Job Qualities, Search Unemployment, and Public Policy*

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Abstract

We analyse the impact of public policy on unemployment and the qualities of jobs created in an economy with directed search frictions. Policy variables include unemployment benefits, job creation subsidies, and a graduated income tax structure with a government budget constraint. Firms choose to create either high or low quality jobs and bid for labor. We find, among other things, that neither the upper tax threshold nor the upper tax rate affect the mix of job qualities or unemployment, and that, while subsidies to high quality jobs affect the mix of job types, they have no effect on unemployment. We also identify a policy configuration that allows for the simultaneous existence of constrained efficiency, *ex post* equity, and a balanced government budget.

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Key words: Directed search, job heterogeneity, public policy.

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1 Introduction

Facing difficult economic times, to reduce unemployment rates and create "quality" jobs, many governments have chosen to adjust their economic policies by introducing austerity programs (which cut social benefits), tax reforms (which reduce tax rates, and the progressivity of the tax structure) and job incentive programs (which subsidize firms to create jobs – particularly "high quality" jobs). These adjustments are often justified with an appeal to common sense, but are also supported by existing economic theory. In particular, these are the prescriptions that come out of the standard Diamond-Mortensen-Pissarides (DMP) model, as exemplified in Pissarides' (1985) classic paper.

In this paper we argue that faith in these policies, both by politicians and by economists, may be misplaced. In fact, under certain conditions (which we consider to be reasonable) unemployment is independent of marginal changes to the tax structure, subsidies to high quality jobs, and (depending on the tax structure) even of unemployment benefits. Similarly, job creation can be independent of the tax structure, and the number of high quality jobs being created can be independent of the generosity of unemployment benefits, while the number of low quality jobs declines – so that average job quality increases with unemployment benefits. To illustrate these points, we construct a simple directed search model, which shares many of the features of the DMP model,[1] but allows for different qualities of jobs to be created, a matching structure with explicit microfoundations, and for a graduated tax and subsidy structure.

The model itself is based on the framework developed by Julien, Kennes and King (2000, 2006) and Julien *et al* (2009), where homogeneous workers apply to multiple vacancies, vacancies approach workers with offers (using mixed strategies) and workers sell their labor to the highest bidder. Firms can create either low quality jobs (with low output and low costs) or high quality jobs (with higher output and costs), in a policy environment with different subsidies for each type of job. Workers choose (and announce) their reserve wage while facing a graduated tax structure and unemployment benefits. The

[1] For example, each worker has one unit of labor to sell, each firm has one vacancy to fill, matching is probabilistic, and entry on the vacancy side is determined by a zero profit condition.

tax structure we consider is progressive and consists of four parameters: two threshold levels of income and two tax rates. The first threshold determines the level of income that is tax-free. The second determines the level where higher incomes pay the higher rate. We identify (and restrict attention to) the region in the parameter space where both types of jobs are created in equilibrium. We characterize the symmetric mixed strategy equilibrium in this region and consider the comparative statics of the endogenous variables (in particular, the creation of high and low quality jobs, and the unemployment rate) as the policy parameters change. Several of the results depend crucially on whether unemployment benefits are above or below the first threshold, and we consider both cases.

Some results, however, are independent of whether or not benefits are taxable. In particular, we find that job creation and unemployment are entirely independent of the upper tax threshold and tax rate. Thus, reducing the tax rates on higher income earners (as in the tax reforms in the US) has no effect on job creation or unemployment. Similarly, the creation of high quality jobs is independent of unemployment benefits, the tax structure, and is influenced by employment subsidies only when these subsidies are different for high quality and low quality jobs. Moreover, subsidizing the creation of high quality jobs (as is done in the "Quality Jobs Programs" in many US states) has no effect on unemployment rates. It does, however, have an effect on the *mix* of different jobs created.

If unemployment benefits are not taxable (as is assumed in, for example, Pissarides (1985)) then, under certain conditions, the unemployment rate is also independent of the benefit rate and the lower tax rate. If, however, benefits are taxable, then both job creation and the unemployment rate are completely independent of the entire tax structure. Thus, in this case, unemployment responds only to the benefit rate (positively) and to the subsidy rate for *low* quality jobs (negatively).

We also characterize the (constrained) efficient allocations in this model, and identify the policy configurations that align the equilibrium allocations with the efficient ones. This is a directed search model, with large numbers of agents, so (with risk-neutral agents) the laissez-faire equilibrium (where all policy parameters are zero) is constrained efficient. We show that many other policy settings also achieve constrained efficiency. In particular, efficiency simply requires that the subsidy rates, the benefit rate, and the tax-free threshold are all equalized. This allows for a wide range of values for the policy parameters.

Finally, we identify policy configurations that preserve efficiency but also completely eliminate the *ex post* risk associated with the laissez faire equilibrium, providing *ex post* equity and balancing the government's budget.

Related Literature

Pissarides (1985) used the DMP model with random matching and generalized Nash bargaining to study policy issues. He introduced lump-sum subsidies to firms, and lump-sum benefits to workers, financed through proportional wage taxes, to this (now standard) environment. Intuitively, job creation and unemployment always move in opposite directions in that model, since job creation is the channel through which equilibrium unemployment changes. In that setting, unemployment is decreasing in firm subsidies, and increasing in unemployment benefits. Somewhat less straightforwardly, since benefits are untaxed in Pissarides' model, increases in the tax rate must be borne, at least partially, by firms. This implies that job creation is decreasing (and unemployment is increasing) in the tax rate.

Pissarides' paper was concerned solely with positive issues, not with the normative questions of optimal allocations or policies. These were taken up in subsequent work by, for example, Boone and Bovenberg (2002), Mortensen and Pissarides (2003), Hungerbühler *et al* (2006), Lehmann and Van Der Linden (2007), Jiang (2014), and Michau (2015). In all of these studies, which use variants of the DMP model, care needs to be taken about the inherent inefficiency of the equilibrium of the DMP model outside of the Hosios rule, and many of the results hinge on whether or not this rule is imposed.

Other researchers have considered both positive and normative questions, using directed search models of the labor market, where the inherent inefficiency of equilibria is not generically an issue. Broadly speaking, directed search models follow two different modelling traditions. In the first, a market is defined with finite numbers of players, and results from this finite game are examined in the limit, as the scale of the market becomes arbitrarily large. The model is solved for exact equilibria of a strategic game in which one side post terms of trade and the other select trading partners using mixed strategies. This generates an endogenous (Binomial) matching process. In the limit as the number of agents is taken to infinity keeping the ratio of vacancies to unemployed constant, this generates a Poisson matching process (also known as urn-ball). Early examples of these types of models are Peters (1991), Montgomery (1991), Burdett, Shi, and Wright (2001) (hereafter, BSW), who have individual firms posting wages, and workers choosing which firm to approach. Julien, Kennes, and King (2000) (hereafter, JKK), have a similar framework, but where individual workers sell their labor through auctions and announce reserve wages. Firms, in that setting, choose which worker to approach, and bid for their labor.[2]

The second approach, based on Moen (1997) and Shimer (1996), starts with the initial assumption that markets are large. They consider measures of agents and an environment in which submarkets can be open freely, each of which has random matching at the local level, but workers are able to choose which island to approach, directed by the wages posted by either market makers, firms or workers. This is known as the competitive search approach. The main difference between the finite and large market approach is that the former generates a matching technology endogenously, while the latter can use any matching technology in any submarkets, (although, typically, pairwise matching is assumed). It is well known that under Poisson matching process used in submarkets, the two approach give the same allocation.[3] In this paper we use the large market approach, while assuming terms of trade to prevail in submarkets are posted by workers as in JKK, and use a Poisson matching process in submarkets.[4]

Julien et al (2009) used the JKK approach, with homogeneous workers and firms, to analyse a policy structure similar to the one found in Pissarides (1985). They pointed

[2] For a synthesis of the BSW and JKK approaches, see Albrecht, Gautier, and Vroman (2006).

[3] For a survey of competitive and directed search models, see Julien, Kircher, and Wright (2016).

[4] It is easy to show that this framework generates the same equilibrium as the limit of the finite strategic approach of Julien, Kennes and King (2000). We use the large market directly because it is simpler to work with.

out, (among other things) that unemployment benefits will pull the economy away from efficiency unless they are paired with employment subsidies of equal value. Golosov, Maziero, and Menzio (2013) used the large market approach, but where each worker's application strategy is private information – policymakers cannot discern whether a worker is unemployed due to bad luck or to an unwillingness to apply. Workers are risk averse in this setting, and the optimal policy involves positive benefits and a regressive labor income tax. Geromichalos (2015) used the BSW approach with firms posting wages, homogeneous firms and workers, and considered different ways of taxing firms to finance benefits. He showed (for example) that lump-sum payments are, in fact, distortionary in this environment, because they lead to firms being too aggressive in their wage competition. None of these studies, however, focus on the main issues that we address in this paper: how the mix of job qualities unemployment, and welfare, respond to policy changes.

The paper is organized as follows. Section 2 details the structure of the model. Section 3 analyzes the decisions of the workers and firms and the equilibrium. Section 4 presents the comparative statics of the equilibrium. Section 5 analyses efficient allocations and Section 6 provides concluding remarks. Proofs of the propositions, where required, are provided in the appendix.

2 The Model

Consider a static economy with a measure N of homogeneous workers and a measure M of homogeneous firms. Each firm can open only one vacancy but can choose a low productivity type, with output $y_1 > 0$, or a high productivity type with output $y_2 > y_1$.[5] Once a type of vacancy is created it is common knowledge. Each type of vacancy creation costs an amount k_i, $i \in \{1, 2\}$, with $k_1 \in (0, y_1)$, and $k_2 \in (k_1, y_2)$. Let M_i be the measure of firms creating type i vacancies. The values of both M_i are determined endogenously by free entry, and aggregate market tightness is defined as $\Theta = \Theta_1 + \Theta_2$, with $\Theta_i = M_i/N$.

The economy has a government which provides employment subsidies σ_i, $i \in \{1, 2\}$ to firms that fill a vacancy of type i. The government levies taxes on workers, with the

[5] We will sometimes refer to these types as "high quality" and "low quality" vacancies, respectively.

following progressive income tax structure. All income less than ω_1 is counted as non-taxable. Income between ω_1 and ω_2 is taxed at rate $\tau_1 \in [0, 1]$. Income in excess of ω_2 is taxed at rate $\tau_2 \in [\tau_1, 1]$. Worker productivities, when operating at differing jobs, are assumed to relate to the tax thresholds in the following manner:

$$\omega_1 < y_1 \leq y_1 + \sigma_1 < \omega_2 < y_2 \leq y_2 + \sigma_2. \tag{1}$$

This assumption makes the progressive income tax structure relevant. The government also provides benefits b to any worker who is unemployed in equilibrium. We assume that $b \in [0, y_1 + \sigma_1)$. To be general enough to consider different policies across existing economies, we consider the possibilities that the unemployment benefits are either taxed or not. In general, after tax benefits, when unemployed, are given by $\min\{(1-\tau_1)b+\tau_1\omega_1,\ b\}$. When $b \geq \omega_1$ the benefits are taxed and the outside option for workers is $(1-\tau_1)b+\tau_1\omega_1$. When $b < \omega_1$, the benefits are not taxed and the outside option for workers is simply b.

