Open Market Operations*

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Abstract

We develop models with liquid government bonds and currency to analyze monetary policy, especially open market operations. Various specifications are considered for market structure, and for the liquidity – i.e., acceptability or pledgeability – of money and bonds in their roles as media of exchange or collateral. Theory delivers sharp policy predictions. It can also generate negative nominal yields, endogenous market segmentation, liquidity traps, and nominal prices or interest rates that appear sluggish. Differences in acceptability or pledgeability are not simply assumed; they are endogenized using information frictions. This naturally generates multiple equilibria, but conditional on selection, we still deliver sharp predictions.

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1 Introduction

This is a paper on monetary economics with the following overall theme: Versions of a commonly-used formal model can speak to many issues of interest to people typically more concerned with policy than theory. Our environment builds on the New Monetarist approach (see Section 2 for a literature review). Some people believe that while such models may be useful for showing why money is useful, they are silent on contemporary policy matters. We want to dispel that belief, and although not the first to try, our approach is somewhat novel. First, we consider changing the growth rate of the money supply, or equivalently, in stationary equilibrium, the inflation rate (by the Quantity Equation) or the nominal interest rate on illiquid assets (by the Fisher Equation). This has been previously analyzed extensively, of course. A bigger novelty is our focus on changing the mix of currency and other assets in the hands of private agents – the classic version of which swaps cash for short-term government bonds, called an *open market operation* or *OMO*, but the analysis can also apply to swaps of cash for other assets, called *quantitative easing* or *QE*. In fact, we show how these policies can be interpreted as implementing targets for alternative interest rates.[1]

The model generates the following results: (1) There is a lower bound for nominal interest rates on liquid bonds, say ρ, which can be $\underline{\rho} = 0$, but $\underline{\rho} > 0$ or $\underline{\rho} < 0$ are also possible. (2) Generally ρ does not move one-for-one with inflation, which is ostensibly a violation of the Fisher Equation, even though that relationship holds exactly for illiquid bonds. (3) Nominal prices do not change one-for-one with money supply changes engineered by OMO's, ostensibly a violation of the Quantity Equation, even though a pure currency injection (the proverbial helicopter drop) is neutral. (4) Equilibria can display all the features of a liquidity trap at $\rho = \underline{\rho}$.

[1] We take for granted that it is generally believed monetary policy has important real effects. See Ramey (2015) for a survey of empirical work on the issue, and any standard macro textbook to support the notion that it constitutes conventional wisdom. To be clear, our goal is to examine policy through the lens of monetary theory with relatively solid microfoundations.

(5) Markets can segment endogenously into those where only currency is used in payments and those where other assets are used. (6) The acceptability and pledgeability of assets can be endogenized based on information frictions, leading naturally to multiple equilibria, with different degrees of liquidity and very different policy implications. While this is an exercise in economic theory, we think that these results considerably enhance our understanding of the role of assets in the exchange process and of the impact of policy in the real world.

The approach strives to be precise about the ways in which assets facilitate transactions, but we also offer alternative heuristics for the model. Thus, the same equations can describe assets serving as media of exchange, or as collateral in secured lending arrangements. There are also different interpretations of the agents, who may be consumers buying goods, firms acquiring inputs, or investors and financial institutions swapping assets. The findings are also shown to be robust in that they hold under different combinations of random or directed search and bargaining or price posting. While flexible, the environment is geared to be rigorous about the frictions that hinder unsecured credit and determine liquidity, which means going beyond the common practice (at least in some circles) of putting assets in utility functions or imposing cash-in-advance and related constraints. In particular, differences in acceptability or pledgeability are not just assumed, they are endogenized using information frictions, because we believe that modeling the transaction process in more detail generates more economic insight.

The paper is organized as follows. Section 2 reviews the literature, although that can be deferred if one prefers seeing the theory first. Section 3 describes the basic environment. Section 4 considers equilibrium with random search, taking liquidity as given, and demonstrates relatively simple versions of the results about nominal interest rates, the appearance of sluggish prices, and liquidity traps. Section 5 endogenizes liquidity and shows how this leads to multiplicity. Section 6 considers directed search and market segmentation. Section 7 concludes.

2 Literature

The New Monetarist approach is discussed in surveys by Williamson and Wright (2010), Nosal and Rocheteau (2011) and Lagos et al. (2015). For those unfamiliar with the method, these models contain elements of general equilibrium theory but also have agents sometimes trading in decentralized markets, as in search theory, where frictions make it interesting to ask *how* they trade. Do they use barter, credit or money? If credit, is it unsecured or secured? Which objects get used as payment instruments or collateral? This work also tries to ascertain why assets may be more or less liquid, and considers various market structures. While the approach can accommodate nominal rigidities, they are not needed to generate interesting results. In this spirit, we avoid sticky prices, but, as mentioned, the outcome is consistent with the appearance of sluggish nominal adjustment.

One motivation for the project is to continue an ongoing effort exploring these kinds of models with different asset combinations.[2] However, a focus on money and bonds is especially relevant for policy. We are not the only ones considering these policies in this class of models. Related work includes Williamson (2012,2014*a*,*b*), Rocheteau and Rodriguez-Lopez (2014), and Dong and Xiao (2015). This paper has various differences, but perhaps the biggest is that we *derive* acceptability and pledgeability using information frictions, which endogenizes liquidity, and can lead to multiple equilibria with different exchange patterns and policy implications. Similar analyses of information frictions include Lester et al. (2012), Li et al. (2012), and references therein, but they do not study policy the way we do.

In terms of different interpretations of traders, many papers call them households, but Silveira and Wright (2010), Williamson and Wright (2010), Chiu and Meh (2011) and Chiu et al. (2016) provide examples where they are producers. Versions where they are financial institutions include Berentsen and Monnet (2008),

[2] See, e.g., Geromichalos et al. (2007), who have money and equity, He et al. (2008), who have money and bank deposits, Lagos and Rocheteau (2008), who have money and capital, or Lagos (2010), who has debt and equity. See the surveys mentioned above for more.

Koeppl et al. (2008,2012), Chapman et al. (2011,2013), Afonso and Lagos (2015a,b), Bech and Monnet (2015) and Chiu and Monnet (2014). Versions where they are investors trading financial assets include Duffie et al. (2005), Lagos and Rocheteau (2009) and many extensions of their work. As regards different interpretations for the role of assets in exchange, our equations apply whether assets get handed over as a medium of exchange, as in Kiyotaki and Wright (1989,1993), or are used as collateral in support of deferred settlement, as in Kiyotaki and Moore (1997,2005). We also discuss an interpretation in terms of stylized repurchase agreements, which is especially relevant for government bonds.[3]

Concerning particular results, He et al. (2008) show how to get negative nominal rates in a model where currency is subject to theft while other assets are not (see also Sanches and Williamson 2010). This is fine, but it seems worth considering alternatives, and our information frictions can be understood as another way to capture safety: assets here cannot be stolen, but can be counterfeited (lemons). On sluggish prices, there is much work, but our results are in the spirit of saying that prices may look sticky without this being exploitable by policy. On multiple equilibria with different transactions patterns and liquidity, many papers on the microfoundations of monetary economics are about exactly this, but OMO's have not been analyzed as they are here. On the liquidity trap, quantitative easing and related issues, again see Williamson's recent work and references therein.[4]

[3]The goal is not to provide a microfounded model of repos, only to propose a flexible mapping between theory and institutions. See, e.g., Vayanos and Weill (2008), Copeland et al. (2012), Antinolfi et al. (2015) and Gottardi et al. (2015) for serious analyses of repos.

[4]It is fair to say there is much overlap between our paper and Williamson's; we think it is unfair to say our results are redundant given his, or vice versa, as it would be unfair to say there is little in all this work that is not in the above-mentioned surveys – or Keynes, Menger and Jevons, for that matter. In this research details matter. To differentiate our product, most importantly, we derive acceptability and pledgeability endogenously using aymmetric information, which yields multiple equilibria with very different policy implications. Also, Williamson uses take-it-or-leave-it offers, while we allow general pricing mechanisms, which is not pseudo generality, but crucial for endogenizing liquidity by having sellers invest in information (they obviously never invest if buyers get the entire surplus). We also consider directed search, which is crucial for endogenizing market segmentation. Williamson does not consider our information frictions, general mechanisms or directed search, but, to his credit, provides more in-depth analyses fiscal policy, private debt and banking. Therefore, rather than redundant, we consider the papers complementary.

Research on market segmentation includes Alvarez et al. (2001,2002,2009). They impose CIA (cash-in-advance) constraints, and assume not everyone is active in all markets or impose a cost to transferring resources between markets. The focus on getting changes in real interest rates and output from monetary injections, negative relationships between inflation and interest rates, and persistent liquidity effects on interest and exchange rates. See Kahn (2006) for an overview and Chiu (2014) for a recent contribution. While the issues are clearly related, our methods are distinct. We have different assets accepted in different submarkets, and buyers choose where to go based on market tightness. There are no CIA constraints – agents can always trade in a cashless submarket – and heterogeneous portfolios are choices, not restrictions. A further contrast is apparent when we mention events like "a type-i buyer meets a type-j seller" or "this agent visits that submarket." There is no notion of who trades with whom in those models, which are Walrasian, except for CIA and related constraints.[5]

Search-based models with bonds and money include Rocheteau (2002) and Shi (2005), where newly-issued bonds cannot be used as to pay for goods. In contrast, our bonds are liquid, making their interest rates lower than those on illiquid bonds, in accordance with evidence in Krishnamurthy and Vissing-Jorgensen (2012). Shi (2008) considers a similar economy with two types of goods, and shows it is socially optimal to restrict the acceptability of nominal bonds, related to Kocherlakota (2003). Also, Shi (2014) studies OMO's in a model where bonds are partially acceptable, a special case of our setup, and where there is temporary separation between the bonds market and the goods market. Our model does not have separation between goods and bond markets, but delivers segmentation endogenously using directed search. While this review could go on much longer, this should suffice to describe how the project fits into the literature.

[5] There is a lot more research using CIA constraints or assets in utility functions; see Bansal and Coleman (1996) for one such model addressing similar issues, and delivering interesting results, taking a different approach from the one used here.

3 Model

As in Lagos and Wright (2005), each period of discrete time has two markets: a decentralized market, or DM, with frictions detailed below; and a frictionless centralized market, or CM. In the CM, a large number of infinitely-lived agents work, consume and adjust their portfolios. In the DM, some of the agents called *sellers* can provide something – a good, service, input or asset – that others called *buyers* would like to acquire (buyer and seller types are permanent, as in Rocheteau and Wright 2005, but not much changes if they are determined randomly each period). Let μ be the measure of buyers and n the ratio of the sellers to buyers in the DM, where they meet pairwise, with α the probability a buyer meets a seller and α/n the probability a seller meets a buyer. This background environment is natural for our purposes because its core element is an asynchronicity between desired expenditures and receipts, the key for any analysis of money or credit.

The period payoffs for buyers and sellers are

$$\mathcal{U}(q,x,\ell)=u(q)+U(x)-\ell \quad \text{and} \quad \widetilde{\mathcal{U}}(q,x,\ell)=-c(q)+\widetilde{U}(x)-\ell, \tag{1}$$

where q is the DM object being traded, x is the CM numeraire good and ℓ is CM labor. For sellers, $c\left(q\right)$ is a cost function, which can be an opportunity cost when q is an asset or input. For buyers, $u\left(q\right)$ can be a utility function, or a production function taking q as input and delivering numeraire in the next CM. The same formulation thus captures DM consumers acquiring output or producers acquiring input, which is relevant to the extent that liquidity considerations impinge on both. Also, as mentioned in Section 2, in some applications the DM traders are financial institutions, e.g., banks in the Fed Funds market, trading to adjust their reserve positions (Afonso and Lagos 2015*a*,*b*). Given this, rather than commit to a particular institutional setting, we keep the formalization abstract so that it can be applied to various types of decentralized trade.

Assume U, $\widetilde{U}$, u and c are twice continuously differentiable with the usual monotonicity and curvature properties. Also, normalize $u(0) = c(0) = 0$, assume there is a $\hat{q} > 0$ such that $u(\hat{q}) = c(\hat{q}) > 0$, and define the efficient q by $u'(q^*) = c'(q^*)$. Quasi-linearity in (1) simplifies the analysis because it leads to a degenerate distribution of assets across agents of a given type at the start of each DM, and makes CM payoffs linear in wealth. There is a discount factor $\beta = 1/(1+r)$, $r > 0$, between the CM and DM, while any discounting between the DM and CM is subsumed in the notation in (1). Also, x and q have limited storability, to hinder barter, and agents are to some degree anonymous in the DM, to hinder unsecured credit. As discussed in detail in the literature mentioned in Section 2, this generates a role for assets in the facilitation of exchange.

