Contingent trade and Monetary Mechanisms (Preliminary, and Very Incomplete)*

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Abstract

We consider contingent price environment akin to [3], and examine the relationship between inflation level and equilibrium price dispersion. It is shown that quantity does not disperse with higher inflation, but the price range shrinks as the inflation level is higher.

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1 Introduction

Trading terms tend to be contingent on the market environment. For example, consider the off-plan housing market. When showing the sample product and advertising the price, sellers often post an announcement stating ”terms and conditions subject to change”. Indeed, price and availability changes frequently on the basis of demand status. In a rational expectation manner, a good analogy is to think about that sellers post a contingent trading terms to attract buyers, and buyers choose a seller to contact. The final trading term is implemented according to the realized demand status.

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In this paper, we consider contingent trade and its monetary implications. We use a monetary environment akin to [7], where buyers have to obtain money by providing their labor-force in a centralized market (CM), and to facilitate the trade in a decentralized market (DM). This branch of literature strives to understand monetary issues with a solid micro foundation. A novelty is we impose minimum assumptions on the matching technology. Instead of specify matching technologies in a ad-hoc way, e.g., urn-ball, Cobb-Douglas, etc., we only impose some reasonable (and quite standard) assumptions to derive the results. Accordingly, the analysis is general enough to cover matching environment, instead of subjecting to a particular one.

We consider sellers commit to a price schedule where both price and production are contingent to matching status. In CM, sellers announce a set of trading terms which specifies price and production along with the corresponding matching state, and then, buyers observing these trading terms, decide which seller they would visit, and how much money to bring to purchase. Once the DM opens, buyers visit the seller, and seller select a buyer should there be more than one, and implement the trade according to the trading terms.

1.1 Literature

A literature examining real market interaction as ours is known as directed search models, e.g., [2, 3, 5]. Key to this analysis is the trade-off between ex-ante trading probability and ex-post trading price. Conventional wisdom shows that real market indeterminacy exists ([2]), where both the single price posting mechanism, as described in [1], and the reserve-auction combination as in [5], belong to the continuum set of equilibria. This result not only fits our observation of price dispersion well, but is also very intuitive. When the prices can be contingent on the matching status, sellers have more degree of freedom of setting prices than buyers have to choose their desired sellers.

However, some results changed dramatically when we consider the problem in environments with money. In [3], it was shown that indeterminacy disappears in a monetary framework akin to [7]. They obtained uniqueness of the equilibrium where sellers post p_1 in a pairwise meeting and $p_2 > p_1$ in a multilateral match. Intuitively, when buyers have to bear the cost of carry money, it presents a new channel of trade-off that helps pin down the equilibrium price.

A new feature of this paper is the seller's endogenous choice of production. The current literature mainly focus on homogeneous goods which generates a uniform utility to all buyers. This capture the indivisible nature of the trade, and focus on the role of friction in determining the allocation between buyers and sellers. However, the apart from the relationship between inflation and price dispersion ([3]), little can be said about the role of monetary mechanism without production choice being explicitly modeled.

The rest of this paper is organized as follows. Section 2 lays out the basic model and introduces key assumptions, section 3 characterize equilibrium and section 4 concludes.

2 Model

We consider contingent pricing in a monetary economy where goods are divisible. Time is discrete and goes on forever. Each period two markets in which agents trade sequentially. There is first a frictionless centralized market (CM), and a decentralized market (DM) with matching frictions. In the DM there are two types of agents, a measure s of sellers and a measure b of buyers, with $N = b/s$ as the buyer-seller ratio. In the DM sellers can produce but do not consume, while buyers want to consume but cannot produce. To generate an essential role of money, there is no record keeping of trades in the DM, so credit cannot be extended ([6]).

Preferences between the CM and DM are separable, and linear in CM labor. The period utility functions of buyers and sellers are

$$\mathscr{U}^b(q,x,l) = u(q) + U(x) - l \text{ and } \mathscr{U}^s(q,x,l) = -q + U(x) - l,$$

where q is the DM good, x is the CM good, and l is labor. One unit of produces one unit of x in the CM, so the real wag is one. In the DM, the marginal cost is one as well. (these are easily relaxed). Constraints $x \geqslant 0$, $q \geqslant 0$ and $l \in [0,1]$ are assumed not to bind, as can be guaranteed in the usual way. Also, U and u are twice continuously differentiable, where $U', u' > 0$, $u'' < 0$ and $u''' > 0$ with at least one of the weak inequalities strict. Also, assume $u(0) = 0$ and[1] $u'(0) > 1$. Agents discount between the CM and DM according to $\beta = 1/(1+r)$, with $r > 0$.

Goods q and x are non-storable. There is a storable asset called fiat money, with supply per buyer M. Assume $M_{+1} = (1+\pi)M$, where subscript +1 indicates next period. Changes in M are accomplished by lump sum transfers if $\pi > 0$ or taxes if $\pi < 0$, but the results also apply if instead government uses new money to buy CM

[1] To use L'hospital, we require $\frac{u'(0)}{c'(0)}\exists$.

goods. For convenience, only buyers pay taxes or get transfers. Let ϕ be the price of money in terms of CM numeraire x. The focus here is on stationary outcomes, where all real variables are constant, including real balances $z = \phi M$. Hence, inflation is $\phi/\phi_{+1} = 1 + \pi$. We also use the Fisher equation, $1 + i = (1 + \pi)(1 + r)$. As standard, i is the nominal return an agent requires in the next CM to give up a dollar in this CM.

The sequence of events is as follows. First, the CM opens and buyers receive a lump sum money injection from the central bank which is common knowledge. Then sellers simultaneously advertise terms of trade contingent on realized demand for the coming DM. The terms of trade consist of a price schedule and the production commitment of the search good: (p_k, q_k) denotes the price and quantity where state k is realized. For simplicity we focus on $k = 1, 2$ throughout the paper. After observing all the price offers, buyers simultaneously choose both which seller to visit and how much money to carry to the coming DM. Money is acquired by producing and selling some general good in exchange for money on the CM. Finally, the CM closes and the DM opens, where buyers visit the seller of their choosing. If a buyer is alone, he pays the pairwise price and consumes the good. If more than one buyers visit the same seller and can afford the multilateral price, one buyer is chosen randomly, pays the multilateral price and consumes the good. If only one buyer can afford the multilateral price then he wins the good and pays the multilateral price. If none of the buyers can afford to pay the high price, no trade takes place. The sequence of events is represented on Figure 1.

The state of an agent in the baseline model is real balances in terms of CM numeraire, $z = \phi m$. The value functions in the CM and DM are $W(z)$ and $V(z)$.

2.1 The CM Problem

The CM problem for an agent of type $j = b, s$ is:

$$W_j(z) = \max_{\langle x,l,\hat{z}\rangle} \left\{ U(x) - l + \beta V_j(\hat{z}) \right\} \text{ s.t. } z + l = x + (1 + \pi)\hat{z},$$

where $\hat{z}$ is real money carried to the next DM. FOCs are:

$$U'(x) - 1 = 0 \tag{1}$$

$$z + l - (1 + \pi)\hat{z} - x = 0 \tag{2}$$

$$-(1 + \pi)U'(x) + \beta V_j'(\hat{z}) \leqslant 0, \; = 0 \text{ if } \hat{z} > 0. \tag{3}$$

From (1), $x = x^*$ where x solves $U'(x^*) = 1$. The envelope condition is $W_j'(z) = 1$. From (3), $\hat{z}$ does not depend on z. As standard, $\hat{z} > 0$ for buyers in monetary equilibrium, while $\hat{z} = 0$ for sellers, since they have no use for liquidity in the DM.

