Incentives for Unaware Agents

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Abstract

The paper introduces the problem of unawareness into Principal-Agent theory and discusses optimal incentive contracts when the Agent is unaware of some dimension of her action space. We introduce two key parameters to describe the problem, the extent and the effect of unawareness, and show under what conditions it is optimal for the Principal to propose an incomplete contract (that keeps the Agent unaware) or a complete contract. The key tradeoff is that of enlarging the Agent’s choice set versus adding costly incentive constraints. If Agents differ in their unawareness, optimal incentive schemes can be distorted for aware and unaware Agents, because, different from standard contract theory, the Single-Crossing Property fails to hold. In this case, even aware Agents can be subject to inefficiently high or low incentives.

Keywords: Incentive design, unawareness, moral hazard, incomplete contracts

JEL Classification: D01, D86, D82, D83
1 Introduction

The classical model of moral hazard between a Principal and an Agent as developed by Mirrlees (1975), Holmström (1979) and Grossman and Hart (1983) assumes that the Agent takes an unobservable action that is typically associated with effort, caution, diligence, time spent, moderation in private consumption, use of efficient technologies, and other decisions. This unobservable action is typically complex and multi-dimensional, and the basic model integrates all these into the one-dimensional "effort" variable. In reality, however, the Agent typically does not understand all these different dimensions and is unaware of some choice possibilities. For example, in an employment relationship, an employee might be unaware of the possibility of using some set of tools to improve her work performance, or another might be unaware of some shirking behavior, such as idling about in "Second Life" in her office, that impairs her performance.

When the Agent is unaware of part of her contracting environment, the standard solution concept for Principal-Agent problems is not satisfactory, since it might not be optimal for the Principal to write all actions of the Agent into the contract and regulate them by means of incentive-compatibility constraints. Incomplete contracts might be a better alternative for the Principal. For instance, if the employer knows that the employee is unaware of an internet cafe near her workplace, then it may not be optimal for the employer to regulate going to the internet cafe explicitly in the contract, since this makes the employee aware of this type of activity and going there is not observable. In the present paper, we therefore consider a generalization of the standard Principal-Agent model in which the Agent is unaware of some dimensions of the effort variable, while being able to optimize over other dimensions.

The paper uses the classical multi-task Principal-Agent model by Holmström and Milgrom (1991) as a starting point. This model is well understood, captures many important contracting considerations by simple parametrization, and is very rich. We first discuss the optimal incentive contracting problem under the assumption that the Principal is fully aware and the Agent is unaware of some dimension of effort choice and therefore takes an unconscious default action in that dimension that does not respond to incentives. In the model, the Principal designs the incentive scheme and contemplates whether to make the Agent aware of the full problem. We show that if the Agent’s default behavior is too lazy or too diligent in this dimension, the Principal will optimally write a complete contract to make the Agent aware. On the other hand, if the Agent’s default behavior is sufficiently efficient, the Principal will write an incomplete contract where this action is missing. This incompleteness increases the total surplus by reducing the incentive component of the Agent’s pay. Compared to the standard optimal incentive contract, there is a new trade-off for the Principal: the benefit of enlarging the choice set of the Agent versus the cost of adding a new incentive constraint.

We then extend the analysis to an environment with heterogeneous awareness of Agents, where the Principal cannot distinguish whether the Agent is aware or not. In such an environment, the contract that is optimal for an unaware Agent is not viable, because aware Agents will exploit its low pay-performance sensitivity. Thus the Principal has to screen Agents.

We characterize the solution to the screening problem in terms of two basic para-
meters of the unawareness problem: the extent of unawareness (how many unaware Agents are there?) and the effect of unawareness (how does unawareness distort the Agent’s action?). Similarly to traditional screening problems (see, e.g., Bolton and Dewatripont (2005)), unaware Agents are kept to their outside utility, i.e. do not receive any rent from the relationship, and the rent received by aware Agents increases in the extent of unawareness. Hence, the existence of unaware Agents exerts a positive externality on the aware Agents. Differently from standard theory, however, in our model the single-crossing property does not hold. This changes the analysis in some important respects. In particular, efficiency losses can arise for both types of Agents, and the incentive-compatibility constraints of both types can bind, leading to a pooling or a constrained separating solution. While unaware Agents always bear too much risk, aware Agents can bear too much or too little risk, depending on the effect of unawareness. Hence, in the parlance of contract theory, there can be “distortion at the top”, and this distortion can even go in both directions.

The biggest difference from standard screening problems, however, is that in our problem the extent of unawareness (the population mix) is not exogenous, because the Principal has the option to make the Agent aware of the full problem by proposing a complete contract. In our final take on the problem, we study this possibility and show that complete or incomplete contracts can both emerge as optimal. Furthermore, all three different sorts of complete contracts obtained earlier can be optimal, separating, pooling, and constrained separating. We further show that the comparative statics of contract incompleteness is surprisingly simple: the larger the extent of unawareness a priori, the more frequent are incomplete contracts at the optimum (where frequency refers to our second key parameter, the measure of the effect of unawareness). This can be interpreted as a self-reinforcing pattern: populations with a large degree of unawareness will operate predominantly with incomplete contracts, thus preserving unawareness. On the other hand, populations with low degrees of unawareness will operate mainly with complete contracts, which eliminate unawareness.

The rest of the paper is organized as follows. The next section describes previous approaches to modelling unawareness and shows how our approach fits into the literature of contracting with non-standard preferences. Section 3 considers the case of homogenous unawareness, in which the Principal faces an Agent whom he knows to be unaware of the full contracting problem. Section 4 introduces heterogenous unawareness, where the Agent may or may not be unaware of the full problem. Section 5 discusses the problem of justifiability of contracts from the unaware Agent’s point of view. Section 6 concludes by discussing a number of conceptual points, such as the robustness against competition, communication and so on.

2 Unawareness and Contract Incompleteness

The difficulties of modelling unawareness by conventional economic information theory have been exposed very clearly by Dekel, Lipman, and Rustichini (1998) who show that it is impossible to describe non-trivial unawareness in the standard state space model. In particular, they argue forcefully that an agent who is unaware of an event should not be aware that he is unaware of this event. In response, Li (2006), Heifetz, Meier and Schipper (2006) and Galanis (2007) have proposed theories that
circumvent the negative result. The shared feature of these papers is that what is
missing in the Agent’s mind is not some points in the state space but a whole di-
mension. Our work focuses on the Agent’s unawareness of her own action set. However,
we build on this work by assuming that what is missing in the Agent’s mind is not
some point in the choice set but one whole dimension of it.

The challenge of incorporating unawareness into dynamic game theory has re-
cently been addressed by Halpern and Rego (2006) and Rego and Halpern (2007)
who provide a general setting for studying games with unawareness of actions. The
Principal-Agent model that we discuss in our paper uses, of course, a simple dynamic
game and fits naturally in the approach proposed by these two authors.

Unawareness of actions requires a theory of restricted decision-making by the
Agent. Does the Agent optimize over a restricted set? Does she follow some heuris-
tics? As discussed above, our theory must assume that a whole dimension of actions
is missing in the Agent’s mind. We can therefore address this problem very sim-
ply, following Hayek (1968), Vanberg (2002) and others, by assuming that the Agent
chooses a default action in the missing dimension and optimizes over the other di-
mension(s). The default action is an instance of rule-guided or automatic behavior
which is not determined by rational choice. There is ample evidence of such behavior
in the sociological and psychological literature that documents various forms of auto-
matic versus controlled behavior (see, e.g., Fiske and Taylor, 2006). This rule-guided
behavior creates a certain exogenous bias in the Agent’s behavior that we take as an
important parameter in our comparative statics.

An important conceptual difficulty in understanding unawareness is the interaction
between fully aware and unaware contracting parties. Gabaix and Laibson (2006)
analyze the interaction between firms and unaware consumers. The consumers who
are unaware of later add-on prices are exploited by the firms. In our paper, the
Agent is only unaware of her own actions, so there is no issue of exploitation. Filiz
(2006) models interaction between a rational insurer and an insuree who is unaware
of some contingencies. In contrast, in our paper, what is missing in the Agent’s mind
is not some future contingencies but her choice possibilities. Perhaps closest to our
theory in this respect is the work by Eliaz and Spiegler (2006) who study a contract-
theoretic model of screening consumers’ awareness of their future changed tastes. In
our paper, the Principal is also confronted with Agents of different awareness and
designs contracts to exploit these differences. But differently from their work, we
focus on the provision of incentives and on how unawareness changes the traditional
Principal-Agent paradigm.

Our work also contributes to the recent literature on the foundations of contract
incompleteness. The literature has proposed several reasons why contracting parties
may not specify everything that is relevant for the interaction in the contract. Most
notable are probably the non-enforceability of some contingencies (Grossman and
Hart (1986), Hart and Moore (1990)), signaling (Aghion and Bolton (1987), Spier
(1992)), and explicit writing cost (Dye (1985), Anderlini and Felli (1999), Battigalli
and Maggi (2002)). Recent approaches endogenize contractual incompleteness by lim-
ited cognition and strategic investment in cognition by the contracting parties (Bolton
and Faure-Grimaud (2007), Tirole (2007)). These papers take a less radical approach
towards unawareness than Dekel, Lipman and Rustichini (1998), as they assume that
agents are aware of the fact that they may be unaware of some relevant elements of the contracting environment. In these theories, as in Gabaix and Laibson (2006) and Filiz (2006), contractual incompleteness arises because better informed agents shroud some contingencies or possibilities. In the present paper, contracts can be incomplete for the same reason: the Principal strategically shrouds a dimension of action choice by the Agent and only announces a compensation scheme. While such shrouding (and hence the distinction between complete and incomplete contracts) is irrelevant in standard Principal-Agent theory, it matters in the context of unawareness, and we characterize its determinants and effects.

