Abstract

The theory of Mechanism Design intends to find ways to implement social choice functions. That is, to characterize rules such that, for any profile of actual preferences, game solutions support the outcomes of those functions. Maskin, in his seminal 1977 paper showed that game forms provide a natural framework to analyze this problem.

We focus here on game forms in which the strategies are declarations of preferences over the outcomes. These game forms are called direct mechanisms. On the space of this kind of game forms we postulate an operation, that given a direct mechanism provides other mechanisms (not necessarily a single one), by optimizing the preferences of the agents. Fixed points under this operation are shown to make each preference profile a Nash equilibrium in the corresponding game. Furthermore, we show that, since those profiles are dominant strategy equilibria, our fixed points support strategy-proof functions.

Keywords: Direct Mechanisms, Fixed Points, Implementability, Strategy-Proofness.

1 Introduction

The theory of Mechanism Design\(^1\), which began with Vickrey’s (1961) analysis of second-price auctions, intends to determine games in which the players, by

\(^1\)\footnote{Thanks are due to Professors Leandro Arozamena and Alejandro Neme for good advice and criticisms of earlier versions of this paper. We have also to thank Prof. Salvador Barberà for the original idea of using fixed points in this field, which we knew through Dr. Neme. Of course all errors are our exclusive responsibility.}

\(^1\)\footnote{See Corchón (1996), Maskin (1999), Palfrey (2002).}
playing according to their own preferences, yield in equilibria the alternatives that a social planner wants. The goals of the planner are completely described by a social choice function that provides, for each profile of preferences, a single outcome. The reason why a game must be found is that the planner does not know the actual types of the agents. The game provides a way to extract information about them.

While the class of possible games is huge, the Revelation Principle allows to cut it down substantively. This principle indicates that, whatever the kind of game solution that is intended to support the outcomes, if there exists a mechanism that accomplishes this, there exists a direct mechanism that yields the same result. In this mechanism the strategy sets consist just of the possible types that agents can declare if asked directly about that. Therefore, if no such ‘direct’ mechanism implements a given social choice function, no other mechanism does so. This means that the analysis of implementability can be reduced to the space of direct mechanisms without loss of generality.\(^2\)

Following Maskin (1977), mechanisms are known as game forms, i.e. they are defined by the strategy sets of the agents and an outcome function, that for every profile of strategies yields an element in the space of alternatives. The planner knows only this much. If she knew the full game, including the valuations of the agents over the alternatives, the problem of design would be trivial. In the case of direct mechanisms, if we identify the type of an agent with her preferences, the outcome function should be just the social choice function that the game forms intends to implement.

In this paper we consider a correspondence of the space of direct mechanisms on itself, such that to each direct game form it assigns some others. We call it a choice improving correspondence. Each mechanism in its range captures the outcomes that may arise if agents maximize their preferences over their option sets left by the initial mechanism (Barberà and Peleg, 1990). That is, the ‘options’ that the declared preferences of the agents leave the others to choose from.

We show that this correspondence is increasing on a very natural ordering of sets of preference profiles. This property is used to show that there exist fixed points for this correspondence, which are shown to yield Nash equilibria in the game, i.e. they are such that no individual deviation from a preference profile can ensure a more preferred outcome. A straightforward argument shows that each of these fixed points are monotonic in Maskin’s (1977) sense.

Furthermore, we show that the fixed points of the choice improving correspondence yield dominant strategy implementations of social choice functions. Gibbard-Satterthwaite’s theorem ensues from a “shrinkage-of-option-sets” property that arises naturally from the definition of the choice improving correspondence.

The originality of this research resides not in its results (known for more than three decades) but in its methods of proof. We think that the line of

\(^2\)Despite this, if other considerations, like the complexity of manipulating outcomes are taken into account, non-direct mechanisms may provide better candidates for implementing social choice functions (Mount and Reiter, 2002; Van Zandt, 2007).
argumentation followed here opens new venues for research in the theory of mechanism design.

This paper is organized as follows. Section 2 introduces the main definitions that will be used in the text. Section 3 presents the proofs of our main claims on Nash implementability. Section 4 does the same for dominant strategies. Finally, section 5 concludes and briefly discusses the prospects for future work.