2.2 The DM Problem

In the DM, contingent on the trading terms, the buyers' surplus and sellers' surplus are, respectively:

$$S_b = u(q_k) + W_b(z_b - p_k) - W_b(z_b) = u(q_k) - p_k, \text{ and}$$

$$S_s = -q_k + W_s(z_s + p_k) - W_s(z_s) = p_k - q_k;$$

where $k = 1, 2$ is the realized price-production pair.

To guarantee gains from trade are positive but finite, assume $u'(0) > 1$ and $\exists \bar{q} > 0$ such that $u(\bar{q}) = \bar{q}$. Define the unconstrained efficient quantity $\tilde{q}$ by $u'(\tilde{q}) = 1$. We also assume $u(\cdot)$ has weekly deacreasing elasticity, i.e., denote $\sigma_u \equiv \frac{u'(q)}{qu(q)}$, as the production elasticity, then $\sigma_u'(q) \leqslant 0$ for any $q > 0$. As standard, we will focus on trading terms that subject to incentive feasible constraint (IFC):

$$p_k \in (q_k, u(q_k)) \text{ for } k = 1, 2. \tag{4}$$

Sellers carry no money to the DM in equilibrium. Still, we write:

$$V_s(z) = W_s(z) + \sum_k \alpha_{s_k}(n, \cdot)\left(p_k - q_k\right),$$

where $\alpha(n, \cdot)$ is the probability of the corresponding trading event. It is twice differentiable w.r.t. local market tightness[2], n, and some other factors which we will detail below. Also, we will denote $\delta_j(n, \cdot)$ as the overall probability of trade, i.e., $\delta_j(n, \cdot) = a_{j_1} + a_{j_2}$ for $j = b, s$. We will also use the notation $\mathbb{E}_k^j(n, \cdot)S_{j_k} \equiv \sum_k\{\alpha_{j_k}(n, \cdot)S_{j_k}\}$ for seller's expected payoff interchangeably.

The buyer's value yields: $V_b(z) = W_b(z) + \mathbb{E}_k^b(n, \cdot)\{[u(q_k) - p_k]\}$, and after simplifying we can write:

$$W_b(z) = \bar{W} + \beta \max_{\langle \hat{z}, q_k \rangle}\{-i\hat{z} + \mathbb{E}_k^b(n, \cdot)\left[u(q_k) - p_k\right]\},$$

where i comes from the Fisher equation, and $\bar{W}$ is a constant irrelevant for the choice of $\hat{z}$. We will focus on seller's and buyer's objective functions: $\hat{W}_s \equiv \mathbb{E}_k^s(n, \cdot)\left[u(q_k) - p_k\right]$ and $\hat{W}_b(\hat{z}) \equiv -i\hat{z} + \mathbb{E}_k^b(n, \cdot)\left[u(q_k) - p_k\right]$. Without loss of generality, we assume $p_2 \geqslant p_1$ hereafter.

[2] n is defined as the buyer-seller ratio in each sub-market. In symmetric equilibrium, it coincides from one to another and converges to the aggregate market tightness, N.

A central trade-off for buyers is how much money to bring. In equilibrium, when $i > 0$, it is clear only two money holdings are possible, $\hat{z}_1 = p_1$ and $\hat{z}_2 = p_2$. This is because no one would bring costly money for no benefit. Call $k \in \{1, 2\}$ buyer's type. From now on, we will write $\hat{z}_k$ and p_k interchangeably, i.e., by posting the price, sellers also post their liquidity requirement. Denote θ as the symmetric mixed strategy profile, i.e., probability that buyer choose to carry z_2, and call it *monetary strategy*.

As type 2 buyer could afford both terms of trade, while type 1 buyer could afford only z_1, trading probability for type 2 trade is dependent on monetary strategy, but in the type 1 trade it is irrelevant[3], i.e., $\alpha_{j_1} = \alpha_{j_1}(n)$ while $\alpha_{j_2} = \alpha_{j_2}(n, \theta)$.

2.3 Matching Probabilities

When setting their trading terms to attract buyers, sellers trade off between ex-ante probability and ex-post profit. Therefore, it is critical to verify some properties about the matching probabilities. In this section, we carefully lay out some assumptions we impose on the matching functions and then, examine the corresponding properties they imply.

First of all, we assume[4] $\partial \alpha_{b_k}(n, \cdot)/\partial n < 0$, i.e., more intensive market impose congestion externality to each buyer in terms of probability of trade; it is also natual to assume the overall trading probability of a seller is increasing w.r.t. n when all buyers bring type 2 amount of money, i.e., $\alpha'_{s_1}(n) + \alpha_{s_{2(1)}}(n, 1) > 0$. (Externality, A1.)

Then, for any $n > 0$, we assume sellers' matching probability to reach a type 2 trade is increasing and concave w.r.t. θ, i.e., $\alpha_{s_{2(2)}} > 0$ and $\alpha_{s_{2(22)}} < 0$ (Concavity, A2.)[5]. This implies a congestion effect on monetary strategy for buyers, i.e., higher θ lowers the trading probability of each buyer. Formally:

Lemma 1 *Under A2.,* $\alpha_{b_{2(2)}} < 0$.

Proof. See appendix. ■

[3]When $z_1 = z_2$, it corresponds to the scenario where $\theta = 1$. We allow this possibility and will characterize corresponding equilibrium endogenously.

[4]Somewhat surprisingly, the urn-ball matching technology with single-multiple matching pricing does not necessarily satisfy this assumption, i.e., for some n and θ, $\alpha_{b_2}(n, \theta)$ is increasing w.r.t. n. However, as we show below, this actually does not affect our main results.

[5]We omit arguments whenever it does not create confusion. On the other hand, notice that potentially n and θ can be connected through some implicit function in equilibrium. To avoid confusion, we will use $\alpha_{s_2(1)}, \alpha_{s_2(2)}$, etc., to represent partial derivatives, instead of more commonly used $\frac{\partial \alpha_{s_2}}{\partial n}, \frac{\partial \alpha_{s_2}}{\partial \theta}$, etc.

Furthermore, notice that trading proceeds in a pairwise manner[6], so the aggregate transaction volume of buyers and sellers for each type of trade should coincide with each other (Market Clearing, A3.). In particular, we assume following condition about matching probabilities:

$$\alpha_{s_1} = n\alpha_{b_1};\ \alpha_{s_2} = n\theta\alpha_{b_2}. \tag{5}$$

Finally, we assume a type 2 buyer's matching probability is invariant of the presence of any type 1 buyer (Invariant, A4.). This means whenever different types of buyers get involved in a multiple match, a type 1 buyer always gets the deal with probability 0. A necessary condition is for a type 2 buyer who enters the market, the overall trading probability is only dependent on "effective competitors" on the market, θn. I.e., $\alpha_{b_1}(n) + \alpha_{b_2}(n, \theta) = \delta_b(\theta n)$, where $\pi(\cdot)$ is type 2 buyer's overall trading probability.

These assumptions enables a more detailed discussion about the matching probabilities. First of all, it is important to verify the monotonic properties of the matching probabilities.