We consider a competitive search framework along the lines of Moen (1997). However, this model differs from his because the heterogeneity of vacancies, (and hence, productivities), reflects *choices* made by firms before their entry decisions.[6] We also differ in the use of the terms of trade mechanism we use. Moen uses wage posting, we use *reserve* wage posting with a second-price auction. In addition workers, (rather than market makers, in Moen), are posting the terms of trade. A continuum of submarkets can be opened freely by a positive measure of workers posting the same reserve wage $w_r \in \mathbb{R}_+$, which they commit to. Workers can exploit ex post opportunities by allowing bidding if a worker gets more than one job offer. This is done via a second-price auction.[7] Firms observe all reserve wages posted in all submarkets, and decide in which submarket to search for a worker. But before doing so, firms choose which type of vacancy to open, pay the cost associated with the vacancy type, and choose a submarket.[8] We focus on symmetric equilibria in which workers and firms are indifferent which submarket to participate in, and where both types of firms participate in all

[6] In Moen's model, productivity comes from random draws from a known distribution upon incurring an entry cost k, which is the same for all vacancies.

[7] In essence this is the structure used in Julien, Kennes and King (2000) where workers post reserve wage and firms choose over workers, in setting with finite numbers of agents, and taken to the limit. Here, we consider the large market directly for simplicity.

[8] Because we focus on symmetric equilibria, the timing of vacancy creation is innocuous. Firms could

submarkets. We restrict the posted reserve wage to be anonymous, in the sense that workers cannot post a menu of reserve wages contingent on the type of firms they meet.[9]

The focus on symmetric equilibria allows us to focus the analysis on only one submarket. There is a measure $n(w_r) \leq N$ of workers active in the submarket, and measures $m_i(w_r)$ of active firms with vacancies of type i, in the submarket in which all workers post the same w_r. We then write $\theta_i(w_r) = m_i(w_r)/n(w_r)$ as the market tightness of a type i vacancy active in the submarket in which n workers post a reserve wage w_r.[10] For ease of notation, we suppress the argument and write $\theta_i \equiv \theta_i(w_r)$ only. In equilibrium, all workers post the same reserve wage across submarkets, workers and firms are indifferent across submarkets, and $\theta_i = \Theta_i$, $i \in \{1, 2\}$, in each submarket, where we use Θ_i to denote the ratio of vacancies of type i to workers, in equilibrium.

Within any submarket, each firm searches for a worker, and we assume that firms of each vacancy type meet workers according to a Poisson distribution with mean θ_i. This process allows multilateral meetings, and is often referred to as urn-ball meeting.[11] The probability that a worker receives the number z_i job offers from type i vacancy is $\Pr\{z_i\} = \frac{\theta_i^{z_i} e^{-\theta_i}}{z_i!}$. Since both types of vacancies search simultaneously, the expected number of type 1 and type 2 offers are independent. The market tightness in the submarket defines all of the relevant probabilities of the meeting process. For the rest of the analysis, the only relevant probabilities are $\Pr\{z_i = 0\} = e^{-\theta_i}$, $\Pr\{z_i = 1\} = \theta_i e^{-\theta_i}$ and $\Pr\{z_i > 1\} = 1 - \theta_i e^{-\theta_i} - e^{-\theta_i}$.

The government operates in all open submarkets, setting the values of policy parameters $\mathcal{P} = (\omega_i, \tau_i, \sigma_i, b)_{i=1,2}$. These, along with the productivity parameters, y_i, and the vacancy creation costs, k_i, constitute the entire set of parameters of the model.

choose a submarket first and then choose a vacancy type before searching for workers in the submarket.

[9] Allowing this type of wage posting could lead to separation, that is, low and high type firms participate in different submarkets, not competing with each other. While interesting, we wish to draw implication in environments in which different types of jobs coexist in the same market.

[10] Note that the tightness is in fact the Radon-Nykodim derivative $\theta_i = dm_i/dn$, and we assume that measure m_i is absolutely continuous in measure n. See Eeckhout and Kircher (2010) for more details.

[11] Peters (1991), Julien, Kennes and King (2000), and Burdett, Shi and Wright (2001) show that in fact the Poisson meeting process is indeed the limit of a sequence of games, for which one set of agents selects the others using a mixed strategy.

Given the multilateral meeting process workers may receive only one offer (in which case they get their reserve wage w_r); or many offers, (where bidding among firms occurs, and the worker chooses the highest offer). The winning vacancy is the highest productivity vacancy, paying a price equal to the productivity of the second-highest vacancy. This is a second-price auction without private information, and it is well known that it is a dominant strategy for bidders to bid their valuations. Here, vacancies bid their surplus.

2.1 Worker Payoffs

Given the progressive income tax structure, when workers choose a reserve wage, that wage can fall between any threshold of the structure and be taxed at different rates. In the case of only one offer, the after tax wage w_a is given by

$$w_a = \begin{cases} w_r & \text{if } w_r \leq \omega_1 \\ (1-\tau_1)w_r + \tau_1\omega_1 & \text{if } \omega_1 \leq w_r \leq \omega_2 \\ (1-\tau_2)(w_r - \omega_2) + (1-\tau_1)(\omega_2 - \omega_1) + \omega_1 & \text{if } \omega_2 \leq w_r. \end{cases} \quad (2)$$

The winning bid is a wage w_i^j offered by firm of vacancy type i with the second highest vacancy being j. The following equation summarizes the possible (winning) after-tax wage bids, which depend on the composition of different vacancy types.

$$w_i^j = \begin{cases} w_0^0 = & \min\{(1-\tau_1)b + \tau_1\omega_1,\ b\} \\ w_1^0 = & w_a \\ w_1^1 = & (1-\tau_1)(y_1+\sigma_1) + \tau_1\omega_1 \\ w_2^0 = & w_a \\ w_2^1 = & (1-\tau_1)(y_1+\sigma_1) + \tau_1\omega_1 \\ w_2^2 = & (1-\tau_2)(y_2+\sigma_2-\omega_2) + (1-\tau_1)(\omega_2-\omega_1) + \omega_1. \end{cases} \quad (3)$$

If a worker is not approached by any firm he receives w_0^0: his unemployment benefit less any tax incurred. If the worker is approached by a single firm (either w_1^0 or w_2^0) the firm extracts the full surplus from production, leaving the worker with his after tax reserve wage w_a. If the worker is approached by more than one firm, then he is able to extract the surplus of the next highest bidder. In particular, if at least two type 1 firms approach the worker then the worker obtains the full surplus from a type 1 job (which includes σ_1) with after-tax payoff w_1^1. If exactly one type 2 firm and at least one type 1 firm approach the worker, then he is paid the full surplus from a type 1 job, with after-tax payoff $w_2^1 = w_1^1$. Finally, if a worker is approached by at least two type 2 jobs, then the worker receives the highest wage: the full surplus from a type 2 job, with after-tax payoff w_2^2.

The (after-tax) expected payoff for a worker is given by

$$\begin{aligned} V(w_r, \theta_1, \theta_2) = {} & e^{-\theta_1-\theta_2} w_0^0 + \theta_1 e^{-\theta_1-\theta_2} w_1^0 + \left(1 - e^{-\theta_1} - \theta_1 e^{-\theta_1}\right) e^{-\theta_2} w_1^1 \\ & + \theta_2 e^{-\theta_1-\theta_2} w_2^0 + \left(1 - e^{-\theta_1}\right) \theta_2 e^{-\theta_2} w_2^1 + \left(1 - e^{-\theta_2} - \theta_2 e^{-\theta_2}\right) w_2^2, \end{aligned} \tag{4}$$

where the w_i^j are given in (2) and (3). The payoff reflects the different state of offers for a worker. If no offers arrive (which occurs with probability $e^{-\theta_1-\theta_2}$), the payoff is w_0^0: the unemployment benefit (which may or may not be taxed). If an offer comes from only one type 1 vacancy, and no type 2 vacancies (which occurs with probability $\theta_1 e^{-\theta_1-\theta_2}$), the worker receives the payoff of w_1^0 (which may or may not be taxed). If offers come from at least two type 1 vacancies, and no type 2 vacancies (which occurs with probability $\left(1 - e^{-\theta_1} - \theta_1 e^{-\theta_1}\right) e^{-\theta_2}$) then the worker's payoff is w_1^1, which is taxed at rate τ_1 for all income above ω_1. If an offer comes from only one type 2 vacancy, and no type 1 vacancies (which occurs with probability $\theta_2 e^{-\theta_1-\theta_2}$), the worker receives payoff w_2^0 (which may or may not be taxed). If an offer from exactly one type 2 vacancy and at least one type 1 vacancy (which occurs with probability $\left(1 - e^{-\theta_1}\right) \theta_2 e^{-\theta_2}$), the worker receives the payoff of w_2^1, (which is taxed at rate τ_1 for all income above ω_1). Finally, if offers come from at least two type 2 jobs (which occurs with probability $\left(1 - e^{-\theta_2} - \theta_2 e^{-\theta_2}\right)$), the worker receives the maximum payoff of w_2^2, (which is taxed at the rate τ_1 for all income between ω_1 and ω_2 and at rate τ_2 for all income greater than ω_2).

2.2 Firm Payoffs

Once the vacancy cost has been paid, the expected payoff from opening a type 1 vacancy is

$$\pi_1(w_r, \theta_1, \theta_2) = e^{-\theta_1} e^{-\theta_2} \left[y_1 + \sigma_1 - w_r \right]. \tag{5}$$

This payoff reflects the fact that if a type 1 vacancy is alone when approaching a worker (which occurs with probability $e^{-\theta_1} e^{-\theta_2}$), it gets the output y_1 and the subsidy σ_1 minus the reservation wage paid to the worker, w_r. If not alone (ie., in all other cases) then either the firm is unsuccessful in hiring the worker or it is successful but all of its surplus is bid away. In either case, if not alone, the firm's *ex post* payoff is zero.

For a type 2 vacancy:

$$\pi_2(w_r, \theta_1, \theta_2) = e^{-\theta_1 - \theta_2} \left[y_2 + \sigma_2 - w_r \right] + \left(1 - e^{-\theta_1}\right) e^{-\theta_2} \left[y_2 + \sigma_2 - y_1 - \sigma_1 \right] \tag{6}$$

If a type 2 vacancy is alone when approaching a worker (which occurs with probability $e^{-\theta_1} e^{-\theta_2}$), it gets the output y_2 and the subsidy σ_2 minus the reservation wage paid to the worker, w_r. If the type 2 vacancy faces no other type 2 vacancies, but at least one type 1 vacancy, when approaching a worker, (which occurs with probability $\left(1 - e^{-\theta_1}\right) e^{-\theta_2}$) it gets the surplus $(y_2 + \sigma_2)$ minus the surplus from a type 1 job $(y_1 + \sigma_1)$, due to the bidding process. If the type 2 vacancy faces at least one other type 2 vacancy when approaching the worker (with the remaining probability), then either the firm is unsuccessful in hiring the worker or it is successful but all of its surplus is bid away. In either case, when facing at least one other type 2 vacancy, the firm's *ex post* payoff is zero.