There are two assets that can potentially serve in this capacity: money, and bonds, meant to be stylized versions of T-bills. Their supplies are A_m and A_b, and their CM prices are ϕ_m and ϕ_b. To ease notation, the benchmark specification has short-term real bonds that, akin to Arrow securities, are issued in one CM and yield a unit of numeraire in the next, but we also consider nominal and long-term bonds. The real value of money and bonds per buyer are denoted z_m and z_b. For money $z_m = \phi_m A_m$; for real bonds $z_b = A_b$; for nominal bonds $z_b = \phi_m A_b$; and for long-term bonds, $z_b = (\phi_b + \delta) A_b$ where δ is a real coupon. In the DM, assets are *partially liquid*, in the sense that they can be used in some transactions, at least up to some limit. There are different ways in which this might work. The simplest is to say that sellers accept some but perhaps not all assets as media of exchange, for immediate settlement, as in Kiyotaki and Wright (1989,1993). However, that may be too restrictive, as we now explain.

Consider describing the DM in terms of deferred settlement. As in Kiyotaki and Moore (1997,2005), a buyer (borrower) getting q promises the seller (creditor) payment in numeraire in the next CM, but due to limited commitment, this must be secured by assets pledged as collateral. If an agent reneges, his assets are seized as punishment to dissuade opportunistic default. This captures the way assets facilitate intertemporal trade, beyond serving as media of exchange in quid pro quo exchange, and is standard in all Kiyotaki-Moore models. Similarly, we can describe the DM in terms of repurchase agreements, where a buyer getting q gives assets to a seller, who gives them back for numeraire at prearranged terms in the next CM. Whether it is important to give back the same assets, or to prearrange the terms, depends on details that merit discussion, but here we simply want to suggest that repos are another realistic way that assets facilitate trade. This is relevant for us because, while individuals may not often use T-bills for shopping, such securities are routinely used by financial institutions.[6]

Kiyotaki-Moore models typically allow only a fraction $\chi_j \in [0, 1]$ of asset j to be used in trade, and we follow the same approach. Section 5 shows how to endogenize χ_j using private information; for now it is exogenous that χ_m and χ_b are the fractions of money and bond holdings that can be used in a DM transaction, with $\chi_j > 0$ unless stated otherwise. In a deferred settlement scheme, e.g., χ_b describes the haircut one takes when using bonds as collateral, often motivated by saying that defaulters can abscond with a fraction $1 - \chi_b$ of their holdings. For χ_m, equally plausible stories include saying that sellers are worried about counterfeiting or, thinking about money broadly to include demand deposits, saying they are worried about bad checks. While $\chi_j = 1$ is of course a fine special case, there is no reason to impose that restriction at this point.

Let α_m be the probability a random seller in the DM accepts only money, α_b the probability he accepts only bonds, and α_2 the probability he accepts both. Assume $\alpha_j > 0$ unless stated otherwise, as in special cases like $\alpha_b = 0$ (no one accepts only bonds), $\alpha_b = \alpha_2 = 0$ (no one accepts bonds), and $\alpha_b = \alpha_m = 0$ (the assets are

[6] The idea is that without being explicit about the mechanism that records, monitors and enforces repos and related arrangements, to better understand assets' role in the transactions process, it is useful to step outside the formal model and recognize that the same equations can describe different ways in which assets facilitate intertemporal trade. Again, see Section 2 for citations to work on repos, and recall our goal is not to provide a theory where repos or collateral are essential, but merely to point out that they are commonly used in financial dealings.

perfect substitutes). While $\alpha_j = 0$ implies χ_j is irrelevant, and vice versa, we include both because α_j and χ_j capture the notion of liquidity on the extensive and intensive margin, respectively. Also, when α_j and χ_j are endogenized below, the microfoundations are somewhat different in each case. As regards $\alpha_b > 0$, it is not hard to imagine situations where bond holdings work well – e.g., they are entries in a spreadsheet that can be transferred electronically between parties in different locations – while having cash in your briefcase, under your mattress, or even in your bank might not do. While sometimes we use $\alpha_b = 0$ below, again, there is no reason to impose that restriction at this point.

In stationarity equilibrium, $z_m = \phi_m A_m$ is constant, and so the money growth rate π equals the inflation rate, $\phi_m/\phi_{m,+1} = 1 + \pi$, where subscript $+1$ indicates next period. As usual, attention is restricted to $\pi > \beta - 1$, or the limit $\pi \to \beta - 1$, which is the Friedman rule. Stationarity also implies z_b is constant, which means A_b is constant for real bonds and $B = A_b/A_m$ is constant for nominal bonds. Policy variables are set by a consolidated monetary-fiscal authority subject to a single budget constraint. In the case of one-period real bonds, e.g., this is

$$G + T - \pi\phi_m A_m + A_b(1 - \phi_b) = 0, \tag{2}$$

where the first term is government consumption of x, the second is a lump-sum transfer, or tax if $T < 0$, the third is seigniorage and the fourth is debt service.

Define the return on an illiquid nominal asset – one that cannot be traded in the DM – by the Fisher equation $1 + \iota = (1 + \pi)/\beta$, where $1/\beta = 1 + r$ is the return on an illiquid real asset. To understand this, think of $1 + \iota$ as the amount of cash in the next CM that makes you willing to give up a dollar today, and $1 + r$ as the amount of x in the next CM that makes you willing to give up a unit of numeraire today (and of course such trades can be priced whether or not they occur in equilibrium). For a real liquid bond, the nominal yield ρ is computed as the amount of cash you can get in the next CM by investing a dollar in the asset today, $1+\rho = \phi_m/\phi_b\phi_{m,+1} = (1+\pi)/\phi_b$. It is convenient to define the spread between the nominal yields on illiquid and liquid bonds by $s = (\iota - \rho)/(1+\rho)$. Intuitively, s is the opportunity cost of the liquidity services embodied in bonds, just like ι is the opportunity cost of the liquidity embodied in currency.

Since $1+\iota = (1+\pi)/\beta$, it is equivalent to describe the Friedman rule by $\pi = \beta - 1$ or $\iota = 0$. Note that there is no equilibrium with $\iota < 0$, but as yet we cannot rule out $\rho < 0$. One experiment considered is a permanent increase in ι, or equivalently π, with fiscal implications offset by T. Another is an OMO involving changes in A_b and A_m to satisfy (2) within the period, with fiscal implications in subsequent periods offset by T. In this case, the new level of A_b on the central bank's balance sheet is permanent, the growth rate of A_b is constant except for a blip at the time of the OMO, and lump-sum taxes cover changes in debt service. Also, it is worth emphasizing that the theory applies to policy changes in *real time* as long as they are unanticipated: since there are no transitional dynamics, the economy simply jumps from one stationary equilibrium directly to another. Of course, it also applies to comparisons across economies at a point in time.

Monetary policy is often described as central banks trying to hit interest rate targets. In our economy there can be two: the nominal rate on illiquid assets ι; and nominal rate on liquid T-bills ρ. The former is connected directly to π via the Fisher Equation, and, as is standard, we need not put illiquid assets into the equations because they do not trade in equilibrium, but we can still price them, making ι perfectly well defined. For the latter, as discussed below, any ρ within a range consistent with the existence of equilibrium can be implemented with a unique A_b that can be achieved through OMO's. This captures well actual policy, at least in a stylized way, given we abstract from commercial banks, the Fed Funds market, and much other institutional detail. Also, as regards realism, while actual T-bills are nominal, Section 4.2 shows that the results do not hinge on this, and so we use real bonds in the baseline specification, mainly to ease notation.

To summarize, agents who can be households, firms, investors or financial institutions interact in decentralized markets where frictions hinder credit and imply a role for assets as media of exchange or collateral. The assets can differ in acceptability and pledgeability. Monetary policy is captured by: (1) the growth rate of the money supply, which pins down π and ι; and (2) the nominal yield ρ on short-term government debt, which can be implemented by choosing A_b through OMO's. While we use short-term real bonds as a benchmark, later we consider nominal and long-term bonds. The focus is on one-time unanticipated policy changes, or equivalently, comparisons across economies with different policies.

4 Random Matching

4.1 Baseline: Short Real Bonds

A buyer's DM state is his portfolio (z_m, z_b), while what matters in the CM is $z = z_m + z_b$. If the CM and DM value functions are $W(z)$ and $V(z_m, z_b)$ then

$$W(z) = \max_{x,\ell,\hat{z}_m,\hat{z}_b} \{U(x) - \ell + \beta V(\hat{z}_m, \hat{z}_b)\} \text{ st } x = z + \ell + T - (1+\pi)\hat{z}_m - \phi_b \hat{z}_b \quad (3)$$

where $\hat{z}_j$ is the real value of asset j taken out of this market, and the real wage is $\omega = 1$ because, for simplicity, 1 unit of ℓ produces 1 unit of x. Assuming $x \geq 0$ and $\ell \in [0, 1]$ are slack, as can be guaranteed in standard ways, the key FOC's are $1 + \pi = \beta V_1(\hat{z}_m, \hat{z}_b)$ and $\phi_b = \beta V_2(\hat{z}_m, \hat{z}_b)$. The envelope condition is $W'(z) = 1$, meaning $W(z)$ is linear. Sellers' CM problem is not shown, but their value function is similarly linear in purchasing power.

Letting p_j denote payment in type-j meetings, and using $W'(z) = 1$, we can write a buyers's DM value function as

$$V(z_m, z_b) = W(z_m + z_b) + \alpha_m[u(q_m) - p_m] + \alpha_b[u(q_b) - p_b] + \alpha_2[u(q_2) - p_2].$$

The first term on the RHS is the continuation value from not trading; the rest are the surpluses from different types of meetings. Payments are constrained by $p_j \leq \bar{p}_j$, where $\bar{p}_j$ is the buyer's *liquidity position* in a type-j meeting: $\bar{p}_m = \chi_m z_m$, $\bar{p}_b = \chi_b z_b$ and $\bar{p}_2 = \chi_m z_m + \chi_b z_b$. Sellers' DM value function is similar, except for their arrival rates and the fact that their surplus is $p_j - c(q_j)$.

The terms of trade are determined by an abstract mechanism: to get q you must pay $p = v(q)$. Kalai's proportional bargaining solution, e.g., implies $v(q) = \theta c(q) + (1-\theta) u(q)$, where θ is the buyer's bargaining power. Nash bargaining is similar, but messier if $p_j \leq \bar{p}_j$ binds. However, other than $v(0) = 0$ and $v'(q) > 0$, all we need for now is this: Let $p^* = v(q^*)$ be the payment that gets the efficient q. Then $p^* \leq \bar{p}_j \Rightarrow p_j = p^*$ and $q_j = q^*$, while $p^* > \bar{p}_j \Rightarrow p_j = \bar{p}_j$ and $q_j = v^{-1}(\bar{p}_j)$. This holds for Kalai and Nash bargaining, Walrasian pricing and many other standard solution concepts, and can also be derived axiomatically (Gu and Wright 2015). Hence, for now we use a generic mechanism, $v(\cdot)$.[7]

As usual, $\iota > 0$ implies buyers cash out (spend all the money they can) in type-m meetings and are still constrained: $p_m = \chi_m z_m < p^*$. Also, they may as well cash out in type-2 meetings, before using bonds, since in these meetings both parties are indifferent between z_m and z_b. Buyers use all the bonds they can in type-2 meetings iff $\bar{p}_2 \leq p^*$, and in type-b meetings iff $\bar{p}_b \leq p^*$. It is obvious that $p_2 \geq p_b$, which leaves three cases: 1. $p_2 = \bar{p}_2$ and $p_b = \bar{p}_b$ (buyers are constrained in all meetings); 2. $p_2 < \bar{p}_2$ and $p_b = \bar{p}_b$ (they are constrained in type-b but not type-2 meetings); or 3. $p_2 < \bar{p}_2$ and $p_b < \bar{p}_b$ (they are not constrained in type-b or type-2 meetings). We consider the cases in turn, always assuming monetary equilibrium exists.[8]

In Case 1 (buyers are always constrained), we have

$$v(q_m) = \chi_m z_m,\ v(q_b) = \chi_b z_b \text{ and } v(q_2) = \chi_m z_m + \chi_b z_b. \tag{4}$$

[7]For simplicity $v(\cdot)$ does not depend on the type of meeting j; this could be relaxed, at a cost of more notation.