Lemma 2 *Seller's matching probabilities, $\alpha_{s_k}(\cdot)$ are increasing/decreasing w.r.t. market tightness, n iff. buyer's corresponding elasticity is less/greater than* 1, *respectively. In particular:*

$$\alpha'_{s_1}, \alpha_{s_2(1)} \gtreqless 0 \text{ iff. } \sigma_{b_1}, \sigma_{b_2(1)} \lesseqgtr 1, \tag{6}$$

where $\sigma_{b_1} \equiv \left|\frac{n\alpha'_{b_1}}{\alpha_{b_1}}\right|$ *and* $\sigma_{b_2(1)} \equiv \left|\frac{n\alpha_{b_2(1)}}{\alpha_{b_2}}\right|$ *are elasticities of matching probabilities.*

Proof. Take the derivatives w.r.t. n for the equations in (5) and rearrange terms, we obtain this result. ■

In turn, obtain uniqueness of the equilibrium, sometimes it is important to verify signs of elasticities. The following lemma shows one about cross elasticity about monetary strategy.

Lemma 3 *Under A2., with any given market tightness n, the buyer-seller cross elasticity for type 2 trade w.r.t. θ, $\sigma_{c_2(2)}(n, \theta) \equiv \left|\frac{\alpha_{s_2}\alpha_{b_2(2)}}{\alpha_{s_{2(2)}}\alpha_{b_2}}\right|$, is increasing. I.e., $\frac{\partial\sigma_{c_2(2)}}{\partial\theta} > 0$.*

Proof. See appendix. ■

Finally, we compare the effect of different forces to the matching probability to sellers and buyers, respectively. Intuitively, larger n increases seller's probability

[6] Although *meeting* could be multilateral.

to reach a deal while higher θ represents a greater likelihood that seller is able to charge a higher price, while both discourage buyers from entering the market due to congestion effect. Some preliminary results about trade offs in between are presented in the following lemma.

Lemma 4 *Under A3., $\frac{\alpha_{s_2(1)}}{\alpha_{s_2(2)}} \gtreqless \frac{\alpha_{b_2(1)}}{\alpha_{b_2(2)}}$ iff. $\frac{\theta}{n} \gtreqless \frac{\alpha_{b_2(1)}}{\alpha_{b_2(2)}}$. In addition, under A4, $\frac{\alpha_{s_2(1)}}{\alpha_{s_2(2)}} - \frac{\alpha_{b_2(1)}}{\alpha_{b_2(2)}} > 0$ for any n and θ.*

Proof. See appendix. ■

We denote $\rho(n, \theta^*) \equiv \alpha_{s_2(2)} \left[\frac{\alpha_{s_2(1)}}{\alpha_{s_2(2)}} - \frac{\alpha_{b_2(1)}}{\alpha_{b_2(2)}} \right]$ as a function of the local market environment parameters (n, θ^*). It represents the net marginal effect of the monetary strategy to seller's trading probability of second type trade. According to calculations in Lemma 4's proof, we further simplify the form: $\rho(n, \theta^*) = \frac{\alpha_{b_2} \alpha'_{b_1}}{\pi'(n\theta)} > 0$.

Now we do a simple check on urn-ball match and pairwise-multiple match pricing strategy, as described in [3]. Agents' matching probabilities are, $\alpha_{s_1}(n) = ne^{-n}$, $\alpha_{s_2}(n, \theta) = 1 - n\theta e^{-n} - e^{-n\theta}$, $\alpha_{b_1}(n) = e^{-n}$, and $\alpha_{b_2}(n, \theta) = \frac{1-e^{-n\theta}}{n\theta} - e^{-n}$, respectively. It is easy to check that these matching technologies satisfy all assumptions proposed here, A1.-A4. After calculations we have: $\alpha'_{s_1}(n) \gtreqless 0 \Leftrightarrow n \lesseqgtr 1$, while $\alpha'_{s_2(1)}(n, \theta) > 0$ always holds. In terms of Lemma 4, we have: $\rho > 0$ always holds for any $n > 0$ and $\theta \in (0, 1)$ with urn-ball technology.

2.4 Monetary Strategy

When thinking about how much money to bring, the buyer trades off gains of type 2 trade with the extra interest cost. In particular, buyers' values of carrying z_1 and z_2 yield, respectively:

$$\hat{W}_b(z_1) = -iz_1 + \alpha_{b_1}(n)[u(q_1) - z_1], \text{ and } \hat{W}_b(z_2) = -iz_2 + \mathbb{E}^b_k(n, \theta)\left[u(q_k) - z_k\right];$$

Given the local market structure (z_1, q_1), (z_2, q_2), n, and other buyers' monetary strategy $\bar{\theta}$, the buyer's best response, θ_{BR}, yields[7]:

$$\theta_{BR} = \begin{cases} = 0, & \text{if } \hat{W}(z_1, \bar{\theta}) < \hat{W}(z_2, \bar{\theta}); \\ \in (0, 1), & \text{if } \hat{W}(z_1, \bar{\theta}) = \hat{W}(z_2, \bar{\theta}); \\ = 1, & \text{if } \hat{W}(z_1, \bar{\theta}) > \hat{W}(z_2, \bar{\theta}). \end{cases}$$

[7] As there is a large number of buyers, each individual buyer's monetary strategy would not affect the aggregate money holding, so the matching probability is unchanged after the buyer joins each group.

A mixed strategy equilibrium, θ^e, exists conditional on two money holdings yielding identical payoff, i.e., $\hat{W}(z_1, \bar{\theta}) = \hat{W}(z_2, \theta)$. This yields:

$$\alpha_{b_2}(n, \theta^e)\left[u(q_2) - z_2\right] = i(z_2 - z_1), \tag{7}$$

i.e., expected payoff from type 2 transaction offsets cost of bringing money. Intuitively, this represents the fact that type 2 trade should be equally attractive for buyer to type 1 trade in a competitive environment. We call this *competitive monetary constraint(CMC)*. In equilibrium, buyer's monetary strategy entails the following proposition.

Proposition 1 *The unique equilibrium monetary strategy, θ^e entails:*

$$\theta^e \begin{cases} = 0, & \textit{if } i(z_2 - z_1) \geqslant \alpha_{b_2}(n, 0)\left[u(q_2) - z_2\right]; & \textit{(I)} \\ s.t.\ \alpha_{b_2}(n, \theta^e) = \Psi, & \textit{if } \Psi \in (\alpha_{b_2}(n, 1), \alpha_{b_2}(n, 0)); & \textit{(II)} \\ = 1, & \textit{if } i(z_2 - z_1) \leqslant \alpha_{b_2}(n, 1)\left[u(q_2) - z_2\right], & \textit{(III)} \end{cases} \tag{8}$$

where $\Psi(z_1, z_2, q_2) = \frac{i(z_2 - z_1)}{\left[u(q_2) - z_2\right]}$ *is defined as the* shadow probabilty *of the type 2 transaction.*

Proof. For case I, as $\hat{W}(z_1) < \hat{W}(z_2)$ for any positive θ, $\theta_{BR} = 0$ for any $\bar{\theta} \in [0, 1]$. The unique symmetric equilibrium is $\theta^* = 0$. Similarly, for case III, as $\hat{W}(z_1) > \hat{W}(z_2)$ for any $\theta < 1$, $\theta_{BR} = 1$ for any $\bar{\theta} \in [0, 1]$, and $\theta^* = 1$. In case II, equalize $\hat{W}(z_1)$ and $\hat{W}(z_2)$, we obtain the equilibrium condition. Because that $\alpha_{b_2}(n, \theta)$ is continuously differentiable, and $\alpha_{b_{2(2)}}(n, \theta) < 0$, according to mean value theorem, there exist a unique θ^* s.t. $\alpha_{b_2}(n, \theta^e) = \frac{i(z_2 - z_1)}{[u(q_2) - z_2]}$. ■