With free entry of each type of vacancy we have, in equilibrium:

$$\pi_i = k_i \quad \forall\ i = 1, 2. \tag{7}$$

3 Competitive Search Equilibrium with Policy

Definition 1. *A symmetric competitive search equilibrium is an allocation defined by the tuple* (V^*, π_1^*, π_2^*), *a choice of reserve wage* ω_r^* *by workers, and submarkets* (Θ_1, Θ_2) *such that*

(i) For all $\omega_r \in [0,\ y_2 + \sigma_2]$, $\pi_i(\omega_r, \theta_1, \theta_2) = k_i$;

(ii) $V^* = \max_{\omega_r, \theta_1, \theta_2} V(\omega_r, \theta_1, \theta_2)$ *s.t.* $\pi_i(\omega_r, \theta_1, \theta_2) \leq k_i$ *and* $\theta_i \geq 0$ *with complementary slackness;*

(iii) $\theta_i = \Theta_i$ *for all* $i \in \{1, 2\}$.

Workers forming the submarket with reserve wage w_r, solve the following program[12]:

$$\max_{w_r, \theta_1, \theta_2} V(w_r, \theta_1, \theta_2) \tag{8}$$

$$\text{s.t. } \pi_i(w_r, \theta_1, \theta_2) \geq k_i, \text{ and } \theta_i \geq 0, \ \forall i \in \{1, 2\}$$

Proposition 3.1. *In equilibrium, each worker in all active submarkets sets the reserve wage equal to*

$$w_r^* = \begin{cases} \min\left\{b + (\theta_1 + \theta_2)\, \tau_1 \left(y_1 + \sigma_1 - \omega_1\right),\ \omega_1\right\} & b < \omega_1 \\ b & b \geq \omega_1 \end{cases}$$

with $\theta_i = \Theta_i$, $i \in \{1, 2\}$, and $\Theta = \Theta_1 + \Theta_2 = M/N$.

The equilibrium reserve wage is either equal to the level of unemployment benefits b – if unemployment benefits are taxed – or exceeds b if unemployment benefits are not taxed. It is always equal to the value of workers' outside option but, if $b < \omega_1$, it never exceeds the first tax threshold ω_1.

[12] A similar formulation is found in Moen (1997), and Acemoglu and Shimer (1999), to name a few. The constraint $\pi_i(w_r, \theta_1, \theta_2) = k_i$ gives a one to one relationship $\theta_i(w_r; k_i)$. This is the reason why we can maximize directly choosing θ_1 and θ_2.

Using the equilibrium reserve wage in the firm payoff functions (5) and (6) in the free entry conditions (7) gives us the equilibrium aggregate tightness conditions:

$$k_1 = e^{-\Theta_1-\Theta_2}\left(y_1+\sigma_1-w_r^*\right)$$
$$k_2 = e^{-\Theta_1-\Theta_2}\left(y_1+\sigma_1-w_r^*\right)+e^{-\Theta_2}\left(y_2+\sigma_2-y_1-\sigma_1\right).$$

A simple substitution then gives us:

$$\begin{aligned} k_1 &= e^{-\Theta_1-\Theta_2}\left(y_1+\sigma_1-w_r^*\right) \\ k_2 &= k_1+e^{-\Theta_2}\left(y_2+\sigma_2-y_1-\sigma_1\right). \end{aligned} \tag{9}$$

The system of equations given in (9) is recursive, with Θ_2 being determined in the second equation, then Θ_1 being determined in the first, with w_r^* given in Proposition (3.1). Due to the piecewise nature of the equilibrium reserve wage w_r^*, there are two cases to analyse when considering Θ_1. In the first case, $\omega_1 \leq b$ and benefits are taxed. In the second case, $b < \omega_1$ and they are not. We establish conditions for existence and uniqueness of $(\Theta_1, \Theta_2) >> 0$ for all cases. These cases are then analyzed separately in the next section, along with all of the comparative statics.

We can rewrite the entry condition for type 2 vacancies, in (9) as

$$\Theta_2 = \ln\left(\frac{y_2+\sigma_2-y_1-\sigma_1}{k_2-k_1}\right). \tag{10}$$

Clearly, there exists a unique $\Theta_2 > 0$ if and only if

$$y_2+\sigma_2-y_1-\sigma_1 > k_2-k_1. \tag{11}$$

This condition represents the marginal productivity of high quality jobs creation when low quality jobs are also created. This places a restriction on the values of σ_1 and σ_2 for high quality jobs to exist. The equilibrium value of Θ_2 does not depend on w_r^* or any policy parameters other than σ_1 and σ_2. Therefore, this condition remains valid for existence of high quality jobs in all cases analyzed in the next section.

To show the existence and uniqueness of low quality job creation, in equilibrium, we use (10) in the entry condition for low quality jobs, given in (9):

$$e^{\Theta_1} = \frac{(y_1 + \sigma_1 - w_r^*)\,(k_2 - k_1)}{(y_2 + \sigma_2 - y_1 - \sigma_1)k_1}.$$

There exists a unique $\Theta_1 > 0$ if and only if

$$\frac{(y_1 + \sigma_1 - w_r^*)\,(k_2 - k_1)}{(y_2 + \sigma_2 - y_1 - \sigma_1)k_1} > 1.$$

We can rewrite this condition as

$$\frac{y_1 + \sigma_1 - w_r^*}{k_1} > \frac{y_2 + \sigma_2 - y_1 - \sigma_1}{k_2 - k_1} > 1, \tag{12}$$

This has the interpretation that the marginal net productivity of a low quality jobs creation must be greater than the marginal productivity of high quality jobs creation.

However, the equilibrium Θ_1 depends on w_r^* which itself, from Proposition (3.1), depends on policy parameters. Therefore, the conditions for uniqueness depends on the cases $\omega_1 \leq b$ and $b < \omega_1$ and the equilibrium reserve wage set by workers w_r^*.

Consider, first, the case of taxed benefits: $\omega_1 \leq b$. From the equilibrium reserve wage in Proposition (3.1), $w_r^* = b$, the existence condition (12) becomes

$$\frac{y_1 + \sigma_1 - b}{k_1} > \frac{y_2 + \sigma_2 - y_1 - \sigma_1}{k_2 - k_1} > 1, \tag{13}$$

which depends only on parameters. Rearranging (13), the existence of equilibrium with high quality and low quality jobs created under the taxed benefits case ($\omega_1 \leq b$) requires the combination of policy parameters $(b, \sigma_1, \sigma_2, \omega_1)$ to satisfy

$$y_1 + \sigma_1 - k_1 \frac{y_2 + \sigma_2 - y_1 - \sigma_1}{k_2 - k_1} > b \geq \omega_1. \tag{14}$$

The intuition behind this condition is that the marginal productivity of low quality job creation must be larger than the marginal productivity of high quality jobs creation. This difference must be large enough relative to the income tax threshold ω_1. This places bounds on the possible values of b to assure existence. In this case, uniqueness is entirely determined by parameter values satisfying the existence condition. Any violation of these conditions takes the equilibrium to a different case.

Consider, now, the case of untaxed benefits: $0 \leq b < \omega_1$. From the equilibrium reserve wage in Proposition (3.1), the existence condition (12) becomes

$$\frac{y_1 + \sigma_1 - \min\left\{b + (\Theta_1 + \Theta_2)\,\tau_1\,(y_1 + \sigma_1 - \omega_1)\,,\ \omega_1\right\}}{k_1} > \frac{y_2 + \sigma_2 - y_1 - \sigma_1}{k_2 - k_1} > 1, \qquad (15)$$

which now also depend on the equilibrium values (Θ_1, Θ_2), and the additional policy parameter τ_1. There are two subcases to consider. In one $w_r^* = \omega_1$ and, in the other, $w_r^* < \omega_1$.

Consider the case where $w_r^* = \omega_1$. In this case, the entire set of policy parameter $\mathcal{P}$ and technological parameters $(y_i, k_i), \forall i$, are such that the existence values of (Θ_1, Θ_2) imply

$$w_r^* = \min\left\{b + (\Theta_1 + \Theta_2)\,\tau_1\,(y_1 + \sigma_1 - \omega_1)\,,\ \omega_1\right\} = \omega_1.$$

The existence condition (15) becomes

$$\frac{y_1 + \sigma_1 - \omega_1}{k_1} > \frac{y_2 + \sigma_2 - y_1 - \sigma_1}{k_2 - k_1} > 1, \qquad (16)$$

Rearranging (13), the existence of equilibrium with high quality and low quality jobs created under the untaxed benefits case $b < \omega_1$ requires the combination of policy parameters $(b, \sigma_1, \sigma_2, \omega_1)$ to satisfy

$$y_1 + \sigma_1 - k_1 \frac{y_2 + \sigma_2 - y_1 - \sigma_1}{k_2 - k_1} > \omega_1 > b. \qquad (17)$$

The intuition behind this condition is similar to the one for condition (14). In this case, uniqueness is entirely determined by parameter values satisfying the existence condition.

Consider now the case where $w_r^* < \omega_1$. Suppose that the set of policy parameters $\mathcal{P}$ and technological parameters $(y_i, k_i), \forall i$, are such that the equilibrium values of (Θ_1, Θ_2) imply

$$w_r^* = b + (\Theta_1 + \Theta_2)\,\tau_1\,(y_1 + \sigma_1 - \omega_1)\,.$$

The existence condition (15) becomes

$$\frac{y_1 + \sigma_1 - b - (\Theta_1 + \Theta_2)\,\tau_1\,(y_1 + \sigma_1 - \omega_1)}{k_1} > \frac{y_2 + \sigma_2 - y_1 - \sigma_1}{k_2 - k_1} > 1, \qquad (18)$$

and condition (10) must hold. Rearranging (18), the existence of equilibrium with high quality and low quality jobs created under the untaxed benefits case ($b < \omega_1$) requires that the combination of policy parameters $\mathcal{P}$ and technological parameters $(y_i, k_i), \forall i$, satisfy

$$\bar{\Theta}_1 \equiv \frac{y_1 + \sigma_1 - b - \Theta_2 \tau_1 (y_1 + \sigma_1 - \omega_1) - k_1 e^{\Theta_2}}{\tau_1 (y_1 + \sigma_1 - \omega_1)} > \Theta_1, \tag{19}$$

with condition (10) on $\Theta_2 > 0$ satisfied. The values of the set of policy parameters $\mathcal{P}$ and $(y_i, k_i), \forall i$, places an upper bound on the admissible value of $\Theta_1 > 0$, to be an equilibrium under the case of untaxed benefits and workers optimally setting an untaxed reservation wage, $\omega_r^* < \omega_1$.