[8]It is standard to show $\alpha_m > 0$ implies monetary equilibrium exists iff $\iota < \bar{\iota}_m$, and $\alpha_m = 0$ implies it exists iff $\alpha_2 > 0$, $\chi_b A_b < p^*$ and $\iota < \bar{\iota}_2$, where $\bar{\iota}_m$ and $\bar{\iota}_2$ may or may not be finite.

Differentiating $V(z_m, z_b)$ using (4) and inserting the results into the FOC's for assets in the CM, it is routine to get the Euler equations

$$1 + \pi = \beta \left[1 + \alpha_m \chi_m \lambda(q_m) + \alpha_2 \chi_m \lambda(q_2) \right] \tag{5}$$

$$\phi_b = \beta \left[1 + \alpha_b \chi_b \lambda(q_b) + \alpha_2 \chi_b \lambda(q_2) \right], \tag{6}$$

where $\lambda(q_j) \equiv u'(q_j)/v'(q_j) - 1$ is the *liquidity premium* in a type-j meeting (i.e., the Lagrange multiplier on $p_j \leq \bar{p}_j$). Using the yield on illiquid bonds ι and the spread s defined above, after some algebra these simplify nicely to

$$\iota = \alpha_m \chi_m \lambda(q_m) + \alpha_2 \chi_m \lambda(q_2) \tag{7}$$

$$s = \alpha_b \chi_b \lambda(q_b) + \alpha_2 \chi_b \lambda(q_2). \tag{8}$$

The LHS of (7) is the marginal cost of holding cash. The RHS is the benefit: with probability α_m a buyer is in a situation where relaxing the constraint $p_m \leq \bar{p}_m$ is worth $\lambda(q_m)$; with probability α_2 he is in a situation where relaxing $p_2 \leq \bar{p}_2$ is worth $\lambda(q_2)$; and in either case he can use a fraction χ_m of z_m. Condition (8) is similar with s the marginal cost of bond liquidity. As a digression, Krishnamurthy and Vissing-Jorgensen (2012) estimate equations like (8) the way people have been estimating money demand equations like (7) for decades. They measure s by the spread between government and corporate yields, motivated by saying T-bills enter utility functions (see also Nagel 2014). While we try to model assets' roles in more detail, their empirical work provides support for the shared concept – that assets have value over and above narrowly measured returns.

As a second digression before the policy analysis, recall that $1+\rho = (1+\pi)/\phi_b$, and thus (5)-(6) immediately imply

$$\rho = \frac{\alpha_m \chi_m \lambda(q_m) - \alpha_b \chi_b \lambda(q_b) + (\chi_m - \chi_b)\alpha_2 \lambda(q_2)}{1 + \alpha_b \chi_b \lambda(q_b) + \alpha_2 \chi_b \lambda(q_2)}. \tag{9}$$

The following result is now obvious, but we highlight it for emphasis:

Proposition 1 *If* $\alpha_b = \alpha_2 = 0$ *or* $\chi_b = 0$ *then* $\rho = \alpha_m \chi_m \lambda(q_m) = \iota$. *In general* $\rho < 0$ *is possible. As special cases,* $\alpha_m \lambda(q_m) = \alpha_b \lambda(q_b) \Rightarrow \rho < 0$ *iff* $\chi_b > \chi$, *and* $\chi_m = \chi_b \Rightarrow \rho < 0$ *iff* $\alpha_b \lambda(q_b) > \alpha_m \lambda(q_m)$.

In the first special case, $\rho < 0$ if assets have the same liquidity premium but A_b is more pledgeable. In the second, $\rho < 0$ if they are equally pledgeable but A_b has a higher liquidity premium, either because $\alpha_b > \alpha_m$ (bonds can be used more frequently) or $\lambda(q_b) > \lambda(q_m)$ (when they can be used they are really valuable). These results are perhaps of interest because it is not easy to get negative nominal rates, and (9) clarifies logically what it takes. It is not easy since negative nominal rates usually violate no-arbitrage, but that is not true here. While any agent can issue bonds – i.e., borrow in the CM – he cannot exploit $\rho < 0$ unless he can guarantee his liabilities will be liquid – i.e., circulate in the DM. Relatedly, when cash is subject to theft, nominal rates can be negative without violating no-arbitrage if issuers must incur costs to guarantee their liabilities will be safe, as with travellers' checks. Here liquidity takes over for safety, but they are related: Section 5 shows $\chi_b > \chi_m$ iff bonds are harder to counterfeit than cash.[9]

It seems pedantic to provide formal definitions; in Case 1, it should be clear that a stationary monetary equilibrium is a list (q_m, q_b, q_2, z_m, s) solving the three equalities in (4), plus the Euler equations (7) and (8), with $z_m > 0$ and $z_b = A_b$.

[9] The relevance of Proposition 1 is to show what it takes to get $\rho < 0$, not to claim the special cases are the most empirically relevant ones; we are happy interpreting it as strengthening the conventional view that it is hard to get $\rho < 0$, even when liquidity is modeled explicitly, for realistic parameters. However, if one wanted to argue for some practical relevance, consider *The Economist* (July 14, 2014): "Not all Treasury securities are equal; some are more attractive for repo financing than others. With less liquidity in the market, those desirable Treasuries can be hard to find: some short-term debt can trade on a negative yield because they are so sought after." Or the Swiss National Bank (2013): "The increased importance of these securities is reflected in the trades on the interbank repo market which were concluded at negative repo rates." The formal theory does not have all the institutional detail, obviously, but in an abstract way this is exactly what is going on: agents are willing to accept negative nominal yields on A_b if it has some advantage in transactions. Potential advantages are that A_b can be used more often (higher α), can be used to a greater extent (higher χ), or is worth a lot when it can be used (higher λ). These possibilities apply to banks, firms, households or any other agents that face liquidity considerations, and at the risk of belaboring the point, we re-emphasize that these or similar equations have been interpreted in terms of all these types of traders.

This has a recursive structure. First use (4) to rewrite (7) as

$$\iota = \alpha_m \chi_m L(\chi_m z_m) + \alpha_2 \chi_m L(\chi_m z_m + \chi_b z_b), \tag{10}$$

where $L(\cdot) \equiv \lambda \circ v^{-1}(\cdot)$. Under standard conditions (see fn. 8), a solution $z_m > 0$ to (10) exists, is generically unique, and entails $L'(\cdot) < 0$ (Gu and Wright 2015). From z_m, (4) determines (q_m, q_b, q_2), then (8) determines s, (9) determines ρ, etc.

Now for policy. First, a one-time increase in A_m reduces ϕ_m to leave $z_m = \phi_m A_m$ the same because (10) solves for z_m independently of A_m. This is classical neutrality. Then the following observations are key (at least they were for us) to understanding OMO's. Suppose A_b changes, with the central bank purchasing or issuing bonds, financed by printing or retiring currency. One cannot ignore the fiscal implications of this policy, but our choice is to sterilize these by adjusting the lump-sum transfer/tax T.[10] So the effect of an OMO is the same as changing A_b with A_m fixed, and dealing with fiscal implications out of general revenue. Again, by classical neutrality, changes in A_m do not matter; changes in A_b might.

Let us write $L'_m = L'(\chi_m z_m)$, etc. Then $\partial z_m / \partial \iota = 1/D_R < 0$, where $D_R \equiv \alpha_m \chi_m^2 L'_m + \alpha_2 \chi_m^2 L'_2 < 0$. In terms of quantities,

$$\frac{\partial q_m}{\partial \iota} = \frac{\chi_m}{v'_m D_R} < 0, \ \frac{\partial q_b}{\partial \iota} = 0 \text{ and } \frac{\partial q_2}{\partial \iota} = \frac{\chi_m}{v'_2 D_R} < 0,$$

where $v'_m = v'(q_m)$, etc. For financial variables, given $\alpha_2 > 0$, we get

$$\frac{\partial s}{\partial \iota} = \frac{\alpha_2 \chi_m \chi_b L'_2}{D_R} > 0, \ \frac{\partial \phi_b}{\partial \iota} = \beta \frac{\alpha_2 \chi_m \chi_b L'_2}{D_R} > 0$$
$$\frac{\partial \rho}{\partial \iota} = \frac{\alpha_m L'_m + \alpha_2 \left[1 - (1+\rho)\chi_b/\chi_m\right] L'_2}{(1+s)(\alpha_m L'_m + \alpha_2 L'_2)} \gtrless 0,$$

If $\alpha_2 = 0$ then $\partial s/\partial \iota = \partial \phi_b / \partial \iota = 0$, since there is no substitution between z_m and

[10] This is especially clean with the preference specification in (1), because changes in T affect labor hours ℓ, but nothing else. See Andolfatto and Williamson (2015) for a related model in which they do something different, but it is worth emphasizing that there is not a unique "correct" way to handle the fiscal implications of monetary policy. While it is interesting to consider alternatives, an advantage of our choice is simplicity.

z_b in DM trade, but with $\alpha_2 > 0$ higher ι raises s and ϕ_b as agents try to move out of A_m and into A_b. The one ambiguous effect is $\partial\rho/\partial\iota$, discussed below in terms of a natural tension between Fisher and Mundell effects. And we emphasize that these portfolio (or liquidity management) effects are as relevant for firms as consumers – i.e., for corporate as well as household finance.

Now for OMO's. First, $\partial z_m/\partial A_b = -\alpha_2\chi_m\chi_b L_2'/D_R < 0$ iff $\alpha_2 > 0$. Then

$$\frac{\partial q_m}{\partial A_b} = -\frac{\alpha_2\chi_b L_2'}{v_m' D_R} < 0,\ \frac{\partial q_b}{\partial A_b} = \frac{\chi_b}{v_b'} > 0 \text{ and } \frac{\partial q_2}{\partial A_b} = \frac{\alpha_m\chi_b L_m'}{v_2' D_R} > 0.$$

Higher A_b decreases z_m and q_m because it makes liquidity less scarce in type-2 meetings, so agents economize on cash, which comes back to haunt them in type-m meetings. Given this, the impact on DM output $\Sigma_j\alpha_j q_j$ is ambiguous. Also, $\chi_b > 0$ implies $\partial s/\partial A_b < 0$, $\partial\phi_b/\partial A_b < 0$ and $\partial\rho/\partial A_b > 0$. In particular, $\partial\rho/\partial A_b > 0$ means there is an invertible mapping between the T-bill rate and the supply, which is important to the extent that policy is described by central banks trying to control interest rates. Here, adjusting A_b can achieve any ρ, at least within certain bounds, since ρ cannot go above ι or below $\underline{\rho}$.

This completes Case 1, which is the most interesting. In Case 2, with $q_2 = q^*$, increasing ι lowers z_m and q_m, does not affect q_b or q_2, raises s and lowers ρ as agents again try to move out of cash and into bonds. Increasing A_b does not affect z_m, q_m or q_2, but increases q_b and ρ and decreases s. For Case 3, with $q_b = q_2 = q^*$, bonds provide no liquidity at the margin, so $\rho = \iota$ and $s = 0$. Then increasing ι reduces z_m and q_m but affects nothing else, while changing A_b affects nothing of interest. Which outcome obtains? If bonds are abundant, in the sense $A_b \geq A_b^* \equiv v(q^*)/\chi_b$, it is Case 3. Otherwise there is $A_b^o < A_b^*$ such that $A_b^o < A_b < A_b^*$ implies Case 2 and $A_b < A_b^o$ implies Case 1.[11] We summarize as follows:

[11] We need not take a stand on which case is the most realistic, but some people argue that there is a scarcity of high-quality liquid assets in real-world markets, which corresponds to Case 1. See BIS (2001), Caballero (2006), Caballero and Krishnamurphy (2006), IMF (2012), Gorton and Ordonez (2013,2014) or Andolfatto and Williamson (2015).