For the seller, a natural option for the type 2 contract is $(z_2, q_2) = (z_1, u^{-1}(z_1))$. This is because when $z_2 = z_1$, there is no extra cost associated with the opportunity type 2 trade and therefore $\theta^* = 1$. Sellers can extract surplus of without suffering from a lower monetary strategy. The minmax profit of type 2 trade is: $\alpha_{s_2}(n, 1)\left[z_1 - u^{-1}(z_1)\right] > 0$. This imposes an incentive constraint on type 2 trade:

$$\alpha_{s_2}(n, \theta)\left[z_2 - q_2\right] \geqslant \alpha_{s_2}(n, 1)\left[z_1 - u^{-1}(z_1)\right]. \tag{9}$$

It is now clear that $q_2 \geqslant u^{-1}(z_1)$, because it ends up with no trade and zero surplus otherwise. For similar reason, $\theta^e = 0$ is eliminated from seller's incentive feasible set. We re-write *CMC* as:

$$z_2^e = \frac{iz_1 + \alpha_{b_2} u(q_2)}{i + \alpha_{b_2}}, \tag{10}$$

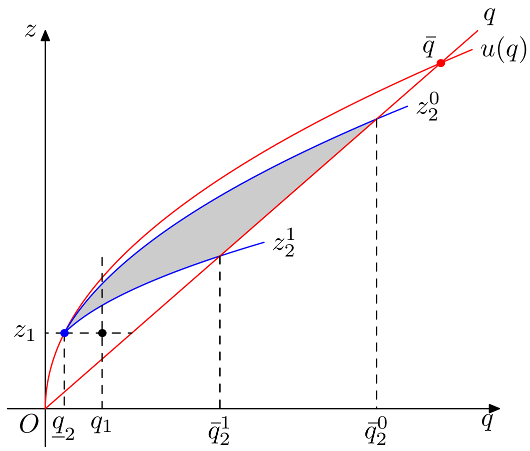

Figure 1: Feasible Trade

for any $n > 0, \theta^e \in (0,1]$ and $q_2 \geqslant u^{-1}(z_1)$.

Figure 1 shows the feasible contract space of the model. The black point represents terms of the type 1 trade, while the feasible contract space of type 2 trade is represented by the shady area. The blue curves, z_2^0 and z_2^1, represent the boundary of the type 2 trade. When we shift z_1 up and down, the feasible space of type 2 shrinks and extends correspondingly.

3 Equilibrium

Now we introduce the equilibrium concept we will characterize for the rest of this paper.

Definition 1 *A monetary competitive search equilibrium is a collection of trading terms*[8], $\{(z_1, q_1), (z_2, q_2)\}$, *and buyers' monetary strategy* θ, *s.t.:*

- *Based on* $\{(z_1, q_1), (z_2, q_2)\}$, *buyer's randomize among sellers in the sub-game equilibrium*
- θ *is the symmetric equilibrium of buyer's monetary strategy;*
- $\{(z_1, q_1), (z_2, q_2)\}$ *is the symmetric sub-game perfect Nash equilibrium price-production pair of sellers; and*

[8]Focusing on stationary equilibrium, we unify buyer's money holding as z_k.

- *Stationary:* π *is constant, and* $\hat{z}_k = z_k$ *every period.*

To characterize the equilibrium in a competitive search manner, suppose a deviating seller posts (p_1, q_1) and (p_2, q_2), and induce a "local BR" of n and θ, while the rest of the market post $(\bar{p}_1, \bar{q}_1)$ and $(\bar{p}_2, \bar{q}_2)$ and provide a "market utility[9]" of $\bar{U}$. In equilibrium, deviating seller matches with the non-deviating ones and provide a market utility:

$$\hat{W}_b(z_1) = \bar{U}, \tag{11}$$

we call this *Competitive Search Constraint (CSC).*

In symmetric equilibrium, $n = N$ for all sellers. We will ommit the arguments of the matching functions in equilibrium, i.e., $\alpha_{j_1} \equiv \alpha_{j_1}(N)$ and $\alpha_{j_2} \equiv \alpha_{j_2}(N, \theta^e)$.

Sellers' problem yields[10]:

$$\max_{\langle z_1, z_2, q_1, q_2, n, \theta \rangle} \{\hat{W}_s = \alpha_{s_1}[z_1 - q_1] + \alpha_{s_2}[z_2 - q_2]\}$$

subject to: competitive constraints, (10), (11), incentive constraints, (4),(9), and feasibility constraint[11]:

$$0 \leqslant q_1; z_1 \leqslant u(q_2) \text{ and } \theta^* \in (0, 1].$$

Construct Largrangian[12]:

$$\begin{aligned}\mathscr{L} = {} & \alpha_{s_1}[z_1 - q_1] + \alpha_{s_2}\left[z_2^e - q_2\right] + \lambda\{-iz_1 + \alpha_{b_1}[u(q_1) - z_1] - \bar{U}\} \\ & + \nu_1[z_1 - q_1] + \nu_2[u(q_2) - z_1] + \nu_3(1 - \theta^e),\end{aligned}$$

with $\lambda > 0, \nu_l \geqslant 0$ and Kuhn-Tucker conditions:

$$\nu_1[z_1 - q_1] = \nu_2[u(q_2) - z_1] = \nu_3(1 - \theta^e)$$

[9]The "market utility" approach we adopt here, which is a standard tool in directed search literature, follows [8]. It has been justified as an effective approximation of the directed search construct when the market is large. See [5], [1] and [4] for details.

[10]As we showed before, z_2^e is defined by a function of (q_2, z_1, θ^*), so we will no longer take it as a choice variable.

[11]Clearly, CSC with $\bar{U} > 0$ implies $q_1 > 0$ and $u(q_1) - z_1 > 0$, so we omit Kuhn-Tucker condition $u(q_1) - z_1 \geqslant 0$ and $q_1 > 0$ here.

[12]For simplicity, the incentive constraint (9) is not explicitly presented in the construction of Lagrangian here. However, as $(z_1, u^{-1}(z_1))$ is included in the feasible type 2 trading term, constraint $z_1 \leqslant u(q_2)$ and $z_2 = z_2^e$ suffices to describe it.

Unlike Kuhn-Tucker multipliers ν_l , the "market utility" multiplier λ, is strictly positive, i.e., the constraint, (11), is always binding. Intuitively, multiplier λ represents the marginal effect of attracting buyers with an increment of $\bar{U}$.

FOCs represent underlying trade-offs when sellers post the trading terms. On the price margin, expand $\frac{\partial \mathscr{L}}{\partial z_1} = 0$ yields:

$$\alpha_{s_1} + \frac{i\alpha_{s_2}}{i + \alpha_{b_2}} + \nu_1 - \nu_2 = \lambda\left(i + \alpha_{b_1}\right), \tag{12}$$

with a marginal increment of z_1, RHS is the cost, i.e., the disutility involved that repels the buyer. Focusing on interior equilibrium ($\nu_1 = \nu_2 = 0$), there are two profits. One is the direct one from type 1 trade represented by α_{s_1}. The other one, $\frac{i\alpha_{s_2}}{i+\alpha_{b_2}}$, represents the easiness of facilitating the type 2 trade. Intuitively, with an increment in z_1, the extra liquidity requirement to facilitate a type 2 trade, $(z_2 - z_1)$ is reduced. In equilibrium, cost and profit offset each other.