Our paper also belongs to the emerging literature on the trade-off of rigidity versus flexibility. Amador, Werning, and Angeletos (2006) study this trade-off in a consumption-saving model. Flexibility is modeled by a large choice set and helps the consumer deal with uncertainty about future preferences, but it exposes the consumer to a self-control problem. Hart and Moore (2007) study the optimality of setting a range of permitted trading prices in a bilateral contracting problem. A larger range promotes more trading ex post when the buyer’s valuation and the seller’s cost are both uncertain ex ante. But it leads to aggrievement of the parties and makes them inefficiently retaliate against each other. The present paper points out a basic trade-off that is due to the provision of incentives: the Principal balances the benefit of enlarging the choice set of the Agent against the cost of adding a new incentive constraint.

3 The Basic Model

There are two parties, a Principal and an Agent. The Principal proposes a contract to the Agent to work for him. The Agent’s work involves effort in two dimensions, \((t_1, t_2) \in \mathbb{R}_+^2\). In our context, the case of higher-dimensional effort is a straightforward extension. The Agent’s effort creates a performance of monetary value \(x = t_1 + t_2 + \epsilon\) where \(\epsilon\) is normally distributed with zero mean and variance \(\sigma^2\). The effort choice \(t_1\) and \(t_2\) is not observable by the Principal, but \(x\) is verifiable. \(t_1\) and \(t_2\) denote different forms of effort spent by the Agent to produce good results that we discuss later in more detail.

By assumption, the Principal remunerates the Agent by a linear compensation rule \(w(x) = \alpha x + \beta\). \(\alpha\) measures the intensity of the incentives provided to the Agent. Hence, \(\alpha x\) is the incentive pay and \(\beta\) represents the base salary.\(^1\) In standard contract theory, a contract is a tuple \((\alpha, \beta, t_1, t_2)\). Although \(t_1\) and \(t_2\) are not observable by the Principal, they can be included in the contract for completeness; their choice, however, must be supported by an appropriate incentive constraint. Alternatively, the parties can write an incomplete contract \((\alpha, \beta)\) that induces the Agent to choose certain levels of effort by virtue of her incentive constraint. If both parties understand the contracting problem, complete and incomplete contracts are equivalent.

The timing is as follows:

\(^1\)We assume this form of contract because it is simple and captures two important elements of incentive contracting. Holmström and Milgrom (1987) provide a foundation for this assumption in a dynamic setting.
1. The Principal proposes a contract or proposing nothing. If a contract is proposed, the Agent decides whether to accept it. If the Principal proposes nothing or the Agent rejects the contract, then the game is over and each party receives the outside payoff zero.

2. If the Agent accepts the contract, the Agent exerts efforts \( t_1 \) and \( t_2 \).

3. The outcome of performance \( x \) is realized and the contractual compensation is paid.

The Agent’s cost of effort can be measured in monetary units and is \( C(t_1, t_2) = \frac{1}{2} t_1^2 + \frac{1}{2} c t_2^2 \) with \( c > 0 \). The smaller \( c \), the more costly the second dimension of effort. The Agent has an exponential von Neuman-Morgenstern utility function over money

\[
u(y) = -e^{-y}
\]

where we have normalized the coefficient of absolute risk aversion to 1. Since \( \epsilon \) is normally distributed, we have the standard result that

\[
u(CE) = E[u(w(x) - C(t_1, t_2))]
\]

where

\[
CE \equiv \alpha(t_1 + t_2) + \beta - C(t_1, t_2) - \frac{1}{2} \sigma^2 \alpha^2.
\]

is the certainty equivalent of the Agent.

The Principal is risk neutral. Hence, his certainty equivalent equals his expected utility

\[
E[x - w(x)] = (1 - \alpha)(t_1 + t_2) - \beta
\]

The total surplus of the Principal and the Agent is the sum of their certainty equivalents

\[
t_1 + t_2 - C(t_1, t_2) - \frac{1}{2} \sigma^2 \alpha^2.
\]

The first best solution maximizes the total surplus and is given by \( t_1^{FB} = 1, t_2^{FB} = c \) and \( \alpha^{FB} = 0 \): the more costly in terms of effort the second task is compared to the first one, the less effort is optimally devoted to it. Additionally, the incentive pay is zero, since we have the full insurance result.

The innovation in our paper is the assumption that the Agent is unaware of \( t_2 \) before contracting. If she is still unaware of \( t_2 \) after contracting, the Agent will choose the default effort, or status quo choice \( t_2 = \tau \geq 0 \) unconsciously in stage 2. The Agent’s choice of the default action is not based on rational calculation. \( \tau \) is only her unconscious rule-guided behavior (see Hayek (1967), Vanberg (2002)).

There are two ways of interpreting the notion of unawareness in our context. The first is narrower and assumes that the Agent is simply unaware of the possibility of choosing the activities summarized by \( t_2 \). This may be the utilization of a certain type of equipment that improves output (in which case the default level \( \tau \) of not using this equipment is probably inefficiently low) or some form of amenity or perquisite that makes work more pleasant (in which case the default level may be too high or too low, depending on its level). A second, broader interpretation of unawareness assumes that the Agent is aware of the activities summarized by \( t_2 \), but unaware of their causes and consequences, or theorems in term of Galanis (2007). In this case, the Agent is
not aware of the true model of the costs and effects of effort and chooses $\tau$ according to some habits, norms or considerations that do not respond to incentives. Examples for this type of activity are unobservable investments into maintaining or improving equipment or the work environment, the Agent’s effort in personal customer relations and other forms of personal conduct, or the Agent’s effort towards helping others in the organization. Other examples are private emails or phone calls during work or the use of a company car for private ends (where more of these activities correspond to lower $t_2$). Here, the Agent is well aware of these activities, but unaware of their costs and benefits and chooses them according to some routine. The essential distinction between activities $t_1$ and $t_2$ is that the former respond to monetary incentives while the latter do not, unless the Agent is made explicitly aware of them.

We summarize this explicit communication by the notion of a complete contract $(\alpha, \beta, t_1, t_2)$ in stage 1. If the Agent sees such a contract she will update her awareness and the new dimension of effort choice comes to her mind. On the other hand, if the Principal proposes an incomplete contract where $t_2$ is missing, the Agent remains unaware of it. Thus, if the Agent is unaware of the full effort problem, complete contracts and incomplete contracts are different instruments.

Under a complete contract, the Principal announces $(t_1, t_2)$, and the Agent is aware of $t_2$. The optimal contract proposed by the Principal is the solution of the following standard problem of multi-task incentive design:

$$\max_{\alpha, \beta, t_1, t_2} (1 - \alpha)(t_1 + t_2) - \beta$$

s.t. $(t_1, t_2) \in \arg\max u(\alpha(t_1 + t_2) + \beta - C(t_1, t_2) - \frac{1}{2}\sigma^2\alpha^2)$

$u(\alpha(t_1 + t_2) + \beta - C(t_1, t_2) - \frac{1}{2}\sigma^2\alpha^2) \geq u(0)$

(3) is the incentive compatibility constraint for the aware Agent, and (4) is her participation constraint.

Because of the simple linear-quadratic form of the problem, the incentive constraint is equivalent to

$$t_1 = \alpha, \quad t_2 = c\alpha$$

Since $u(\cdot)$ is strictly increasing, the participation constraint must be binding, and the Principal maximizes the total surplus subjective to the Agent’s incentive constraint (5). The solution is

$$\alpha^A = \frac{1 + c}{1 + c + \sigma^2},$$

$$\beta^A = \frac{1}{2}(\sigma^2 - 1 - c)(\frac{1 + c}{1 + c + \sigma^2})^2,$$

$$t_1^A = 1 - \frac{\sigma^2}{1 + c + \sigma^2},$$

$$t_2^A = c - \frac{c\sigma^2}{1 + c + \sigma^2}.$$
The superscript \( A \) means that the Agent is “Aware”. The Principal’s expected profit is
\[
\pi^A = \frac{(1 + c)^2}{2(1 + c + \sigma^2)}
\] (10)
which is positive. Thus proposing a complete contract is always better than proposing nothing for the Principal.

If the Principal proposes an incomplete contract, then the Agent is still unaware of the second effort dimension and will choose the default action \( \tau \) in stage 2. In this section, we assume that the Principal knows \( \tau \), interpreted as a typical status quo choice taken by unaware Agents. Thus the Principal solves the following problem:
\[
\max_{\alpha, \beta, t_1} (1 - \alpha)(t_1 + \tau) - \beta
\] (11)
\[
\text{s.t. } t_1 \in \arg\max u(\alpha(t_1 + \tau) + \beta - C(t_1, \tau) - \frac{1}{2}\sigma^2\alpha^2)
\] (12)
\[
u(\alpha(t_1 + \tau) + \beta - C(t_1, \tau) - \frac{1}{2}\sigma^2\alpha^2) \geq u(0)
\] (13)

(12) and (13) are the incentive compatibility constraint and participation constraint, respectively, for the unaware Agent. Since \( \tau \) is exogenous, there is no incentive-compatibility constraint for \( t_2 \).\(^3\) As in the case of awareness, the incentive constraint on \( t_1 \) reduces to \( t_1 = \alpha \), and the solution to the optimization problem then is
\[
\alpha^U = \frac{1}{1 + \sigma^2},
\] (14)
\[
\beta^U = \frac{\tau^2}{2c} - \frac{\tau}{1 + \sigma^2} + \frac{\sigma^2 - 1}{2(1 + \sigma^2)^2},
\] (15)
\[
t_1^U = 1 - \frac{\sigma^2}{1 + \sigma^2}
\] (16)

**Observation 1** \( \alpha^U < \alpha^A \).

If the Principal does not mention \( t_2 \) in the contract, then the incentive component plays a less significant role in the wage structure. The intuition is that if the Agent is unaware of one dimension of effort, she is more restricted in adjusting her effort choice. Hence, the outcome \( x \) is less sensitive to effort and the Agent’s pay should be less sensitive to \( x \).