Proposition 3.2. *There exist values for the set of policy parameters $\mathcal{P}$ and technological parameters $(y_i, k_i), \forall i$, and $b < \omega_1$, for which there exist a unique equilibrium $0 < \Theta_1 < \bar{\Theta}_1$ and $0 < \Theta_2$ if and only if*

$$\frac{y_1 + \sigma_1 - b - \ln \frac{y_2 + \sigma_2 - y_1 - \sigma_1}{k_2 - k_1} \tau_1 (y_1 + \sigma_1 - \omega_1)}{k_1} > \frac{y_2 + \sigma_2 - y_1 - \sigma_1}{k_2 - k_1} > 1.$$

The intuition for the existence and uniqueness of equilibrium with both types of vacancies created when benefits are untaxed and workers set an optimal reservation wage that is untaxed is similar to the previous cases. However, there is an additional factor affecting the net marginal productivity of low quality jobs creation, which is equal to $\Theta_2 \tau_1 (y_1 + \sigma_1 - \omega_1)$. We have now established all of the conditions for existence and uniqueness of equilibrium under all possible cases. The next section considers equilibrium comparative static analysis with the policy parameters.

4 Job Creation and Unemployment in Equilibrium

We have seen that the equilibrium number of low quality vacancies per worker, Θ_1, depends crucially on whether or not unemployment benefits b are taxed. We must, therefore, consider both cases, when determining the effects of policy. The equilibrium value of Θ_2 in (9), however, does not depend on w_r^* and, thus, we can characterize its properties prior to considering the different cases. From the second equation in (9) we obtain, immediately, the following results.

Proposition 4.1. *In an equilibrium with both high quality and low quality jobs operating, market tightness for high quality jobs is given by*

$$\Theta_2 = \ln(y_2 + \sigma_2 - y_1 - \sigma_1) - \ln(k_2 - k_1)$$

with comparative statics:

$$\begin{aligned} \partial\Theta_2/\partial\sigma_1 &= -(y_2 + \sigma_2 - y_1 - \sigma_1)^{-1} < 0 \\ \partial\Theta_2/\partial\sigma_2 &= (y_2 + \sigma_2 - y_1 - \sigma_1)^{-1} > 0 \\ \partial\Theta_2/\partial b = \partial\Theta_2/\partial\omega_1 &= \partial\Theta_2/\partial\omega_2 = \partial\Theta_2/\partial\tau_1 = \partial\Theta_2/\partial\tau_2 = 0. \end{aligned} \tag{20}$$

Intuitively, increments to subsidies for low quality job creation σ_1 increase their profitability and induce further entry of low quality jobs. This reduces the profitability of high quality jobs and, consequently, high quality job creation. Analogously, increments to subsidies for high quality job creation σ_2 increase their profitability and induce further entry of high quality jobs. Interestingly, though, the influence of the job subsidies on high quality job creation depend only on the *difference* $(\sigma_2 - \sigma_1)$, reflecting the marginal productivity of a high quality job, as discussed in the previous section.

Any effects of increments in benefits b, or the tax structure (ω_i, τ_i) can only operate through influencing the reserve wage w_r^*. However, from (9), any change in w_r^* will be absorbed by the equilibrium condition for Θ_1, represented by the parameter k_1, leaving the number of high quality jobs unchanged in equilibrium.

We now turn to analyse the impacts of the policy parameters on the creation of low quality jobs Θ_1, and unemployment – both of which depend largely on whether or not benefits are taxed.

4.1 When Benefits are Taxed

By combining the free entry conditions (9) with Proposition 3.1 we obtain the following:

Proposition 4.2. *In a symmetric equilibrium where both types of vacancies are created and $b \geq \omega_1$, the equilibrium values of Θ_1 and unemployment rate are given, respectively,* *by:*

$$\Theta_1 = \ln\left(\frac{y_1 + \sigma_1 - b}{y_2 + \sigma_2 - y_1 - \sigma_1}\right) + \ln\left(\frac{k_2}{k_1} - 1\right)$$
$$U = e^{-(\Theta_1+\Theta_2)} = \frac{k_1}{y_1 + \sigma_1 - b}$$

and comparative statics[13]

	σ_1	σ_2	b	ω_1	ω_2	τ_1	τ_2
Θ_1	$+$	$-$	$-$	0	0	0	0
U	$-$	0	$+$	0	0	0	0

Any increase in unemployment benefits raises the outside option of workers and, hence, their reserve wages. This drives down the profitability of producing low quality jobs, which reduces their entry, Θ_1. Given that high quality job creation Θ_2 is independent of benefits (from Proposition 4.1) this implies that total job creation $\Theta_1 + \Theta_2$ falls and, hence, the unemployment rate rises.

Increases in the subsidies for low quality jobs σ_1 increase their profitability and, thereby, induce more entry of low quality jobs Θ_1. Proposition 4.1 tells us that this also reduces the entry of high quality jobs Θ_2, since high quality jobs and low quality jobs compete with each other for workers. However the direct increase in Θ_1 is larger than the indirect decrease in Θ_2, and so $\Theta_1 + \Theta_2$ rises and, hence, the unemployment rate falls.

From Proposition 4.1, we know that increases in the subsidies for high quality jobs σ_2 increase their profitability and, thereby, induce more entry of high quality jobs Θ_2. This also reduces the entry of low quality jobs Θ_1, due to the increased competition for workers. Here, though, the increase in Θ_2 is perfectly offset by the decrease in Θ_1, and so $\Theta_1 + \Theta_2$ is unchanged, and unemployment rate is unaffected by changes in σ_2.

Since, in this case, benefits are taxed at the same rate as wages for those who work in low quality jobs, no marginal change in either the threshold value ω_1 or the tax rate τ_1 will affect the outside option for workers and so the reserve wage will remain unchanged. This

[13] The exact derivations are in the Appendix.

implies that the profitability of low quality jobs is unchanged and, hence, the entry of low quality jobs Θ_1. From Proposition 4.1, we also know that Θ_2 is unchanged by any change in ω_1 or τ_1. Therefore, total job creation $\Theta_1 + \Theta_2$ is unchanged, and unemployment is unaffected by changes in either ω_1 or τ_1.

Given the maintained assumption that $\omega_2 \in (y_1+\sigma_1, y_2+\sigma_2)$, then ω_2 and τ_2 are relevant only to those workers that have at least two high quality jobs approach them. Whenever this occurs, firms must pay $y_2+\sigma_2$, regardless of the value of either ω_2 or τ_2. Thus, the profitability and entry of high quality jobs is independent of them. Given this, the profitability of low quality jobs, and the entry of low quality jobs Θ_1 is independent of ω_2 and τ_2. From Proposition 4.1, we also know that Θ_2 is independent of ω_2 and τ_2. Therefore, total job creation $\Theta_1+\Theta_2$ is unchanged, and unemployment is unaffected by changes in either ω_2 or τ_2.

As mentioned above, the results in this section are conditional on workers attracting positive numbers of high quality and low quality firms, which requires (9) to hold for positive values of Θ_1 and Θ_2 when $w_r^* = b$. In particular, $\Theta_2 > 0$, holds if and only if condition (11) is satisfied. For $\Theta_1 > 0$ this requires condition (13) to be satisfied. A government would choose σ_1 and b such that $y_1 + \sigma_1 - b > 0$, otherwise no low quality jobs would be created and accepted by workers. Thus, $\Theta_1 > 0$ if and only if the condition (11) holds. This places a bound on the subsidies for both high quality and low quality jobs to be created.

4.2 When Benefits are Untaxed

Unemployment benefits are untaxed when $b < \omega_1$. As in the previous subsection, we analyze the equilibrium when both types of vacancies are created. Combining the free entry conditions in (9) with the appropriate expression for w_r^* in Proposition 3.1 produces two equations determining the tightness ratios $(\Theta_i)_{i=1,2}$, and consequently, unemployment $e^{-(\Theta_1+\Theta_2)}$ in equilibrium.

However, since the reserve wage w_r^* is the minimum of a linear function of the tightness ratios and the non-taxable income threshold ω_1 in this case, there are two sub cases to consider depending upon whether the reserve wage is strictly less than the non-taxable threshold or not.

We find that if the structural and policy parameters of the model imply that the reserve wage is strictly less than the non-taxable threshold, then no analytic closed form solution for market tightness of low quality job (and, hence, unemployment) exists. (However, as shown in Proposition 3.2, the equilibrium is unique.) If, instead, the reserve wage equals the non-taxable threshold, then closed form solutions for Θ_1 and unemployment do exist. (However, as shown above, in both cases, the expression for Θ_2 is the same.) This is summarized in the following proposition.

Proposition 4.3. *In the symmetric equilibrium where both vacancy types are created and $b < \omega_1$, for a given set of policy parameter values $\mathcal{P}$:*

i. If

$$\Theta_1 < \frac{y_1 + \sigma_1 - b - \Theta_2 \tau_1 (y_1 + \sigma_1 - \omega_1) - k_1 e^{\Theta_2}}{\tau_1 (y_1 + \sigma_1 - \omega_1)} \equiv \bar{\Theta}_1, \tag{21}$$

with Θ_2 determined as in Proposition 4.1, then the equilibrium reserve wage w_r^* is strictly less then the non-taxable income threshold ω_1. Moreover, the equilibrium market tightness for low quality jobs satisfies

$$k_1 e^{\Theta_1} = \frac{k_2 - k_1}{y_2 + \sigma_2 - y_1 - \sigma_1} \cdot \left(y_1 + \sigma_1 - b - (\ln \left(\frac{y_2 + \sigma_2 - y_1 - \sigma_1}{k_2 - k_1} \right) + \Theta_1) \tau_1 (y_1 + \sigma_1 - \omega_1) \right)$$

while equilibrium unemployment rate is

$$U = e^{-(\Theta_1 + \Theta_2)} = \frac{k_1}{y_1 + \sigma_1 - b - (\Theta_1 + \Theta_2) \tau_1 (y_1 + \sigma_1 - \omega_1)};$$

ii. If $\Theta_1 \geq \bar{\Theta}_1$, then the equilibrium reservation wage $w_r^* = \omega_1$, and the equilibrium market tightness for low quality jobs is given by

$$\Theta_1 = \ln \left(\frac{y_1 + \sigma_1 - \omega_1}{y_2 + \sigma_2 - y_1 - \sigma_1} \right) + \ln \left(\frac{k_2}{k_1} - 1 \right)$$

while equilibrium unemployment rate is

$$U = e^{-(\Theta_1 + \Theta_2)} = \frac{k_1}{y_1 + \sigma_1 - \omega_1}.$$

Proposition 4.3 allows us to analyse how market tightness and unemployment rate respond to policy changes in two separate (sub)cases: when reserve wages are lower than the non-taxable threshold ($w_r^* < \omega_1$) and when reservation wages equal the threshold ($w_r^* = \omega_1$). Condition (21) determines which of these equilibria is realized, given the parameter settings of the model. Case i. in Proposition 4.3 occurs when the values of the technological parameters $\{y_i, k_i\}$, $i = 1, 2$ and the set of policy parameter values $\mathcal{P}$ leads to a relatively low rate of low quality job creation ($\Theta_1 < \bar{\Theta}_1$). Otherwise for other parameter values, a relatively high rate of low quality jobs are created. We now consider these two subcases.

