \Pi_i(t_1^1, \alpha(s), b_{-i}) + \delta f_s(\alpha(s)), \quad (13)
\]

where \(f_s(\alpha(s))\) denotes the vector \((f_{s_i,1}(\alpha_1(s)), \ldots, f_{s_i,K}(\alpha_K(s)))\). The assumption on the distribution implied by \(b_{-i}\) allow to write the difference \(\Pi_i(t_1^2, b_1^1, b_{-i}) -
\]

\(^{18}\)See, for instance, Kallenberg (2002), Theorem 6.1, p. 104.
\[ \Pi_i(t_i^2, b_i^2, b_{-i}) \] as an integral:

\[
\Pi_i(t_i^2, b_i^1, b_{-i}) - \Pi_i(t_i^2, b_i^2, b_{-i}) = \int_{[0,1]} \nabla b_i \Pi_i(t_i^2, b_0, \alpha(s), b_{-i}) \cdot \alpha'(s) \, ds \\
> \int_{[0,1]} \nabla b_i \Pi_i(t_i^1, b_0, \alpha(s), b_{-i}) \cdot \alpha'(s) \, ds + \delta \sum_j \int_{[0,1]} f_{s_{i,j}} (\alpha_j (s)) \alpha_j' (s) \, ds \\
\geq \delta \sum_j \int_{[0,1]} f_{s_{i,j}} (\alpha_j (s)) \alpha_j' (s) \, ds \\
\geq 0,
\]

where the first inequality comes from (10) and (13); the second comes from the fact that \( b_i^1 \in \Theta_i (t_i^1, b_{-i}) \), that is,

\[
\Pi_i(t_i^1, b_i^1, b_{-i}) - \Pi_i(t_i^1, b_i^2, b_{-i}) = \int_{[0,1]} \nabla b_i \Pi_i(t_i^1, b_0, \alpha(s), b_{-i}) \cdot \alpha'(s) \, ds \geq 0;
\]

and the third comes from \( \alpha_j' (s) \geq 0 \) for all \( j \). Now, this implies that \( \Pi_i (t_i^2, b_i^1, b_{-i}) > \Pi_i (t_i^2, b_i^2, b_{-i}) \), which contradicts the fact that \( b_i^2 \in \Theta_i (t_i^2, b_{-i}) \).

### References


