Strategic Complexity in Repeated Extensive Games

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Abstract

This paper studies a two-player machine (finite automaton) game in which an extensive-form game with perfect information is infinitely repeated, with preferences depending both on the payoffs and the complexity of strategies. We introduce a new measure of strategic complexity called multiple complexity, which incorporates the responsiveness to information within the stage game as well as the number of states of a machine. We completely characterize Nash equilibria of the machine game, and demonstrate that, in contrast to Piccione and Rubinstein (1993), it has non-trivial Nash equilibria in general. In the sequential-move prisoner’s dilemma, cooperation can be sustained in an equilibrium of the machine game.

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1 Introduction

Strategic complexity is an important concept in the theory of bounded rationality, based on the idea that people prefer less "complex" strategies. It is, however, not necessarily obvious which strategies are considered to be less complex than others. In the repeated game context, Aumann (1981) first proposed a measure of strategic complexity using finite automata (Moore machines), followed by the seminal work of Neyman (1985) and Rubinstein (1986). In a machine game, each player employs a finite automaton, which has a finite set of states, each assigning an action to play. Rubinstein (1986) and Abreu and Rubinstein (1988) assumed that the preferences depend both on the payoffs in the infinitely repeated game and the complexity of the automaton, where the complexity is measured by the number of states the automaton possesses. We refer to this measure of complexity as counting-states complexity (CS-complexity).

As shown by Abreu and Rubinstein (1988), complexity consideration significantly shrinks the set of Nash equilibria. Let us consider a repeated game $G^\infty$ of a two-player simultaneous-move prisoner’s dilemma $G$ with an action set $\{C, D\}$. Abreu and Rubinstein (1988) showed that the outcome of any Nash equilibrium path consists of action profiles selected from the set $\{(C, C), (D, D)\}$, or from the set $\{(C, D), (D, C)\}$. This is due to the fact that automata of the two players must have the same number of states in any Nash equilibrium. They also showed that, with the lexicographic preferences, mutual cooperation at every period (except for first periods) is still realized in a Nash equilibrium for a discount factor sufficiently close to one.

Directly applying CS-complexity, Piccione and Rubinstein (1993) analyzed two-player repeated games in which the stage game has an extensive form with perfect information. They proved that, regardless of the discount factor, any Nash equilibrium of the machine game consists of an infinite repetition of a Nash equilibrium of the stage game. In the sequential-move version of the prisoners’ dilemma, this result implies that the only Nash equilibrium of the repeated game is to always play $D$, and thus cooperation is never realized. The failure in achieving cooperation derives from the fact that CS-complexity does not account for the complexity of utilizing information in a period (i.e., within the stage game). This paper proposes a new measure of complexity, multiple complexity ($M$-complexity), which incorpo-

\footnote{See Piccione (1992) for an improved version of the proof.}
rates the complexity of the behavior rule in a period as well as that across periods. With this complexity measure, we show that machine games with a sequential-move stage game generally have a broader set of Nash equilibria than with CS-complexity. In particular, in the repeated sequential-move prisoner’s dilemma, mutual cooperation can be sustained in an equilibrium of the machine game.

To motivate our concept of M-complexity, consider the sequential-move prisoner’s dilemma $\Gamma$ of which the game tree is shown in Figure 1. The unique Nash equilibrium of $\Gamma$ is for both players always to deceive. Let us first review the result of Piccione and Rubinstein (1993) with CS-complexity. It says that in the sequential-move repeated game $\Gamma^\infty$ the only action profile played in the equilibrium with CS-complexity is $(D, D)$, the unique stage-game Nash equilibrium outcome. Note that in this equilibrium in $\Gamma^\infty$ the automaton of each player has only one state. From an automaton with more than one states, player 2 profitably deviates to an automaton with a single state yielding the same payoffs.

Consider an automaton pair $(M_1, M_2)$ in $\Gamma^\infty$ which generates only the action profiles $(C, C)$ and $(D, D)$ on the path. Suppose that $M_2$ has two or more states. With CS-complexity, let us see that any such automaton pair is not an equilibrium of the machine game. The crucial deviation of player 2 from automaton $M_2$ is an automaton $M'_2$ with the single state that adopts the stage-game strategy $s_2$ defined by $s_2(C) = C$ and $s_2(D) = D$, where player 2 cooperates when she observes player 1 to cooperate, and deceives when
she observes player 1 to deceive. This automaton $M_2$ generates the action profile $(C, C)$ or $(D, D)$ at every period, keeping unchanged the outcome in the repeated game. Since automaton $M_2'$ has a strictly lower CS-complexity than $M_2$, this deviation is profitable for player 2. Player 1 in turn deviates to always playing $C$, arriving at an automaton with a single state.

The above conclusion critically depends on the fact that CS-complexity assigns the lowest complexity to any strategy that plays the same stage-game strategy at every period, which is represented by a machine with a single state. Generally, any strategies implemented by automata with a same number of states are considered equally complex under the CS-complexity measure. Our concept of M-complexity, in contrast, accounts for how the stage-game strategy in each state of an automaton exploits the information structure within the stage game, as well as how the automaton reacts to the histories of the repeated game. The M-complexity of an automaton is defined to be the sum of the cardinalities of the ranges of all the stage-game strategies played by the automaton. Note that the M-complexity of any automaton is larger than or equal to its CS-complexity, and the M-complexity coincides with the CS-complexity if and only if each state assigns a constant stage-game strategy that plays only one action independent of the opponent's previous action within the period. In the above example, automaton $M_2$ has M-complexity two or larger, as it has more than one states, while the M-complexity of $M_2'$ is two, as the range of $s_2$ is $\{C, D\}$. Hence, when the M-complexity of $M_2$ is two, the above deviation to $M_2'$ is not profitable under M-complexity. In Section 4, we will in fact construct a Nash equilibrium with two states (and M-complexity two) which sustains mutual cooperation for a discount factor sufficiently close to one.