4.2.1 When Condition (21) Holds

We analyze the (sub)case when $w_r^* < \omega_1$ first. From Proposition 4.3, with Θ_2 given in (10) market tightness for low quality firms Θ_1 is pinned down by

$$k_1 e^{\Theta_1 + \Theta_2} = y_1 + \sigma_1 - b - (\Theta_1 + \Theta_2)\tau_1(y_1 + \sigma_1 - \omega_1). \tag{22}$$

With Θ_1 thus determined, and Θ_2 determined in (10), equilibrium unemployment rate in this case is given by

$$U = e^{-(\Theta_1 + \Theta_2)} = \frac{k_1}{y_1 + \sigma_1 - b - (\Theta_1 + \Theta_2)\tau_1(y_1 + \sigma_1 - \omega_1)}. \tag{23}$$

There are no closed form solutions for Θ_1 or U in this case. However, comparative statics can be found by totally differentiating equations (22) and (23), using equation (10).

Proposition 4.4. *In the symmetric equilibrium where both types of vacancies are created, benefits are untaxed ($b < \omega_1$) and Condition (21) holds*[14] *the following comparative static results apply:*

	σ_1	σ_2	b	ω_1	ω_2	τ_1	τ_2
Θ_1	$+/-$	$-$	$-$	$+$	0	$-$	0
U	$+/-$	0	$+$	$-$	0	$+$	0

where

[14] Exact derivations are in the Appendix.

$$\partial\Theta_1/\partial\sigma_1 = [1-(\Theta_1+\Theta_2)\,\tau_1]\left[k_1e^{\Theta_1+\Theta_2}+\tau_1(y_1+\sigma_1-\omega_1)\right]^{-1}+(y_2+\sigma_2-y_1-\sigma_1)^{-1}$$
$$\partial U/\partial\sigma_1 = -e^{-(\Theta_1+\Theta_2)}\,[1-(\Theta_1+\Theta_2)\,\tau_1]\left[k_1e^{\Theta_1+\Theta_2}+\tau_1(y_1+\sigma_1-\omega_1)\right]^{-1}.$$

As in the case where benefits are taxed, an increase in benefits b raises the outside option for workers and, hence, their reserve wages. This increases the expected wage that low quality jobs will have to offer, which makes these jobs less profitable, reducing their entry, Θ_1. With high quality job creation Θ_2 being independent of b (From Proposition 4.1) this implies that total job creation $\Theta_1+\Theta_2$ falls, and unemployment rises.

An increase in the subsidies for low quality jobs σ_1 has both positive and negative effects on their entry. First, the subsidy has a direct positive effect on profitability, driving up entry as in the case where benefits are taxed, above. However, when benefits are untaxed, and the reserve wage $w_r^* < \omega_1$, then the reserve wage has a premium above b, by the amount $(\Theta_1+\Theta_2)\tau_1(y_1+\sigma_1-\omega_1)$, which is increasing in σ_1. Thus, in this case, with the reserve wage increasing in σ_1, the overall effect of an increase in σ_1 on Θ_1 cannot be signed without further restrictions. One sufficient condition, for $\partial\Theta_1/\partial\sigma_1$ to be positive, is that $(\Theta_1+\Theta_2)\tau_1 < 1$.[15] By using Proposition 4.1 to find the effect on Θ_2, we can see that this particular condition is also both necessary and sufficient for unemployment to fall as σ_1 rises.

Exactly as in the case where benefits are taxed, an increase in the subsidies for high quality jobs σ_2 increases their profitability, and the entry of high quality jobs Θ_2. This reduces the profitability of low quality jobs, and their entry Θ_1. These two effects exactly offset each other, leaving total job creation and unemployment unchanged.

Unlike the case where benefits are taxed, here an increase in the lower tax threshold ω_1 increases the entry of low quality jobs Θ_1. Increases in this threshold reduces the amount of taxes that workers must pay in low quality jobs, which reduces the reserve wage premium $(\Theta_1+\Theta_2)\tau_1(y_1+\sigma_1-\omega_1)$ above benefits b that firms must pay. This makes low quality jobs more profitable and increases their entry. From Proposition 4.1, we know that the entry of high quality jobs is independent of ω_1. Thus, an increase in ω_1 raises $\Theta_1+\Theta_2$ and reduces unemployment.

[15] We believe that this condition typically holds because both terms on the left side are usually less that unity, for most countries.

For precisely the same reasons as in the case where benefits are taxed, as described above, the entry of low quality jobs Θ_1 and high quality jobs Θ_2 is independent of both the upper tax threshold ω_2 and rate τ_2. Thus, in this case too, unemployment in independent of both ω_2 and τ_2.

Increases in the lower tax rate τ_1, in this case, increase the reserve wage premium $(\Theta_1 + \Theta_2)\tau_1(y_1 + \sigma_1 - \omega_1)$ above benefits b that firms must pay. This makes low quality jobs less profitable and decreases their entry. From Proposition 4.1, we know that the entry of high quality jobs is independent of τ_1. Thus, an increase in τ_1 decreases $\Theta_1 + \Theta_2$ and increases unemployment.

4.2.2 When Condition (21) Does Not Hold

Now consider the case when (21) fails to hold and so $w_r^* = \omega_1$ is realized in equilibrium. The following results are immediate from part (ii) of Proposition 4.3.

Proposition 4.5. *In the symmetric equilibrium where both types of vacancies are created, benefits are untaxed ($b < \omega_1$) and Condition (21) does not hold, the following comparative static results apply:*

	σ_1	σ_2	b	ω_1	ω_2	τ_1	τ_2
Θ_1	$+$	$-$	0	$-$	0	0	0
U	$-$	0	0	$+$	0	0	0

When examining this case, it is important to understand a key feature of the equilibrium in this region of the parameter space. Whenever $b < \omega_1$, benefits are untaxed, the reserve wage is greater than b – because workers who take employment will pay a tax, whereas those who collect benefits will not. In order to be induced to choose to work, workers must be paid a premium above the amount b, to cover the tax penalty from working. The value of this premium, from Proposition 3.1, is the minimum of either $(\Theta_1+\Theta_2)\tau_1(y_1+\sigma_1-\omega_1)$ (if Condition (21) holds) or $\omega_1 - b$ (if Condition (21) does not hold). Thus, if Condition (21) holds, the reserve wage is $b + (\Theta_1 + \Theta_2)\tau_1(y_1 + \sigma_1 - \omega_1)$, which is increasing in b; and when Condition (21) does not hold, the reserve wage is simply equal to ω_1, which is independent of b. Starting from a point where Condition (21) holds, as b increases, it reduces $\bar{\Theta}_1$ – eventually to where Condition (21) no longer holds. At this point, the reserve wage premium $(\Theta_1+\Theta_2)\tau_1(y_1+\sigma_1-\omega_1)$ is exactly equal to ω_1-b, and the reserve wage is ω_1. Further increments in b leave the reserve wage unchanged at ω_1, while the premium ω_1-b shrinks – eventually to zero, at the point where $b=\omega_1$. From that point onwards benefits are taxed, the reserve wage is equal to b, and the reserve wage premium no longer exists.

This explains the (otherwise curious) result that, in this region of the parameter space, where Condition (21) does not hold, and reserve wage is equal to ω_1, the entry of low quality jobs Θ_1 is independent of the benefit rate. In this region, the reserve wage (and, hence, the expected wage) is independent of b. Thus, b plays no role in the entry decision of low quality vacancies. For reasons described above, from Proposition 4.1, we also know that high quality vacancies Θ_2 are independent of b. Thus, the total number vacancies $\Theta_1+\Theta_2$ and the unemployment rate are both independent of benefits b.

In this case, any increase in the subsidies for low quality jobs σ_1 has an unambiguously positive effect on the entry of these jobs Θ_1. Whereas, in the case where Condition (21) holds (discussed in subsection 4.2.1), the reserve wage has a premium $(\Theta_1+\Theta_2)\tau_1(y_1+\sigma_1-\omega_1)$ – which is increasing in σ_1 – here, the reserve wage premium is simply ω_1-b, which is independent of σ_1. Thus, there is no indirect effect to offset the positive effect of σ_1 on Θ_1 in this case (as there is when Condition (21) holds). Here, the effect is unambiguously positive. From Proposition 4.1 we can also see that the effect on Θ_2 is negative. The overall effect on $\Theta_1+\Theta_2$ is unambiguously positive and, thus, any increase in σ_1 reduces unemployment, as is seen clearly from Part (ii) of Proposition 4.3.

When considering the effects of an increase in the subsidies to high quality jobs σ_2, in this case, the results are exactly the same as when benefits are taxed and when they are not taxed and Condition (21) holds (in Propositions 4.2 and 4.4), for precisely the same reasons as those presented in the discussions following those propositions.

Increases in the lower tax threshold ω_1,in this case, have effects that are quite different from the other cases. Recall, from Proposition 4.2, that changes in this parameter have no effect at all on Θ_1 or unemployment rate when benefits are taxed. Also, from Proposition 4.4, when benefits are untaxed but Condition (21) holds, increases in ω_1 *increase* Θ_1 and *reduce* the unemployment rate, through their effects on the reserve wage premium $(\Theta_1+\Theta_2)\tau_1(y_1+\sigma_1-\omega_1)$. In this case, however, with untaxed benefits but where Condition (21) does *not* hold, increases in ω_1 *decrease* Θ_1 and *increase* the unemployment rate. The reason for this is that, in this case, the reserve wage is set *equal* to ω_1. Any marginal increment in ω_1 will therefore increase the reserve wage, which increases expected wages, reducing the profitability of low quality jobs, and reducing their entry Θ_1. Since, from Proposition 4.1, there is no effect on Θ_2, then the total number of vacancies $\Theta_1+\Theta_2$ falls, and the unemployment rate rises.

In this case, unlike the case where Condition (21) holds, changes in τ_1 have no effect on Θ_1. Once again, this is because the reserve wage is equal to ω_1 in this case, which is independent of τ_1. Thus, no changes in τ_1 have any effect on the profitability or entry of low quality jobs. Also, from Proposition 4.1, we know that the entry of high quality jobs is independent of τ_1. Thus, both total vacancies $\Theta_1+\Theta_2$ and the unemployment are independent of the lower tax rate τ_1.