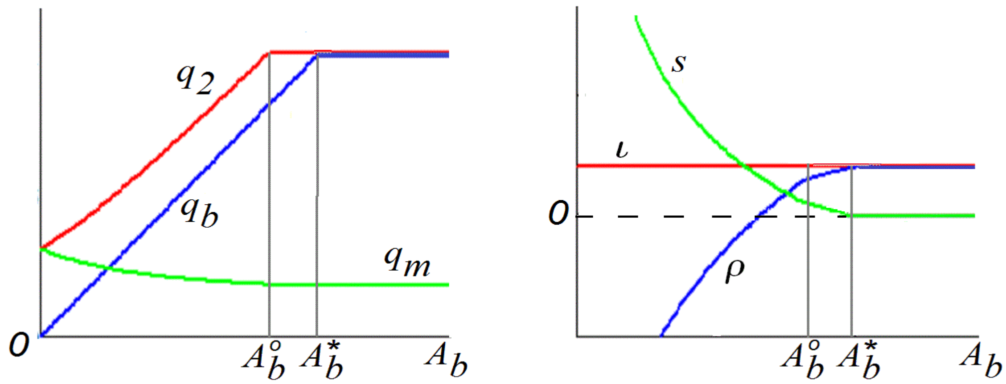

Figure 1: Effects of A_b

Proposition 2 *Consider an OMO that injects A_m. If $A_b < A_b^o$, ρ, q_2 and q_b decrease while s and q_m increase. If $A_b^o < A_b < A_b^*$, ρ and q_b decrease, s increases, while q_m and q_2 stay the same. If $A_b > A_b^*$, these variables all stay the same.*

In Figure 1, an OMO injecting A_m moves us from right to left, going from Case 3 to Case 2 to Case 1. Again, the real effects are due to decreasing A_b, not increasing A_m, which is neutral absent ad hoc restrictions on price adjustment. Yet notice that in Case 1 it *looks like* prices are sluggish: the net impact of an OMO that injects currency is to increase z_m, because money demand rises when liquid bonds become more scarce, so the value of money ϕ_m must go down by less than the supply A_m goes up. One might mistake this for failure of the Quantity Equation, but recall that a cash injection by lump-sum transfer still implies ϕ_m falls so that $\phi_m A_m$ does not change. Hence, it may not be easy to test neutrality by looking at changes in A_m without conditioning on how the changes are implemented.[12]

Similarly, for the Fisher Equation, one should not look at the effect of π on ρ, because theory predicts that is nonmonotone. In examples ρ increases with π when π is low or high, and decreases when π is in between. This nonmonotonicity arises because inflation tends to raise nominal returns for a given real return, by the Fisher effect, but also tends to lower real returns, by the Mundell effect. To

[12] There is no way to resurrect a Quantity Equation for OMO's by saying nominal prices are proportional to some aggregate of A_m and A_b – the equilibrium effects are not that simple, because while the two assets are substitutes, in general, they are not perfect substitutes.

test the Fisher Equation one should compare π and ι, not π and ρ, where ι is the nominal rate on an illiquid asset.

These are the benchmark results.[13] Clearly one can get similar propositions by putting (A_m, A_b) in the utility function, as if assets were goods, like apples. But unlike apples, assets are valued for their liquidity services, plus returns, of course, and liquidity is not a primitive the way the utility of apples might be. Now, some assets are somewhat like apples, such as apple trees, and it is only fiat money that has no intrinsic worth. Still, if assets are valued for liquidity, why not model this explicitly? A reason to do so is that, with assets in utility, there is no discipline concerning when agents satiate in liquidity, yet in the above analysis the points A_b^o and A_b^* at which $\lambda(q_2)$ and $\lambda(q_b)$ hit 0 are endogenous. Another is that the liquidity value of assets depends on policy, but it is hard to specify how if one puts them in consumers' utility functions. Of course similar comments apply to producers' or bankers' production functions. Given this, and recognizing it is to some extent a matter of taste, we prefer to derive the payoff from (A_m, A_b) indirectly by modeling assets' role in exchange.

4.2 Variations: Nominal or Long Bonds

Now consider nominal bonds, paying 1 dollar in the next CM. Assume A_b and A_m grow at the same rate, so $B = A_b/A_m$, $z_m = \phi_m A_m$ and $z_b = Bz_m$ are stationary. As in the benchmark model, in Case 1, $\partial z_m/\partial\iota = 1/D_N$ where $D_N < 0$. Also,

$$\frac{\partial q_m}{\partial \iota} = \frac{\chi_m}{v_m' D_N} < 0,\ \frac{\partial q_b}{\partial \iota} = \frac{B\chi_b}{v_b' D_N} < 0 \text{ and } \frac{\partial q_2}{\partial \iota} = \frac{\chi_m + B\chi_b}{v_2' D_N} < 0.$$

The only qualitative difference is now ι affects q_b. For OMO's, consider changing B. Then $\partial z_m/\partial B = -\alpha_2 \chi_m \chi_b z_m L_2'/D_N < 0$, and for a constant $C > 0$

$$\frac{\partial q_m}{\partial B} = -\frac{\alpha_2 C L_2'}{v_m' D_N} < 0,\ \frac{\partial q_b}{\partial B} = \frac{C\left(\alpha_m L_m' + \alpha_2 L_2'\right)}{v_b' D_N} > 0,\ \frac{\partial q_2}{\partial B} = \frac{\alpha_m C L_m'}{v_2' D_N} > 0.$$

[13] There are other effects that we find interesting, but neglect, to maintain focus on policy – e.g., Appendix B considers changes in the α's and χ's, interpretable in terms of financial innovation.

We can also derive effects on ρ, consider Cases 2 and 3, etc. As these results are all similar to Section 4.1, let us revert to real bonds, which involve less notation. However, first we clarify one point: A one-time increase in A_m reduces ϕ_m through the usual channel, and if we hold constant the nominal bond supply A_b, the real supply $\phi_m A_b$ falls, which has real effects. In this sense money is not neutral, because $B = A_b/A_m$ changes, but this is akin to saying money is not neutral when there are fixed nominal tax rates – it's true, but not remarkable. Increasing A_m holding B constant remains neutral.

Now consider long-term bonds, say consols paying δ in CM numeraire in perpetuity. Then $z_b = (\phi_b + \delta) A_b$ is endogenous due to the bond's resaleability in the CM. In Case 1, the Euler equations for money and bonds are

$$\iota = \alpha_m \chi_m L(\chi_m z_m) + \alpha_2 \chi_m L(\chi_m z_m + \chi_b z_b) \tag{11}$$

$$r = \frac{\delta(1+r)A_b}{z_b} + \alpha_b \chi_b L(\chi_b z_b) + \alpha_2 \chi_b L(\chi_m z_m + \chi_b z_b). \tag{12}$$

Appendix A shows the effects of i and A_b are qualitatively similar but messier than in the benchmark model, and so we revert to short-term bonds. However, it is first useful to depict (11)-(12) in (z_m, z_b) space as the EM and EB curves in the upper part of Fig. 2. One can show both are downward sloping, but EB has a greater slope, so they cross uniquely. Also, changes in A_b shift EB but not EM. For comparison the bottom panels show the picture for one-period bonds.

In the lower right, with short bonds, EB shifts down after an OMO that increases A_m, as shown by the arrows. Since z_m increases, prices look sluggish. The upper right, with long bonds, is similar but has additional multiplier effects. After A_b falls, ϕ_b rises because bonds are more scarce, which partially offsets the impact but on net z_b falls. As with short bonds, lower z_b raises z_m as agents try to substitute across assets. But higher z_m makes lower z_b not as bad, so the demand for and thus the price of bonds fall, and z_b falls further. That leads to an additional rise in z_m, an additional fall in z_b etc. We summarize as follows.

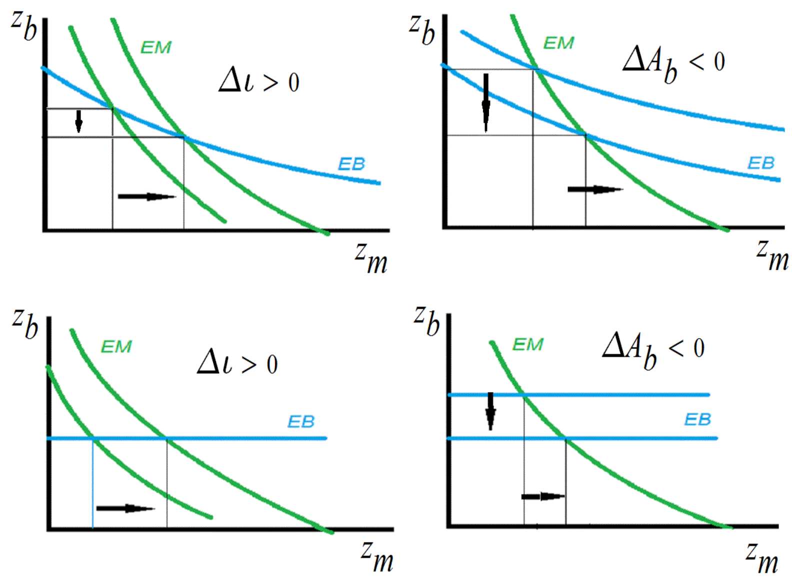

Figure 2: Increase in ι and decrease in A_b with long and short bonds

Proposition 3 *With nominal or long bonds, the qualitative results are the same.*

4.3 Liquidity Trap

A liquidity trap is defined by Keynes (1936) as follows: "after the rate of interest has fallen to a certain level, liquidity-preference may become virtually absolute in the sense that almost everyone prefers cash to holding a debt which yields so low a rate of interest. In this event the monetary authority would have lost effective control over the rate of interest." This does *not* correspond to the case $A_b \geq A_b^*$ in Fig. **??**, where agents are satiated in bond liquidity and ρ is at its upper bound. We now show how the economy can get into a different situation, where OMO's do not affect ρ or output, which are at their lower bounds. For this exercise we add heterogeneous buyers: type-j have probabilities α_m^j, α_b^j and α_2^j of a type-m, type-b and type-2 meeting, and $\mu_j > 0$ is the fraction of type-j (the reason for this is explained in fn. 14). In terms of economics, one type can represent households that use cash as a means of payment, e.g., while another can represent financial institutions or other firms that use bonds as collateral. This captures the idea that different types of agents can value instruments with different liquidity properties, and further illustrates the flexibility of the formalization.

Suppose there is a type j with $\alpha_m^j = \alpha_b^j = 0 < \alpha_2^j$ (i.e., money and bonds are perfect substitutes for j). If they choose $\hat{z}_m, \hat{z}_b > 0$, then

$$1 + \pi = \beta \left[1 + \alpha_2^j \chi_m \lambda(q_2^j) \right] \text{ and } \phi_b = \beta \left[1 + \alpha_2^j \chi_b \lambda(q_2^j) \right]. \tag{13}$$

Moreover, given $\alpha_m^j = \alpha_b^j = 0$, (9) implies $\rho = \underline{\rho}$, where

$$\underline{\rho} = \iota \left(\chi_m - \chi_b \right) / \left(\chi_m + \iota \chi_b \right). \tag{14}$$

This pins down ρ in terms of ι and the χ's, *independent of* A_b. Intuitively, for type j, A_b and A_m are perfect substitutes since 1 unit of $\hat{z}_m$ in the DM gets them the same as χ_b / χ_m units of $\hat{z}_b$. Hence, if $\hat{z}_m^j, \hat{z}_b^j > 0$, the assets must have the same pledgeability-adjusted return .