2 The Choice-Improving Correspondence

We will consider here a society of $n$ agents, each one endowed with a preference ordering, $\preceq_i$, over a set of alternatives, $X$, with $|X| > 2$. We denote by $R$ the class of complete and transitive orderings over $X$. We denote by $\tau(\preceq_i) \in X$ the set of top elements of $\preceq_i$, i.e. for all $x \in X$, $x \preceq_i y$ for any $y \in \tau(\preceq_i)$. If the preference of $y$ over $x$ is strict we denote this by $x < y$.

A social choice function $f : R^n \rightarrow X$ assigns to each profile of preferences $\preceq = (\preceq_1, \ldots, \preceq_n)$ a single $x \in X$. The range of $f$ is denoted $r_f$.

To implement $f$ a game form can be postulated, namely a $G = \langle \{S_i\}_{i=1}^n, \hat{f} \rangle$, with $\hat{f} : \prod_{i=1}^n S_i \rightarrow X$. Together with the profile of preferences $\preceq \in R^n$, $\hat{f}$ determines a game $G^\preceq = \langle \{S_i\}_{i=1}^n, \hat{f}, \preceq \rangle$. In this game, given a solution notion, i.e. a class $S \subseteq \prod_{i=1}^n S_i$, we intend that for every $(s_1, \ldots, s_n) \in S$, $\hat{f}(s_1, \ldots, s_n) = f(\preceq_1, \ldots, \preceq_n)$ (Osborne and Rubinstein, 1994).

We restrict ourselves to the case in which each $S_i = R$. If so, a game form $G$ is called a direct mechanism. The class of direct mechanisms for $n$ agents over the set of alternatives $X$ can be written in simplified form $F = \{f|f : R^n \rightarrow X\}$.

Notice that if a direct mechanism given by $\hat{f}$ implements a social choice function $f$, for any given $S \subseteq R^n$, $\hat{f}(\preceq_1, \ldots, \preceq_n) = f(\preceq_1, \ldots, \preceq_n)$ where $(\preceq_1, \ldots, \preceq_n) \in S$ while $(\preceq_1, \ldots, \preceq_n)$ is the true profile of preferences. The implementation is said truthful if $(\preceq_1, \ldots, \preceq_n) \in S$.

Our goal is to characterize direct mechanisms that ensure truthful implementations for $S$ being the family of Nash equilibria. That is, if $\preceq$ is a Nash equilibrium in the game $(\hat{f}, \preceq)$ then $f(\preceq) = \hat{f}(\preceq)$. In order to find the family of game forms that ensure this Nash implementability let us introduce two notions that will be heavily used in our argumentation:

- For $\preceq_i \in R$ and $Y \subseteq X$, the choice set of $\preceq_i$ in $Y$ is $C(\preceq_i, Y) = \{y \in Y : x \preceq_i y$ for all $x \in Y\}$.
- Given $\hat{f} \in F$, and $(\preceq_i, \preceq_{-i}) \in R^n$, for each $i$ the set of options left to $i$ by $\preceq_{-i}$ is $o^\hat{f}_i(\preceq_{-i}) = \{x \in X : \text{there exists } \preceq_i \text{ such that } \hat{f}(\preceq_i, \preceq_{-i}) = x\}$.
- Given $\hat{f} \in F$, the class of profiles for which its outcome is chosen by every agent is $M_f = \{\preceq \in R^n : f(\preceq) = \cap_{i=1}^n C(\preceq_i, o^\hat{f}_i(\preceq_{-i}))\}$.

We will endow the class of direct mechanisms with a binary relation $\sqsubseteq$ defined as follows:
\[ \hat{f} \subseteq \hat{f}' \text{ if and only if } M_{\hat{f}} \subseteq M_{\hat{f}'} \]

where \( \subseteq \) is the set inclusion relation.

**Proposition 1** The relation \( \sqsubseteq \) on \( \mathcal{F} \) is a preorder.

**Proof:** Reflexivity and transitivity follow immediately from the subset relation. Antisymmetry does not follow: just consider \( \hat{f}_1 \) and \( \hat{f}_2 \), two mechanisms that yield, for every \( \preceq \) the most preferred alternative of agents 1 and 2, respectively. That means that \( \hat{f}^k(\preceq) \in \bigcap_{i=1}^n C(\preceq_i, o_i^k(\preceq_{\neq i})) \) for \( k = 1, 2 \) and every \( \preceq \in \mathcal{R}^n \). Then, \( M_{\hat{f}_1} = M_{\hat{f}_2} \) but \( \hat{f}_1 \neq \hat{f}_2 \). \( \square \)