More generally, for the class of simple sequential-move games with two players, we give a necessary and sufficient condition for a strategy profile to be a Nash equilibrium of the machine game under M-complexity. In any Nash equilibrium, the automaton assigns distinct actions in distinct states, and thus the number of the states must be smaller than or equal to the number of the actions available to the player. This condition restricts the number of states in equilibria, resulting in further refinement of the set of equilibria than in Abreu and Rubinstein (1988). Nevertheless, in contrast to Piccione and Rubinstein (1993), Pareto efficient pure-action outcomes are supported in Nash equilibria as long as the deviation gains are not too large (when the discount factor is close to one).

This paper proceeds as follows. In section 2, we present necessary def-
initions. In section 3, we show lemmas. In section 4, we prove the main
theorem and present examples. In section 5, we discuss an extension of our
model.

2 Definitions

Let $G = (A_1, A_2; u_1, u_2)$ be a two-player strategic game, $A_i$ be a finite set of
pure actions for player $i$ ($i = 1, 2$), and $u_i$ be player $i$'s payoff function defined
on $A = A_1 \times A_2$. Let $G^\infty$ be the infinitely repeated game of $G$. If player
$i$'s stage-game payoff in period $t$ is $u_i^t$, then player $i$'s payoff $\pi_i$ in $G^\infty$ is
$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i^t$ for a discount factor $0 < \delta < 1$.

In the repeated game $G^\infty$, player $i$'s finite automaton is represented by a
four-tuple $M_i = (Q_i, q_i^0, \lambda_i, \mu_i)$ in which $Q_i$ is a finite set of states, $q_i^0 \in Q_i$ is
the initial state, $\lambda_i : Q_i \to A_i$ is the output function, and $\mu_i : Q_i \times A \to Q_i$ is
the transition function. Let $M_i$ be the set of all automata for player $i$, and let
$M = M_1 \times M_2$. The number of states in the automaton $M_i \in M_i$, i.e. $\#Q_i$, is
called the counting-states (CS-) complexity and denoted by $\text{comp}_{cs}(M_i)$.

A machine game of $G^\infty$ is a game in which each player chooses her automa-
ton and obtains the repeated-game payoffs in $G^\infty$. Players in the machine
game incorporate the number of states in their automata as well as their
payoffs. We assume the following class of preferences defined by Abreu and

Definition 1. A preference relation $\succ_i$ of player $i$ in the machine game
satisfies all of the following criteria.

For $M_1, M'_1 \in M_1$ and $M_2, M'_2 \in M_2$,

1. If $\pi_i(M_1, M_2) = \pi_i(M'_1, M'_2)$ and $\text{comp}_{cs}(M_i) = \text{comp}_{cs}(M'_i)$,
then $(M_1, M_2) \sim_i (M'_1, M'_2)$.

2. If $\pi_i(M_1, M_2) > \pi_i(M'_1, M'_2)$ and $\text{comp}_{cs}(M_i) = \text{comp}_{cs}(M'_i)$,
then $(M_1, M_2) \succ_i (M'_1, M'_2)$.

3. If $\pi_i(M_1, M_2) = \pi_i(M'_1, M'_2)$ and $\text{comp}_{cs}(M_i) < \text{comp}_{cs}(M'_i)$,
then $(M_1, M_2) \succ_i (M'_1, M'_2)$.

\footnote{For a finite set $S$, $\# S$ denotes the cardinality of $S$.}
Various preference relations satisfy the above criteria. For instance, the lexicographic preference, which is defined by the above three criteria except that the second one is replaced by “if $\pi_i(M_1, M_2) > \pi_i(M'_1, M'_2)$, then $(M_1, M_2) \succ_i (M'_1, M'_2)$,” is a special case of Definition 1.

In the machine game, the following result is fundamental.

**Proposition 1** (Abreu and Rubinstein, 1988). Suppose that a pair of automata $(M_1, M_2) \in \mathcal{M}$ is a Nash equilibrium of the machine game. Let $q_i^t$ be player $i$’s state in period $t$, and $a_i^t$ be player $i$’s action in period $t$. Then the following three statements are true.

1. $\text{comp}_{cs}(M_i) = \text{comp}_{cs}(M_2)$.
2. $q_1^t = q_1^{t'}$ if and only if $q_2^t = q_2^{t'}$ for any two periods $t, t'$.
3. $a_1^t = a_1^{t'}$ if and only if $a_2^t = a_2^{t'}$ for any two periods $t, t'$.