**Observation 2** \( t_1^U < t_1^A < t_1^{FB} \) and \( t_2^A < t_2^{FB} \).

Because of the problem of hidden action, the effort levels that the Agent controls are less than the first best levels. Moreover, \( t_1^U < t_1^A \), i.e., the aware Agent will work harder even in the dimension in which awareness makes no difference. This is due to Observation 1, \( \alpha^U < \alpha^A \): the pay-performance sensitivity plays a more important role in the wage structure when the Agent is fully aware.

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\(^3\)We assume that the aware Agent and the unaware Agent derive the same utility level from their outside option, say staying at home. In particular, we rule out the possibility that the Agent can improve the value of her outside option when being aware of \( t_2 \).
When the Agent is unaware, the expected profit of the Principal is
\[ \pi^U = \frac{1}{2(1 + \sigma^2)} + \tau - \frac{\tau^2}{2c}. \]  
(17)

Combining (10) and (17), we have
\[ \pi^A - \pi^U = \frac{\tau^2}{2c} - \tau + \frac{(1 + c)^2}{2(1 + c + \sigma^2)} - \frac{1}{2(1 + \sigma^2)}. \]  
(18)

The right-hand side of (18) is quadratic in \( \tau \). Solving this quadratic equation (and ignoring the case of indifference) yields the following proposition.

**Proposition 1** If the Principal knows that the Agent is unaware of \( t_2 \), he optimally proposes \( (\alpha^U, \beta^U, t^U_1) \) for values
\[ \tau \in \left( c - \frac{c\sigma^2}{\sqrt{(1 + \sigma^2 + c)(1 + \sigma^2)}}, c + \frac{c\sigma^2}{\sqrt{(1 + \sigma^2 + c)(1 + \sigma^2)}} \right). \]  
(19)

Otherwise, he proposes \( (\alpha^A, \beta^A, t^A_1, t^A_2) \).

In other words, the Principal will write an incomplete contract without mentioning \( t_2 \) if and only if \( \tau \) is in the interval given in (19).

Proposition 1 implies that if the Agent is unaware that she is too lazy or too diligent in some dimension of the effort choice, the Principal will optimally make her aware of this effort dimension. It is quite plausible that if the Agent is unconsciously very lazy, it is better for the Principal to make the Agent aware of it and subject her to explicit incentives. Interestingly, however, even if the Agent is too diligent, say spending an enormous amount of time helping others on the job, the Principal also has an incentive to make the Agent aware of this dimension of her job. The reason is that the Agent bears the cost even of the actions she undertakes unconsciously. Hence, if \( \tau \) is too far away from the efficient level, in either case, the Agent’s allocation of efforts reduces total surplus, which ultimately hurts the Principal. However, making the Agent aware of this problem also has a cost: it adds an incentive constraint to the Agent’s choice problem, with a corresponding reduction of surplus.

Remembering that \( t^E_B = c \), Proposition 1 reflects the fact that if the Agent’s default action is sufficiently close to the first best one, then the Principal will optimally be silent on \( t_2 \). If the Principal announces \( t_2 \) in the contract, the Principal is forced to provide explicit incentives to the Agent, which in this case is more costly than having the Agent operate at the status-quo level \( \tau \). An interesting observation is that even if \( \tau \in \left( c - c\sigma^2/\sqrt{(1 + \sigma^2 + c)(1 + \sigma^2)}, t^E_2 \right) \), that is, if after making the Agent aware, the Agent will work harder, the Principal still prefers writing an incomplete contract. The reason is that the high effort level \( t^A_2 \) comes at the expense of a high incentive pay, which hurts the Principal.

\[^4\text{Because } \pi^A > 0, \text{ whenever } \pi^U > \pi^A, \text{ we get } \pi^U > 0. \text{ Hence, the Principal always gains from proposing a contract.}\]
As a general rule, the Principal’s decision between a complete and an incomplete contract balances the benefit of enlarging the Agent’s choice set against the cost of adding additional incentive constraints.

The comparative statics of Proposition 1 show that when \( c \) decreases, the range of \( \tau \) for which the optimal contract is incomplete shifts to the left and shrinks. Hence, if \( t_2 \) becomes more and more costly, it is less probable that the optimal contract is incomplete and high default effort levels \( \tau \) are more likely to lead to contract incompleteness. Similarly, when \( \sigma^2 \) increases, the interval gets larger. Thus the noisier the environment, the more probable it is that optimal contract is incomplete.

Finally, it should be noted that there is no need for policy intervention to promote the Agent’s awareness. Since the Principal maximizes total surplus (subject to the incentive constraint of the Agent), when he prefers an incomplete contract, the total surplus is larger than under a complete contract.

4 Heterogeneous Awareness

In the previous section, we have assumed that the Principal knows whether the Agent is aware of her full effort problem. We now generalize the analysis by assuming that the Principal does not know this. Formally, we assume that there are a fraction \( \lambda \) of the Agents who are fully aware (type \( A \)) and \( 1 - \lambda \) of the Agents who are unaware of the second dimension of the effort problem (type \( U \)), but the Principal cannot distinguish them. To simplify the exposition, we set \( c = 1 \) (the whole analysis extends to arbitrary \( c > 0 \)). To make the problem interesting, we assume that the status-quo effort level of the unaware Agent lies in the interval (19) in which it is better for the Principal to keep the unaware Agent unaware:

\[
\tau \in (\tau_{\text{min}}, \tau_{\text{max}}) \equiv \left( 1 - \frac{\sigma^2}{\sqrt{(2 + \sigma^2)(1 + \sigma^2)}} \right) \left( 1 + \frac{\sigma^2}{\sqrt{(2 + \sigma^2)(1 + \sigma^2)}} \right) \tag{20}
\]

In the previous section, we have identified the contracts \((\alpha^A, \beta^A, t^A_1, t^A_2)\) for the aware Agent and \((\alpha^U, \beta^U, t^U_1)\) for the unaware Agent that the Principal would optimally offer each of these two types if he knew their type. However, as the following observation shows, if the Principal offers both these contracts Agents of different types do not self-select:

**Observation 3** If the Principal proposes the contracts \((\alpha^A, \beta^A, t^A_1, t^A_2)\) and \((\alpha^U, \beta^U, t^U_1)\), the aware Agent will choose \((\alpha^U, \beta^U, t^U_1)\).

To show Observation 3 note that if the aware Agent chooses \((\alpha^A, \beta^A, t^A_1, t^A_2)\), she receives a certainty equivalent of 0, since her participation constraint (4) binds. But if she chooses \((\alpha^U, \beta^U, t^U_1)\), she receives

\[
\max_{t_1, t_2} \{ \alpha^U(t_1 + t_2) + \beta^U - C(t_1, t_2) - \frac{1}{2} \sigma^2 (\alpha^U)^2 \}
\]

\[
= \frac{1}{2} (2 - \sigma^2)(\alpha^U)^2 + \beta^U
\]

\[
= \frac{(\tau + \sigma^2 \tau - 1)^2}{2(\sigma^2 + 1)^2} \geq 0.
\]
Hence, we need to determine the menu of contracts into which the Agents select themselves according to their type. Yet, there is a second problem. If the Principal proposes a menu of contracts of which one specifies two effort levels \((t^A_1, t^A_2)\), then unaware Agents will become aware of the second dimension \(t_2\), because \(t_2\) is explicitly announced in the menu of contracts. But the Principal can easily circumvent this problem by proposing two incomplete contracts \((\alpha^A, \beta^A)\) and \((\alpha^U, \beta^U)\) without mentioning the Agent’s effort obligations. As discussed in the previous section, there is no conceptual difference between an incomplete and a complete contract in our setting if the Agent is aware of the full effort problem. The corresponding efforts are automatically implied by the Agent’s optimization given the contracts. In fact, from (5) with \(c = 1\) we know that \(t^A_1 = t^A_2 = \alpha^A\) and \(t'^U_1 = \alpha^U\).

From now on we shall therefore only consider menus \(C^A = (\alpha^A, \beta^A), C^U = (\alpha^U, \beta^U)\) of incomplete contracts.

The Principal now solves the following fairly standard screening problem, where we already replace the Agent’s effort incentive constraint by the corresponding first-order condition:

\[
\max_{\alpha^A, \beta^A, t^A_1, t^A_2, \alpha^U, \beta^U, t'^U_1} \lambda[(1 - \alpha^A)(t^A_1 + t^A_2) - \beta^A] + (1 - \lambda)[(1 - \alpha^U)(t'^U_1 + \tau) - \beta^U] \\
\text{s.t. } t^A_1 = t^A_2 = \alpha^A \\
\quad t'^U_1 = \alpha^U \\
\quad \alpha^A(t^A_1 + t^A_2) + \beta^A - C(t^A_1, t^A_2) - \frac{1}{2}\sigma^2(\alpha^A)^2 \geq 0 \\
\quad \alpha^U(t'^U_1 + \tau) + \beta^U - C(t'^U_1, \tau) - \frac{1}{2}\sigma^2(\alpha^U)^2 \geq 0 \\
\quad \alpha^A(t^A_1 + t^A_2) + \beta^A - C(t^A_1, t^A_2) - \frac{1}{2}\sigma^2(\alpha^A)^2 \geq \max_{t_1, t_2} \{\alpha^U(t_1 + t_2) + \beta^U - C(t_1, t_2) - \frac{1}{2}\sigma^2(\alpha^U)^2\} \\
\quad \alpha^U(t'^U_1 + \tau) + \beta^U - C(t'^U_1, \tau) - \frac{1}{2}\sigma^2(\alpha^U)^2 \geq \max_{t_1} \{\alpha^A(t_1 + \tau) + \beta^A - C(t_1, \tau) - \frac{1}{2}\sigma^2(\alpha^A)^2\}
\]

Here, \((PCA)\) and \((PCU)\) are the Agent’s participation constraints, for the aware and the unaware type, respectively, and \((ICA)\) and \((ICU)\) the incentive-compatibility constraints that make sure that the aware and the unaware Agent select the appropriate contracts. Although the unaware Agent is rational in the sense that she optimizes her effort choice given the compensation rule and chooses her preferred compensation rule, she does not know why the Principal proposes the menu in question. Thus the assumption of mutual knowledge of the interaction does not hold in our model. We assume that the unaware Agent is not only unaware of the full effort problem, but also boundedly rational in the sense that she cannot infer from the menu that she is unaware of this problem. We come back to this question of the justifiability of contracts in section 5.