\( \langle \mathcal{F}, \sqsubseteq \rangle \) is a non-trivial preordered set. To see this, just notice that there exist at least two different mechanisms \( \hat{f} \) and \( \hat{f}' \) such that \( \hat{f} \sqsubseteq \hat{f}' \). It suffices to consider \( \hat{f} \) the anti-dictatorial mechanism, that selects the least preferred alternative for a given agent \( i \), and \( \hat{f}' \) the dictatorial mechanism, that yields the most preferred alternative of an agent \( j \). Then, it is easy to check that \( M_{\hat{f}} = \emptyset \) while \( M_{\hat{f}'} = \mathcal{R}^n \).

Now consider the following correspondence, which we call *choice-improving*:

\[ \phi : \mathcal{F} \rightarrow \mathcal{F} \]

that for every \( \hat{f} \in \mathcal{F} \) and every profile \( \preceq \in \mathcal{R}^n \), yields a family of direct mechanisms such that each \( \hat{f}' \in \phi(\hat{f}) \) verifies:

\[ \hat{f}'(\preceq) = \begin{cases} \hat{f}(\preceq), & \text{if } \preceq \in M_{\hat{f}} \\ x \in \bigcap_{i=1}^n o_i^k(\preceq_{\neq i}) \setminus \{\hat{f}(\preceq)\}, & \text{otherwise.} \end{cases} \]

That is, \( \phi \) takes a direct mechanism \( \hat{f} \) and returns all the direct mechanisms such that for any possible profile of preferences, yield either the alternatives that maximize the individual preferences over the options left by \( \hat{f} \) or a feasible outcome other than that of \( \hat{f} \). One observation is immediate:

**Proposition 2** Given \( \hat{f} \in \mathcal{F} \), \( |\phi(\hat{f})| = \prod_{\preceq \notin M_{\hat{f}}} (|\bigcap_{i=1}^n o_i^k(\preceq_{\neq i})| - 1) \).

A simple example may give a better idea of the huge number of alternative direct mechanisms obtained through \( \phi \):

**Example**
Consider \( \hat{f} \), the Borda mechanism over \( X = \{a, b, c, d, e\} \), \( n = 3 \) and \( \mathcal{LR} \), the class of linear orderings of \( X \).

\(^3\text{That is, } \hat{f} \sqsubseteq \hat{f}' \text{ while } \hat{f}' \nsubseteq \hat{f}. \)
Given these two profiles in $\mathcal{LR}^3$, 

\[
\preceq^I = (\preceq^I_1, \preceq^I_2, \preceq^I_3) \quad \preceq^I = (\preceq^I_1, \preceq^I_2, \preceq^I_3)
\]

it follows that, $\hat{f}(\preceq^I) = a$. Since, for $j = 1, \ldots, 3$, $a \in o_j^I(\preceq^I_j)$, then $\hat{f}(\preceq^I) = \bigcap_{i=1}^3 C(\preceq^I_j, o_j^I(\preceq^I_j))$.

On the other hand $\hat{f}(\preceq^{II}) = b$, while $o_j^I(\preceq^{II}_1) = \{a, b, c, d\}$ and $o_j^I(\preceq^{II}_2) = \{a, b, c\}$ and $o_j^I(\preceq^{II}_3) = \{a, b, c\}$. Therefore, $C(\preceq^{II}, o_j^I(\preceq^{II}_j)) = \{a\}$, $C(\preceq^{II}, o_j^I(\preceq^{II}_j)) = \{b\}$, and $C(\preceq^{II}, o_j^I(\preceq^{II}_j)) = \{c\}$. That is, $\hat{f}(\preceq^{II}) \neq \bigcap_{i=1}^3 C(\preceq^I_j, o_j^I(\preceq^I_j))$.

Then, any $\hat{f}^\prime \in \phi(\hat{f})$ will verify that $\hat{f}^\prime(\preceq^I) = \hat{f}(\preceq^I)$. But $\hat{f}^\prime(\preceq^{II})$ can yield any element in $\bigcap_{j=1}^3 o_j^I(\preceq^{II}_j) \setminus \{\hat{f}(\preceq^{II})\} = \{a, c\}$.