Let $\Gamma$ be a two-player sequential-move game in which player 1 first chooses an action $a_1 \in A_1$ and player 2 chooses an action $a_2 \in A_2$ after observing $a_1$. Let $S_i$ ($i = 1, 2$) be the set of player $i$’s strategies in $\Gamma$. Since player 1 is the first-mover and player 2 is the second-mover, we have $S_1 = A_1$ and $S_2 = \{s_2 \mid s_2 : A_1 \rightarrow A_2\}$. The constant strategy $s_2(\cdot) = a_2$ is occasionally denoted by $a_2$, as a slight abuse of the notation. Let $\Gamma^\infty$ be the infinitely repeated game of $\Gamma$. The set of automata for player $i$ in $\Gamma^\infty$ is denoted by $\mathcal{M}_i^\Gamma$ and defined below. Let $\mathcal{M}_i^\Gamma = \mathcal{M}_i^1 \times \mathcal{M}_i^2$. Since player 1, who moves first, chooses an action in $\Gamma$ in the same way as in $G$, the formulation of automata for player 1 in $\Gamma^\infty$ and in $G^\infty$ are identical, and thus we regard $\mathcal{M}_1^\Gamma = \mathcal{M}_1$. An automaton for player 2 in $\Gamma^\infty$ is defined by $M_2 = (Q_2, q_2, \lambda_2, \mu_2) \in \mathcal{M}_2^\Gamma$, in which the output function $\lambda_2$ is a function from $Q_2$ to $S_2$. The range of $\lambda_2$ is the set of stage-game strategies of player 2 in $\Gamma$. Under CS-complexity, the result of Piccione and Rubinstein (1993) implies that the machine game for $\Gamma^\infty$ has only trivial equilibria consisting of an automaton pair with a single state, which induces an infinite repetition of a stage-game Nash equilibrium in $\Gamma$.

We are now ready to introduce the new measure of complexity for automata in $\Gamma^\infty$.

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3Notice that $G$ is not the corresponding strategic-form of $\Gamma$. While one might argue that actions having a same label but belonging to different information sets have to be considered different ones, we identify them since our main purpose here is to compare the behavior in the sequential-move game $\Gamma$ with that in the simultaneous-move game $G$. 

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Definition 2. For player $i$’s automaton $M_i = (Q_i, q_i^0, \lambda_i, \mu_i) \in \mathcal{M}_1^\Gamma$, the multiple (M-) complexity of $M_i$ is defined by
\[
\text{comp}_m(M_i) = \sum_{q_i \in Q_i} \# \{ a_i(\lambda_i(q_i), s_j) | s_j \in S_j \} \quad (i \neq j)
\]
where $a_i(s_i, s_j) \in A_i$ is player $i$’s action induced by the pair of stage-game strategies $(s_i, s_j)$ in $\Gamma$.

Note that $a_i(\lambda_i(q_i), s_2) = \lambda_i(q_1)$ for any $s_2 \in S_2$, and that $a_2(a_1, \lambda_2(q_2)) = (\lambda_2(q_2))(a_1)$ for every $a_1 \in A_1$. Therefore $\text{comp}_m(M_i) = \text{comp}_{cs}(M_i)$ for any player 1’s automaton $M_1 \in \mathcal{M}_1^\Gamma$. For a stage-game strategy $s_2 : A_1 \rightarrow A_2$, let us define $c(s_2) = \# \{ s_2(a_1) \in A_2 | a_1 \in A_1 \}$, namely the cardinality of the range of $s_2$. Then $\text{comp}_m(M_2) = \sum_{q_2 \in Q_2} c(\lambda_2(q_2))$ for player 2’s automaton $M_2 \in \mathcal{M}_2^\Gamma$.

Multiple complexity incorporates a “complexity” of outputs as well as the CS-complexity of the automaton.\(^4\) In other words, $c(s_2)$ is considered to be a measure of a complexity of $s_2$. The intuition is that when player 2 has more opportunities to change her choices according to player 1’s actions, her automaton is measured to be more complex. The simplest output of player 2’s automata is the stage-game strategy which always plays the same action. In $\Gamma^\infty$, we will assume the preference relation in Definition 1 with respect to the M-complexity instead of the CS-complexity.

Suppose that $M_2 = (Q_2, q_2^0, \lambda_2, \mu_2) \in \mathcal{M}_2^\Gamma$ satisfies $c(\lambda_2(q_2)) = 1$ for all $q_2 \in Q_2$. Then, at every state, player 2’s choice of the actions is independent of player 1’s. This may be interpreted that player 2 moves without observing player 1’s output. Thus player 2 moves as if she played the corresponding simultaneous-move game $G$. In other words, an automaton in the simultaneous-move game is regarded as an automaton in the sequential-move game. More formally, for a given automaton $M_2 = (Q_2, q_2^0, \lambda_2, \mu_2) \in \mathcal{M}_2$ in the simultaneous-move game, we define an output function in the sequential-move game $\tilde{\lambda}_2 : Q_2 \rightarrow \{ s_2 : A_1 \rightarrow A_2 \}$ to be $(\tilde{\lambda}_2(q_2))(a_1) = \lambda_2(q_2)$ for every $a_1 \in A_1$. Then the mapping $(Q_2, q_2^0, \lambda_2, \mu_2) \mapsto (Q_2, q_2^0, \tilde{\lambda}_2, \mu_2)$ is an injection from $\mathcal{M}_2$ to $\mathcal{M}_2^\Gamma$. In this way, we henceforth regard $M_2$ as a subset of $\mathcal{M}_2^\Gamma$.

\(^4\)Multiple complexity does not incorporate complexity of transition functions. When the stage game is in a strategic form, Banks and Sundaram (1990) showed that a stage-game Nash equilibrium is played in every period on any equilibrium of the repeated game with a class of complexity which takes into account transition functions.

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3 Preliminary results

In this section, we present preliminary lemmas. First, we show a relation between CS-complexity and M-complexity which is shown directly from Definition 2.