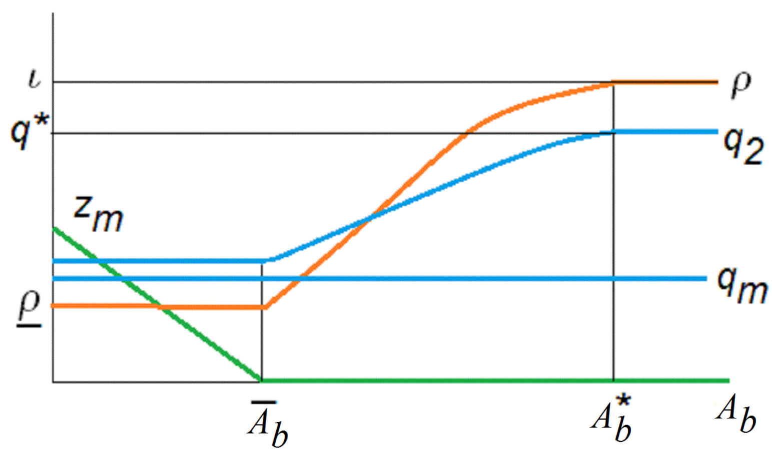

Figure 3: Effects of A_b, with a liquidity trap in $(0, \bar{A}_b)$

Fig. 3 shows the case of two types, type-m with $\alpha_b^m = \alpha_2^m = 0 < \alpha_m^m$, and type-2 with $\alpha_m^2 = \alpha_b^2 = 0 < \alpha_2^2$ (e.g., households and banks). Type-m hold $\hat{z}_m^m > 0$ and $\hat{z}_b^m = 0$, as they have no use for liquid bonds. Type-2 hold $\hat{z}_b^2 = A_b / \mu_2 > 0$, and maybe some cash, $z_m^2 \geq 0$. There are three possibilities. If $A_b \geq A_b^*$ then $\hat{z}_m^2 = 0$ and $q_2 = q^*$. If $A_b < A_b^*$ then there are two subcases. One has $\hat{z}_m^2 = 0$ and occurs if $A_b \geq \bar{A}_b \in (0, A_b^*)$. In this situation type-2 cannot get q^*, but get close enough that it is not worth holding cash to top up their liquidity. The other has $\hat{z}_m^2 > 0$ and occurs if $A_b < \bar{A}_b$. In this situation total liquidity for type-2 is independent of A_b because, at the margin, it's money that matters. This is the liquidity trap: marginal changes in A_b induce changes in real money balances that leave total liquidity the same, while ρ and q stick at their lower bounds. Importantly, the way out of this trap is to raise of A_b. While one may not see this in macro textbooks, our micro-based theory predicts that injecting cash and withdrawing bonds by an OMO makes the situation worse.[14]

Proposition 4 *For $A_b < \bar{A}_b$, OMO's that change A_b crowd out z_m to leave total liquidity the same, with no effect on ρ or the q's, which are at lower bounds.*

5 Endogenous Liquidity

The next step is to endogenize α_j and χ_j by appealing to recognizability – i.e., information frictions.[15]

5.1 Acceptability

As in Lester et al. (2012), assume that some sellers cannot distinguish high- from low-quality versions of certain assets, and low-quality assets can be produced on

[14] To explain why we have two types, if A_m and A_b are perfect substitutes, monetary equilibrium otherwise collapses at $A_b > A_b^*$ – i.e., we add type-m to prop up money demand. As long as A_b is not too big, we do not need type-m, but this heterogeneity is actually desirable for its own sake, as it captures the idea that some agents will always use cash, barring extreme situations like hyperinflation. We also mention the results are related to Wallace (1981), but he has no such heterogeneity, and assumes money and bonds are always perfect substitutes, implying OMO's must be irrelevant; they are irrelevant here if $A_b \in (0, \bar{A}_b)$ but not if $A_b \in (\bar{A}_b, A_b^*)$.

[15] The notion that assets or means of payment can be lemons has a history going back at least to Law, Jevons and Menger, and has become even more popular recently (see the Nosal and Rocheteau 2011 or Lagos et al. 2015 survey). One, but not the only, interpretation concerns counterfeiting. Importantly, it is wrong to dismiss this as unimportant, even if it is not a big problem in reality, because the *threat* of counterfeiting can impinge on liquidity even if it does not occur in equilibrium. With a broad interpretation of money, one version reflects sellers worrying about bad checks. In any case, whether or not cash or bonds can be counterfeited, the bigger idea is that counterparties may not know their true value.

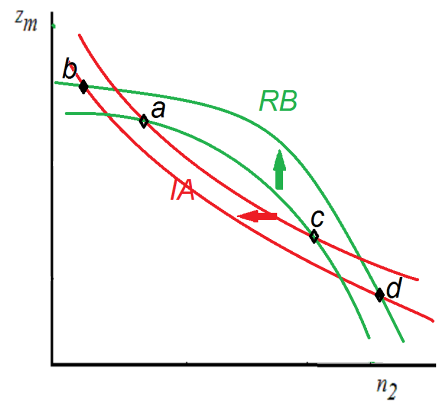

Figure 4: Equilibria with endogenous α's

the spot for free. Also suppose low quality assets have literally 0 value (as in Nosal and Wallace 2007; this can be relaxed as in Li and Rocheteau 2011). Then sellers unable to recognize quality reject these assets outright since, if they were to accept them buyers would print up and hand over worthless paper.

Also, for simplicity, set $\chi_j = 1$, consider Kalai bargaining, and assume all sellers can recognize A_m but to recognize A_b they must pay an individual-specific cost with CDF $F(\kappa)$. Let n_2 be the measure of sellers that pay the cost and hence accept bonds. The marginal seller is one with $\kappa = \Delta$, where

$$\Delta = \alpha (1 - \theta) [u(q_2) - c(q_2) - u(q_m) + c(q_m)] \tag{15}$$

is the increase in profit from information. Equilibrium solves $n_2 = F(\Delta)$, with $\Delta = \Delta(z_m)$ because the q's depend on z_m. In Fig. 4, $n_2 = F \circ \Delta(z_m)$ defines a curve in (n_2, z_m) space called IA for *information acquisition*. It slopes down, shifts right with A_b and is independent of ι. If buyers are constrained in all meetings – i.e., we are in Case 1 – the Euler equation for z_m defines a curve called RB, for *real balances*. It slopes down, and shifts down with A_b or ι. Equilibrium is where the curves cross. As in Fig. 4, RB might cut IA from below or above.

Combining these relationships, equilibrium involves $n_2 = F \circ \Delta \circ z_m(n_2) \equiv \Upsilon(n_2)$. One can show $\Upsilon : [0, 1] \to [0, 1]$ is increasing (this is where Kalai bargaining helps). Existence then follows by Tarski's fixed-point theorem, even if $F(\cdot)$ is not continuous, as when there is a mass of sellers with the same κ. Equilibrium may entail $n_2 = 0$, $n_2 = 1$ or $0 < n_2 < 1$. Since Υ is increasing, *multiplicity* can easily emerge, as one might expect when payments methods are endogenous. Intuitively, higher n_2 decreases z_m, since it makes buyers less likely to encounter a seller that accepts only cash; then lower z_m raises the relative profitability of recognizing bonds, and this increases the measure of sellers investing in information.[16]

Despite multiplicity, the model has sharp predictions conditioning on selection. For inflation, using '$x \backsimeq y$' to mean 'x and y take the same sign,' we have

$$\begin{aligned}
\frac{\partial z_m}{\partial \iota} &= \frac{v'_m v'_2}{D_\alpha} \backsimeq D_\alpha,\ \frac{\partial q_m}{\partial \iota} = \frac{v'{}_2}{D_\alpha} \backsimeq D_\alpha,\ \frac{\partial q_2}{\partial \iota} = \frac{v'_m}{D_\alpha} \backsimeq D_\alpha \\
\frac{\partial n_2}{\partial \iota} &= \frac{\alpha(1-\theta)(c'_m u'{}_2 - c'_2 u'{}_m) F'}{D_\alpha} \backsimeq -D_\alpha,
\end{aligned}$$

where $D_\alpha = \alpha^2 (1-\theta)(\lambda_2 - \lambda_m)(c'_m u'_2 - c'_2 u'_m) F' + \alpha n_m \lambda'_m v'_2 + \alpha n_2 \lambda'_2 v'{}_2$. One can check $\partial s / \partial \iota \backsimeq \partial \phi_b / \partial \iota \backsimeq -D_\alpha$, while $\partial \rho / \partial \iota$ is ambiguous, as discussed above, due to Fisher and Mundell effects. Similarly, the effects of OMO's are

$$\begin{aligned}
\frac{\partial z_m}{\partial A_b} &= -\frac{\alpha v'_m n \left[{}_2 \lambda'{}_2 + \alpha(1-\theta) F' (u'_2 - c'{}_2)(\lambda_2 - \lambda_m)\right]}{n D_\alpha} \backsimeq D_\alpha \\
\frac{\partial q_m}{\partial A_b} &= -\frac{\alpha \left[n_2 \lambda'{}_2 + \alpha(1-\theta) F' (u'{}_2 - c'{}_2)(\lambda_2 - \lambda_m)\right]}{n D_\alpha} \backsimeq D_\alpha \\
\frac{\partial q_2}{\partial A_b} &= \frac{\alpha \left[(n - n_2)\lambda'{}_m + \alpha(1-\theta) F' (u'{}_m - c'{}_m)(\lambda_2 - \lambda_m)\right]}{n D_\alpha} \backsimeq -D_\alpha \\
\frac{\partial n_2}{\partial A_b} &= \frac{\alpha\alpha(1-\theta) F' \left[(n - n_2)(u'{}_2 - c'{}_2)\lambda'_m + n(u'_m - c'{}_m)\lambda'_2\right]}{n D_\alpha} \backsimeq -D_\alpha.
\end{aligned}$$

One can also check $\partial s / \partial A_b \backsimeq \partial \phi_b / \partial A_b \backsimeq -\partial \rho / \partial A_b$.

Note $D_\alpha < 0$ iff RB cuts IA from below, and so as usual the results alternate across multiple equilibria. In Fig. 4, if $D_\alpha < 0$, as at point a, an OMO that injects

[16] Endogenous recognizability also generates *fragility*, with small parameter changes potentially causing big jumps in endogenous variables, and *hysteresis*, as in dollarization episodes where higher domestic inflation leads to locals adopting foreign currency but subsequent disinflations do not lead to reversals. See Lester et al. (2012) for details.

currency shifts RB up and IA left, increasing z_m and decreasing n_2; if $D_\alpha > 0$ the effects are reversed. There is no particular reason to reason to select one type of equilibrium, and indeed, it is not hard to have a unique equilibrium of one type or the other. This is important, because it means that to make reasonable policy predictions we need to know the fundamental parameters, *plus* the type of equilibrium, which may be difficult in practice. This may be problematic for policy makers, but it is a natural result of endogenizing liquidity, and ignoring the problem does not make it go away.

Proposition 5 *With endogenous α's, equilibrium exists, but in general it is not unique. The effects of OMO's depend on whether $D_\alpha < 0$ or $D_\alpha > 0$, but given D_α they are all determined as described above.*

5.2 Pledgeability

Next, as in Rocheteau (2011) or Li et al. (2012), assume that to produce low-quality assets agents must pay costs proportional to their values: for money the cost is $\gamma_m \phi_m$; and for bonds it is γ_b. Also, all sellers are uninformed, and fraudulent assets are produced in the CM before visiting the DM. As is standard in signaling models, we use bargaining with $\theta = 1$, and as a first pass, let $\alpha_2 > 0 = \alpha_m = \alpha_b$, so there is only one type of meeting, but we still must distinguish payments made in money and bonds, say d_m and d_b, not just the sum, for incentive reasons.

The incentive conditions for d_m and d_b are:

$$(\phi_{m,-1} - \beta\phi_m)\, a_m + \beta\alpha_2\phi_m d_m \leq \gamma_m \phi_m a_m \tag{16}$$

$$(\phi_{b,-1} - \beta)\, a_b + \beta\alpha_2 d_b \leq \gamma_b a_b. \tag{17}$$

See Li et al. (2012) for details, but the intuition is simple: The RHS of (16) is the cost of counterfeiting a_m; the LHS is the cost of acquiring a_m genuine dollars $(\phi_{m,-1} - \beta\phi_m)a_m$, plus the cost of trading away d_m with probability α_2. Sellers believe a_m is genuine since, e.g., who would spend \$20 to make a phony \$10 bill?