The choice-improving correspondence is increasing:

**Proposition 3** Given $\hat{f} \in \mathcal{F}$, any $\hat{f}^\prime \in \phi(\hat{f})$ is such that $\hat{f} \subseteq \hat{f}^\prime$.

**Proof:** Consider any $\preceq \in M_j$. We have to prove that $\preceq \in M_{\hat{f}}$. Since $\preceq \in M_{\hat{f}}$, $\hat{f}(\preceq) \in \bigcap_{i=1}^n C(\preceq_i, o_j^I(\preceq_i))$. By definition of $\phi$ we have that $\hat{f}(\preceq) = \hat{f}(\preceq)$. Now suppose that $\preceq \notin M_{\hat{f}}$. This means that exists a $j$ and $x \in o_j^I(\preceq_j)$ such that $\hat{f}(\preceq) \not\prec_j x$. Then, there exists $\preceq_j \in \mathcal{R}$ such that $x = \hat{f}(\preceq_j, \preceq_j)$ (since $x \in o_j^I(\preceq_j)$). By definition of $\phi$ either $\hat{f}(\preceq_j, \preceq_j) = \hat{f}(\preceq_j, \preceq_j)$ or $\hat{f}(\preceq_j, \preceq_j) \in \bigcap_{i=1}^n o_j^I(\preceq_i) \setminus \{\hat{f}(\preceq_i, \preceq_j)\}$. In either case, $x \in o_j^I(\preceq_j)$. But then, since in particular $\hat{f}(\preceq) = \hat{f}(\preceq) \in C(\preceq_j, o_j^I(\preceq_j))$ we have that $\preceq \preceq_j \not\prec_j (\preceq_j)$. Contradiction. Then $\preceq \in M_{\hat{f}}$. □

It is well known that an increasing function on a partially ordered set in which each chain has a least upper bound has a fixed point (Davey and Priestley, 2002). We will see that that is also true in our case:\footnote{We follow closely the proof presented in the entry “Bourbaki-Witt Theorem” in http://en.wikipedia.org/wiki/}

**Theorem 1** If we assume the Axiom of Choice (AC), there exists at least a $\hat{f} \in \mathcal{F}$ such that $\hat{f} \in \phi(\hat{f})$.

**Proof:** Let us start with a direct mechanism $\hat{f}$ and define, by recursion over the ordinals, a function $\Phi : \text{ORD} \rightarrow \mathcal{F}$ as

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\( \Phi(0) = \hat{f} \),

- \( \Phi(\alpha + 1) \in \phi(\Phi(\alpha)) \), for any successor ordinal \( \alpha + 1 \),
- \( \Phi(\alpha^*) = \sup\{\Phi(\alpha) : \alpha < \alpha^*\} \) for any limit ordinal \( \alpha^* \).

Notice that this definition requires the AC since \( \phi(\cdot) \) is a correspondence, and only one element in \( \phi(\Phi(\alpha)) \) can be selected. On the other hand, for any limit ordinal \( \alpha^* \), \( \{\Phi(\alpha) : \alpha < \alpha^*\} \) is a chain in which, according to Proposition 3, \( \Phi(\alpha) \sqsubseteq \Phi(\alpha + 1) \). Since each chain in \( F \) has an upper bound (a \( \hat{f}^\top \) such that \( M_{\hat{f}^\top} = \mathbb{R}^n \)) then \( F \) is a well-defined set. In that case, and since ORD is not a set, there cannot exist a one-to-one function of ORD on \( F \). Then, there has to exist an \( \alpha \) such that \( \Phi(\alpha) \in \phi(\Phi(\alpha)) \). If \( \Phi(\alpha) = \hat{f}^* \) this means that \( \hat{f}^* \in \phi(\hat{f}^*) \).

\[ \square \]

### 3 Fixed Points of \( \phi \) and Nash Implementation

The following properties characterize a \( \hat{f}^* \in F \) such that \( \hat{f}^* \in \phi(\hat{f}^*) \):

**Proposition 4** If \( \hat{f}^* \in \phi(\hat{f}^*) \), \( M_{\hat{f}^*} = \mathbb{R}^n \).

**Proof:** Suppose not. Then, there exists \( \leq \) such that \( \hat{f}^*(\leq) \notin \bigcap_{i=1}^n C(\leq; o_i^\hat{f}^* (\leq -i)) \). Then, for any \( \hat{f}' \in \phi(\hat{f}^*) \), \( \hat{f}' \in \bigcap_{i=1}^n o_i^{\hat{f}^*} (\leq -i) \setminus \{\hat{f}^*(\leq)\} \).