Now consider the production margin. Expand $\frac{\partial \mathscr{L}}{\partial q_1} = 0$ yield:

$$\lambda\alpha_{b_1} u'(q_1) = \alpha_{s_1} + \nu_1,$$

when choosing the production level, sellers trade off between the effect to attract more buyers (LHS) and the cost of production (RHS). Combine this with (12) yields:

$$q_1^* : u'(q_1) - 1 = \frac{i}{\alpha_{b_1}} + \frac{1}{\lambda\alpha_{b_1}}\left(\nu_2 - \frac{i\alpha_{s_2}}{i + \alpha_{b_2}}\right). \tag{13}$$

The RHS represents the "liquidity premium" of the monetary environment. It consists of the liquidity cost (normalized by trading probability) and the effect on the easiness of type 2 trade.

Then, expand $\frac{\partial \mathscr{L}}{\partial q_2} = 0$ yields:

$$q_2^* = \max\left\{q_2^e, u^{-1}(z_1)\right\}\ ; \tag{14}$$

where $q_2^e(\theta^e) : u'(q_2^e) - 1 = \frac{i}{\alpha_{b_2}}$.

Under a certain market tightness, seller's production in a type 2 trade is dependent on two key factors, the equilibrium monetary strategy, θ^e, and the type 1 price, z_1.

Next, the seller also chooses equilibrium monetary strategy, θ^e, and local market tightness, n. $\frac{\partial \mathscr{L}}{\partial \theta^e} = 0$ and $\frac{\partial \mathscr{L}}{\partial n} = 0$ yield, respectively:

$$\alpha_{s_2(2)}\left(z_2^e - q_2\right) + \frac{i\alpha_{b_2(2)}\alpha_{s_2}\left[u(q_2) - z_1\right]}{\left(i + \alpha_{b_2}\right)^2} - \nu_3 = 0; \tag{15}$$

$$\alpha'_{s_1}\left(z_1 - q_1\right) + \lambda\left\{\alpha'_{b_1}\left[u\left(q_1\right) - z_1\right]\right\} + \alpha_{s_2(1)}\left(z_2^e - q_2\right) + \frac{i\alpha_{b_2(1)}\alpha_{s_2}\left[u(q_2) - z_1\right]}{\left(i + \alpha_{b_2}\right)^2} = 0. \tag{16}$$

When competing among sellers, both on the margins of θ^e and n, the seller trades off between greater probability and the congestion effect on the buyers' side that deters more buyer from entering. In equilibrium, these two effects offsets each other. For any $\theta^e \in (0,1)$, $\alpha_{s_2(2)}\left(z_2^e - q_2\right) = -\frac{i\alpha_{b_2(2)}\alpha_{s_2}\left[u(q_2)-z_1\right]}{\left(i+\alpha_{b_2}\right)^2}$, so we can further simplify (16) as:

$$\alpha'_{s_1}\left[z_1 - c(q_1)\right] + \lambda\left\{\alpha'_{b_1}\left[u(q_1) - z_1\right]\right\} + \left[z_2^e - c(q_2)\right]\rho(N, \theta^e) = 0. \tag{17}$$

4 Friedman's Rule ($i = 0$)

In this section, we consider the special case where $i = 0$, commonly referred to as "Friedman's Rule". An observation on the buyer's market utility constraint yields the following lemma.

Lemma 5 *Sellers post $z_2 = u(q_2)$ in all equilibria.*

Proof. Substitute $i = 0$ in competitive constraint, (7), the result follows straightforwardly. ■

At Friedman's Rule, there is no cost associated with extra liquidity. Therefore, in competitive equilibria, type 2 sellers extract all surplus and buyers have no gain in type 2 trade.

In the context of existing directed search literature, this is a novel result, and it is in sharp contrast to existing ones. In particular, [2] shows that real indeterminacy exist when sellers can post a menu of price. Equilibria not only include single price posting ($z_1 = z_2 = z$) as in [1] and the reserve- auction combination ($z_1 = q$ and $z_2 = u(q)$) as in [5], but a continuum set in between as well, where none of them are payment equivalent with each other. Here, with a monetary environment, we find that the high price extracts all surplus and indeterminacy disappears. This is not associated with introducing to the model the cost of carrying money, as $i = 0$. Rather, it is because the buyers decides how much money to bring *before* entering the market, and thus, the equilibrium must maintain *ex-ante* payment equivalency of different money holdings. As a result, any *ex-post* trading opportunism disappears here.

As trade-offs in buyer's money holding eliminates, buyer's monetary strategy now appears to be payment-irrelevant.

Lemma 6 *A continuum of equilibrium monetary strategies exists:* $\{\theta^* : \theta^* \in [0, 1]\}$.

Now we consider production choice in the environment. It appears that productions in both type 1 and type 2 trade render social efficiency.

Proposition 2 *Sellers post* $q_1 = q_2 = q^*$ *in all equilibria.*

Proof. First, according to condition (14), $q_2 = q^*$.

Then, consider the choice of q_1. Notice that $z_1 = u(q_1)$ is not a possibility as that yields $\bar{U} = 0$. In turn, as $q_2 = q^* \geqslant q_1$, we have: $u(q_2) > z_1$, implying $\nu_4 = 0$. According to condition (13), $q_1 = q^*$. ■

With a competitive environment, Friedman's rule implies efficient production. This result is consistant with monetary models, e.g., [9], [7], etc. We show this wisdom is robust in the competitive monetary environment with trading contingency.

We now characterize the equilibrium price of type 1 trade. There are two possible forms of pricing schemes, reserve price ($z_1 = q_1$) or sale price ($z_1 \in (q_1, u(q_1))$). The following proposition shows under which condition each appears in equilibrium.

Proposition 3 *If* $\frac{\alpha_{s_2(1)}}{\alpha_{s_1}} \geqslant -\frac{\alpha'_{b_1}}{\alpha_{b_1}}$, *then* $z_1 = q^*$ *in equilibrium; otherwise* $z_1 \in (q^*, u(q^*))$, *and it is given by (19).*

Proof. Recall (16) and substitute $q_1 = q_2 = q^*$:

$$z_1 = \frac{\alpha'_{s_1} q^* - \lambda \alpha'_{b_1} u\left(q^*\right) - \alpha_{s_2(1)} \left[u\left(q^*\right) - q^*\right]}{\alpha'_{s_1} - \lambda \alpha'_{b_1}}, \tag{18}$$

where according to (12), $\lambda = \frac{\alpha_{s_1} - \nu_2}{\alpha_{b_1}}$.