Lemma 1. Every automaton \( M_2 = (Q_2, q_2^0, \lambda_2, \mu_2) \in \mathcal{M}_2^\Sigma \) for player 2 satisfies the property \( \text{comp}_m(M_2) \geq \text{comp}_c(M_2) \). The equality holds if and only if \( c(\lambda_2(q_2)) = 1 \) for all \( q_2 \in Q_2 \).

Proof. The above properties are easily shown from the definition of M-complexity, as \( \text{comp}_m(M_2) = \sum_{q_2 \in Q_2} c(\lambda_2(q_2)) \), and \( c(\lambda_2(q_2)) \geq 1 \).

Next, under the assumption that player 1 plays distinct actions in distinct states, we show a possibility of player 2’s deviation from an automaton in \( \Gamma^\infty \) to the one in \( G^\infty \) with the equal M-complexity, keeping the repeated-game outcome unchanged. For a pair of automata \((M_1, M_2) \in \mathcal{M}^\Gamma\), let \( a_t = (a_t^1, a_t^2) \in A_1 \times A_2 \) and \( q_t = (q_t^1, q_t^2) \in Q_1 \times Q_2 \) be the pair of outputs and states, respectively, induced by \((M_1, M_2)\) in period \( t \).

Lemma 2. For \((M_1, M_2) \in \mathcal{M}^\Gamma\), suppose that \( a_t^1 = a_t'^1 \) implies \( q_t^1 = q_t'^1 \) for all periods \( t \) and \( t' \). Then, there exists an automaton \( \tilde{M}_2 \in \mathcal{M}_2 \) for player 2 such that \( \text{comp}_m(M_2) = \text{comp}_c(\tilde{M}_2) \) and \((M_1, M_2)\) and \((M_1, \tilde{M}_2)\) generate the same action profiles in every period.

Proof. Let \( M_1 = (Q_1, q_1^0, \lambda_1, \mu_1) \in \mathcal{M}_1 \) and \( M_2 = (Q_2, q_2^0, \lambda_2, \mu_2) \in \mathcal{M}_2^\Sigma \). Define \( \tilde{M}_2 = (Q_2, \tilde{q}_2^0, \tilde{\lambda}_2, \tilde{\mu}_2) \in \mathcal{M}_2 \) as follows:

\[
\tilde{Q}_2 = \bigcup_{q_2 \in Q_2} \{ (q_2, a_2) \in Q_2 \times A_2 \mid a_2 = \lambda_2(q_2)(a_1) \text{ for some } a_1 \in A_1 \},
\]

\[
\tilde{q}_2^0 = (q_2^0, a_2^0),
\]

\[
\tilde{\lambda}_2(q_2, a_2)(a_1) = a_2 \text{ for all } a_1 \in A_1,
\]

\[
\tilde{\mu}_2((q_2, a_2), a) = \begin{cases} 
(q_2^{t+1}, a_2^{t+1}) & \text{if } ((q_2, a_2), a) = ((q_2^t, a_2^t), a^t) \text{ for some } t \\
& \text{and there is a period } t' \leq t - 1 \\
& \text{such that } ((q_2^t, a_2^t), a') = ((q_2^{t'}, a_2^{t'}), a'^t), \\
(q_2^{t+1}, a_2^{t+1}) & \text{if } ((q_2, a_2), a) = ((q_2^t, a_2^t), a^t) \text{ for some } t \\
& \text{and the above does not hold,} \\
\text{arbitrary} & \text{otherwise.}
\end{cases}
\]
From the definition of \( \bar{Q}_2 \), \( \text{comp}_m(M_2) = \text{comp}_v(\bar{M}_2) \) is easily verified.

Let \((\bar{a}^t_1, \bar{a}^t_2)\) be the action profile in period \( t \) induced by \((M_1, \bar{M}_2)\). We will prove \((\bar{a}^t_1, \bar{a}^t_2) = (a^t_1, a^t_2)\) for all \( t \) by induction. For \( t = 1 \), \( \bar{a}^1_1 = a^1_1 \) and \( \bar{a}^1_2 = \lambda_2(q^1_2, a^1_2) = a^1_2 \). Next fix \( t \) and assume that \((\bar{a}^k_1, \bar{a}^k_2) = (a^k_1, a^k_2)\) for all \( k = 1, \ldots, t \). For player 1, \( a^{t+1}_1 = a^{t+1}_1 \) directly follows from the assumption.

If there is no period \( t' \) \((t' \leq t - 1)\) such that \((q^t_1, a^t_1), a^t'\) = \((q^{t'}_2, a^{t'}_2), a^{t'}\) then the assumption of the lemma implies that \( q^t_1 = q^{t'}_1 \). Therefore \( q^{t+1}_1 = q^{t'+1}_1 \) and \( a^{t+1}_1 = a^{t'+1}_1 \). This yields

\[
\bar{a}^{t+1}_2 = \lambda_2(q^{t+1}_2, a^{t+1}_2) = a^{t+1}_2.
\]

If there exists a period \( t' \) \((t' \leq t - 1)\) such that \((q^t_1, a^t_1), a^t'\) = \((q^{t'}_2, a^{t'}_2), a^{t'}\) then the assumption of the lemma implies that \( q^t_1 = q^{t'}_1 \). Therefore \( q^{t+1}_1 = q^{t'+1}_1 \) and \( a^{t+1}_1 = a^{t'+1}_1 \). This yields

\[
a^{t+1}_2 = \lambda_2(q^{t+1}_2, a^{t+1}_2) = a^{t+1}_2.
\]

On the other hand,

\[
\bar{a}^{t+1}_2 = \lambda_2(q^{t+1}_2, a^{t+1}_2) = a^{t+1}_2.
\]

Thus \( \bar{a}^{t+1}_2 = a^{t+1}_2 \) is proved. Two automaton profiles \((M_1, M_2)\) and \((M_1, \bar{M}_2)\) generate the same action profiles in every period. \( \square \)

In the following Lemma we use an analogous argument to the lemma in Piccione and Rubinstein (1993).