But this means that for any \( \hat{f}' \in \phi(\hat{f}^*) \), \( \hat{f}^*(\leq) \neq \hat{f}'(\leq) \), and therefore, \( \hat{f}^* \notin \phi(\hat{f}^*) \). Absurd. \( \square \)

**Proposition 5** A direct mechanism \( \hat{f}^* \) is a fixed point of \( \phi \) iff it is a maximal element in \( (F, \sqsubseteq) \).

**Proof:** \( \Rightarrow \) Assume that \( \hat{f}^* \in \phi(\hat{f}^*) \) but it is not maximal, i.e. there exists \( \hat{g} \in F \) such that \( \hat{g} \nsubseteq \hat{f}^* \). This means that either \( M_{\hat{g}} \nsubseteq M_{\hat{f}^*} \). But, by Proposition 4, \( M_{\hat{g}} = \mathbb{R}^n \) while by definition \( M_{\hat{g}} \subseteq \mathbb{R}^n \). Contradiction.

\( \Leftarrow \) If \( \hat{f}^* \) is a maximal element of \( F \) it means that for any \( \hat{g} \in F \), \( \hat{g} \subseteq \hat{f}^* \). Then, \( M_{\hat{g}} \subseteq M_{\hat{f}^*} \). Suppose that \( \hat{f}^* \notin \phi(\hat{f}^*) \). This means that \( M_{\hat{f}^*} \neq \mathbb{R}^n \), since for every \( \hat{f}' \in \phi(\hat{f}^*) \) there exists \( \leq \in \mathbb{R}^n \) such that \( \hat{f}'(\leq) \neq \hat{f}^*(\leq) \) and then, \( \leq \notin M_{\hat{f}^*} \).

On the other hand, by Theorem 1 there exists a fixed point of \( \phi \), say \( \hat{f}^{**} \) and by Proposition 4, \( M_{\hat{f}^{**}} = \mathbb{R}^n \), but then \( M_{\hat{f}^{**}} \subset M_{\hat{f}^*} \). That is, \( \hat{f}^{**} \nsubseteq \hat{f}^* \). Contradiction. \( \square \)

These properties of fixed points of \( \phi \) lead to the Nash-implentability\(^5\) of the social choice functions captured by those direct mechanisms:

\(^5\)Notice that this differs from usual treatments of Nash implementability by direct mechanisms, in which is understood that agents select preference profiles instead of relations (see, for instance, Chap. 10 of Osborne and Rubinstein, 1994).
Theorem 2 A direct mechanism $\hat{f}$ is fixed point of $\phi$ ($\hat{f}^* \in \phi(\hat{f}^*)$) if and only if each profile $\preceq \in \mathbb{R}^n$ is a Nash equilibrium in the corresponding game $\langle \hat{f}^*, \preceq \rangle$.

**Proof:** $\Rightarrow$ Assume that $\hat{f}^*$ is a fixed point for $\phi$. That is, $\hat{f}^* \in \phi(\hat{f}^*)$. Then, for every profile $\preceq \in \mathbb{R}^n$, $\hat{f}^*(\preceq, \preceq) \in \cap_{i=1}^n C(\preceq_i, o^I_i(\preceq_i))$, i.e. for each $i$, $\hat{f}^*(\preceq) \in C(\preceq_i, o^I_i(\preceq_i))$. This means that $\hat{f}^*(\preceq_i, \preceq_i) \preceq_i \hat{f}^*(\preceq_i)$. That is, $\preceq$ is a Nash equilibrium in the game $\langle \hat{f}^*, \preceq \rangle$.

$\Leftarrow$ Assume that each $\preceq$ is a Nash equilibrium in the $\langle \hat{f}^*, \preceq \rangle$. Then, for any alternative preference $\preceq_i'$, we have $\hat{f}^*(\preceq_i', \preceq_i) \preceq_i \hat{f}^*(\preceq_i)$. That is, $\hat{f}^*(\preceq) \in C(\preceq_i, o^I_i(\preceq_i))$ for every $i$. Then, $\hat{f}^*(\preceq) \in \cap_{i=1}^n C(\preceq_i, o^I_i(\preceq_i))$. Since this is true for every $\preceq$, $\hat{f}^* \in \phi(\hat{f}^*)$. $\square$

Maskin (1977) has shown that implementability in Nash equilibria implies the following property of monotonicity of the social choice function, also called strong positive association (Muller and Satterthwaite, 1977):

For all $a \in X$ and for any pair $\preceq, \preceq' \in \mathbb{R}^n$, if $a = \hat{f}^*(\preceq)$ and for each $i = 1, \ldots, n$, and for each $b \in X$, $b \preceq_i a \Rightarrow b \preceq_i' a$ then $a = \hat{f}^*(\preceq')$.