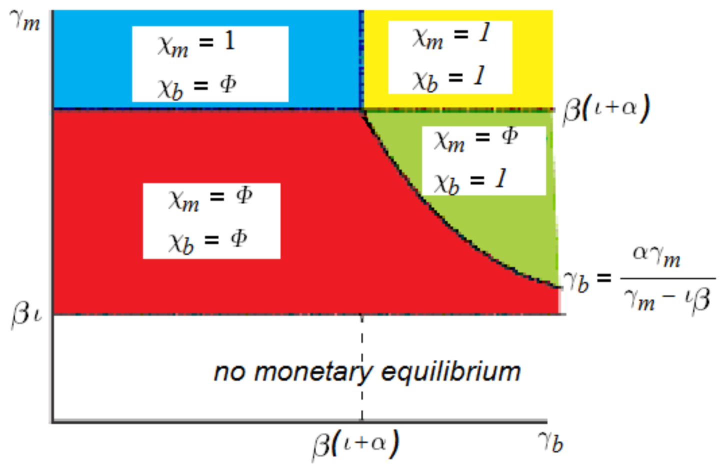

Figure 5: Different regimes with endogeneous χ's

Now DM trade has multiple constraints: bargaining implies $c(q_2) = \phi_m d_m + d_b$; feasibility implies $\phi_m d_m \leq z_m$ and $d_b \leq z_b$; and (16)-(17) imply $d_j \leq \chi_j z_j$ where

$$\chi_m = \frac{\gamma_m - \beta\iota}{\beta\alpha_2} \text{ and } \chi_b = \frac{\gamma_b - \beta s}{\beta\alpha_2}. \tag{18}$$

The outcome, or regime, depends on which constraint binds. Consider first $\chi_m, \chi_b \in (0,1)$. Then (7)-(8) reduce to

$$\beta\iota = (\gamma_m - \beta\iota)\,\lambda(q_2) \text{ and } \beta s = (\gamma_b - \beta s)\,\lambda(q_2). \tag{19}$$

The first condition yields q_2, then $s = \iota\gamma_b/\gamma_m$ and $\chi_b = \gamma_b\,(\gamma_m - \beta\iota)\,/\alpha_2\gamma_m$. This regime is consistent with equilibrium iff $\gamma_m > \beta\iota$, $\gamma_m < \beta\,(\iota + \alpha_2)$ and $\gamma_b < \beta\alpha_2\gamma_m/\,(\gamma_m - \iota)$.[17] Consider next $\chi_m = 1$ and $\chi_b \in [0,1)$. Then

$$\iota = \alpha_2\lambda(q_2) \text{ and } s = (\gamma_b - \beta s)\,\lambda(q_2). \tag{20}$$

One can check this is consistent with equilibrium iff $\gamma_m > \beta\,(\iota + \alpha_2) > \gamma_b$. Other regimes are similar, and the regions where they emerge are depicted by Fig. 5 in

[17] In this regime $1 + \rho = \chi_m/\chi_b = \gamma_m/\gamma_b$, and so $\rho < 0$ iff A_m is easier to counterfeit than A_b. This goes a level deeper than Proposition 1, and is arguably a realistic scenario. As mentioned above, the risk of accepting counterfeit currency is similar to the risk of having genuine currency stolen, which as in He et al. (2008) can also lead to $\rho < 0$.

(γ_b, γ_m) space, using Φ to mean any number in $(0,1)$, i.e., a mixed strategy. While a unique monetary equilibrium exists if $\gamma_m > \beta\iota$, money cannot be valued if $\gamma_m \leq \beta\iota$, in which case there is a nonmonetary equilibrium.

These results are relatively straightforward because so far A_m and A_b are perfect substitutes, but that also makes OMO's irrelevant. To change this, let $\alpha_m > 0$, and consider the natural regime where $\chi_m = 1$ and $\chi_b \in (0,1)$. The equilibrium conditions reduce to

$$\iota = \alpha_m L(z_m) + \alpha_2 L(z_m + \chi_b z_b) \tag{21}$$

$$\gamma_b = \alpha_2 \chi_b [1 + L(z_m + \chi_b z_b)], \tag{22}$$

defining two curves in (χ_b, z_m) space labeled RB and IC in Fig. 6. While RB slopes down, IC can be nonmonotone,

$$\frac{\partial z_m}{\partial \chi_b}_{|IC} \backsimeq \Theta \equiv 1 + L_2 + \chi_b z_b L_2'. \tag{23}$$

It is not hard to build examples with multiplicity. Intuitively, if agents think χ_b is low then q_2 is low and s is high, which gives a big incentive to create fraudulent bonds, and hence the endogenous χ_b is low.

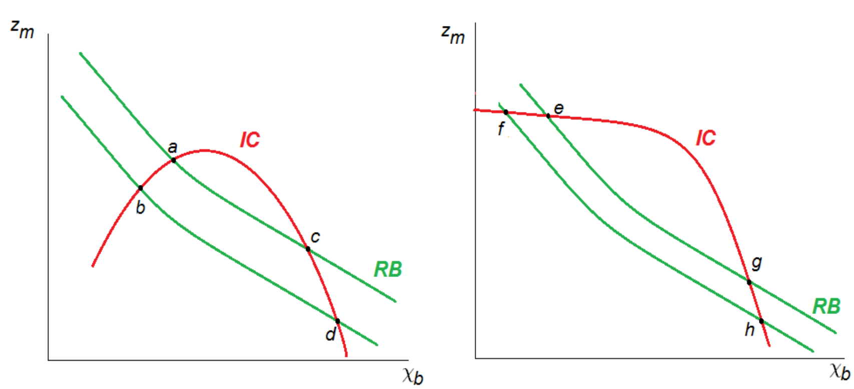

Figure 6: Different configurations with endogenous χ's

The results now depend on $D_\chi = (1 + L_2)(\alpha_m L'_m + \alpha_2 L_2') + \alpha_m \chi_b z_b L_2' L_m'$. From (23), $\Theta > 0$ implies $D_\chi < 0$, and there are three relevant configurations:

(i) $\Theta > 0$ implies IC is upward sloping and cuts RB from below, as at point a in the left panel of Fig. 6; $\Theta < 0$ implies IC is downward sloping and either (ii) cuts RB from below, as at e in the right panel, or cuts it from above, as at point c or g. An increase in ι shifts RB down. This implies $\partial z_m/\partial\iota = \Theta/D_\chi > 0$ when $\Theta < 0$, as in the move from e to f, or can imply $\partial z_m/\partial\iota < 0$, as in the other cases. Similarly, $\partial\chi_b/\partial\iota \simeq D_\chi$ depends on the configuration of RB and IC. One can check the effects on ϕ_b and s depend on $D_o = \alpha_m\alpha_2\chi_b z_b\lambda_m'\lambda_2' + \alpha_2(1+\lambda_2)(\alpha_m\lambda_m' v_2' + \alpha_2\lambda_2' v_m')$, which depends on the configuration, but $\partial\rho/\partial\iota$ is ambiguous even given D_o for reasons discussed above.

In terms of OMO's,

$$\frac{\partial z_m}{\partial A_b} = -\frac{\alpha_2^2\chi_b(1+\lambda_2)v_m'\lambda_2'}{D_o} \simeq D_o,\ \frac{\partial q_m}{\partial A_b} = -\frac{\alpha_2^2\chi_b(1+\lambda_2)\lambda_2'}{D_o} \simeq D_o$$
$$\frac{\partial q_2}{\partial A_b} = \frac{\alpha_m\alpha_2\chi_b(1+\lambda_2)\lambda_m'}{D_o} \simeq -D_o,\ \frac{\partial \chi_b}{\partial A_b} = -\frac{\alpha_m\alpha_2\chi_b\lambda_m'\lambda_2'}{D_o} \simeq -D_o.$$

Also, $\partial s/\partial A_b \simeq D_o$, $\partial\phi_b/\partial A_b \simeq D_o$ and $\partial\rho/\partial A_b \simeq -D_o$. As was the case with acceptability, OMO's affect pledgeability endogenously, but the sign of the effect of A_b on χ_b depends on the configuration. We think this would be hard to understand if one started with assets in utility or production functions. Also, again, predictions depend on knowing the type of equilibrium and that may be nontrivial in practice. We repeat that this may be problematic for policy policy analysts, but it is hard to avoid when liquidity is endogenous. We pursued one route to this end, recognizability, but alternative approaches are available. Indeed, endogenous liquidity and multiplicity are recurrent themes the in microfoundations of monetary economics, exemplified, e.g., by Kiyotaki and Wright (1989).

Proposition 6 *With endogenous χ's, for $\gamma_m > \beta\iota$ monetary equilibrium exists uniquely when $\alpha_m = \alpha_b = 0$, but it is not generally unique when $\alpha_m > 0$. The effects of OMO's depend on the configuration of IC and RB, as shown in Fig. 6, but given this they are all determined as described above.*

6 Directed Matching

Here we sketch two models with directed search, which is not a minor detail, but a fundamentally different way of conceptualizing the exchange process. In this extension, there are two types of sellers: a measure n_m accept A_m; and a measure n_2 accept A_m and A_b. For now, fix n_m, $n_2 = 1 - n_m$ and $\chi_j = 1$. Define submarket j as a set of type-j sellers with measure n_j, and a set of buyers searching for them with measure μ_j, with *SM* the submarket where A_m is accepted and *S2* the one where A_m and A_b are accepted. Assume $\mu_m + \mu_2 = \mu$ is not too large, so all buyers participate (see below), and let $\sigma_j = n_j/\mu_j$ be market tightness. The probability a buyer meets a seller in submarket j is $\alpha(\sigma_j)$, and the probability a seller meets a buyer is $\alpha(\sigma_j)/\sigma_j$, with the former increasing and the latter decreasing in σ_j.

6.1 Bargaining

Consider first Kalai bargaining.[18] Buyers going to *SM* take $\hat{z}_m^m > 0$ and $\hat{z}_b^m = 0$, and those going to *S2* take $\hat{z}_b^2 = A_b/\mu_2 > 0$ and $\hat{z}_m^2 \geq 0$. Then $\hat{z}_m^m = v(q_m)$ and $\hat{z}_b^2 + \hat{z}_m^2 \geq v(q_2)$, where the latter holds with equality iff $q_2 < q^*$ and $s > 0$. Given $q_2 < q^*$, we can have $\hat{z}_m^2 = 0$ or $\hat{z}_m^2 > 0$, with $\iota > s$ in the former case and $\iota = s$ in the latter. Since buyers now meet only one type of seller,

$$\iota = \alpha(\sigma_m)\lambda(q_m) \text{ and } s = \alpha(\sigma_2)\lambda(q_2). \tag{24}$$

If *SM* and *S2* are both open, buyers must be indifferent between them,

$$\alpha(\sigma_m)\left[u(q_m) - v(q_m)\right] - \iota z_m^m = \alpha(\sigma_2)\left[u(q_2) - v(q_2)\right] - s z_b^2. \tag{25}$$

Given the total measure of buyers μ,

$$n_m/\sigma_m + n_2/\sigma_2 = \mu. \tag{26}$$

[18] Note that directed search with bargaining, rather than posting, makes sense when buyers care about the kind of seller they meet – i.e., which payment methods are accepted.

A monetary equilibrium is now a list (q_j, σ_j, s) solving (24)-(26). Again there are three regimes: bonds are scarce, so type-2 carry cash; bonds are less scarce, so type-2 carry no cash even though $q_2 < q^*$; bonds are plentiful, so type-2 carry no cash and get $q_2 = q^*$. The first regime has $\rho = 0$ and $s = \iota$. Then (25)-(26) imply $\alpha(\sigma_2) = \alpha(\sigma_m)$ and $\sigma_m = \sigma_2 = \sigma = 1/\mu$. From (24) $q_m = q_2 = q$, where q solves $\iota = \alpha(\sigma)\lambda(q)$. Since z_b and z_m are perfect substitutes in this regime, the two submarkets are essentially the same, and increases in A_b merely crowd out real money balances in *S2*. This outcome is an equilibrium iff $A_b \leq \bar{A}_b = n_2\mu v(q)$.

Consider next $\hat{z}_m^2 = 0$, with $s = 0$ and $q_2 = q^* > q_m$. From (25)

$$\alpha(\sigma_2) = \frac{\max_q \{-\iota v(q_m) + \alpha(\sigma_m)\left[u(q_m) - v(q_m)\right]\}}{u(q^*) - v(q^*)}, \tag{27}$$

so $\sigma_2 < \sigma_m$ given $\iota > 0$, which says buyers trade with a higher probability in *SM*, to make up for the lower return on A_m than A_b. One can show there is a unique (σ_m, σ_2) solving (26)-(27), and this is an equilibrium iff $A_b \geq A_b^*$. Similarly, consider $z_m^2 = 0$, $0 < s < \iota$ and $q_2 < q^*$, where buyer indifference means

$$\begin{aligned} -\iota z_m^m + \alpha(\sigma_m)\left[u(q_m) - v(q_m)\right] &= -s z_b^2 + \alpha(\sigma_2)\left[u(q_2) - v(q_2)\right] \\ &> -\iota z_m + \alpha(\sigma_2)\left[u(q_m) - v(q_m)\right] \end{aligned}$$

because A_m has a lower return, and $q_m < q_2$, from (24). Again buyers' trading probability is higher in *SM*, and this is an equilibrium iff $A_b \in \left(\bar{A}_b, A_b^*\right)$.