Finally, in this case as in all the other cases we have considered, and for the reasons discussed above, neither the upper tax threshold ω_2 nor the upper tax rate τ_2 have any effect on Θ_1, Θ_2, or, consequently, the unemployment rate.

Notice that unemployment in this (sub)case is unresponsive to marginal income tax rates $(\tau_i)_{i=1,2}$ for both high and low income workers, or the upper income tax threshold ω_2. In contrast, Pissarides (1985) finds that a higher wage tax rate increases unemployment rate when the workers' outside option remains untaxed. The (sub)case is essentially characterized by a corner solution to the workers' optimization problem where each worker equates his reserve with the non-taxable threshold.

As mentioned above, all of the results in this subsection have been derived under the restriction that both types of vacancies are created in equilibrium. This holds when conditions (11) and (15) are satisfied where $(\Theta_i)_{i=1,2}$ are evaluated at their expressions implied by Proposition 4.3 under both the (sub)case when (21) holds and when it is violated.

Having characterized the comparative static effects of changes in the policy parameters on the creation of different qualities of jobs and the unemployment rate, we now turn to consider issues of efficiency in this environment.

5 Constrained Productive Efficiency

We consider a social planner who wishes to maximize *ex ante* surplus per worker. Although the planner is capable of controlling the number of high quality and low quality vacancies created, she is constrainded by being required to conform to the urn-ball matching technology present in the economy. This means each vacancy approaches each worker with equal probability. The planner thus solves

$$\max_{\theta_1,\theta_2} \left(1 - e^{-\theta_2}\right) y_2 + e^{-\theta_2}\left(1 - e^{-\theta_1}\right) y_1 - \theta_1 k_1 - \theta_2 k_2$$

subject to $\theta_1, \theta_2 \geq 0$. [16]

There are two trade-offs facing the planner. As is the case when vacancies are homogeneous, increasing the number of entrants by one type i vacancy increases the probability, for each worker of being matched to a vacancy. However, each new vacancy adds a cost k_i, and reduces the probability of surplus being created when another vacancy of its own type is matched to a worker. In addition to this trade-off, the planner must also weigh the positive effect on total production by creating an additional *high quality* vacancy, against the loss of production resulting from the fact that a worker matched to this high quality vacancy can no longer be paired with a *low quality* one. This trade-off highlights an additional cost of creating a high quality vacancy not present in a homogeneous job economy.

[16] The planner's objective as stated above assumes it is socially optimal for a worker to be matched with a good vacancy instead of a bad one when approached by vacancies of both types. Clearly, this assumption is made with no loss of generality since good vacancies produce a larger surplus than bad ones when paired with any worker.

Proposition 5.1. *The solution to the social planner's problem is*

$$\Theta_1 = \max\left\{\ln\left(\frac{y_1}{y_2-y_1}\right)+\ln\left(\frac{k_2}{k_1}-1\right),\ 0\right\}$$
$$\Theta_2 = \max\left\{\ln\left(y_2-y_1\right)-\ln\left(k_2-k_1\right),\ 0\right\},$$

and is implementable under any policy setting that satisfies

$$\sigma_1 = \sigma_2 = b = \omega_1, \quad (\tau_1, \tau_2, \omega_2) \in [0,\ 1]^2 \times \mathbb{R}_+,$$

$$y_2 + \sigma_2 - y_1 - \sigma_1 > k_2 - k_1, \tag{24}$$

and

$$\frac{k_2 - k_1}{k_1} > \frac{y_2 + \sigma_2 - y_1 - \sigma_1}{y_1 + \sigma_1 - b}. \tag{25}$$

5.1 Constrained Efficiency of the Decentralized Equilibrium

Clearly, the planner's solution coincides exactly with the equilibrium allocations when all of the policy parameters are set to zero – the laissez-faire equilibrium. The efficiency of this equilibrium stems from the way the auction process delivers to each firm the expected social benefits from creating a new vacancy, for each type of vacancy, once they have incurred the entry costs. Consider, for example, a firm that creates a new low quality job. If the firm is alone when it approaches a worker, then the social benefit from creating that vacancy is y_1, which is precisely the payoff the firm gets in this case. If, however, the firm is not alone when approaching the worker, then the social benefit from producing the vacancy is zero – which, again, is the payoff the firm gets in this case. Consider now a firm that creates a new high quality job. If the firm is alone when it approaches the worker then the social benefit from creating the vacancy is y_2, which is exactly the payoff the firm gets in this case. If, alternatively, the firm finds that at least one low quality vacancy but no other high quality vacancy has approached the worker, then the social benefit from creating the vacancy is $y_2 - y_1$, which is what is delivered by the auction mechanism in this case. Finally, if a high quality vacancy faces at least one other high quality vacancy when approaching the worker, then the social benefit from creating that vacancy is zero – once again, this is the payoff delivered by the auction in this case.

In each case, the expected payoff to the firm is equated to the social benefit from the creation of the vacancy. In equilibrium, these expected payoffs (i.e., social benefits) are equated to the private costs of vacancy creation, which are also the social costs: k_1 and k_2.

It is, of course, no surprise that the decentralized equilibrium of this directed search model, in the absence of a government, is constrained efficient. However, Proposition 5.1 also implies that many other policy configurations are also efficient, as long as all firms are paid a subsidy that is equal to unemployment benefits, and benefits are taxed. Given this restriction, the tax rates themselves can take any value, as can the upper tax threshold. Notice that, when we impose the condition $\sigma_1 = \sigma_2 = b = \omega_1$, from Proposition 5.1, the equilibrium wage distribution becomes:

$$w_i^j = \begin{cases} w_0^0 = & b \\ w_1^0 = & b \\ w_1^1 = & (1-\tau_1)y_1 + b \\ w_2^0 = & b \\ w_2^1 = & (1-\tau_1)y_1 + b \\ w_2^2 = & (1-\tau_2)y_2 + (1-\tau_1)\omega_2 + \tau_1 b. \end{cases} \tag{26}$$

Wage dispersion still exists in this equilibrium, which can be eliminated entirely by the additional restriction that $\tau_1 = \tau_2 = 1$. With this further restriction imposed, the after-tax wage distribution collapses down to $w_i^j = b$. If we then impose the further restriction that the government balances its budget, *ex ante*, then we can determine the value of b from the government budget constraint. Subsidies per worker are simply b (If a worker is unemployed, he receives b, if he is employed in a low quality job, then this job receives subsidy $\sigma_1 = b$, if he is employed in a high quality job, then this job receives subsidy $\sigma_2 = b$.) Expected tax revenues per worker in this case are given by:

$$t = [e^{-\Theta_2}(1 - e^{-\Theta_1} - \Theta_1 e^{-\Theta_1}) + \Theta_2 e^{-\Theta_2}(1 - e^{-\Theta_1})]y_1 + (1 - e^{-\Theta_2} - \Theta_2 e^{-\Theta_2})y_2 \tag{27}$$

This leads, immediately, to the following result:

Proposition 5.2. *The following policy configuration induces equilibrium allocations that satisfy constrained efficiency, ex post equity, and balance the government's budget.*

$$\begin{aligned} \sigma_1 &= \sigma_2 = \omega_1 = b \\ \tau_1 &= \tau_2 = 1 \\ b &= [e^{-\Theta_2}(1 - e^{-\Theta_1} - \Theta_1 e^{-\Theta_1}) + \Theta_2 e^{-\Theta_2}(1 - e^{-\Theta_1})]y_1 + (1 - e^{-\Theta_2} - \Theta_2 e^{-\Theta_2})y_2 \end{aligned}$$

where Θ_1 *and* Θ_2 *are given in Proposition 5.1.*

Of course, with risk neutral workers, there is no gain to be had by eliminating the risk associated with the ex post wage distribution in this way. We have found, though, that analogous results apply when workers are risk averse.[17]

6 Conclusion

In this paper we have examined the implications of different policy configurations in an environment with is similar to the standard DMP one, but which has some important differences and quite different policy implications. Both models are, loosely speaking, "supply side" models where firm entry decisions are key for the determination of output and unemployment. Both models are similar, also, in the sense that each worker has one unit of labor to sell, each firm has one vacancy to sell, and the matching process faces a similar friction. However, whereas the DMP model can be used to justify austerity programs, reductions in tax rates, and employment subsidies, this cannot be said (in general) of the model we have considered here.

In all the cases we consider here, marginal changes to *upper* tax thresholds and tax rates have no effect on output, job qualities, or unemployment. Similarly, changes in subsidies for *high* quality jobs have no effect on unemployment – but they do affect the *mix* of job qualities: an increase in these subsidies increases the creation of high quality jobs, but also reduces the creation of low quality jobs by the same amount, leaving the unemployment rate unchanged. Also, in all cases, changes to unemployment benefits have no effect on the creation of high quality jobs.

[17] This material is available upon request.

The tax structure, in general, has no effect on output, job qualities, or unemployment, if benefits are taxed. If they are not taxed then the effects of the lower tax threshold and rate are more nuanced. While the creation of high quality jobs is unaffected by the lower tax rate and threshold, the creation of low quality jobs (and, hence, unemployment) depends on the initial values of these policy variables. If the lower tax rate is relatively small (according to Condition (21), above) then reductions in this tax rate will increase the creation of low quality jobs and reduce unemployment (in a way similar to that in Pissarides (1985)). However if the lower tax rate is large then no marginal reduction in this rate will have any effect on output, job qualities, or unemployment. Moreover, somewhat counterintuitively, in this case, an increase in the lower tax threshold will *decrease* the creation of low quality jobs and increase unemployment.

Increments to subsidies to low quality jobs increase the creation of these jobs, and decrease the creation of high quality jobs, when the benefits are taxed or if they are untaxed and the lower tax rate is relatively large (violating Condition (21)). In both cases, the increase in low quality job creation is larger than the decrease in high quality job creation and, so, unemployment falls. If, however, the lower tax rate is relatively small, then an increase in the subsidy to low quality jobs has an ambiguous effect on the quantity of these jobs, and unemployment – depending on parameter values.

While changes in benefits to do not affect the creation of high quality jobs, the effects of these changes on low quality jobs and unemployment depend on the tax structure. If benefits are taxed, or they are untaxed and the lower tax rate is small, then reducing benefits will increase the creation of low quality jobs and reduce unemployment (in a way similar to that in Pissarides (1985)). However, if benefits are untaxed and the lower tax rate is relatively large, then changes in benefits have no effect on output, the job quality mix, or unemployment.

The laissez-faire equilibrium in the DMP model is not, in general, constrained-efficient (unless the Hosios rule is imposed) but, in *this* model, it is. However, as we have seen here, the laissez-faire equilbrium is not the only constrained-efficient one. A broad array of other policy configurations are also efficient – including one that induces both *ex ante* and *ex post* equitable outcomes, and which balances the government's budget. Thus, although this model, like the DMP model, is a "supply side" model, it can be used to justify policies that redistribute income from the rich to the poor.