Upon substituting \((ICA')\) and \((ICU')\) into the other expressions of the above
problem, the contract design problem becomes

\[
\max_{\alpha^A, \beta^A, \alpha^U, \beta^U} \lambda [2(\alpha^A - (\alpha^A)^2) - \beta^A] + (1 - \lambda) [(1 - \tau)\alpha^U - (\alpha^U)^2 + \tau - \beta^U]
\]  
\[\text{s.t. } \frac{1}{2}(2 - \sigma^2)(\alpha^A)^2 + \beta^A \geq 0 \quad (PCA)\]

\[
\frac{1}{2}(1 - \sigma^2)(\alpha^U)^2 + \tau\alpha^U + \beta^U - \frac{\tau^2}{2} \geq 0 \quad (PCU)
\]

\[
\frac{1}{2}(2 - \sigma^2)(\alpha^A)^2 + \beta^A \geq \frac{1}{2}(2 - \sigma^2)(\alpha^U)^2 + \beta^U \quad (ICA)
\]

\[
\frac{1}{2}(1 - \sigma^2)(\alpha^U)^2 + \tau\alpha^U + \beta^U \geq \frac{1}{2}(1 - \sigma^2)(\alpha^A)^2 + \tau\alpha^A + \beta^A \quad (ICU)
\]

\[\text{Figure 1: Second best solution when } \sigma^2 > 2\]

Figures 1 (drawn for \(\sigma^2 > 2\)), 2 (drawn for \(1 < \sigma^2 < 2\)), and 3 (drawn for \(\sigma^2 < 1\)) depict the problem graphically in the space of all contracts \((\alpha, \beta)\). The figures show the optimal full-information contracts \((C_A^F, C_U^F)\) for each type in isolation, as derived in (6)-(7) and (14)-(15), respectively, in the last section:

\[
C_A^F = \left(\frac{2}{2 + \sigma^2}, \frac{2(\sigma^2 - 2)}{(2 + \sigma^2)^2}\right) \quad (22)
\]

\[
C_U^F = \left(\frac{1}{1 + \sigma^2}, \frac{\tau^2}{2} - \frac{\tau}{1 + \sigma^2} + \frac{\sigma^2 - 1}{2(1 + \sigma^2)^2}\right) \quad (23)
\]

Here the index \(F\) stands for full-information contracts (full information about the Agent’s awareness on the side of the Principal). As noted in Observation 3, these contracts are not incentive-compatible.

Let

\[
v^A(\alpha, \beta) = \frac{1}{2}(2 - \sigma^2)\alpha^2 + \beta
\]

\[
v^U(\alpha, \beta) = \frac{1}{2}(1 - \sigma^2)\alpha^2 + \tau\alpha + \beta - \frac{\tau^2}{2}
\]
Figure 2: Second best solution when $1 < \sigma^2 < 2$

Figure 3: Second best solution when $\sigma^2 < 1$
denote the certainty equivalents of the aware and the unaware Agent, respectively, under contracts \((\alpha, \beta)\). Figures 1, 2, and 3 show the participation boundaries \(u'(\alpha, \beta) = 0\) and an indifference curve \(u^A(\alpha, \beta) = \text{const} > 0\). Because the indifference curves are quadratic, they can intersect twice. Because of this failure of the single-crossing property to hold, the local incentive analysis of traditional screening problems (see, e.g., Bolton and Dewatripont, 2005) will not suffice in this problem.

The figures also show that the participation boundary of the unaware Agent lies above that of the aware Agent, with a point of tangency at \(\alpha = \tau\). Starting with this observation, we can simplify the contracting problem by a sequence of arguments that are familiar from standard screening theory.

**Lemma 1** (PCA) is redundant in the solution of problem (21)-(ICU).

**Proof.** Direct calculation shows that the right-hand-side of (ICA) satisfies

\[
\frac{1}{2}(2 - \sigma^2)(\alpha^U)^2 + \beta^U \geq \frac{1}{2}(1 - \sigma^2)(\alpha^U)^2 + \tau \alpha^U + \beta^U - \frac{\tau^2}{2}
\]

with equality at \(\alpha = \tau\). (ICA) and (PCU) therefore imply (PCA).

Figures 1, 2 and 3 illustrate Lemma 1 graphically.

**Lemma 2** (ICA) binds at the optimum.

**Proof.** If not, then the Principal can raise his profit by slightly lowering \(\beta^A\). This violates none of the constraints, since (PCA) is redundant by Lemma 1.

In Figures 1, 2, and 3, Lemma 2 shows the contract for the aware Agent must lie on the Agent’s indifference curve through \(C^U = (\alpha^U, \beta^U)\).

**Lemma 3** (PCU) binds at the optimum.

**Proof.** Suppose not. Then the Principal can raise his expected profit by slightly lowering \(\beta^A\) and \(\beta^U\) by the same amount. This will trivially not violate (ICA) and (ICU), and does not not violate (PCA) since it is redundant.

In Figures 1, 2, and 3, Lemma 3 shows that \(C^U\) must lie on the participation boundary of the unaware Agent.

**Lemma 4** (ICU) is equivalent to \((\alpha^A - \tau)^2 \geq (\alpha^U - \tau)^2\).

**Proof.** (ICU) is equivalent to

\[
\beta^U - \beta^A \geq \frac{1}{2}(1 - \sigma^2)((\alpha^A)^2 - (\alpha^U)^2) + \tau(\alpha^A - \alpha^U).
\]

By Lemma 2, we have

\[
\beta^U - \beta^A = \frac{1}{2}(2 - \sigma^2)((\alpha^A)^2 - (\alpha^U)^2).
\]

Hence, (ICU) is equivalent to

\[
(\alpha^A)^2 - (\alpha^U)^2 \geq 2\tau(\alpha^A - \alpha^U)
\]
which is equivalent to
\[(\alpha^A - \tau)^2 \geq (\alpha^U - \tau)^2.\]

Graphically in Figures 1, 2, and 3, Lemma 4 says that \(\alpha^U\) must be closer to \(\tau\) than \(\alpha^A\) on the horizontal axis.

In Figures 1, 2, and 3, Lemmas 1-3 imply that the contract \(C^U\) lies on the participation boundary for the unaware Agent, and the contract for the aware Agent must lie on the bold segments of the aware Agent’s indifference curve through \(C^U\).

Given the lemmas, problem (21)-(ICU) is reduced to the following problem (24)-(ICU).

\[
\begin{align*}
\max_{\alpha^A, \beta^A, \alpha^U, \beta^U} & \quad \lambda [2(\alpha^A - (\alpha^A)^2) - \beta^A] + (1 - \lambda)[(1 - \tau)\alpha^U - (\alpha^U)^2 + \tau - \beta^U] \\
\text{s.t.} & \quad \frac{1}{2}(1 - \sigma^2)(\alpha^U)^2 + \tau\alpha^U + \beta^U - \frac{\tau^2}{2} = 0 \\
& \quad \frac{1}{2}(2 - \sigma^2)(\alpha^A)^2 + \beta^A = \frac{1}{2}(2 - \sigma^2)(\alpha^U)^2 + \beta^U \\
& \quad (\alpha^A - \tau)^2 \geq (\alpha^U - \tau)^2
\end{align*}
\]

Substituting out for \(\beta^U\) and \(\beta^A\) using (PCU) and (ICA) yields a quadratic maximization problem in \(\alpha^U\) and \(\alpha^A\) subject to the inequality constraint (ICU). In standard screening problems that satisfy the single-crossing property, the incentive constraint (ICU) would not bind, and the (second-best) optimal menu of contracts would generically be separating. Here, things are different in two ways. First, the incentive constraint (ICU) may bind. And second, if it binds this does not necessarily entail pooling \((\alpha^A - \tau = \alpha^U - \tau)\), but can also lead to a constrained separating outcome \((\alpha^A - \tau = \tau - \alpha^U)\). The following proposition shows that all these cases may in fact occur at the optimum.

An important reference point in this proposition is
\[t_{2F}^A = \frac{2}{2 + \sigma^2},\]
the Agent’s optimal choice of \(t_2\) under the optimal contract when the Agent is aware of both dimensions of the contracting problem. \(t_{2F}^A\) has been derived in (9) in the last section.\(^5\)

**Proposition 2** Let
\[
\begin{align*}
\tau &= \tau(\lambda) = \frac{3(1 - \lambda)(\sigma^2 + 1) + 1 + \lambda}{(\sigma^2 + 2)(\lambda + 2(1 - \lambda)(\sigma^2 + 1))} \\
\bar{\tau} &= \bar{\tau}(\lambda) = \frac{(1 - \lambda)(\sigma^2 + 1) + 3\lambda - 1}{\lambda(\sigma^2 + 2)}
\end{align*}
\]
We have \(\frac{d\tau}{d\lambda} > 0, \frac{d\bar{\tau}}{d\lambda} < 0\) for all \(\lambda \in [0, 1]\), \(\tau(0) < \tau_{\min} < \tau_{\max} < \bar{\tau}(0)\) and \(\tau(1) = \bar{\tau}(1) = t_{2F}^A\). The solution of the problem (24)-(ICU) is unique and given as follows:

\(^5\)Note that \(2/(2 + \sigma^2) \in (\tau_{\min}, \tau_{\max})\) defined in (20).
1. If $\tau < \underline{\tau}$ or $\tau > \overline{\tau}$ the incentive constraint (ICU) is slack and the solution is separating.