As Fig. 7 shows, OMO's are neutral when A_b is below $\bar{A}_b$ (liquidity trap) or above A_b^* (liquidity satiation), but not in between, similar to random search. But now σ_2 and σ_m depend on A_b since tightness in each submarket is endogenous. In particular, when A_b increases, buyers in *SM* get better terms and a higher probability of trade, even though A_b is not actually used in that submarket. The result that now q_2 and q_m both increase with A_b, while q_2 increases but q_m decreases with random search, may not be a huge difference, but we still find directed search a novel and attractive way to think about market segmentation.

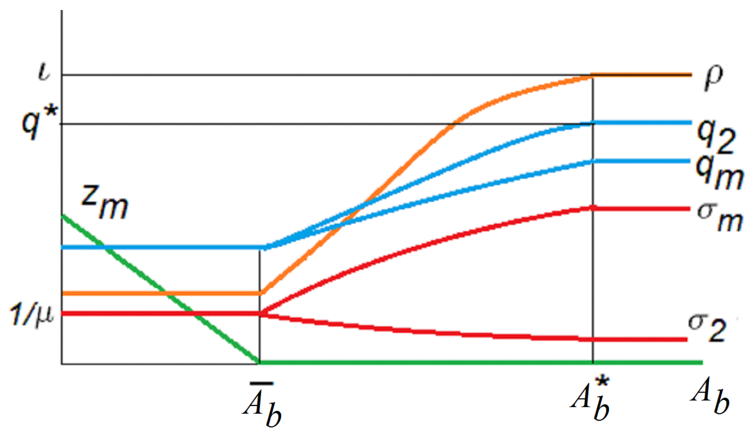

Figure 7: Effects of A_b with directed search

6.2 Posting

Instead of bargaining, suppose third parties called market makers set up submarkets to attract buyers and sellers, who then meet bilaterally according to the matching technology.[19] In the CM, market makers post $(q_j, \hat{z}_m^j, \hat{z}_b^j, \sigma_j)$ for the next DM, where traders commit to swapping q_j for $\hat{z}_m^j$ and $\hat{z}_b^j$ if they meet. As is standard, we can find $(q_j, \hat{z}_m^j, \hat{z}_b^j, \sigma_j)$ by maximizing buyers' surplus, given sellers get Π_j, with Π_j in equilibrium dictated by the market.

The problem for submarket SM can be written

$$U^b(\iota, \Pi_m) = \max_{q,\hat{z},\sigma} \{\alpha(\sigma)[u(q) - \hat{z}] - \iota\hat{z}\} \text{ st } \frac{\alpha(\sigma)}{\sigma}[\hat{z} - c(q)] = \Pi_m. \tag{28}$$

Generically (28) has a unique solution, with $U^b(\iota, \Pi_m)$ decreasing in ι and Π_m. Hence, all type-m submarkets are the same, so by constant returns, we need just one. Eliminating $\hat{z}_m$ and taking FOC's wrt q_m and σ_m, we get

$$\frac{u'(q)}{c'(q)} - 1 = \frac{\iota}{\alpha(\sigma)} \tag{29}$$

$$\alpha'(\sigma)[u(q) - c(q)] = \Pi_m \left\{1 + \frac{\iota[1 - \varepsilon(\sigma)]}{\alpha(\sigma)}\right\}, \tag{30}$$

[19]This market maker story (from Moen 1997 and Mortensen and Wright 2002) is merely a convenient way to describe the equilibrium outcome: the results are the same if sellers post terms of trade to attract buyers, or vice versa.

where $\varepsilon(\sigma) \equiv \sigma\alpha'(\sigma)/\alpha(\sigma) \in (0,1)$ is the elasticity of matching. We can similarly analyze *S2*, with ι replaced by s.

Monetary equilibrium is a list $(q_j, \hat{z}_m^j, \hat{z}_b^j, \sigma_j, s_j, \Pi_j)$ such that: $(q_j, \hat{z}_m^j, \hat{z}_b^j, \sigma_j)$ solves the above problems; $s_m = \iota$ and $s_2 = s$ are determined by market clearing; (26) holds; and $U^b(i, \Pi_m) = U^b(s, \Pi_2)$. As in many directed search models, with posting, technical complications make it hard to prove monotonicity of the endogenous variables with respect to parameters, or even continuity, although some progress can be made using the methods of monotone comparative statics (see Choi 2015). To avoid these technicalities, consider here a special matching function $\alpha(\sigma) = \min\{1, \sigma\}$ (see Appendix C for the general version). Also, suppose $\mu > n_m + n_2$, so buyers only participate in the DM up to the point where

$$\min\{1, \sigma_j\}\left[u(q_j) - \hat{z}_m^j - \hat{z}_b^j\right] - \iota\hat{z}_m^j - s\hat{z}_b^j = 0.$$

In *SM*, with this matching function, $\alpha(\sigma_m) = \min\{1, \sigma_m\}$ and $\alpha(\sigma_m)/\sigma_m = \min\{1/\sigma_m, 1\}$. In equilibrium $\sigma_m = 1$ and $\mu_m = n_m$.[20] Then, from the buyer's participation constraint, $\hat{z}_m^m = u(q_m)/(1+\iota)$. Substituting this into the seller's surplus and maximizing, we get $u'(q_m)/c'(q_m) = 1+\iota$. In *S2*, similarly, $\sigma_2 = 1$ and $\mu_2 = n_2$. Again, if $z_m^2 > 0$ then $\rho = 0$, $q_2 = q_m$ and the submarkets are essentially the same. This is an equilibrium iff $z_b = A_b/n_2 \leq u(q_m)/(1+\iota)$. If instead $z_m^2 = 0$ then q_2 solves $u'(q_2)/c'(q_2) = 1+s$ and $z_b = A_b/n_2 = u(q_2)/(1+s)$ makes buyers indifferent. Then $q_2 = q^*$ obtains if $s = 0$, which requires $A_b/n_2 \geq u(q^*)$. If $u(q_m)/(1+\iota) < A_b/n_2 < u(q^*)$ then $q_2 < q^*$ and $s \in (0, \iota)$. Now the outcome looks like Fig. 3, instead of Fig. 7, since $\sigma_m = \sigma_2 = 1$ and q_m are independent of z_b with the special matching technology, although in this formulation buyers choose which assets to use by choosing the submarket in which to search.

Finally, because this is a novel aspect of the paper, consider once more endogenizing seller types by making them pay κ if they want to recognize bonds. As

[20] Tightness $\sigma_m > 1$ is inconsistent with equilibrium, e.g., because we can increase sellers' expected surplus by attracting additional buyers without harming those already there.

above, they invest in information iff $\kappa \leq \Delta = \Pi_2 - \Pi_m$. However, different from Section 5.1, equilibrium is now unique, because market makers internalize the complementarities between sellers' information and buyers' portfolio decisions.[21] Also, $\iota = s$ and $\rho = 0$ cannot occur here if $F(0) = 0$, since if it is costly for all sellers to invest in information, no one chooses *S2* when $\Pi_2 = \Pi_m$. Yet if $F(0) > 0$, $\iota = s$ and $n_2 = F(0)$ become possible. For low A_b, sellers participate in *S2* iff $\kappa = 0$, in which case $\iota = s$ and $\rho = 0$. As A_b increases, Δ becomes positive and some sellers with $\kappa > 0$ join *S2*. Output increases in both submarkets but less in *SM*. Intuitively, q_m increases because σ_m rises, while q_2 increases because ρ rises even though σ_2 falls.

This sketch of directed search shows how the general framework is flexible, and for some issues the results are different from random search. While this is also partly a matter of taste, for us, a nice feature of directed search is that it generates endogenous market segmentation, with sellers choosing which assets to accept, and buyers choosing where to go and what to hold. This contrasts with many models where agents are constrained in their market participation, the portfolios they hold, and the payment methods they use.

7 Conclusion

This paper analyzed monetary policy in economies where assets explicitly facilitate trade, either as media of exchange or as collateral, and traders can be interpreted as consumers, producers or financial institutions. There are many predictions, some of which are consistent with conventional wisdom, although sometimes for different reasons – e.g., sluggish prices appear as an outcome not an assumption, and OMO's affect ρ due to changes in A_b not A_m. Other predictions differ from conventional wisdom, such as the finding that injecting currency with an OMO is

[21] This is related to the intuition in Rocheteau and Wright (2005) in a different context – there sellers pay an entry cost but there is no recognizability friction.

exactly the wrong remedy for a liquidity shortage. The model generated endogenous segmentation, liquidity traps, and negative nominal yields, even if $\rho < 0$ may use unrealistic parameters. Some results are quite robust, such as the finding that ρ not only does not move one-for-one with π, it is nonmonotone, due to Fisher and Mundell effects. Specifications for market structure included bargaining as well as posting. Information theory was used to explain differences in acceptability and pledgeability. This allowed us to capture liquidity on the extensive and intensive margins, endogenize transactions patterns, and generate multiple equilibria with different policy implications.

To be clear, the objective was to provide a progress report on our research into environments with alternative specifications for policy, search behavior, pricing mechanisms and information frictions. The goal was not a definitive treatment of OMO's in some model-free sense. In future work it will be important to confront the data, of course, even if we concentrated for now on theory.[22] Alternative microfounded monetary environments can also be considered. We conjecture the results would be similar in Shi's (1997) model. We are less sure about the results, or even how to set up the problem, in Molico's (2006) model, but a version of Chiu and Molico (2010,2011) could work. The difficulty with those environments is that asset holdings are history dependent. Eliminating that complication, which allows us to get analytical (not only numerical) results, is an advantage of the Lagos and Wright (2005) setup, although not everyone agrees the net benefit is positive (e.g., see Wallace 2014).

We wrap up by borrowing from Hahn (1973), who prefaced his discussion on the foundations of monetary economics with this suggestion: "The natural place to start is by taking the claim that money has something to do with the activity of exchange, seriously." We agree, although a friendly amendment would be to gener-

[22] Empirical work by Krishnamurthy and Vissing-Jorgensen (2012) and Nagel (2014) provides support for the key concept, that assets other than cash convey liquidity. For a more structural approach, models similar to the one here are not hard to estimate or calibrate (see, e.g., Aruoba and Schorfheide 2011 or Venkateswaran and Wright 2013).

alize “money” to “liquid assets.” He eventually concluded with this: “I should like to end on a defensive note. To many who would call themselves monetary economists the problems which I have been discussing must seem excessively abstract and unnecessary... Will this preoccupation with foundations, they may argue, help one iota in formulating monetary policy or in predicting the consequences of parameter changes? Are not IS and LM sufficient unto the day? ... It may well be that the approaches here utilized will not in the event improve our advise to the Bank of England; I am rather convinced that it will make a fundamental difference to the way in which we view a decentralized economy.”

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Appendix A: Long Bonds (not necessarily for publication)

Consider long-term bonds, as in Section 4.2. In Case 1 (the constraints bind in all meetings), the effects of ι are:

$$\frac{\partial z_m}{\partial \iota} = \frac{\chi_b^2(\alpha_b L_b' + \alpha_2 L_2') - \delta(1+r)A_b/z_b^2}{D_l} < 0, \ \frac{\partial z_b}{\partial \iota} = \frac{-\chi_m \chi_b \alpha_2 L_2'}{D_l} > 0$$
$$\frac{\partial q_b}{\partial \iota} = \frac{-\chi_m \chi_b \alpha_2 L_2'}{v_b' D_l} > 0, \ \frac{\partial q_2}{\partial \iota} = \frac{\chi_m \chi_b^2 \alpha_b L_b' - \delta(1+r)A_b/z_b^2}{v_2' D_l} < 0$$

where $D_l = \chi_m^2 \left(D_m \alpha_m L_m' + D_2 \alpha_2 L_2'\right) > 0$, with $D_m = \chi_b^2 \left(\alpha_b L_b' + \alpha_2 L_2'\right) - \delta(1+r)A_b/z_b^2$ and $D_2 = \chi_b^2 \alpha_b L_b' - \delta(1+r)A_b/z_b^2$. The effect on q_m is basically the same as the effect on z_m. The main difference from short bonds is that $z_b = \phi_b A_b$ is now endogenous, because of the price ϕ_b. For financial variables:

$$\frac{\partial s}{\partial \iota} = \frac{-\delta(1+r)A_b \chi_m \chi_b \alpha_2 L_2'}{z_b^2 D_l} > 0, \ \frac{\partial \phi_b}{\partial \iota} = \frac{-\chi_m \chi_b \alpha_2 L_2'}{A_b D_l} > 0$$
$$\frac{\partial \rho}{\partial \iota} = \frac{1}{1+s} + \frac{(1+\rho)\,\delta(1+r)A_b \chi_m \chi_b \alpha_2 L_2'}{z_b^2 (1+s) D_l} \gtreqless 0$$

The one ambiguous result is $\partial\rho/\partial\iota$, the same as the baseline model, due to the combination of Fisher and Mundell effects.