How robust should we expect these results to be? The model in this paper is very simple: it is static, it considers only two types of jobs, and it has homogeneous workers. Extending the model, to make it dynamic, would be an interesting exercise – particularly to consider issues of the optimal length of benefits, and the influences of policy parameters on on-the-job search, raiding, and counteroffers from incumbent employers.[18] However, if policy parameters are stationary, we believe that it is reasonable to presume that the main results here would be preserved. We also expect that, while allowing for more types of jobs would certainly complicate the analysis, analogous results would emerge. For example, with three types of jobs (assuming that all exist in equilibrium) we expect that subsidies to the top two jobs would have no effect on the unemployment rate. Allowing for different types of workers would also complicate things, since we would expect that worker productivities would now enter into the probabilities that firms assign when choosing workers to approach.[19] These would, in general, be functions of the tax parameters – which implies that these parameters could enter into the equilibrium matching process itself. We consider all of these extensions to be interesting and worth exploring.

Appendix

Proof of Proposition 3.1. Note first and define $w^0 = w_1^0 = w_2^0 = w_a$, which is the only expression containing w_r, and $w^1 = w_1^1 = w_2^1$. In addition, redefine $\hat{y}_i = y_i + \sigma_i$ to simplify notation. Using these and rewriting the workers' payoff, we can form the Lagrangian

$$\mathcal{L} = \max_{w_r,\theta_1,\theta_2} e^{-(\theta_1+\theta_2)}[w_0^0 - w^1 + (\theta_1+\theta_2)(w^0 - w^1)] + e^{-\theta_2}(1+\theta_2)[w^1 - w_2^2] + w_2^2$$

$$+\lambda_1[e^{-(\theta_1+\theta_2)}(\hat{y}_1 - w_r) - k_1] + \lambda_2[e^{-\theta_2}(\hat{y}_2 - \hat{y}_1) + e^{-(\theta_1+\theta_2)}(\hat{y}_1 - w_r) - k_2]$$

[18] Julien, Kennes, and King (2006) consider a model with these features, but without policy parameters. Most of the essential properties of the static model are preserved in the dynamic extension, including the efficiency results.

[19] See Basov, King, and Uren (2014), for example.

The necessary condition for w_r is given by

$$(\theta_1+\theta_2)\frac{\partial w^0}{\partial w_r}=\lambda_1+\lambda_2$$

where

$$\frac{\partial w^0}{\partial w_r}=\begin{cases}1 & \text{if } w_r\leq\omega_1\\ 1-\tau_1 & \text{if } w_r\in(\omega_1,\omega_2)\\ 1-\tau_2 & \text{if } w_r\geq\omega_2.\end{cases}$$

By complementary slackness, $\lambda_1[e^{-(\theta_1+\theta_2)}(\hat{y}_1-w_r)-k_1]=0$ and $\lambda_2[e^{-\theta_2}(\hat{y}_2-\hat{y}_1)+e^{-(\theta_1+\theta_2)}(\hat{y}_1-w_r)-k_2]=0$. From free entry, the constraints are always binding and $\lambda_1>0,\lambda_2>0$. Since $\frac{\partial w^0}{\partial w_r}>0$, $\theta_1+\theta_2>0$.
For θ_1 we have

$$[w_0^0-w^1+(\theta_1+\theta_2)(w^0-w^1)]+(w^0-w^1)=(\lambda_1+\lambda_2)(\hat{y}_1-w_r),$$

and for θ_2

$$\theta_2(w_2^2-w^1)=\lambda_2(\hat{y}_2-\hat{y}_1).$$

Since $(w_2^2-w^1)>0$, $(\hat{y}_2-\hat{y}_1)>0$ and $\lambda_2>0$, then $\theta_2>0$, which implies $\theta_1>0$ as well, and both firms participate in the submarket off equilibrium path.
Note that after simplifications

$$(w_2^2-w^1)=(\hat{y}_2-\hat{y}_1)+(\tau_2-\tau_1)\omega_2,$$

therefore

$$\theta_2((\hat{y}_2-\hat{y}_1)+(\tau_2-\tau_1)\omega_2)=\lambda_2(\hat{y}_2-\hat{y}_1),$$

and $\lambda_2>\theta_2$. Without tax policy, $\tau_i=0$, we get $\lambda_2=\theta_2$ as in standard competitive search.
Because of the progressivity of the income tax structure, we must consider several cases for w_r. Since we consider only equilibria where both types of vacancies open, we rule out cases where $w_r\geq y_1+\sigma_1$.
Case 1: $b\leq w_r\leq\omega_1<y_1+\sigma_1<\omega_2$. This is a situation where unemployment benefits are untaxed, and a reserve wage under the first threshold. Workers pay taxes only when selected by more than one firm:

$$w_0^0=b\quad w_1^0=w_2^0=w_r\quad w_a'=\frac{\partial w_a}{\partial w_r}=1. \tag{28}$$

This immediately implies, from the necessary conditions, that $\theta_1 + \theta_2 = \lambda_1 + \lambda_2$, and $\lambda_1 < \theta_1$. Using this in the condition for θ_1, and simplifying, gives us

$$w_r^* = b + (\theta_1 + \theta_2)\,\tau_1\,(y_1 + \sigma_1 - \omega_1)\,,$$

recognizing this expression is valid whenever $b \le w_r^* < \omega_1$.

Case 2: $b \le \omega_1 \le w_r < y_1 + \sigma_1 < \omega_2$. This is a situation of untaxed unemployment benefits, but where the reserve wage is taxed when employed:

$$w_0^0 = b \quad w_1^0 = w_2^0 = (1-\tau_1)w_r + \tau_1\omega_1 \quad w'(r) = 1 - \tau_1 \tag{29}$$

Substituting these into the condition for θ_1, and rearranging, gives us $w_r^* = \frac{b-\tau_1\omega_1}{1-\tau_1}$. However, w_r^* in this case is suboptimal since

$$\omega_1 \le (1-\tau_1)w_r^* + \tau_1\omega_1 = b \le \omega_1,$$

with strict inequalities when $b < \omega_1$. The worker sets the reserve wage so that his after tax reserve wage equals his outside option. But the resulting expression above implies that $\omega_1 < \omega_1$, a contradiction. Hence, this case is only valid for $b = \omega_1$.

Case 3: $w_r \le \omega_1 < b \le y_1 + \sigma_1 < \omega_2$. In this situation, the workers' payoffs satisfy:

$$w_0^0 = (1-\tau_1)b + \tau_1\omega_1 \qquad w_1^0 = w_2^0 = w_r \qquad w_a' = 1 - \tau_1.$$

This implies that the optimal reservation wage in this interval exceeds a convex combination of b and ω_1. But $\omega_1 \le b$ implies $\omega_1 < w_r$, a contradiction.

Case 4: $\omega_1 < w_r \le b \le y_1 + \sigma_1 \le \omega_2$. Workers pay taxes regardless of their employment situation:

$$w_0^0 = (1-\tau_1)b + \tau_1\omega_1 \quad w_1^0 = w_2^0 = (1-\tau_1)w_r + \tau_1\omega_1 \quad w_a' = 1 - \tau_1.$$

Using this in the first necessary condition for θ_1, this implies $w_r^* = b$.

Case 5: $\omega_1 < b \le w_r \le y_1 + \sigma_1 \le \omega_2$. Workers pay taxes regardless of their employment situation:

$$w_0^0 = (1-\tau_1)b + \tau_1\omega_1 \quad w_1^0 = w_2^0 = (1-\tau_1)w_r + \tau_1\omega_1 \quad w_a' = 1 - \tau_1. \tag{30}$$

Using this in the first necessary condition for θ_1, this implies $w_r^* = b$.

Second order conditions are cumbersome, and available upon request. □

Proof of Proposition 3.2. To prove existence, consider the two equilibrium entry conditions (9) under case $b < \omega_1$ and $w_r^* < \omega_1$. Subtituting in the equilibrium reserve wage $w_r^* = b + (\Theta_1 + \Theta_2)\tau_1(y_1 + \sigma_1 - \omega_1)$ and rearranging yields

$$L(\Theta_1) \equiv e^{\Theta_1} = \frac{y_1 + \sigma_1 - \omega_1 - b - (\Theta_1 + \Theta_2)\tau_1(y_1 + \sigma_1 - \omega_1)}{k_1 e^{\Theta_2}} \equiv R(\Theta_1).$$

Assume that condition for $\Theta_2 > 0$ holds. We have $\lim_{\Theta_1 \to 0} L(\Theta_1) = 1$, $\lim_{\Theta_1 \to \infty} L(\Theta_1) = \infty$, with $\lim_{\Theta_1 \to 0} R(\Theta_1) = \frac{y_1+\sigma_1-\omega_1-b-\Theta_2\tau_1(y_1+\sigma_1-\omega_1)}{k_1 e^{\Theta_2}}$ and $\lim_{\Theta_1 \to \infty} R(\Theta_1) = -\infty$. Note further that $L'(\Theta_1) = e^{\Theta_1} = L''(\Theta_1) > 0$, so $L(\Theta_1)$ is strictly convex, and $R'(\Theta_1) = -\tau_1(y_1 + \sigma_1 - \omega_1) < 0$ with $R(\Theta_1)$ linear in Θ_1. Therefore, there exists a $\Theta_1 > 0$ if and only if $\lim_{\Theta_1 \to 0} R(\Theta_1) > 1$ or

$$y_1 + \sigma_1 - \omega_1 - b - \Theta_2\tau_1(y_1 + \sigma_1 - \omega_1) > k_1 e^{\Theta_2}. \tag{31}$$

Substituting the values of Θ_2 and e^{Θ_2} from (10), gives

$$\frac{y_1 + \sigma_1 - b - \ln \frac{y_2+\sigma_2-y_1-\sigma_1}{k_2-k_1}\tau_1(y_1 + \sigma_1 - \omega_1)}{k_1} > \frac{y_2 + \sigma_2 - y_1 - \sigma_1}{k_2 - k_1},$$

the condition in the Proposition. Finally, note that (31) is equivalent to

$$\bar{\Theta}_1 = \frac{y_1 + \sigma_1 - \omega_1 - b - \Theta_2\tau_1(y_1 + \sigma_1 - \omega_1)}{k_1 e^{\Theta_2}} > 1.$$

Therefore, from the properties of $L()$ and $R()$ functions, there exist a unique $\Theta_1 > 0$, when there exist a unique $\Theta_2 > 0$, such that $\Theta_1 < \bar{\Theta}_1$.