**Lemma 3.** Suppose that \((M_1, M_2) \in \mathcal{M}^\Gamma\) is an equilibrium in \( \Gamma^\infty \) under \( M\)-complexity. Then

\[
\text{comp}_{c,s}(M_1) = \text{comp}_{c,s}(M_2) = \text{comp}_m(M_2).
\]

Particularly, the second equality implies that player 2 moves independently of player 1’s action in any state of \( M_2 \).
Proof. Let $M_1 = (Q_1, q_1^t, \lambda_1, \mu_1)$ and $M_2 = (Q_2, q_2^t, \lambda_2, \mu_2)$. For given $M_1$, consider player 2’s payoff maximization problem in the repeated game. This Markovian decision problem has a stationary solution $\sigma_2 : Q_1 \to A_2$. Then, define player 2’s automaton $M_2' = (Q_1, q_1^t, \lambda_2', \mu_2')$ with $\lambda_2'(q_1)(\cdot) = \sigma_2(q_1)$ and $\mu_2'(q_1, \cdot) = \mu_1(q_1, (\lambda_1(q_1), \sigma_2(q_1)))$. Since $c(\lambda_2'(q_1)) = 1$ for any $q_1 \in Q_1$, $\text{comp}_m(M_2')$ is equal to $\text{comp}_e(M_2')$, and to $\text{comp}_e(M_1)$ by the definition of $M_2'$. Since $M_2$ is a best reply to $M_1$, $(M_1, M_2) \succeq_2 (M_1, M_2')$. On the other hand, by the definition of $M_2'$, $\pi_2(M_1, M_2') \leq \pi_2(M_1, M_2')$. Therefore by Definition 1, it must hold that $\text{comp}_m(M_2) \leq \text{comp}_e(M_2')$. Hence, $\text{comp}_m(M_2) \leq \text{comp}_e(M_1)$.

Considering player 1’s Markovian decision problem for given $M_2$ shows $\text{comp}_e(M_1) \leq \text{comp}_e(M_2)$. By combining all the inequalities above and in Lemma 1, the lemma is proved.

Recall that we regard $M_2$ as a subset of $M_2^\Gamma$. When $(M_1, M_2) \in M^\Gamma$ is an equilibrium in $\Gamma^\infty$ under multiple complexity, this lemma implies that $M_2 \in M_2^\Gamma$. Therefore $(M_1, M_2)$ is an equilibrium also in $G^\infty$. By this fact, Proposition 1 can be applied to $\Gamma^\infty$, leading to the following lemma.

**Lemma 4.** Suppose that $(M_1, M_2) \in M^\Gamma$ is an equilibrium in $\Gamma^\infty$ under multiple complexity. Let $(a_1^t, a_2^t)$ be the action profile in period $t$, and $(q_1^t, q_2^t)$ be the pair of states in period $t$. In any two periods $t, t'$,

1. $q_1^t = q_1^t$ if and only if $q_2^t = q_2^t$,
2. $a_1^t = a_1^{t'}$ if and only if $a_2^t = a_2^{t'}$.

4 Main Theorem

We now obtain a necessary and sufficient condition of Nash equilibria in the machine game of $\Gamma^\infty$ under multiple complexity.

**Theorem 1.** For a pair of automata $(M_1, M_2) \in M^\Gamma$, let $(a_1^t, a_2^t)$ and $(q_1^t, q_2^t)$ be the pair of actions and states, respectively, induced by $(M_1, M_2)$ in period $t$. Then $(M_1, M_2)$ is a Nash equilibrium in $\Gamma^\infty$ under multiple complexity if and only if (i) $M_2 \in M_2^\Gamma$ (i.e., player 2 moves independently of the opponent’s previous action within the period), (ii) $(M_1, M_2)$ is a Nash equilibrium in $G^\infty$, and furthermore (iii) $a_1^t = a_1^{t'}$ implies $q_1^t = q_1^{t'}$ for all periods $t$ and $t'$ ($i = 1, 2$).
Proof. First suppose that \((M_1, M_2)\) is an equilibrium in \(\Gamma^\infty\) under multiple complexity. By Lemma 3, \((M_1, M_2)\) is an equilibrium in \(G^\infty\). Let \(M_2 = (Q_2, q_2^0, \lambda_2, \mu_2) \in \mathcal{M}_2\). Note that \(\text{comp}_m(M_2) = \text{comp}_{cs}(M_2) = \#\{q_2^t| t = 1, 2, \ldots\}\). Define an automaton \(M'_2 = (Q_2, q'_2, \lambda'_2, \mu'_2) \in \mathcal{M}'_2\) such that \(\text{comp}_{cs}(M'_2) = 1\) as follows:

- \(Q'_2 = \{q'_2\}\),
- \(\lambda'_2(q'_2)(a'_1) = a'_2\),
- \(\mu'_2(q'_2, \cdot) = q'_2\).