The effects of OMO's are:

$$\frac{\partial z_m}{\partial A_b} = \frac{\chi_m \chi_b \alpha_2 L_2' \gamma(1+r)}{z_b D_l} < 0, \ \frac{\partial z_b}{\partial A_b} = \frac{-\chi_m^2 \delta(1+r)(\alpha_m L_m' + \alpha_2 L_2')}{z_b D_l} > 0$$
$$\frac{\partial q_b}{\partial A_b} = \frac{-\chi_m^2 \delta(1+r)(\alpha_m L_m' + \alpha_2 L_2')}{z_b v_b' D_l} > 0, \ \frac{\partial q_2}{\partial A_b} = \frac{-\chi_m^2 \chi_b \delta(1+r)\alpha_m L_m'}{z_b v_2' D_l} > 0$$

and again the effect on q_m is the same as z_m. For financial variables:

$$\frac{\partial s}{\partial A_b} = \frac{-\chi_m^2 \chi_b^2 \delta(1+r)\left[\alpha_m L_m'(\alpha_b L_b' + \alpha_2 L_2') + \alpha_b \alpha_2 L_b' L_2'\right]}{z_b D_l} < 0,$$
$$\frac{\partial \rho}{\partial A_b} = \frac{\chi_m^2 \chi_b^2 \delta(1+r)\,(1+\rho)\left[\alpha_m L_m'(\alpha_b L_b' + \alpha_2 L_2') + \alpha_b \alpha_2 L_b' L_2'\right]}{z_b\,(1+s)\,D_l} > 0,$$
$$\frac{\partial \phi_b}{\partial A_b} = \frac{-\phi_b \chi_m^2 \chi_b^2 \delta(1+r)\,(1+\rho)\left[\alpha_m L_m'(\alpha_b L_b' + \alpha_2 L_2') + \alpha_b \alpha_2 L_b' L_2'\right]}{z_b\,(\rho-\pi)\,(1+s)\,D_l} < 0.$$

One can similarly consider Cases 2 and 3. ■

Appendix B: Additional Effects (not necessarily for publication)

In the baseline model, with buyers constrained in all meetings, the effects of the arrival rates on quantities are:

$$\frac{\partial q_m}{\partial \alpha_m} = \frac{-L_m}{(\alpha_m L_m' + \alpha_2 L_2')v_m'} > 0,\ \frac{\partial q_m}{\partial \alpha_b} = 0,\ \frac{\partial q_m}{\partial \alpha_2} = \frac{-L_2}{(\alpha_m L_m' + \alpha_2 L_2')v_m'} > 0,$$
$$\frac{\partial q_2}{\partial \alpha_m} = \frac{-L_m}{(\alpha_m L_m' + \alpha_2 L_2')v_2'} > 0,\ \frac{\partial q_2}{\partial \alpha_b} = 0,\ \frac{\partial q_2}{\partial \alpha_2} = \frac{-L_2}{(\alpha_m L_m' + \alpha_2 L_2')v_2'} > 0,$$

plus $\partial q_b/\partial \alpha_m = \partial q_b/\partial \alpha_b = \partial q_b/\partial \alpha_2 = 0$. For the other (financial) variables:

$$\frac{\partial z_m}{\partial \alpha_m} = \frac{-L_m}{\alpha_m \chi_m L_m' + \alpha_2 \chi_m L_2'} > 0,\ \frac{\partial z_m}{\partial \alpha_b} = 0,\ \frac{\partial z_m}{\partial \alpha_2} = \frac{-L_2}{\alpha_m \chi_m L_m' + \alpha_2 \chi_m L_2'} > 0$$
$$\frac{\partial s}{\partial \alpha_m} = \frac{-\alpha_2 \chi_b L_2' L_m}{\alpha_m L_m' + \alpha_2 L_2'} < 0,\ \frac{\partial s}{\partial \alpha_b} = \chi_b L_b > 0,\ \frac{\partial s}{\partial \alpha_2} = \frac{\alpha_m \chi_b L_m' L_2}{\alpha_m L_m' + \alpha_2 L_2'} > 0$$
$$\frac{\partial \rho}{\partial \alpha_m} = \frac{\alpha_2 L_2' R L_m}{D_D} > 0,\ \frac{\partial \rho}{\partial \alpha_b} = \frac{-(\alpha_m L_m' + \alpha_2 L_2')RL_b}{D_D} < 0,\ \frac{\partial \rho}{\partial \alpha_2} = \frac{-\alpha_m L_m' R L_2}{D_D} < 0$$

where $R = (1+\rho)\chi_b$ and $D_D = (\alpha_m L_m' + \alpha_2 L_2')(1 + \alpha_b \chi_b L_b + \alpha_2 \chi_b L_2)$. The effects on ϕ_b are similar to ρ with the opposite sign.

The effects of pledgeability on quantities are:

$$\frac{\partial q_m}{\partial \chi_m} = \frac{-(\alpha_m L_m + \alpha_2 L_2)}{\chi_m v_m'(\alpha_m L_m' + \alpha_2 L_2')} > 0,\ \frac{\partial q_b}{\partial \chi_m} = 0,\ \frac{\partial q_2}{\partial \chi_m} = \frac{-(\alpha_m L_m + \alpha_2 L_2)}{\chi_m v_2'(\alpha_m L_m' + \alpha_2 L_2')} > 0$$
$$\frac{\partial q_m}{\partial \chi_b} = \frac{-\alpha_2 A_b L_2'}{\alpha_m L_m' + \alpha_2 L_2'} < 0,\ \frac{\partial q_b}{\partial \chi_b} = \frac{A_b}{v_b'} > 0,\ \frac{\partial q_2}{\partial \chi_b} = \frac{\alpha_m L_m' A_b}{(\alpha_m L_m' + \alpha_2 L_2')v_2'} > 0$$

For other variables:

$$\frac{\partial z_m}{\partial \chi_m} = -\frac{z_m}{\chi_m} - \frac{\iota}{\chi_m^3(\alpha_m L_m' + \alpha_2 L_2')} \gtrless 0, \frac{\partial z_m}{\partial \chi_b} = \frac{-\alpha_2 A_b L_2'}{\chi_m(\alpha_m L_m' + \alpha_2 L_2')} < 0$$
$$\frac{\partial s}{\partial \chi_m} = \frac{-\alpha_2 \chi_b L_2'(\alpha_m L_m + \alpha_2 L_2)}{\alpha_m \chi_m L_m' + \alpha_2 \chi_m L_2'} < 0,\ \frac{\partial s}{\partial \chi_b} = \frac{s}{\chi_b} + \alpha_b \chi_b A_b L_b' + \frac{\alpha_2 \chi_2 A_b L_2' \alpha_m L_m'}{\alpha_m L_m' + \alpha_2 L_2'} \gtrless 0$$
$$\frac{\partial \rho}{\partial \chi_m} = \frac{\alpha_2 \chi_b L_2'(\alpha_m L_m + \alpha_2 L_2)(1+\rho)}{(\alpha_m \chi_m L_m' + \alpha_2 \chi_m L_2')(1+s)} > 0,\ \frac{\partial \rho}{\partial \chi_b} = \frac{-(1+\rho)}{1+s} D_A \gtrless 0$$

where $D_A = s/\chi_b + \alpha_b \chi_b A_b L_b' + (\alpha_2 \chi_b A_b L_2' \alpha_m L_m') / (\alpha_m L_m' + \alpha_2 L_2')$. The effects on ϕ_b are similar to ρ with opposite sign. Note the impact of χ_m on z_m is ambiguous, as is the impact of χ_b on s and ρ. ■

Appendix C: Directed Search (not necessarily for publication)

Since Section 6.2 focused mainly on examples, here we present a more general directed search model with posting when there is one asset z, with a spread s between the return on it and on an illiquid bond; a special case is fiat money where $s = \iota$. Market makers post $(q, \hat{z}, \sigma)$ to solve a version of (28), with s instead of ι and Π instead of Π_m. Generically there is a unique solution, with $U^b(s, \Pi)$ decreasing in both s and Π (we assume Π is not too big, so the market can open). The FOC's wrt q and σ are given by (29)-(30) except with s and Π instead of ι and Π_m. This generates a correspondence $\sigma(\Pi)$, similar to a demand correspondence, with σ the quantity and Π the price, and one can show $\sigma(\Pi)$ is decreasing (Rocheteau and Wright 2005, Lemma 5).

Let us normalize the measure of buyers to $\mu = 1$. One approach in the literature assumes that n is fixed, and therefore in equilibrium $\sigma = n$ (the seller-buyer ratio in the representative submarket is the population ratio). Then $\sigma(\Pi) = n$ pins down Π. In this case,

$$\frac{\partial q}{\partial s} = \frac{c'}{\alpha u'' - (\alpha + s) c''} < 0, \text{ and } \frac{\partial q}{\partial n} = -\frac{\alpha' (u' - c')}{\alpha u'' - (\alpha + s) c''} > 0.$$

Also, suppose ε is constant, as it is with a Cobb-Douglas matching function (truncated to keep probabilities between 0 and 1). Then we derive

$$\frac{\partial \hat{z}}{\partial s} = \frac{\alpha \{u'c' [\alpha + s(1-\varepsilon)] - \varepsilon(1-\varepsilon)(u-c) [\alpha u'' - (\alpha + s) c'']\}}{[\alpha + s(1-\varepsilon)]^2 [\alpha u'' - (\alpha + s) c'']} < 0$$
$$\frac{\partial \hat{z}}{\partial n} = \frac{\iota \alpha' \{\varepsilon(1-\varepsilon)(u-c) [\alpha u'' - (\alpha + s) c''] - u'c' [\alpha + s(1-\varepsilon)]\}}{[\alpha + s(1-\varepsilon)]^2 [\alpha u'' - (\alpha + s) c'']} > 0.$$

Another approach in the literature assumes a perfectly-elastic supply of homogeneous sellers, with fixed cost of entry κ, so that in equilibrium $\Pi = \kappa$ and $\sigma = \sigma(\kappa)$ is endogenous. In this case,

$$\frac{\partial q}{\partial s} = \frac{c' \alpha'' (u - c)}{D} < 0, \text{ and } \frac{\partial q}{\partial \kappa} = -\frac{\alpha' [1 + s(1-\varepsilon)/\alpha](u' - c')}{D} < 0.$$

with $D = [\alpha u'' - (\alpha + s) c''] [\alpha''(u - c) + s\kappa(1-\varepsilon)\alpha'/\alpha^2] - \alpha'^2 (u' - c')^2 > 0$ (while

D cannot be signed globally, except in special cases like $\iota = 0$, in equilibrium $D > 0$ by the SOC's). Also, if ε is constant, then

$$\frac{\partial \sigma}{\partial s} = \frac{[\alpha u'' - (\alpha + s)c'']\kappa(1-\varepsilon)/\alpha - \alpha'(u' - c')c'}{D} < 0$$
$$\frac{\partial \sigma}{\partial \kappa} = \frac{[\alpha u'' - (\alpha + s)\, c''][1 + s(1-\varepsilon)/\alpha]}{D} < 0$$
$$\frac{\partial \hat{z}}{\partial s} = \frac{\kappa(1-\varepsilon)^2 \left[\alpha u'' - (\alpha + s)\, c''\right] + c'^2 \left[\alpha(u - c)\alpha'' - s\kappa(1-\varepsilon)\alpha'\right]}{\alpha^2 D} < 0$$
$$\frac{\partial \hat{z}}{\partial \kappa} = -\frac{\iota\alpha' \left\{u' \left[\alpha + s(1-\varepsilon)\right] + \varepsilon(1-\varepsilon)c \left[\alpha u'' - (\alpha + s)\, c''\right]\right\}}{\alpha \left[\alpha + s(1-\varepsilon)\right] D} \gtrless 0.$$