Now we consider two cases, $\nu_2 = 0, z_1 \in (q_1, u(q_1))$ and $\nu_2 > 0, z_1 = q_1$. The first case yields:

$$z_1^e = \frac{\left(-\frac{\alpha_{s_1}\alpha'_{b_1}}{\alpha_{b_1}} - \alpha_{s_2(1)}\right)u(q^*) + \left(\alpha'_{s_1} + \alpha_{s_2(1)}\right)q^*}{\alpha'_{s_1} - \frac{\alpha_{s_1}\alpha'_{b_1}}{\alpha_{b_1}}}. \tag{19}$$

Notice $\left(\alpha'_{s_1} + \alpha_{s_2(1)}\right) > 0$ and $\alpha'_{s_1} - \frac{\alpha_{s_1}\alpha'_{b_1}}{\alpha_{b_1}} > 0$ according to A1. Therefore, z_1^e is strictly convex combination of $u(q^*)$ and q^*, i.e., $z_1^e \in (q_1, u(q_1))$, iff. $\left(-\frac{\alpha_{s_1}\alpha'_{b_1}}{\alpha_{b_1}} - \alpha_{s_2(1)}\right) > 0$. Otherwise, $z_1^e < q^*$ and incentive constraint $z_1 = q_1^*$ holds. Simplifying the condition yields the property stated. ■

If the matching technology for type 2 trade is sufficiently sensitive, equilibrium pricing scheme renders reserve-auction combination which coincides with the one analyzed in [5]. Otherwise, the equilibrium price is higher than q^*.

Notably, as indeterminacy of monetary strategy exists according to lemma 6, it appears in price z_1 as well. As $\alpha_{s_1(1)} > 0$ and $\alpha_{s_1(11)} < 0$, z_1^e is increasing w.r.t. θ^*. This indeterminacy is different to the one presented in [2]. In particular, [2] showed *real* indeterminacy in the sense that a continumm of equilibrium price exists, each is payment inequivalent to another. This indeterminacy is subject to urn-ball match technology, and it disappears in either a large market or a convex payment structure ([2]). Ours is one associated with strategic money holding. It appears only when Friedman's rule is implimented and therefore monetary strategy payment irrelevant.

To conclude this section, with Friedman's rule, efficient production is implemented in equilibrium. Equilibrium price shows indeterminancy because of payment irrelevancy of monetary strategy. Instead of having to subject to a particular matching technology, this form of indeterminacy subjects to the monetary environment, and only appears when Friedman's rule is implemented.

5 General Case ($i > 0$)

In this section, we move on to characterize equilibrium under more general case where $i > 0$. Key is to understand the connection between type 1 trade and the type 2. Intuitively, type 1 price imposes a lower bound to the type 2 production and price, and also, gains in type 1 trade presents a benchmark when seller is choosing the monetary strategy. Simplifying (15) yields the following proposition.

Proposition 4 *With interior solution of equilibrium monetary strategy, i.e., if $\theta^* \in (0,1)$, then,*

$$\frac{z_2 - q_2}{u(q_2) - q_2} = \left(\frac{\sigma}{\frac{i}{\alpha_{b_2}} + 1 + \sigma} \right). \tag{20}$$

Furthermore, denote $G(i, \theta^e) \equiv \left[\frac{\alpha_{b_2}}{i+\alpha_{b_2}}\right]\left[1 - \frac{i\sigma}{i+\alpha_{b_2}}\right]$. For any $z_1 > 0$ and $q_2 > u^{-1}(z_1)$:

$$\theta^* \begin{cases} = 1, & \text{if } \frac{q_2 - z_1}{u(q_2)-z_1} \leqslant G(i,1); \\ \in (0,1) \text{ s.t. } \frac{q_2-z_1}{u(q_2)-z_1} = G(i,\theta^e), & \text{otherwise.} \end{cases} \tag{21}$$

Proof. See appendix. ■

We will write $G(i, \theta^e)$ and $G_i(\theta^e)$ interchangeably. The first result specifies seller's share of surplus in a type 2 trade. It is commonly understood that in competitive search environment, sellers and buyers share surplus according to elasticity of the trading probabilities, respectively. In here, apart from (normalized) liquidity cost, $\frac{i}{\alpha_{b_2}}$, which buyers bare to facilitate the trade, the shares between buyer and seller are propotional to their elasticity respectively. The second part reveals the connection between q_2 and θ^e on the equilibrium path based on the type 1 price. Intuitively, $G(i, \theta^e)$ represents seller's incentive to provide extra production. Among its components, $\left[\frac{\alpha_{b_2}}{i+\alpha_{b_2}}\right]$ is the net profit created by each production. While $\frac{i\sigma}{i+\alpha_{b_2}}$ represents the burden created by the liquidity cost to the equilibrium monetary strategy, subtracting that from the unit, $\left[1 - \frac{i\sigma}{i+\alpha_{b_2}}\right]$ represents the net gain for seller. In equilibrium, $G(i, \theta^e)$ coincides with the cost-effective ratio of extra production, $\frac{q_2-z_1}{u(q_2)-z_1}$. Observe that $\lim_{q_2 \to [u^{-1}(z_1)]^+} \frac{q_2-z_1}{u(q_2)-z_1} = -\infty$, necessary condition for $\theta^* = 1$ is that $G(i, 1)$ is lower bounded. In light of that, urn-ball match does not have boundary solution for monetary strategy unless $u(q_2) = z_1$, because $G(i, 1) = -\infty$ for any positive i.

The production margin of type 2 trade, q_2, affects two lines of trade off for the profit maximizing seller. On the extensive margin, it maximizes the net profit of producing, represented by condition (14); on the intensive margin, it also imposes a distributional effect on the extra surplus created, represented by proposition **??**. In equilibrium, two trades off coincides with each other. Combining proposition 4 with condition (14), we are able to establish connection between z_1 and $\left(\theta^*, q_2^*\right)$.

Proposition 5 *Given z_1, equilibrium of $\left(\theta^*, q_2^*\right)$ entails:*

- *If* $\sigma_u(q_2^e(1)) \leqslant 1 - \frac{i\sigma_{c_2(2)}(1)}{i+\alpha_{b_2}(N,1)}$, *then for all* $z_1 > 0$, $\theta^* = 1$, *and*

$$q_2^* = \begin{cases} q_2^e(1), & \textit{for any } z_1 \in \left[0, u\left(q_2^e(1)\right)\right]; \\ u^{-1}(z_1) & \textit{for } z_1 > u\left(q_2^e(1)\right); \end{cases}$$

- *If* $\sigma_u(q_2^e(1)) > 1 - \frac{i\sigma_{c_2(2)}(1)}{i+\alpha_{b_2}(N,1)}$, *then* $\exists! \tilde{\theta} \in (0,1)$ *s.t.* $z_1^e(\tilde{\theta}) = 0$. *Furthermore, equilibrium entails:*

$$(\theta^*, q_2^*) \begin{cases} = \left(\theta, q_2^e(\theta)\right), & \textit{for any } z_1 = z_1^e(\theta) \textit{ where } \theta \in \left[\tilde{\theta}, 1\right]; \\ = \left(1, q_2^e(1)\right), & \textit{for } z_1 \in (z_1^e(1), u\left(q_2^e(1)\right)); \\ = \left(1, u^{-1}(z_1)\right), & \textit{for } z_1 \geqslant u\left(q_2^e(1)\right), \end{cases}$$

where $z_1^e(\theta) \equiv \frac{q_2^e(\theta) - G_i(\theta) u\left(q_2^e(\theta)\right)}{1 - G_i(\theta)}$.

Proof. Claim I: $z_1^e(0) < 0$. Proof. As $\sigma_{bs_2(2)}(0) = 0$, $G_i(0) = \frac{\alpha_{b_2}}{i+\alpha_{b_2}} = \frac{1}{u'\left[q_2^e(0)\right]}$ and

$$z_1^e(0) = \left(1 + \frac{\alpha_{b_2}}{i}\right)\left\{q_2^e(0) - \frac{u\left[q_2^e(0)\right]}{u'\left[q_2^e(0)\right]}\right\};$$

Notice that due to Lagrange's mean value theorem, $u\left[q_2^e(0)\right] = u'(\zeta)\, q_2^e(0)$ where $\zeta \in (0, q_2^e(0))$, and also knowing that $u''(\cdot) < 0$, $u'\left[q_2^e(0)\right] < u'(\zeta)$. This implies $\left\{q_2^e(0) - \frac{u\left[q_2^e(0)\right]}{u'\left[q_2^e(0)\right]}\right\} < 0$, so $z_1^e(0) < 0$.