Note that this definition is well-defined by the second condition in Lemma 4, and that \((M_1, M'_2)\) and \((M_1, M_2)\) generate the same action profiles in every period. By the definition of multiple complexity, \(\text{comp}_m(M'_2) = \#\{a'_2\}\). Suppose that \(a'_2 = a''_2\) but \(q'_2 \neq q''_2\) for some \(t, t'\). Then \(\#\{a'_2\} < \#\{q'_2\}\), and therefore \(\text{comp}_m(M'_2) < \text{comp}_m(M_2)\), which implies that \(M_2\) is not a best reply to \(M_1\). By contradiction, we have shown that \(a'_2 = a''_2\) implies \(q'_2 = q''_2\). With this fact, Lemma 4 implies that if \(a'_i = a''_i\) then \(q'_i = q''_i\).

Second suppose that \((M_1, M_2)\) is an equilibrium in \(G^\infty\), and that \(a'_i = a''_i\) implies \(q'_i = q''_i\) at any two periods \(t, t'\). Since every state of \(M_1\) appears on the equilibrium path of \((M_1, M_2)\), the second supposition means that the automaton \(M_1\) assigns distinct actions in distinct states. Therefore for any \(M'_2 \in \mathcal{M}'_2\), the condition in Lemma 2 holds true with respect to \((M_1, M'_2)\). Assume that there is \(M'_2 \in \mathcal{M}'_2\) such that \((M_1, M'_2) \succeq (M_1, M_2)\). Then there is \(\tilde{M}'_2 \in \mathcal{M}_2\) as in Lemma 2 such that \((M_1, \tilde{M}'_2) \succeq (M_1, M_2)\), contradicting the assumption that \((M_1, M_2)\) is an equilibrium in \(G^\infty\). Hence \((M_1, M_2)\) is an equilibrium in \(\Gamma^\infty\) under multiple complexity.

The third condition in Theorem 1 asserts that each player plays distinct actions in distinct states in an equilibrium under M-complexity. Therefore, the number of states in any equilibrium must be smaller than or equal to the number of actions available to each player.

**Corollary 1.** If \((M_1, M_2)\) is an equilibrium in \(\Gamma^\infty\) under M-complexity. Then

\[
\text{comp}_{cs}(M_1) = \text{comp}_{cs}(M_2) \leq \min(\#A_1, \#A_2).
\]

Proof. Since \((M_1, M_2)\) is an equilibrium in \(\Gamma^\infty\), \(\text{comp}_{cs}(M_1) = \text{comp}_{cs}(M_2)\) by Lemma 3. By the third condition in Theorem 1, \(\text{comp}_{cs}(M_i) \leq \#A_i\) for \(i = 1, 2\).

\[\square\]
Table 2: Two-player prisoner’s dilemma ($x > 2, y < 0, 0 < x + y < 4$)

<table>
<thead>
<tr>
<th></th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>2, 2</td>
<td>$y, x$</td>
</tr>
<tr>
<td>$D$</td>
<td>$x, y$</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Figure 2: Two-player prisoner’s dilemma ($x > 2, y < 0, 0 < x + y < 4$)

Figure 3: The Tit-for-Tat automaton

The first and the second condition in Theorem 1 assert that the set of equilibria in $\Gamma^\infty$ under M-complexity is a subset of equilibria in $G^\infty$. Thus, the conditions of the equilibria in $G^\infty$, investigated in Abreu and Rubinstein (1988), help the analysis of equilibria in $\Gamma^\infty$ under M-complexity. On the other hand, the third condition in Theorem 1 refines the set of equilibria in $G^\infty$. The following two examples show how these conditions specify the set of equilibria.

**Example 1.** Let $G$ be the simultaneous-move prisoner’s dilemma whose payoff matrix is given by Figure 2, and let $\Gamma$ be the corresponding sequential-move prisoner’s dilemma. Assume that the preferences are lexicographic. By Corollary 1, any Nash equilibrium in the machine game of $\Gamma^\infty$ under M-complexity consists of a pair of automata with at most two states, as there are only two actions $C, D$ available in the game. When the number of states is one, it is obvious that an equilibrium must be such that both players infinitely repeat $D$.

We here consider the case $x < 4$. Suppose that the number of states is two. By Lemma 4, the history of the actions and the states must be cyclic with period two or cyclic with period one after the second period. First consider the case of period one. Since we have already considered
the repetition of $D$, consider the repetition of $C$. One example is when both players implement a well-known automaton called Tit-for-Tat\textsuperscript{5} shown in Figure 3.\textsuperscript{6} This automaton pair generates the history $((D, D), (C, C), (C, C), \ldots)$, and yields the repeated-game payoff $2\delta$. Consider player 1’s deviation from this automaton to a single-state automaton which always plays $D$. Then this deviation generates the history $((D, D), (D, C), (D, D), (D, C), \ldots)$, and gives her a repeated-game payoff $\delta x / (1 + \delta)$. From the assumption $x < 4$, there is a discount factor $\delta$ such that $(x - 2) / 2 \leq \delta < 1$, with which player 1 has no incentive of such a deviation. Then it is easily shown that both players playing Tit-for-Tat is an equilibrium in the machine game of $G^\infty$. By Theorem 1, it is also an equilibrium in $\Gamma^\infty$ under M-complexity. In this equilibrium $(C, C)$ is played repeatedly after the second period. When CS-complexity is adopted, player 2 deviates to an automaton with only one state which outputs the stage-game strategy playing $C, D$ when player 1 chooses $C, D$, respectively. Under M-complexity, the complexity of this automaton is two. Hence player 2 has no incentive of deviation, and the equilibrium survives.