2. If $\underline{\tau} \leq \tau \leq t_{2F}^{A}$ the incentive constraint (ICU) is binding, and the solution is constrained separating with $\alpha^A - \tau = \tau - \alpha^U$.

3. If $t_{2F}^{A} \leq \tau \leq \overline{\tau}$ the solution is pooling.

The proof is given in the appendix, where we also provide the explicit solutions to the optimal contracts $(\alpha^A, \beta^A, \alpha^U, \beta^U)$.

---

**Figure 4: Graphical illustration of the three regimes of Proposition 2**

Figure 4 provides a graphical illustration of the different regions identified by Proposition 2. Note that $\tau(0) < \tau_{\min} < \tau(1)$ and $\tau(0) > \tau_{\max} > \tau(1)$. The figure focuses on the two key parameters that describe the unawareness problem: $\tau$, the effect of unawareness, and $1 - \lambda$, the extent of unawareness. In order to interpret the proposition, we distinguish whether $\tau$ is greater or smaller than $t_{2F}^{A}$.

If $\tau > t_{2F}^{A}$ the Agent unconsciously works harder than the optimal full-awareness level. In terms of the graphical description of the contracting problem in $(\alpha, \beta)$—space in Figures 1, 2, and 3, this is the case where the full information contract $C^A_F$ lies to the left of the point of tangency of the two participation boundaries (which is at $\alpha = \tau$).

Suppose first that $\lambda$ is sufficiently large, meaning $\lambda > \tau^{-1}(\tau)$ in terms of Proposition 2 and Figure 4. In this case, the intuition can be understood from Figure 5, which develops Figure 2. The points $C^A_F$ and $C^U_F$ are the full-information contracts for the aware and unaware Agents, respectively. The problem for the Principal is to choose a point $C^U$ on the unaware Agent’s participation boundary ($PCU$) and a point $C^A$ on the aware Agent’s indifference curve through $C^U$ ($ICA$) such that $\alpha^A$ is farther from $\tau$ than $\alpha^U$ ($ICU$). Since $\lambda$ is large, it is optimal for the Principal to have $C^A$ close to the full-information solution for the aware Agent. Therefore, it is no big distortion to have $C^U$ to the right of $C^A$. This means that (ICU) is slack.

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6 Without loss of generality, our figures focus on the case $1 < \sigma^2 < 2$. 

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and there is separation of aware and unaware Agents. The aware Agent’s optimal pay-performance sensitivity $\alpha^A$ is equal to the full-information value $\alpha^A_F = \frac{2}{2 + \sigma^2}$ identified in the previous section and in (22). Yet, the aware Agent receives a larger share of the surplus because her participation constraint does not bind. In Figure 5, the loss of the Principal compared to the full-information case is represented by the distance $L_A$.

On the other hand, the unaware Agent’s pay-performance sensitivity $\alpha^U$ is larger than her full-information pay-performance sensitivity. To prevent the aware Agent from mimicking the unaware Agent, there is an efficiency loss. This is represented by the distance $L_U$ in Figure 5.

In essence, there is a trade-off of $L_A$ versus $L_U$ for the Principal, that is, a high base salary for the aware Agent versus an inefficient pay-performance sensitivity for the unaware Agent.

When, on the other hand, $\lambda$ is small, the intuition is in Figure 6. Now there are few aware Agents in the population, so the loss $L_A$ can be larger while $L_U$ should be smaller. This means that $C^U$ is closer to $C^U_F$ and $C^A$ lies on a higher indifference curve of the aware Agent. But the unaware Agent’s incentive constraint ($ICU$) imposes a limit on how much $C^U$ can be moved towards $C^U_F$; if $\alpha^U$ and $\alpha^A$ are close to each other, $C^U$ must lie (weakly) to the right of $C^A$. The optimum therefore pools the aware and unaware Agents and furthermore distorts the aware Agent’s pay-performance sensitivity inefficiently below $\alpha^A_F$. Compared to the trade-off of $L_A$ versus $L_U$ in Figure 5, now, when $\lambda$ is small, there is a new efficiency loss: insufficient risk bearing by the aware Agent because of a reduced pay-performance sensitivity, which is represented by the distance $L'_A$ in Figure 6.

The second parameter constellation to consider is the case $\tau < t^A_{2F}$, which means...
that the Agent unconsciously works less hard than the optimal full-awareness level. In contrast to the graphical description of the contracting problem in \((\alpha, \beta)\)–space in Figures 1, 2, and 3, this is the case where the full information contract \(C_{\Lambda}^{A}\) lies to the right of the point of tangency of the two participation boundaries (which is at \(\alpha = \tau\)).

When \(\lambda\) is large, the intuition is as in the case \(\tau > t_2^F\) described above and given in Figure 7. Since there are many aware Agents, the contract \(C_{\Lambda}^{A}\) should be close to the full-information contract \(C_{\Lambda}^{F}\), which does not conflict with the requirement that \(\alpha^{U}\) must be closer to \(\tau\) than \(\alpha^{A}\) (the ICU-constraint). The Principal’s loss \(L_{U}\) on the unaware Agent is relatively large, and the solution is separating as before. The pay-performance sensitivity \(\alpha^{A}\) for the aware Agent is equal to the full-information level and her base salary \(\beta^{A}\) is relatively low, but the pay-performance sensitivity \(\alpha^{U}\) for the unaware Agent is inefficiently large.

When \(\lambda\) is small, the reasoning changes and the intuition is given in Figure 8. Now the Principal wants to keep the loss \(L_{U}\) small and can tolerate a higher loss \(L_{A}\). Moving \(C_{U}\) towards \(C_{F}^{U}\) reduces the unaware Agent’s incentive pay \(\alpha^{U}\) towards the full-information level (without reaching it) and increases the aware Agent’s base salary \(\beta^{A}\). This movement along the unaware Agent’s participation boundary is again restricted by the incentive constraint (ICU), but instead of pooling this implies a “far” separating solution. The reason is that in order to keep the unaware Agent from mimicking the aware one, the aware Agent’s pay-performance sensitivity \(\alpha^{A}\) must be sufficiently large compared to \(\alpha^{U}\) (because \(C_{U}^{F}\) and \(C_{F}^{A}\) lie on different sides of \(\tau\)). Since the loss from inefficiently distorting the pay-performance sensitivity of
Figure 7: Solution of problem (24) when $\lambda$ is large and $\tau < t_{2F}^1$.

Figure 8: Solution of problem (24) when $\lambda$ is small and $\tau < t_{2F}^1$. 
the aware Agent ($L_A'$ in the figure) is relatively small (because of the small $\lambda$), it is optimal to do so, necessitating an increase of $\alpha^A$ for an even stronger separation of the two types. Now the unaware Agent’s pay-performance sensitivity $\alpha^U$ is close to the full-information level, while the aware Agent’s pay-performance sensitivity $\alpha^A$ is distorted upwards compensated by a high base salary.

Interestingly, when $\tau$ is close to the full-information level $t_{2F}^A$, the solution is typically constrained by the unaware incentive constraint (ICU). As Figure 4 shows, only when $\lambda$ is very close to 1, the unconstrained maximum is optimal. The reason is that, in Figures 1 - 8, the point $C^A$ is so close to the point of tangency of the two participation boundaries that when the indifference curve of the aware Agent moves slightly upwards (i.e. her base salary increases), the unaware Agent’s (ICU) becomes binding.

Much of the preceding analysis is as in standard screening models. For example, the “good” type (the aware Agent) gets a positive rent, while the “bad” type is kept to her reservation utility. Furthermore, the aware Agent’s rents are decreasing in its population share $\lambda$. On the other hand, the failure of the single-crossing property to hold changes things in some important respects. In particular, efficiency losses (i.e. distortions in $\alpha$) can arise for both types, and the incentive-compatibility constraints of both types can bind in a separating solution.

Yet, the analysis in this section so far is incomplete. In contrast to the standard screening problem, the Principal has another option: making the unaware Agent aware. Thus $\lambda$ cannot be regarded as an exogenous variable. To wit, in the full-information solution ($C^A, C^U_F$) derived in Section 3 it is not wise for the Principal to make the unaware Agent aware when condition (20) holds. But if the Principal does not know whether the Agent is aware of the full effort problem, it might be better for the Principal to announce the full effort problem, in order to avoid the screening costs associated with the allocation in Proposition 2.

As noted before, the aware Agent’s rents are decreasing in $\lambda$. Yet, these rents are paid with probability $\lambda$, hence, it is not clear what the impact of $\lambda$ on total rents is, and whether this affects the Principal’s announcement decision. The following proposition provides a surprisingly clear-cut answer.

**Proposition 3** There are bounds $\tau_L = \tau_L(\lambda) < \tau_R = \tau_R(\lambda)$, with $\frac{d\tau_L}{d\lambda} > 0$, $\frac{d\tau_R}{d\lambda} < 0$, $\tau_L(1) = \tau_R(1) = t_{2F}^A$, $\tau_L(0) = \tau_{\min}$ and $\tau_R(0) = \tau_{\max}$, such that the Principal optimally proposes the contracts identified in Proposition 2 if $\tau \in (\tau_L, \tau_R)$, and makes the Agent aware otherwise.