Claim II: z_1^e is single crossing, i.e., if $z_1^e(1) > 0$, then $\exists! \tilde{\theta}$ s.t. $z_1^e(\tilde{\theta}) = 0$ and $\theta \gtreqless \tilde{\theta} \iff z_1^e(\tilde{\theta}) \gtreqless 0$. Otherwise $z_1^e(\theta) < 0$ for any $\theta \in (0,1)$.

Proof. Combining terms, we have:

$$z_1^e \equiv \frac{q_2^e\left\{1 - \left(1 - \frac{i\sigma_{c_2(2)}}{i+\alpha_{b_2}}\right)\frac{1}{\sigma_u}\right\}}{1 - G_i};$$

This boils down to the problem of $z_1^e \gtreqless 0 \iff 1 - \frac{i\sigma_{c_2(2)}}{i+\alpha_{b_2}} \lesseqgtr \sigma_u$.Notice that *LHS* is decreasing while *RHS* is increasing (as q_2^e is decreasing and $\sigma_u'(q) \leqslant 0$) w.r.t. θ, single crossing property holds, leading to the single crossing property.

If $\sigma_u(q_2^e(1)) \leqslant 1 - \frac{i\sigma_{c_2(2)}(1)}{i+\alpha_{b_2}(N,1)}$, then $z_1^e < 0$ for all $\theta \in [0,1]$, thus, for all $z_1 > 0$, $\frac{q_2^e(1)-z_1}{u(q_2^e(1))-z_1} \leqslant G_i(1)$ holds. This implies $\theta^* = 1$. Combining with (14) yields the property stated.

If $\sigma_u(q_2^e(1)) > 1 - \frac{i\sigma_{c_2(2)}(1)}{i+\alpha_{b_2}(N,1)}$, then $\exists! \tilde{\theta}$ s.t. $z_1^e(\tilde{\theta}) = 0$ and $\theta \gtreqless \tilde{\theta} \Longleftrightarrow z_1^e(\tilde{\theta}) \gtreqless 0$. Notice that $u(q) > q$ for any $q = q_2^e$, so $z_1^e < u(q_2^e)$ for all $\theta \in [0,1]$. For any $z_1 = z_1^e(\theta)$ for any $z_1 = z_1^e(\theta)$ where $\theta \in \left[\tilde{\theta}, 1\right], \left(\theta^*, q_2^*\right) = \left(\theta, q_2^e(\theta)\right)$ is the equilibrium; for $z_1 \in (z_1^e(1), u\left(q_2^e(1)\right)); \frac{q_2^e - z_1}{u(q_2^e)-z_1} \leqslant G(i,1)$ holds and $\left(\theta^*, q_2^*\right) = \left(1, q_2^e(1)\right)$; for $z_1 \geqslant u\left(q_2^e(1)\right)$, $q_2^* = u^{-1}(z_1)$ and $\theta^* = 1$. ■

Now we are able to boil down the problem to the choice of z_1 and q_1. Depends on the nature of matching technologies, we separate the situations in two scenarios: $\sigma_u(q_2^e(1)) \leqslant 1 - \frac{i\sigma_{c_2(2)}(1)}{i+\alpha_{b_2}(N,1)}$ or $\sigma_u(q_2^e(1)) > 1 - \frac{i\sigma_{c_2(2)}(1)}{i+\alpha_{b_2}(N,1)}$.

If $\sigma_u(q_2^e(1)) \leqslant 1 - \frac{i\sigma_{c_2(2)}(1)}{i+\alpha_{b_2}(N,1)}$, then $\theta^* = 1$ for all $z_1 \geqslant 0$. Equilibrium $\left(z_1^*, q_1^*\right)$ solves:

$$z_1^* = \frac{\alpha'_{s_1} q_1 - \lambda \alpha'_{b_1} u(q_1) - \alpha_{s_2(1)} \left(\frac{\alpha_{b_2} u(q_2)}{i+\alpha_{b_2}} - q_2\right) - \frac{i\alpha_{b_2(1)}\alpha_{s_2} u(q_2)}{\left(i+\alpha_{b_2}\right)^2}}{\alpha'_{s_1} - \lambda \alpha'_{b_1} + \frac{i\alpha_{s_2(1)}}{i+\alpha_{b_2}} - \frac{i\alpha_{b_2(1)}\alpha_{s_2}}{\left(i+\alpha_{b_2}\right)^2}}; \tag{22}$$

where

$$\lambda = \frac{1}{(i + \alpha_{b_1})} \left(\alpha_{s_1} + \frac{i\alpha_{s_2}}{i + \alpha_{b_2}} + \nu_1 - \nu_2\right), \tag{23}$$

and

$$q_1^* : u'(q_1) - 1 = \frac{i}{\alpha_{b_1}} + \frac{1}{\lambda \alpha_{b_1}} \left(\nu_2 - \frac{i\alpha_{s_2}}{i + \alpha_{b_2}}\right)$$

.

In this case, q_2^e reaches z_1^e before θ reaches 1, and therefore θ jumps to 1 after $\hat{\theta}$ (Figures 3 and 4). Notice that $z_1^e(0) < 0$ which implies there exists a $\theta > 0$ s.t. $z_1^e(\theta) = 0$.

6 Appendix

Proof of Lemma 1. According to Lagrange's mean value theorem, given n, for any $\theta \in (0,1)$, $\exists! \theta_\epsilon \in (0,\theta)$ s.t.

$$\theta \alpha_{s_2(2)}(n,\theta_\epsilon) = \alpha_{s_2}(n,\theta). \tag{24}$$

As $\alpha_{s_2(22)}(n,\theta) < 0$ for any $\theta \in (0,1)$, we have that $\alpha_{s_2(2)}(n,\theta_\epsilon) > \alpha_{s_2(2)}(n,\theta)$. Combine with (24), this implies $\alpha_{s_2(2)}(n,\theta) < \frac{\alpha_{s_2}(n,\theta)}{\theta} = n\alpha_{b_2}(n,\theta)$. Furthermore, differentiate equation (5) w.r.t. θ, we have: $\alpha_{s_2(2)}(n,\theta) = n\alpha_{b_2}(n,\theta) + n\theta\alpha_{b_2(2)}(n,\theta)$, rearrange terms, we have:

$$\alpha_{s_2}(n,\theta) < n\alpha_{b_2}(n,\theta) \Longrightarrow \alpha_{b_2(2)}(n,\theta) < 0.$$