Second consider the case of period 2. By Theorem 1, a Nash equilibrium in $\Gamma^\infty$ is regarded as that in $G^\infty$. Thus the action profiles are the repetition either of $(C, C), (D, D)$ or of $(C, D), (D, C)$. In the former case, either player deviates to an automaton which always yields $D$, gaining strictly higher payoffs. The latter case is realized by the pair of automata shown in Figure

\textsuperscript{5}We adopt the terminology “Tit-for-Tat” from Binmore and Samuelson (1992).

\textsuperscript{6}The symbols associated with arrows denote the opponent’s action. We omit the symbols of the player’s own actions, on which the transition function of the automaton does not depend.
Figure 5: The equilibrium payoffs in the sequential-move prisoner’s dilemma.

In summary, there are three kinds of equilibrium outcomes in game $\Gamma^\infty$: the infinite repetition of $(D, D)$; $(D, D)$ in the first period followed by the repetition of $(C, C)$; and the alternating play of $(C, D), (D, C)$ (or of $(D, C), (C, D)$). For the payoffs given in Figure 1 (i.e., $x = 3, y = -1$), the set of equilibrium payoffs in $\Gamma^\infty$ is described in Figure 5 when the discount factor $\delta$ is almost equal to one.

Finally, consider the case $x \geq 4$. Since there is no discount factor $\delta < 1$ such that $\delta > (x - 2)/2$, players profitably deviate from Tit-for-Tat to the automaton always to play $D$. Therefore, in this case, no equilibrium exists which induces an infinite repetition of $C$. We will later provide a general argument about a sufficient condition for the existence of such equilibria.

**Example 2.** Next consider the battle of the sexes game $G$ with a payoff matrix shown in Figure 6. The pure-action Nash equilibria of this game are $(B, B)$ and $(O, O)$. In the equilibria in the machine game of $G^\infty$, if $(B, O)$ or $(O, B)$ is played in some period, then only the action profile either $(B, O)$ or $(O, B)$ appears in every period on the equilibrium path. Therefore the player can deviate to the automaton always playing the same action, gaining strictly positive payoff. Thus players play either $(B, B)$ or $(O, O)$ in any equilibrium.

Let $\Gamma$ be the sequential-move game corresponding to $G$. By Theorem 1,
automata with one or two states must be used in the equilibria in $\Gamma^\infty$ under M-complexity. When the number of state is one, the equilibrium outcome is an infinite repetition of $(B, B)$ or of $(O, O)$. When the number of states is two, there is an equilibrium in which $(B, B)$ and $(O, O)$ are played alternately. Therefore, when the discount factor $\delta$ is sufficiently close to one, there are three types of equilibrium payoffs: The repetition of $(B, B)$, the repetition of $(O, O)$, and the alternating play of $(B, B), (O, O)$ (or of $(O, O), (B, B)$).

Theorem 1 describes clear-cut conditions for an automaton pair to be a machine-game equilibrium in $\Gamma^\infty$ under M-complexity. Those conditions refine the set of equilibria in $G^\infty$ shown in Abreu and Rubinstein (1988). However, as Example 1 shows, even a Pareto efficient pure-action stage-game outcome is not always implemented by an equilibrium under M-complexity. In this section, we provide a sufficient condition ensuring that a stage-game outcome is supported in an equilibrium. Let $\underline{u}_i = \min_{a_i \in A_i} \max_{a_j \in A_j} u_i(a_1, a_2)$ be the pure-action minmax payoff for player $i$ in the stage game $G$, and let $\hat{a}_j$ be the action of player $j$ used to minmax the opponent's payoff. Let $\overline{v}_i(a_j) = \max_{a_i} u_i(a_1, a_2)$ be the maximum one-shot payoff that player $i$ gains by deviating from $(a_1, a_2)$.

**Proposition 2.** Suppose that the players’ preferences are lexicographic. If $u_i(\tilde{a}_1, \tilde{a}_2) > \underline{u}_i$ and $u_i(\tilde{a}_1, \tilde{a}_2) > (u_i(\hat{a}_1, \tilde{a}_2) + \overline{v}_i(\hat{a}_j))/2$ for $i = 1, 2$, then, for a sufficiently large discount factor, there exists a Nash equilibrium in $\Gamma^\infty$ under M-complexity that sustains an infinite repetition of $(\tilde{a}_1, \tilde{a}_2)$ after the second period.

**Proof.** The proof is by construction of a pair of automata. Consider player $i$’s automaton $M_i = (Q_i, q_i^0, \lambda_i, \mu_i)$ with two states defined as follows:

$Q_i = \{q_i^1, q_i^2\}$, 
$\lambda_i(q_i^1) = \hat{a}_i$, $\lambda_i(q_i^2) = \hat{a}_i$,
\[ \mu_i(q_i^1, (a_i, a_j)) = \begin{cases} q_i^2 & \text{if } a_j = \hat{a}_j, \\ q_i^1 & \text{otherwise,} \end{cases} \]

\[ \mu_i(q_i^2, (a_i, a_j)) = \begin{cases} q_i^2 & \text{if } a_j = \hat{a}_j, \\ q_i^1 & \text{otherwise.} \end{cases} \]

This automaton pair \((M_1, M_2)\) generates the infinite repetition of \((\hat{a}_1, \hat{a}_2)\) after the second period. If player \(i\) deviates from the action \(\hat{a}_i\) to \(a_i\) in state \(q_i^1\), then she obtains payoff \(u_i(a_i, \hat{a}_j) \leq v_i\). The first inequality in the assumption ensures that such a deviation is not profitable.