In our case we have a similar result:

**Theorem 3** Given a game form $\hat{f}^*$, such that every $\preceq \in \mathbb{R}^n$ is a Nash equilibrium of the corresponding game $\langle \hat{f}^*, \preceq \rangle$, if $\hat{f}^*(\preceq) = x$ and for every other $y \in X$, if for each $i$, $y \preceq_i x$ implies that $y \preceq_i^* x$, then $\hat{f}^*(\preceq') = x$.

**Proof:** According to Theorem 2, every $\preceq \in \mathbb{R}^n$ is a Nash equilibrium of the corresponding game $\langle \hat{f}^*, \preceq \rangle$. We will show that this implies that $\hat{f}^*(\preceq') = x$ for every profile $\preceq'$ such that for every $y \in X$, $\neq x$, if for each $i$, $y \preceq_i x$ implies that $y \preceq_i^* x$.

We will proceed inductively on the number of agents who change preferences from $\preceq$ to $\preceq'$. So, let $\preceq_i'$ be such that for every $y \in X$, $y \preceq_i x$, $y \preceq_i^* x$ implies $y \preceq_i^* x$. Assume that $\hat{f}^*(\preceq_i', \preceq_i) \neq x$. But then, $\hat{f}^*(\preceq_i', \preceq_i) \preceq_i \hat{f}^*(\preceq_i, \preceq_i)$. This contradicts that $\langle \preceq_i', \preceq_i \rangle$ is a Nash equilibrium of the game when the preferences are $\langle \preceq_i', \preceq_i \rangle$. Therefore, for each $i$, $\hat{f}^*(\preceq_i', \preceq_i) = x$.

Now suppose that $\hat{f}^*(\preceq_i') = x$, where $J \subseteq \{1, \ldots, n\}$, $|J| > 1$, and for each $j \in J$, $\preceq_j$ verifies that for every $y \in X$ $y \neq x$, $y \preceq_j x$ implies $y \preceq_j^* x$.

Then, $\langle \preceq_i', \preceq_i \rangle$ is a Nash equilibrium in the game upon $\hat{f}^*$.

Consider now $\hat{f}^* \in \{1, \ldots, n\} \setminus J$. Let us see that $\hat{f}^*(\preceq_i', \preceq_i) = x = \hat{f}^*(\preceq_i', \preceq_i)$. Suppose not. That is, that $\hat{f}^*(\preceq_i', \preceq_i) \neq x$. But then, $\hat{f}^*(\preceq_i', \preceq_i) \neq x$. But this contradicts that $\langle \preceq_i', \preceq_i \rangle$ is a Nash equilibrium. Therefore, by induction on the size of $J$ we have that $\hat{f}^*(\preceq') = x$. $\square$
4 Strategy-Proofness

The choice improving correspondence has an interesting property, namely that a sequence generated upon an arbitrary direct mechanism by the iterated application of $\phi$ is accompanied by a reduction in the number of options common to all agents:

**Theorem 4** Consider a chain in $\mathcal{F}$, $\hat{f}^0, \ldots, \hat{f}^\alpha, \ldots, \hat{f}^*$ such that either $\hat{f}^\alpha \in \phi(\hat{f}^{\alpha-1})$, if $\alpha$ is a successor ordinal, or $\hat{f}^\alpha \in \sup\{\phi(\hat{f}^\beta) : \beta < \alpha\}$, if $\alpha$ is a limit ordinal, while $\hat{f}^*$ is a fixed point of $\phi$. Then, for every $\zeta \in \mathcal{X}$ and for $\beta < \alpha$, $\bigcap_{i=1}^n o_i^{f^\alpha}(\zeta_{-i}) \subseteq \bigcap_{i=1}^n o_i^{f^\beta}(\zeta_{-i})$.