The proof of the proposition is in the appendix, where we also derive the bounds $\tau_L, \tau_R$ explicitly. Figure 9 illustrates Proposition 3 in terms of the $\tau-\lambda$ diagram of Figure 4. In the shadowed area, the solution of problem (24)-(ICU) is dominated by making all Agents aware.

In the extreme case $\lambda = 0$, we are back to the case of the unaware Agent of Section 2. In terms of Figure 6 and 8, the distance $L_U$ can be reduced to zero at no cost., and indeed the condition $\tau \in (\tau_L(0), \tau_R(0))$ of the proposition is the same as no-announcement condition (20).

In the other extreme case $\lambda = 1$, the Principal always prefers making the Agent aware, since making them aware is nothing but the optimal solution for the aware
Agent and there are only aware Agents in the population. If $\tau = t^A_{2F}$, both aware Agent and unaware Agent choose the same level of effort in the second dimension. Thus $\tau = t^A_{2F}$ is the only situation where the Principal is indifferent between making them aware and separating solution when $\lambda = 1$.

The intuition why the interval $(\tau_L, \tau_R)$ shrinks as $\lambda$ increases is following. By revealed preferences, the Principal is more willing to contract with the unaware Agent, because otherwise he simply makes the unaware Agent aware (which he is always able to do). The only problem for the Principal is that the aware Agent wants to mimic the unaware Agent. Thus the Principal has to bear a cost of screening for the existence of aware Agents. Note that making all Agents aware always leads to a constant profit for the Principal. Hence the more frequent the aware Agent type is, the less the Principal can benefit from contracting with the unaware Agents. Therefore, the requirement for the default action $\tau$ is stricter to guarantee that incomplete contracting is still better than making all Agents aware. Thus the larger the population of aware Agents, the more likely it is that the Principal prefers making all Agents aware.

5 Justifiability of Contracts

Filiz (2006) and Ozbay (2006) argue that in the context of unawareness a reasonable equilibrium concept should include the requirement that the Agent thinks that the contract is justifiable in the sense that the contract is optimal for the Principal also from the Agent’s point of view. In this section, we explore the ramifications of this added requirement for our analysis.

First, it is simple to see the solution of the basic contracting problem in section 3 is justifiable in this sense. If the optimal contract is complete the Agent is aware, and the problem reduces to the standard Principal-Agent problem, the solution of
which is even robust to common knowledge of rationality and the contractual setting. If the optimal contract is incomplete, the Agent remains unaware and unconsciously believes that $t_2 = \tau$. Then the Agent’s objective function (1) includes a fixed-cost element, and again the contract is optimal in the Agent’s mind.

However, the solution of the heterogenous awareness problem in section 4 is not necessarily justifiable. When the Principal prefers incomplete contracts (the case $\tau \in (\tau_L, \tau_R)$ in Proposition 3), the unaware Agent does not understand why there are two different contracts (separating or constrained separating solution) or a single contract designed the way it is (pooling solution). If we require the contract to be justifiable, the solution is either the complete contract outcome (full awareness outcome) or the incomplete contract $C_U$ that makes sense for the unaware Agent.\footnote{Here, we assume that the aware Agent understands the contracting problem as well as the Principal. If the aware Agent does not know that other Agents may be unaware, this contract is not justifiable either.}

The additional justifiability constraint therefore reduces the Principal’s pro-f t from proposing incomplete contracts and makes him more likely to make all Agents aware. This is because we add an additional justifiability constraint in the problem (24)-(ICU). Furthermore, with $C_U$, the aware Agent obtains a maximal rent as noted in Observation 3.

Yet, it can be argued that justifiability is too strong a requirement. If one acknowledges that the model necessarily only describes a simplified snapshot of a full (highly multi-dimensional) contracting problem, it may well be reasonable to assume that the Agent does not want or need to understand the reason for what she sees, as long as what she chooses is optimal for her.

In what follows we therefore propose a weaker justifiability restriction than that of Filiz (2006) and Ozbay (2006). In the spirit of Bolton and Faure-Grimaud (2007) and Tirole (2007), we assume that if the observed contracts are not justifiable, the unaware Agent becomes aware that something might be wrong with her view and starts thinking about it. This cognitive effort leads to her full awareness with probability $\delta$. With probability $1 - \delta$ the Agent remains unaware and chooses (one of) the proposed contracts without further ado. This extension can be easily integrated into the analysis of the preceding section. In fact, after seeing a non-justifiable contract, i.e., a menu of incomplete contracts that is different from the single contract $C_U$, the fraction of aware Agents increases to $\lambda' \equiv \lambda + (1 - \lambda)\delta$. Hence, non-justifiable contracts promote awareness. Thus making all Agents aware is more likely to be better than non-justifiable contracts according to Figure 9, as both $\tau_L$ and $\tau_R$ are monotone.

Now there are three alternatives for the Principal: (i) making all Agents aware, (ii) proposing $C_U$ alone, (iii) proposing the menu of contracts identified in Proposition 2, with the fraction of aware Agents replaced by $\lambda'$. From Figure 9, we see that now the incomplete contracting solution (iii) now becomes less attractive than the complete contracting solution (i). Yet, alternatives (ii) and (iii) may be also optimal in some circumstances. For example, when $\lambda$ is small, $\tau$ is far from $t_2^A$ and $\delta$ is large, proposing $C_U$ alone is optimal. In Figure 9, we can see that in this case non-justifiable contracts (alternative (iii)) tend to be worse than the complete contracts (i) since $\lambda'$ is large. Furthermore, proposing $C_U$ alone (alternative (ii)) is better than the first alternative: since $\lambda$ is small, the loss from
the rent of the aware Agent is small for the Principal. On the other hand, when $\delta$ is small enough, the third alternative can be optimal, as $\lambda'$ is not significantly greater than $\lambda$.

6 Concluding Remarks

This paper provides a model of incentive design for unaware Agents. It is worth pointing out the following issues.

**Competition.** The result is not significantly changed if we introduce Bertrand-competition among homogeneous Principals. In this case, the Principal proposes the same contracts as before, except that the base salaries $\beta^A$ and $\beta^U$ are both increased by an amount such that the Principal earns zero expected profits. Since this leads to the maximal profit for each Principal, no one has incentive to deviate from it. In particular, if they all propose incomplete contracts, no one has an incentive to make all Agents aware, since this only decreases their profits. Thus competition cannot promote awareness of the Agents.

**Communication-Proofness.** We have shown that under certain conditions, the Principal prefers to leave the Agent unaware of the full contracting problem. Interestingly, even the aware Agent has no incentive to make her unaware colleague aware through communication. First, the aware Agent has no incentive to do so before contracting, because, in the solution, the unaware Agent exerts a positive externality on the aware Agent by conferring a positive rent on her. Second, also after contracting, the aware Agent has no incentive to make the unaware Agent aware, because making the unaware Agent aware cannot create any extra rent for herself but only hurts the Principal. Hence, the analysis in section 4 is robust to the possibility of internal communication among Agents.

**Dynamic stability.** Our analysis has been static, but it lends itself to an interesting dynamic interpretation. In Proposition 3 we have shown that the optimal contracts is the more likely to leave unaware Agents unaware, the smaller $\lambda$ (where the measure of Agents is taken with respect to $\tau$). Hence, the more Agents are likely to be unaware, the more will remain unaware after contracting, and vice versa. This suggests a certain stability of unawareness. This observation may be particular important as it suggests that deviations from full rationality are not necessarily doomed to die out in the long run.

**Welfare Implication of Public Announcements.** Is there room for a benevolent policy maker to improve the outcome through promoting the Agent’s awareness? In section 3 we have argued that when there are only unaware Agents in the population, a public policy of making Agents aware is not welfare enhancing. The reason is that such an announcement would force the Principal to provide explicit incentives to the Agent, which is costly. Since the Principal maximizes total surplus, the fact that he does not choose to make the Agent aware shows that the costs outweigh the benefits. In section 4, when there are heterogeneous Agents, this conclusion still holds. If the Principal prefers to leave the Agent unaware, the outcome Pareto-dominates the outcome of making all Agents aware. To see this point, note that when the Principal prefers to leave the Agent unaware, he must earn a higher expected profit than making her aware. Furthermore, the aware Agent earns a positive rent
while making all Agents aware leads to a zero rent for her. Finally, the unaware Agent earns a zero rent in any case. Thus there is no need for the policy maker to intervene in the heterogeneous environment as well. Of course, this no-intervention recommendation crucially depends on the assumptions that the Principal knows all the choice possibilities of the Agent and what the Agent is unaware of is only her own actions.

Awareness of Unawareness. In many contracting problems, it is interesting to include the possibility that an Agent is aware that she might be unaware of something but does not know what it is. This is particularly important when the Agent is unaware of some exogenous contingencies (move of the nature). Then she has to choose some vague term in the contract to avoid being trapped ex post. Furthermore, if the Agent is unaware of some action of the Principal, her awareness of unawareness is also important, because, in both cases, the Agent is aware that she might be surprised ex post by some state of nature or some action of the Principal. However, when the Agent is unaware of her own actions, it is not necessary to model her awareness of unawareness, because she will be never surprised by her own actions. We can therefore safely ignore the possibility that the Agent is aware that she might be unaware of some action of herself.