If player \(i\) deviates from the action \(\hat{a}_i\) in state \(q_i^2\), then the sequence of the highest payoffs for player \(i\) after deviation is at most \(\bar{v}_i(\hat{a}_j), u_i(\hat{a}_1, \hat{a}_2), \bar{v}_i(\hat{a}_j), u_i(\hat{a}_1, \hat{a}_2), \ldots\). If \(u_i(\hat{a}_1, \hat{a}_2) > (u_i(\hat{a}_1, \hat{a}_2) + \bar{v}_i(\hat{a}_j))/2\) then player \(i\) has no incentive of such deviation with a sufficiently large discount factor. Hence, \((M_1, M_2)\) is an equilibrium in \(G^\infty\). Since \(u_i(\hat{a}_1, \hat{a}_2) > v_i\), we must have \(\hat{a}_i \neq \hat{a}_i\), and thus \(\lambda_i(q_i^1) \neq \lambda_i(q_i^2)\). This implies that, by Theorem 1, \((M_1, M_2)\) is an equilibrium in \(\Gamma^\infty\) under M-complexity.

Note that the automaton constructed in the above proof is an analogue of the Tit-for-Tat automaton illustrated in Example 1. The difficulty to implement the pure-action outcome is derived from the third condition in Theorem 1, that is, an automaton should play distinct actions in distinct states. This condition, which prohibits using strategies that repeats the same action finitely many times, restricts possibilities to construct a punishment. If the stage game possesses several actions that can be used to punish the opponent, then the criterion in the above proposition will be relaxed.

5 Discussion

If we introduce multiple complexity in the machine game of the repeated sequential-move game, there exists a Nash equilibrium of the machine game consisting of automata with more than one state. This result is different from that of Piccione and Rubinstein (1993) in which counting-states complexity is adopted. An intuition for this result is that player 2 can reduce the number of states in her automaton by employing a stage-game strategy depending on player 1’s actions, keeping the counting-states complexity unchanged.

In this paper, we have formulated the output function of an automaton as a mapping from the set of states to the set of stage-game strategies, based on the reduction of the extensive-form stage-game to its normal form. Let us discuss how multiple complexity is related to an alternative formulation in
an automaton of an extensive game introduced in Piccione and Rubinstein (1993).

Let $\Gamma$ be a two-player extensive game with perfect recall. Let $U_i$ be the set of player $i$'s information sets in $\Gamma$, $A(u_i)$ be the set of actions available at $u_i \in U_i$, and $A(U_i) = \bigcup_{u_i \in U_i} A(u_i)$. Let $E$ be the set of end-nodes in $\Gamma$. A player $i$'s automata in $\Gamma^\infty$ is defined by $(Q_i, q_i^0, \lambda_i, \tilde{\lambda}_i, \tilde{\mu}_i)$ with output function $\lambda_i : Q_i \times U_i \to A(U_i)$ and transition function $\tilde{\mu}_i : Q_i \times (U_i \cup E) \to Q_i$. Transition occurs at the time the information set is reached and before the action is taken.

When $\Gamma$ is the simple sequential-move game with perfect information, $U_1$ consists of a single information set and $U_2$ can be identified with $A_1$. Moreover, $A(U_2)$ is identified with $A_2$. Thus we can have $\tilde{\lambda}_1 : Q_1 \to A_1$, $\tilde{\lambda}_2 : Q_2 \times A_1 \to A_2$ in the above definition. For an output function $\tilde{\lambda}_2$ and $S_2 = \{A_1 \to A_2\}$, let us define $\lambda_2 : Q_2 \to S_2$ to be $\lambda_2(q_2)(a_1) = \tilde{\lambda}_2(q_2, a_1)$. Then $\tilde{\lambda}_2$ and $\lambda_2$ assign the same action for a pair $(q_2, a_1)$. With this formulation, the multiple complexity is given by

$$\text{comp}_m(M_2) = \sum_{q_2 \in Q_2} \# \{\tilde{\lambda}_2(q_2, a_1) \in A_2 | a_1 \in A_1\}.$$ 

In this representation, multiple complexity is complementary to the notion of response complexity introduced by Chatterjee and Sabourian (2000). While their response complexity counts the cardinality of the range of the mapping $\tilde{\lambda}_2(\cdot, a_1)$, our multiple complexity counts the cardinality of the range of the mapping $\tilde{\lambda}_2(q_2, \cdot)$.

Finally, it might be a challenging problem to characterize a Nash equilibrium of the machine game for a general extensive game. The identification above can be applied to a repeated game of an extensive-form stage game $\Gamma$ with perfect recall, that is, $\lambda_i : Q_i \to S_i$ with $S_i = \{s_i : U_i \to A(U_i)\}$. Thus defining $M$-complexity is possible in $\Gamma^\infty$ if $\tilde{\lambda}_i$ is defined everywhere on $Q_i \times U_i$. However, when players move more than once in a stage game, one has to be careful in formulating transition functions. A further analysis is left to future research.

References

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