**Proof:** Notice that, as seen in the proof of Proposition 3, no matter if $\alpha$ is a successor or limit ordinal, it is true that there exists a profile $\zeta \in \mathcal{X}$ such that:

$$\bigcap_{i=1}^n o_i^{f^\alpha}(\zeta_{-i}) = \bigcap_{i=1}^n o_i^{f^\beta}(\zeta_{-i}) \setminus \{\hat{f}^\beta(\zeta)\}_{\beta < \alpha}$$

Suppose not. That is, for every $\zeta$, there exists an ordinal $\gamma \leq \alpha$ such that $\bigcap_{i=1}^n o_i^{f^\alpha}(\zeta_{-i}) \subseteq \bigcap_{i=1}^n o_i^{f^\beta}(\zeta_{-i}) \setminus \{\hat{f}^\beta(\zeta)\}_{\beta < \gamma}$. Then, take $\gamma = \sup_{\zeta \in \mathcal{X}} \{\zeta\}_{\zeta \leq \alpha}$. By definition of $\phi$, for every $\zeta \in \mathcal{X}$, $\hat{f}^\alpha(\zeta) = \hat{f}^\beta(\zeta)$ for every $\alpha > \gamma$. That is, $\hat{f}^*$ is a fixed point of $\phi$. Contradiction. This means that for every $\zeta \in \mathcal{X}$ and for $\beta < \alpha$, $\bigcap_{i=1}^n o_i^{f^\alpha}(\zeta_{-i}) \subseteq \bigcap_{i=1}^n o_i^{f^\beta}(\zeta_{-i})$. $\Box$

This result is particularly relevant considering that $M_f = \{\zeta \in \mathcal{X} : \hat{f}(\zeta) = \bigcap_{i=1}^n C(\zeta, o_i^f(\zeta_{-i}))\}$ has a clear interpretation:

**Proposition 6** A profile $\zeta$ is in $M_f$ if and only if it is a dominant strategy equilibrium in the corresponding game $\langle \hat{f}, \zeta \rangle$.

**Proof:** $\Rightarrow$ Consider $\zeta \in M_f$. Then, for each $i$ we have that $\hat{f}(\zeta_i, \zeta_{-i}) \in C(\zeta_i, o_i^f(\zeta_{-i}))$. But this means that for any feasible profile $\zeta'$ (i.e. such that $\hat{f}(\zeta'_i, \zeta_{-i}) \in o_i^f(\zeta_{-i}))$ we have that $\hat{f}(\zeta'_i, \zeta_{-i}) \leq_i \hat{f}(\zeta_i, \zeta_{-i})$, independently of what $\zeta_{-i}$ is. This means that $\zeta$ is a dominant strategy equilibrium in the game $\langle \hat{f}, \zeta \rangle$.

$\Leftarrow$ Assume that each $\zeta$ is a dominant strategy equilibrium in $\langle \hat{f}, \zeta \rangle$. Then, for any alternative preference $\zeta'_i$ we have that for each $i$ and each $\zeta'_{-i}$, $\hat{f}(\zeta'_i, \zeta'_{-i}) \leq_i \hat{f}(\zeta_i, \zeta_{-i})$. That is, $\hat{f}(\zeta) \in C(\zeta_i, o_i^f(\zeta'_{-i}))$ for every $i$ and every $\zeta'_{-i}$. Then, $\hat{f}(\zeta) \in \bigcap_{i=1}^n C(\zeta_i, o_i^f(\zeta_{-i}))$. That is, $\zeta \in M_f$. $\Box$

**Example**
Consider $\hat{f}$, the dictatorial mechanism for 1 over $X = \{a, b, c\}$, $n = 3$ and $\mathcal{L}R$,
the class of linear orderings of $X$.

Given the following profile in $\mathcal{L}R^3$,

$$\preceq = \langle a \preceq_1 b, \preceq_2 c, \preceq_3 a \rangle$$

we have that, $\hat{f}(\preceq) = a$. Notice that $o_1^f(\preceq_2, \preceq_3) = \{a, b, c\}$, $o_2^f(\preceq_1, \preceq_3) = \{a\} = o_3^f(\preceq_1, \preceq_2)$. Then, the choice of each of the three agents is $a$. Therefore, $\preceq \in M_f$.