A Appendix

A.1 Proof of Proposition 2

Proposition 2 (with explicit expressions for the optimal contracts)

Let

\[ \tau = \tau(\lambda) = \frac{3(1 - \lambda)(\sigma^2 + 1) + 1 + \lambda}{(\sigma^2 + 2)(\lambda + 2(1 - \lambda)(\sigma^2 + 1))} \]

\[ \tau = \tau(\lambda) = \frac{(1 - \lambda)(\sigma^2 + 1) + 3\lambda - 1}{\lambda(\sigma^2 + 2)} \]

We have \( \frac{d}{d\lambda}\tau > 0, \frac{d}{d\lambda}\tau < 0 \) for all \( \lambda \in [0, 1] \), \( \tau(0) < \tau_{\text{min}} < \tau_{\text{max}} < \tau(0) \) and \( \tau(1) = \tau(1) = t_{2F}^A \). The solution of the problem (24)-(ICU) is unique and given as follows:

1. If \( \tau < \tau \) or \( \tau > \tau \) the incentive constraint (ICU) is slack and the solution is separating, with

\[ \alpha^A = \frac{2}{2 + \sigma^2}, \quad \beta^A = \frac{4(\sigma^2 - 2)}{2(2 + \sigma^2)^2} \frac{1}{2} (1 - \lambda)^2 (1 - (\sigma^2 + 1)\tau - 1)^2 (1 + \sigma^2(1 - \lambda))^2, \]

\[ \alpha^U = \frac{1 + \lambda(\tau - 1)}{1 + \sigma^2(1 - \lambda)}, \quad \beta^U = \frac{2}{\sigma^2 - \tau} \frac{1 + \lambda(\tau - 1)}{1 + \sigma^2(1 - \lambda)} - \frac{(1 - \sigma^2)(1 + \lambda(\tau - 1))^2}{2(1 + \sigma^2(1 - \lambda))^2}, \]

2. If \( \tau \leq \tau \leq \frac{2}{2 + \sigma^2} \), the incentive constraint (ICU) is binding, with \( \alpha^A - \tau = \tau - \alpha^U \)
and
\[
\alpha^A = \frac{1}{1+2\lambda+\sigma^2} (2\tau(1+\sigma^2)(1-\lambda) - 1 + 3\lambda + \tau \lambda),
\]
\[
\beta^A = \frac{1}{2} \left[ \frac{1}{1+2\lambda+\sigma^2} (1-3\lambda + (5+2\sigma^2)\tau \lambda) \right]^2 - \frac{\tau}{1+2\lambda+\sigma^2} (1-3\lambda + (5+2\sigma^2)\tau \lambda)
+ \frac{\tau^2}{2} \frac{1}{2} (2 - \sigma^2) \left[ \frac{1}{1+2\lambda+\sigma^2} (2\tau(1+\sigma^2)(1-\lambda) - 1 + 3\lambda + \tau \lambda) \right]^2,
\]
\[
\alpha^U = 2\tau - \alpha^A = \frac{1}{1+2\lambda+\sigma^2} (1-3\lambda + (5+2\sigma^2)\tau \lambda),
\]
\[
\beta^U = \frac{1}{2} \tau^2 - \frac{1}{2} (1 - \sigma^2) \left[ \frac{1}{1+2\lambda+\sigma^2} (1-3\lambda + (5+2\sigma^2)\tau \lambda) \right]^2
- \frac{\tau}{1+2\lambda+\sigma^2} (1-3\lambda + (5+2\sigma^2)\tau \lambda).
\]

3. If \( \frac{2}{2+\sigma^2} \leq \tau \leq \bar{\tau} \) the solution is pooling, with
\[
\alpha^A = \alpha^U = \frac{1+\lambda+\tau \lambda}{1+2\lambda+\sigma^2},
\]
\[
\beta^A = \beta^U = \frac{1}{2} \tau^2 - \frac{1}{2} (1 + \lambda + \tau \lambda) \left[ \frac{1}{1+2\lambda+\sigma^2} (1-3\lambda + (5+2\sigma^2)\tau \lambda) \right]^2
- \frac{1}{2} \frac{(1 + \lambda + \tau \lambda)^2}{(1+2\lambda+\sigma^2)} (1-\sigma^2).
\]

**Proof.** Using Lemmas 2 and 3, one can eliminate the fixed payment \( \beta \) from the problem and express the contracting problem solely in terms of the incentive component \( \alpha \):
\[
\max_{\alpha^A, \alpha^U} \lambda \left[ 4\alpha^A - (\sigma^2 + 2)(\alpha^A)^2 + 2\tau \alpha^U - (\alpha^U)^2 \right] + (1 - \lambda) \left[ 2\alpha^U - (\sigma^2 + 1)(\alpha^U)^2 \right]
\]
\[
\text{s.t. } (\alpha^A - \tau)^2 \geq (\alpha^U - \tau)^2
\]
By straightforward differentiation, the unconstrained solution to the maximization problem (25) is
\[
\alpha^A = \frac{2}{2+\sigma^2}, \quad \alpha^U = \frac{1+\lambda(\tau-1)}{1+\sigma^2(1-\lambda)}
\]
(27)
This solution satisfies the constraint (26) strictly if and only if
\[
(\tau(\sigma^2 + 2) - 2)(\lambda + (1 - \lambda)(\sigma^2 + 1))^2 > (1 - \lambda)^2(\tau(\sigma^2 + 1) - 1)^2(\sigma^2 + 2)^2
\]
Viewed as a quadratic inequality in \( \tau \), this is equivalent to \( \tau < \underline{\tau} \) or \( \tau > \bar{\tau} \). Hence, (27) yields the separating solution of the proposition.
If \( (ICU) \) is binding, there are two possibilities: \( \alpha^U = \alpha^A \) (pooling) or \( \alpha^U + \alpha^A = 2\tau \) (constrained separating). Direct comparison shows that when \( \frac{2}{2+\sigma^2} \leq \tau \leq \bar{\tau} \lambda \) we have the pooling solution, and when \( \underline{\tau}(\lambda) \leq \tau \leq \frac{2}{2+\sigma^2} \) we have the constrained separating solution.

The monotonicity of \( \underline{\tau}(\lambda) \) and of \( \bar{\tau}(\lambda) \) follows by differentiation, and the statements about \( \underline{\tau}(0), \bar{\tau}(0), \underline{\tau}(1) \), and \( \bar{\tau}(1) \) by direct computation. □
A.2 Proof of Proposition 3

Proof. Firstly, we analyze the case 1 in Proposition 2. In this case, we plug the solution under separating solution into the objective function. The profit for the Principal is

\[
\pi^S \equiv \lambda \left( \frac{4}{\sigma^2+2} - \frac{8}{(\sigma^2+2)^2} - 2 \frac{\sigma^2-2}{(\sigma^2+2)^2} - \frac{1}{2} (\lambda - 1)^2 \right) \left( \frac{\sigma^2-2}{(\sigma^2+2)^2} \right)^2 \\
+ (1 - \lambda) \left( \tau - \frac{1}{2} \lambda \right) \left( \frac{\lambda - 1}{\sigma^2+2} \right)^2 + \left( \frac{\lambda - 1}{(\sigma^2+2)^2} \right) + \left( 1 - \tau \right) \frac{\lambda - 1}{\sigma^2+2} + \frac{1}{2} (\lambda - \tau - 1)^2 \left( \frac{\sigma^2-2}{(\sigma^2+2)^2} \right).
\]

On the other hand, if the Principal makes all Agents aware, he earns a profit \( \pi^A \equiv \frac{2}{2+\sigma^2} \).

We get \( \pi^A - \pi^S = (\lambda-1)(2\lambda-2\tau-2\tau+3\sigma^2-3\sigma^2\lambda-3\sigma^2\tau-4\sigma^2\tau+3\sigma^2\lambda+2\sigma^4\lambda+2) \).

We further gain that \( \pi^A > \pi^S \) if and only if \( \tau > R_1 \equiv \frac{(2+\sigma^2)(\sigma^2\lambda-\sigma^2-1) + \sigma^2 \sqrt{(2+\sigma^2)(\lambda-1)(\sigma^2+2)(\lambda^2-\sigma^2-1)}}{(2+\sigma^2)(\lambda+\sigma^2+1)} \).

Now we want to check if \( R_1 \) lies to the left or to the right of the boundary of this case. Thus we check the sign of the following term

\[
R_1 - \tau = \frac{(\lambda-1)(2\lambda-2\tau-2\tau+3\sigma^2-3\sigma^2\lambda-3\sigma^2\tau-4\sigma^2\tau+3\sigma^2\lambda+2\sigma^4\lambda+2)}{(2+\sigma^2)(\lambda+\sigma^2+1)}.
\]

Since \(-\sigma^2(\lambda-1)(\sigma^2-1)\) is negative, \( R_1 > \tau \) if and only if
\[
\sigma^2(\lambda-1)^2(\sigma^2\lambda-\sigma^2-1) - \sigma^2\lambda(\lambda-1)(\sigma^2+2)(\sigma^2\lambda-\sigma^2-1) = \sigma^4(2\lambda-1)(\lambda-1)(\sigma^2\lambda-\sigma^2-1)(\lambda+\sigma^2+1) > 0.
\]

Thus if \( \tau < R_1 \), the Principal still uses the solution of separating solution. Otherwise the Principal makes all Agents aware.

Now we want to check if \( L_1 \) lies to the left or to the right of the boundary of this case. Thus we check the sign of the following term

\[
L_1 - \tau = \frac{(\lambda-1)(2\lambda-2\tau-2\tau+3\sigma^2-3\sigma^2\lambda-3\sigma^2\tau-4\sigma^2\tau+3\sigma^2\lambda+2\sigma^4\lambda+2)}{(2+\sigma^2)(\lambda+\sigma^2+1)}.
\]

Since \( (-3+2\sigma^2)(\lambda-1)(\sigma^2\lambda-\sigma^2-1) + \sigma^2(2\lambda-1)(1+\sigma^2) - \lambda^2(\lambda-1)^2(2+\sigma^2)(\lambda^2-\sigma^2-1) = \lambda + \sigma^2 + 1 > 0 \) and \( \lambda > \frac{1}{2} \), the Principal still uses the solution of separating solution. Otherwise the Principal makes all Agents aware.