□

Proof of Proposition 4.2. Simply taking the derivatives of the equilibrium conditions gives us:

$$\partial\Theta_1/\partial b = -(y_1 + \sigma_1 - b)^{-1} < 0$$
$$\partial\Theta_1/\partial\sigma_1 = (y_1 + \sigma_1 - b)^{-1} > 0$$
$$\partial\Theta_1/\partial\sigma_2 = -(y_2 + \sigma_2 - y_1 - \sigma_1)^{-1} < 0$$
$$\partial\Theta_1/\partial\omega_1 = \partial\Theta_1/\partial\omega_2 = \partial\Theta_1/\partial\tau_1 = \partial\Theta_1/\partial\tau_2 = 0$$
$$\partial U/\partial b = k_1 (y_1 + \sigma_1 - b)^{-2} > 0$$
$$\partial U/\partial\sigma_1 = -k_1 (y_1 + \sigma_1 - b)^{-2} < 0$$
$$\partial U/\partial\sigma_2 = \partial U/\partial\omega_1 = \partial U/\partial\omega_2 = \partial U/\partial\tau_1 = \partial U/\partial\tau_2 = 0$$

□

Proof of Proposition 4.3. We prove the proposition in two separate (sub) cases. In each case, we first obtain expressions pinning down $(\Theta_i)_{i=1,2}$. Then we solve for unemployment, $e^{-\Theta_1-\Theta_2}$. Finally, we infer a necessary and sufficient condition that distinguishes (sub) case 1 from 2 under a two-job equilibrium.

(Sub) Case 1: $w_r^* < \omega_1$. From row 2 of (9), it follows that

$$\Theta_2 = \ln(y_2+\sigma_2-y_1-\sigma_1) - \ln(k_2-k_1) \tag{32}$$

in equilibrium. Substitute $w_r^* = b + \tau_1(\Theta_1+\Theta_2)(y_1+\sigma_1-\omega_1)$ from Proposition 3.1, and then (32), into row 1 of (9) to obtain

$$k_1 e^{\Theta_1} = \frac{k_2-k_1}{y_2+\sigma_2-y_1-\sigma_1}\left(y_1+\sigma_1-b-(\ln\left(\frac{y_2+\sigma_2-y_1-\sigma_1}{k_2-k_1}\right)+\Theta_1)\tau_1(y_1+\sigma_1-\omega_1)\right). \tag{33}$$

Substitute $w_r^* = b+\tau_1(\Theta_1+\Theta_2)(y_1+\sigma_1-\omega_1)$ into row 1 of (9) to obtain given expression for equilibrium unemployment $e^{-\Theta_1-\Theta_2}$.

(Sub) Case 2: $w_r^* = \omega_1$. Substitute $w_r^* = \omega_1$ (9) to obtain:

$$\begin{aligned} k_1 &= e^{-\Theta_1-\Theta_2}(y_1+\sigma_1-\omega_1) \\ k_2-k_1 &= e^{-\Theta_2}(y_1+\sigma_1-y_2-\sigma_2). \end{aligned}$$

Rearranging yields:

$$\begin{aligned} \Theta_1 &= \ln\left(\frac{y_1+\sigma_1-\omega_1}{y_2+\sigma_2-y_1-\sigma_1}\right) + \ln\left(\frac{k_2}{k_1}-1\right) \\ \Theta_2 &= \ln(y_2+\sigma_2-y_1-\sigma_1) - \ln(k_2-k_1). \end{aligned} \tag{34}$$

Substitute into $e^{-\Theta_1-\Theta_2}$ to get the stated expression for equilibrium unemployment.

Proof of Proposition 4.4:
Simply taking the derivatives of the equilibrium conditions gives us:

$$\partial\Theta_1/\partial b = -\left(k_1 e^{\Theta_1+\Theta_2} + \tau_1(y_1+\sigma_1-\omega_1)\right)^{-1} < 0$$
$$\partial\Theta_1/\partial\sigma_1 = \left[1-(\Theta_1+\Theta_2)\tau_1\right]\left[k_1 e^{\Theta_1+\Theta_2} + \tau_1(y_1+\sigma_1-\omega_1)\right]^{-1} + (y_2+\sigma_2-y_1-\sigma_1)^{-1}$$
$$\partial\Theta_1/\partial\sigma_2 = -(y_2+\sigma_2-y_1-\sigma_1)^{-1} < 0$$
$$\partial\Theta_1/\partial\omega_1 = (\Theta_1+\Theta_2)\left[k_1 e^{\Theta_1+\Theta_2} + \tau_1(y_1+\sigma_1-\omega_1)\right]^{-1} > 0$$
$$\partial\Theta_1/\partial\tau_1 = -(\Theta_1+\Theta_2)(y_1+\sigma_1-\omega_1)\left[k_1 e^{\Theta_1+\Theta_2} + \tau_1(y_1+\sigma_1-\omega_1)\right]^{-1} < 0$$
$$\partial\Theta_1/\partial\omega_2 = \partial\Theta_1/\partial\tau_2 = 0$$
$$\partial U/\partial b = e^{-(\Theta_1+\Theta_2)}\left[k_1 e^{\Theta_1+\Theta_2} + \tau_1(y_1+\sigma_1-\omega_1)\right]^{-1} > 0$$
$$\partial U/\partial\sigma_1 = -e^{-(\Theta_1+\Theta_2)}\left[1-(\Theta_1+\Theta_2)\tau_1\right]\left[k_1 e^{\Theta_1+\Theta_2} + \tau_1(y_1+\sigma_1-\omega_1)\right]^{-1}$$
$$\partial U/\partial\omega_1 = -e^{-(\Theta_1+\Theta_2)}(\Theta_1+\Theta_2)\left[k_1 e^{\Theta_1+\Theta_2} + \tau_1(y_1+\sigma_1-\omega_1)\right]^{-1} < 0$$
$$\partial U/\partial\tau_1 = e^{-(\Theta_1+\Theta_2)}(\Theta_1+\Theta_2)(y_1+\sigma_1-\omega_1)\left[k_1 e^{\Theta_1+\Theta_2} + \tau_1(y_1+\sigma_1-\omega_1)\right]^{-1} > 0$$
$$\partial U/\partial\sigma_2 = \partial U/\partial\omega_2 = \partial U/\partial\tau_2 = 0$$

Derivation of the Derivative of Θ_1 wrt σ_1:

Substitute $w_r^* = b + (\Theta_1+\Theta_2)\tau_1(y_1+\sigma_1-\omega_1)$ into (9) to get

$$\begin{aligned} k_1 &= e^{\Theta_1+\Theta_2}(y_1+\sigma_1-b-(\Theta_1+\Theta_2)\tau_1(y_1+\sigma_1-\omega_1)) \\ k_2-k_1 &= e^{-\Theta_2}(y_2+\sigma_2-y_1-\sigma_1) \end{aligned}$$

Multiply row 1 by e^{Θ_1} and totally differentiate the system with respect to σ_1 to obtain:

$$\begin{aligned} k_1 e^{\Theta_1}\frac{\partial\Theta_1}{\partial\sigma_1} &= -e^{-\Phi_2}\frac{\partial\Theta_2}{\partial\sigma_1}(y_1-b-(\Theta_1+\Theta_2)\tau_1(y_1-\omega_1)) \\ &\quad + e^{-\Theta_2}\left(1-\left(\frac{\partial\Theta_1}{\partial\sigma_1}+\frac{\partial\Theta_2}{\partial\sigma_1}\right)\tau_1(y_1-\omega_1)-(\Theta_1+\Theta_2)\tau_1\right) \\ \frac{\partial\Theta_2}{\partial\sigma_1} &= -\frac{1}{y_2-y_1}. \end{aligned}$$

We have replaced $y_i+\sigma_i$ with y_i for brevity. Then:

$$\left(k_1 e^{\Theta_1+\Theta_2}+\tau_1(y_1-\omega_1)\right)\frac{\partial\Theta_1}{\partial\sigma_1} = -\frac{\partial\Theta_2}{\partial\sigma_1}\left(y_1-b+(1-\Theta_1-\pi_2^*)\tau_1(y_1-\omega_1)\right) + 1-(\Theta_1+\Theta_2)\tau_1$$

after some algebra. Substitute derivative of Θ_2^* back into the above expression and simplify to obtain:

$$\begin{aligned} \left(k_1 e^{\Theta_1+\Theta_2}+\tau_1(y_1-\omega_1)\right)\frac{\partial\Theta_1}{\partial\sigma_1} &= \frac{y_1-b+(1-\Theta_1-\Theta_2)\tau_1(y_1-\omega_1)}{y_2-y_1} + 1-(\Theta_1+\Theta_2)\tau_1 \\ &= \frac{y_2-w_r+\tau_1(y_1-\omega_1)-(\Theta_1+\Theta_2)\tau_1(y_2-y_1)}{y_2-y_1}. \end{aligned}$$

But rearranging expression for w_r^* in Proposition 3.1 when $w_r^* \in [b,\ \omega_1)$ yields $\Theta_1 + \Theta_2 = \frac{w_r - b}{\tau_1(y_1 - \omega_1)}$. Substituting into the above obtains

$$\left(k_1 e^{\Theta_1+\Theta_2} + \tau_1(y_1-\omega_1)\right)\frac{\partial\Theta_1}{\partial\sigma_1} = \frac{y_2 - w_r + \tau_1(y_1-\omega_1) - \frac{(w_r-b)(y_2-y_1)}{y_1-\omega_1}}{y_2-y_1}$$
$$= \frac{\left(y_2 - w_r + \tau_1(y_1-\omega_1)\right)(y_1-\omega_1) - (w_r-b)(y_2-y_1)}{(y_2-y_1)(y_1-\omega_1)}.$$

Replace y_i with $y_i + \sigma_i$ to obtain expression for the derivative found in the paper. □

Proof of Proposition 4.5. Simply taking derivative of equilibrium conditions give:

$$\partial\Theta_1/\partial\sigma_1 = (y_2+\sigma_2-\omega_1)/[(y_1+\sigma_1-\omega_1)(y_2+\sigma_2-y_1-\sigma_1)]^{-1} > 0$$
$$\partial\Theta_1/\partial\sigma_2 = -(y_2+\sigma_2-y_1-\sigma_1)^{-1} < 0$$
$$\partial\Theta_1/\partial\omega_1 = -(y_1+\sigma_1-\omega_1)^{-1} < 0$$
$$\partial\Theta_1/\partial b = \partial\Theta_1/\partial\omega_2 = \partial\Theta_1/\partial\tau_1 = \partial\Theta_1/\partial\tau_2 = 0$$
$$\partial U/\partial\sigma_1 = -k_1(y_1+\sigma_1-\omega_1)^{-2} < 0$$
$$\partial U/\partial\omega_1 = k_1(y_1+\sigma_1-\omega_1) > 0$$
$$\partial U/\partial b = \partial U/\partial\sigma_2 = \partial U/\partial\omega_2 = \partial U/\partial\tau_1 = \partial U/\partial\tau_2 = 0$$

□

Proof of Proposition 5.1. The first order conditions imply

$$\theta_1: \quad e^{-\theta_1-\theta_2}y_1 = k_1$$
$$\theta_2: \quad e^{-\theta_2}y_2 - e^{-\theta_2}\left(1-e^{-\Theta_1}\right)y_1 = k_2.$$

Solving simultaneously for $(\theta_i)_{i=1,2}$ and imposing non-negativity constraints completes the proof. □

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