This completes the proof. ■

Proof of Lemma 3. First, notice that $\frac{\alpha_{s_2}(n,\theta)\alpha_{b_2(2)}(n,\theta)}{\alpha_{b_2}(n,\theta)\alpha_{s_2(2)}(n,\theta)} < 0$, so $\sigma = -\frac{\alpha_{s_2}(n,\theta)\alpha_{b_2(2)}(n,\theta)}{\alpha_{b_2}(n,\theta)\alpha_{s_2(2)}(n,\theta)}$. This implies $\frac{\partial\sigma}{\partial\theta} > 0$ iff. $\partial\left(\frac{\alpha_{s_2}(n,\theta)\alpha_{b_2(2)}(n,\theta)}{\alpha_{b_2}(n,\theta)\alpha_{s_2(2)}(n,\theta)}\right)\Big/\partial\theta < 0$. Using (5) , we have:

$$\frac{\alpha_{s_2}(n,\theta)\alpha_{b_2(2)}(n,\theta)}{\alpha_{b_2}(n,\theta)\alpha_{s_2(2)}(n,\theta)} = \frac{n\theta\left(\alpha_{s_{2(2)}}(n,\theta) - n\alpha_{b_2}(n,\theta)\right)}{\alpha_{s_{2(2)}}(n,\theta)} = n\left(\theta - \frac{\alpha_{s_2}(n,\theta)}{\alpha_{s_{2(2)}}(n,\theta)}\right).$$

Now we require:

$$\partial\left(\frac{\alpha_{s_2}(n,\theta)}{\alpha_{s_{2(2)}}(n,\theta)}\right)\Bigg/\partial\theta > 1 \tag{25}$$

as necessary and sufficient condition. Expand LHS, we have:

$$\partial\left(\frac{\alpha_{s_2}(n,\theta)}{\alpha_{s_{2(2)}}(n,\theta)}\right)\Bigg/\partial\theta = 1 - \frac{\alpha_{s_2}(n,\theta)\alpha_{s_2(22)}(n,\theta)}{\left(\alpha_{s_{2(2)}}(n,\theta)\right)^2}.$$

Notice that $\alpha_{s_2}(n,\theta) > 0$ and $\alpha_{s_2(22)}(n,\theta) < 0$ for any $\theta \in (0,1)$, inequality (25) holds. ■

Proof of Lemma 4. Using equation (5), we have:

$$\frac{\alpha_{s_2(1)}}{\alpha_{s_2(2)}} = \frac{\theta\alpha_{b_2} + n\theta\alpha_{b_2(1)}}{n\alpha_{b_2} + n\theta\alpha_{b_2(2)}} = \frac{\theta}{n} + \frac{n\theta\alpha_{b_2(2)}}{\alpha_{s_2(2)}}\left[\frac{\alpha_{b_2(1)}}{\alpha_{b_2(2)}} - \frac{\theta}{n}\right],$$

Notice that $\frac{n\theta\alpha_{b_2(2)}}{\alpha_{s_2(2)}} < 0$, so, if $\frac{\alpha_{b_2(1)}}{\alpha_{b_2(2)}} > \frac{\theta}{n}$, then $\frac{\alpha_{s_2(1)}}{\alpha_{s_2(2)}} < \frac{\theta}{n} < \frac{\alpha_{b_2(1)}}{\alpha_{b_2(2)}}$, and vice versa. The property stated then follows. Now, under A4., $\alpha_{b_2}(n,\theta) = \pi(n\theta) - \alpha_{b_1}(n)$, where

$\pi(n\theta)$ is an arbitrary decreasing function. This implies:

$$\frac{\alpha_{b_2(1)}(n,\theta)}{\alpha_{b_2(2)}(n,\theta)} = \frac{\theta\pi'(n\theta) - \alpha'_{b_1}(n)}{n\pi'(n\theta)} = \frac{\theta}{n} - \frac{\alpha'_{b_1}(n)}{n\pi'(n\theta)} < \frac{\theta}{n};$$

Combine with the proven property, we have:

$$\frac{\alpha_{s_2(1)}}{\alpha_{s_2(2)}} - \frac{\alpha_{b_2(1)}}{\alpha_{b_2(2)}} > 0$$

for any n and θ, under assumptions A3. and A4. ■

Proof of Proposition 4. The aim is to aggregate terms so that one side of the equation is a (monotonic) function of θ only. To do that, first, assemble $\sigma(N,\theta^*)$ on the RHS and yields:

$$\Phi \gtreqless 0 \Leftrightarrow \frac{i + \alpha_{b_2}(n,\theta)}{\alpha_{b_2}(n,\theta)} \gtreqless \sigma(n,\theta)\left[\frac{u(q_2) - z_2^{\theta}}{z_2^{\theta} - c(q_2)}\right]; \tag{26}$$

Then, substitute (**??**), it further equivalents to:

$$\begin{aligned}
&\Leftrightarrow \frac{i + \alpha_{b_2}(N,\theta^*)}{\alpha_{b_2}(N,\theta^*)} \gtreqless \frac{i\sigma(N,\theta^*)\left[u(q_2) - z_1\right]}{\{i\left[z_1 - c(q_2)\right] + \alpha_{b_2}(N,\theta^*)\left[u(q_2) - c(q_2)\right]\}} \\
&\Leftrightarrow \frac{\left[i + \alpha_{b_2}(N,\theta^*)\right]\left[z_1 - c(q_2)\right]}{\alpha_{b_2}(N,\theta^*)\left[u(q_2) - z_1\right]} \gtreqless \frac{i\sigma(N,\theta^*)}{i + \alpha_{b_2}(N,\theta^*)} - 1 \\
&\Leftrightarrow \frac{c(q_2) - z_1}{u(q_2) - z_1} \lesseqgtr \left[\frac{\alpha_{b_2}(N,\theta^*)}{i + \alpha_{b_2}(N,\theta^*)}\right]\left[1 - \frac{i\sigma(N,\theta^*)}{i + \alpha_{b_2}(N,\theta^*)}\right];
\end{aligned}$$

The LHS is the diagnose function we defined, $G(i,\theta^*)$. It is not difficult to check, according to Lemma 3, that $G_2(i,\theta^*) < 0$ for any $i > 0$ and $N > 0$. In turn, this implies:

$$\begin{cases}
\Phi > 0 \text{ for all } \theta^* \in (0,1), & \text{if } \frac{c(q_2)-z_1}{u(q_2)-z_1} < G(i,1); \\
\Phi < 0 \text{ for all } \theta^* \in (0,1), & \text{if } \frac{c(q_2)-z_1}{u(q_2)-z_1} > G(i,0); \\
\Phi = 0 \text{ obtains interior solution within } \theta^* \in (0,1), & \text{otherwise.}
\end{cases}$$

This completes the proof. ■

References

[1] K. Burdett, S. Shi, and R. Wright. Pricing and matching with frictions. *Journal of Political Economy*, 109(5):1060–1085, 2001.

[2] M. G. Coles and J. Eeckhout. Indeterminacy and directed search. *Journal of Economic Theory*, 111(2):265–276, 2003.

[3] R. Dutu, S. Huangfu, and B. Julien. Contingent prices and money. *International economic review*, 52(4):1291–1308, 2011.

[4] M. Galenianos and P. Kircher. On the game-theoretic foundations of competitive search equilibrium. *International economic review*, 53(1):1–21, 2012.

[5] B. Julien, J. Kennes, and I. King. Bidding for labor. *Review of Economic Dynamics*, 3(4):619–649, 2000.

[6] N. R. Kocherlakota. Money is memory. *Journal of Economic Theory*, 81(2):232–251, 1998.

[7] R. Lagos and R. Wright. A unified framework for monetary theory and policy analysis. 2003.

[8] J. D. Montgomery. Equilibrium wage dispersion and interindustry wage differentials. *The Quarterly Journal of Economics*, pages 163–179, 1991.

[9] G. Rocheteau and R. Wright. Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium. *Econometrica*, 73(1):175–202, 2005.