Then it follows that we can show that every Nash equilibrium in $\langle \hat{f}^*, \preceq \rangle$ is actually a dominant strategy equilibrium:

**Proposition 7** A direct mechanism $\hat{f}^*$ is fixed point of $\phi$ if and only if each profile $\preceq \in R^n$ is a dominant strategy equilibrium of $\langle \hat{f}^*, \preceq \rangle$.

**Proof:** $\Rightarrow$) If $\hat{f}^* \in \phi(\hat{f}^*)$ it means that each $\preceq \in R^n$ verifies that $\preceq \in M_f$.

Then, by Proposition 6, $\preceq$ is a dominant strategy equilibrium in the game $\langle \hat{f}^*, \preceq \rangle$.

$\Leftarrow$) Assume that each $\preceq$ is a dominant strategy equilibrium in $\langle \hat{f}^*, \preceq \rangle$. Then, each $\preceq$ is trivially a Nash equilibrium of the game and by Theorem 2, $\hat{f}^*$ it is a fixed point of $\phi$. $\square$

Any social choice function implemented by means of dominant strategy equilibria is said strategy-proof. A seminal result for those functions is Satterthwaite’s theorem. The corresponding version in this setting is as follows:

**Theorem 5** If a direct mechanism $\hat{f}^*$ is fixed point of $\phi$ then, there exists an agent $i$ such that $\hat{f}^*(\preceq_i, \preceq_{-i}) = C(\preceq_i, o^f_i(\preceq_{-i}))$, for each $\preceq_i \in R$ and each $\preceq_{-i} \in R^{n-1}$.

**Proof:** First, let us see that for any $\preceq \in R^n$, $|\bigcap_{i=1}^n o^f_i(\preceq_{-i})| = 1$. Because of Theorem 4, we have that the size of $\bigcap_{i=1}^n o^f_i(\preceq_{-i})$ is minimal. Suppose that there exist $x, y \in \bigcap_{i=1}^n o^f_i(\preceq_{-i})$, with $x \neq y$. Since $\hat{f}^*(\preceq) \in \bigcap_{i=1}^n o^f_i(\preceq_{-i})$, say $x = \hat{f}^*(\preceq)$ while $y = \hat{f}^*(\preceq', \preceq_{-i})$ for at least one $i$. Then, we would have that $y \preceq_i x$ since $\hat{f}^*$ is a fixed point of $\phi$. But then, $\langle \preceq_i', \preceq_{-i} \rangle \notin M_f$, and therefore for each $\hat{f}' \in \phi(\hat{f}^*)$, $\hat{f}'(\preceq_i', \preceq_{-i}) \neq \hat{f}^*(\preceq_i', \preceq_{-i})$. But this contradicts that $\hat{f}^*$ is a fixed point of $\phi$.

Let us now see that there exists $i$ such that for every $\langle \preceq_i, \preceq_{-i} \rangle \in R^n$, $\hat{f}^*(\preceq_i, \preceq_{-i}) = C(\preceq_i, o^f_i(\preceq_{-i}))$. Suppose not. Then, for every $i$ there is a $\preceq_i \in R$ such that $\hat{f}^*(\preceq_i, \preceq_{-i})$ $\neq \hat{f}_i^*(\preceq_{-i})$ $x$ for every $x \in o^f_i(\preceq_{-i})$, in particular for the unique $\hat{x} \in o^f_i(\preceq_{-i}) \cap \bigcap_{j\neq i} o^f_j(\preceq_{-i,j})$. Then, $\langle \preceq_i, \preceq_{-i} \rangle$ is not a Nash
equilibrium in the game. But then, \( \hat{f}^* \not\in \phi(\hat{f}^*) \). Contradiction. \( \Box \)

5 Discussion

The main goal of this paper has been to characterize, in the class of direct mechanisms, the fixed points of a correspondence that to each such game form assigns other direct game forms that in some form “improve” upon the original one. More precisely, it provides those alternatives that maximize the declared preferences on the sets of options left by the declarations of all other agents. The fixed points are shown to make each preference profile a Nash equilibrium, i.e. such that no individual finds it profitable to deviate from her true preferences while the others keep theirs.

Moreover, we have shown that these fixed points make each preference profile a dominant strategy equilibrium. That is, that no individual agent has incentives to deviate, no matter what the others have chosen. For both notions of implementation we have shown that the well known properties of, respectively, monotonicity and dictatoriality, ensue.

Our goal is to extend this kind of analysis to the whole variety of implementation settings, particularly those in which information is incomplete.

References