Secondly, we analyze case 3 in Proposition 2. In this case, we plug the solution under pooling solution into the objective function. The profit for the Principal is \( \pi^P \equiv \frac{2\lambda+2\lambda^2+3\lambda^2+\sigma^2\lambda\lambda^2-2\sigma^2\lambda^2\lambda+1}{4\lambda+2\sigma^2+2} \).

If the Principal makes all Agents aware, he earns a profit \( \pi^A \equiv \frac{2}{2+\sigma^2} \).

We get \( \pi^A - \pi^P = \frac{2}{2+\sigma^2} - \frac{2\lambda+2\lambda^2+3\lambda^2+\sigma^2\lambda\lambda^2-2\sigma^2\lambda^2\lambda+1}{4\lambda+2\sigma^2+2} \).

We further gain that \( \pi^A > \pi^P \) if and only if \( \tau > R_2 \equiv \frac{(2+\sigma^2)(\lambda^2-\sigma^2-2\lambda\sigma^2\lambda-2\lambda^2\lambda+1)}{(2+\sigma^2)(\lambda+\sigma^2+1)} \).

or \( \tau < L_2 \equiv \frac{(2+\sigma^2)(\lambda^2-\sigma^2-2\lambda^2\lambda^2\lambda-2\lambda\sigma^2\lambda+1)}{(2+\sigma^2)(\lambda+\sigma^2+1)} \).

Because \( t_{21}^f - L_2 = \frac{(2+\sigma^2)(\lambda^2-\sigma^2-2\lambda\sigma^2\lambda-2\lambda^2\lambda+1)}{(2+\sigma^2)(\lambda+\sigma^2+1)} \) and \( 2 + 4\lambda + 3\sigma^2 + \sigma^4 + 2\sigma^2\lambda - (\sigma^2 + \lambda + 1)^2 = -(\lambda^2 - \sigma^2 - 2\lambda - 1)^2 > 0 \), we gain that \( \tau < L_2 \) is impossible in this case.
The same as in case 1, now we want to check if \( R_2 \) lies to the left or to the right of the boundary of this case. Thus we check the sign of the following term:

\[
R_2 - \tau = \frac{\sigma^2(1-\lambda)(2\lambda^2 - \sigma^2 - \lambda + \sigma^2\lambda - 1) - 10\lambda^2 - 7\sigma^2 + 2\sigma^2\lambda + 2\sigma^2 + \lambda + \sigma^2\lambda + 2) \sqrt{(4\lambda^4 L^2 + 3\sigma^2 + 4\sigma^2\lambda + 2)}< 0 \text{ if and only if } (2\lambda^2 - \sigma^2 - \lambda + \sigma^2\lambda - 1) - 10\lambda^2 - 7\sigma^2 + 2\sigma^2\lambda + 2\sigma^2 + \lambda + \sigma^2\lambda + 2) \sqrt{(4\lambda^4 L^2 + 3\sigma^2 + 4\sigma^2\lambda + 2)} > 0.
\]

Thus if \( \tau > R_2 \) and \( \lambda < \frac{1}{2} \), the Principal makes all Agents aware. Otherwise the Principal still uses the solution of pooling solution.

Thirdly, we analyze case 2 in Proposition 2. In this case, we plug the solution under constrained separating solution into the objective function. The profit for the Principal is

\[
\pi^C = \frac{1}{(2+\sigma^2+\lambda^2+1)(2\tau - 6\lambda + 26\lambda^2 + 9\lambda^2 - \tau^2 + 2\sigma^2\tau - 10\lambda^2 - 9\lambda^2 - \tau^2)}
\]

Thus if \( \tau > R_2 \) and \( \lambda < \frac{1}{2} \), the Principal makes all Agents aware. Otherwise the Principal still uses the solution of pooling solution.

Now we want to check if \( L_4 \) lies to the left or to the right of the boundary of this case. Thus we check the sign of the following term:

\[
L_4 - \tau = \frac{(3 + 2\sigma^2)(1 - \lambda)(2\sigma^2 + 1) - (\lambda - 2\sigma^2 - 2\sigma^2\lambda - 2)^2(2 + 4\lambda + 3\sigma^2 + \sigma^4 + 2\sigma^2\lambda)}{(2\lambda^2 - \sigma^2 - 10\lambda - 12\sigma^2\lambda - 4\sigma^4\lambda + 12\sigma^2\lambda^2 + 4\sigma^4\lambda^2 - 1)} > 0.
\]

Thus if \( \tau > L_1 \) and \( \lambda < \frac{1}{2} \), the Principal makes all Agents aware. Otherwise the Principal still uses the solution of constrained separating solution.

In summary, we have \( \tau_1(\lambda) = L_1 \) and \( \tau_R(\lambda) = R_1 \) when \( \lambda \geq \frac{1}{2} \) and \( \tau_L(\lambda) = L_4 \) and \( \tau_R(\lambda) = R_2 \) when \( \lambda \leq \frac{1}{2} \) such that the Principal optimally proposes the contracts identified in Proposition 2 if \( \tau \in (\tau_L, \tau_R) \), and makes all Agents aware otherwise.

It is left to show that \( \frac{d\tau}{d\lambda} > 0 \) and \( \frac{d\tau}{d\lambda} < 0 \) with \( \tau_L(\lambda) = \tau_R(\lambda) = \tau_{\text{max}} \) and \( \tau(0) = \tau_{\min} \).

First, we show \( R_1 \) is decreasing in \( \lambda \), because

\[
\frac{d}{d\lambda} (R_1) = \frac{-2\lambda^2 - 2\sigma^2 - 2\sigma^2\lambda - 2\sigma^2\lambda - 2\sigma^2 + \lambda + \sigma^2\lambda + 2)(1+\sigma^2)}{2\lambda^2 - 2\sigma^2 - 2\sigma^2\lambda - 2\sigma^2\lambda - 2\sigma^2 + \lambda + \sigma^2\lambda + 2) < 0.
\]

Second, we show \( L_1 \) is increasing in \( \lambda \).
\[
\frac{d}{d\lambda} (L_1) = -\frac{\sigma^2(\lambda - 5\sigma^2 - 2\sigma^4 + 4\sigma^2\lambda + 2\sigma^4\lambda - 3 + 2(1 + \sigma^2)\sqrt{3\sigma^2 - 2\lambda + \sigma^4 - 5\sigma^4\lambda - 2\sigma^4\lambda^2 + \sigma^4\lambda^2 + 2})}{2(\lambda + \sigma^2 + 1)^2 \sqrt{3\sigma^2 - 2\lambda + \sigma^4 - 5\sigma^4\lambda - 2\sigma^4\lambda^2 + \sigma^4\lambda^2 + 2}}.
\]

Since \(\lambda < 3 + 4\sigma^2\lambda - 5\sigma^2 + 2\sigma^4\lambda - 2\sigma^4 < 0\), \(\frac{d}{d\lambda} (L_1) > 0\) if and only if \((\lambda - 3 + 4\sigma^2\lambda - 5\sigma^2 + 2\sigma^4\lambda - 2\sigma^4)^2 - 4(1 + \sigma^2)^2(3\sigma^2 - 2\lambda + \sigma^4 - 5\sigma^4\lambda - 2\sigma^4\lambda^2 + \sigma^4\lambda^2 + 2) = (\lambda + \sigma^2 + 1)^2 > 0\) which is always true.

Third, we show \(R_2\) is decreasing in \(\lambda\), because
\[
\frac{d}{d\lambda} (R_2) = -\frac{\sigma^2(3\lambda + 3\sigma^2 + \sigma^4 + \lambda^3 + \sigma^2\lambda + \sigma^4\lambda^2 + 2 + (1 + \sigma^2 + \lambda^2)\sqrt{4\lambda + 3\sigma^2 + \sigma^4 + 2\sigma^4\lambda + 2})}{(\lambda^2 - \sigma^2 - 2\lambda - 1)^2 \sqrt{4\lambda + 3\sigma^2 + \sigma^4 + 2\sigma^4\lambda + 2}} < 0.
\]

Fourth, we show \(L_4\) is increasing in \(\lambda\).
\[
\frac{d}{d\lambda} (L_4) = -\frac{\sigma^2((11\sigma^2 - 3\lambda^2 + 4\sigma^2 - 20\sigma^2\lambda - 8\sigma^4\lambda + 8\sigma^2\lambda^2 + 4\sigma^4\lambda^2 + 7)\sqrt{4\lambda + 3\sigma^2 + \sigma^4 + 2\sigma^4\lambda + 2}}{\sqrt{9\lambda^2 - \sigma^2 - 10\lambda - 12\sigma^2\lambda - 4\sigma^2\lambda^2 + 12\sigma^2\lambda^2 + 4\sigma^2\lambda^2 - 1} \sqrt{4\lambda + 3\sigma^2 + \sigma^4 + 2\sigma^4\lambda + 2}} > 0.
\]

Thus \(\frac{d}{d\lambda} (L_4) > 0\).

Therefore, we have \(\frac{d\tau}{d\lambda} > 0\) and \(\frac{d\tau}{d\lambda} < 0\).

Obviously, \(R_1 = \frac{2}{2 + \sigma^2} = t_{2F}^s\) when \(\lambda = 1\) and \(L_1 = \frac{2}{2 + \sigma^2} = t_{2F}^s\) when \(\lambda = 1\). Thus \(\tau_L(1) = \tau_R(1) = t_{2F}^s\).

\(R_2 = \tau_{\text{max}}\) when \(\lambda = 0\) and \(L_4 = \tau_{\text{min}}\) when \(\lambda = 0\). Thus \(\tau_L(0) = \tau_{\text{min}}\) and \(\tau_R(0) = \tau_{\text{max}}\).
References


