How do economic variables affect the pricing of commodity derivatives and insurance?

Hirbod Assa, Meng Wang, Gabriel J. Power, and Philippe Grégoire

Version dated October 1, 2017.

Abstract

This paper focuses on designing and pricing commodity derivatives and insurance within a novel financial engineering framework that is subsequently tested on commodity price data. These results contribute to a better understanding of the “financialization” of commodities. We quantify explicitly how futures and option prices, as well as insurance premiums, are affected by economic variables linked to commodity supply and demand (e.g., their price elasticity). Our approach is therefore structural instead of reduced-form. Our results generalize existing commodity derivative pricing models and further show under which conditions derivative instruments and insurance will not be offered.

Keywords: option pricing; risk management; insurance; derivatives; commodities; demand; elasticity;

1 Affiliations: Assa is Lecturer (Assistant Professor), Institute for Risk and Uncertainty, Chadwick Building, Room G62, University of Liverpool, U.K. Ph: +44-151-7944367, em: assa@liverpool.ac.uk

Wang is a PhD candidate, University of Liverpool, U.K. Em: simon.wang.uk@gmail.com

Power is Associate Professor of Finance, FSA Business School, Université Laval, Quebec City QC Canada G1V0A6. Ph: 418-656-2131 ext 4619, em: gabriel.power@fsa.ulaval.ca

Grégoire is Professor of Finance and Industrielle-Alliance Chairholder, FSA Business School, Université Laval, Quebec City QC Canada G1V0A6. Ph: 418-656-2131 ext. 5828, em: Philippe.Gregoire@fsa.ulaval.ca
How do economic variables affect
the pricing of commodity derivatives and insurance?

Hirbod Assa, Meng Wang, Gabriel J. Power, and Philippe Grégoire

1. Introduction

Unlike financial security prices, commodity prices are mainly influenced by commodity-specific economic variables, which are themselves affected by fundamental weather, production, and storage variables. Commodity prices result from demand, supply, inventory, and economic risk factors. Although a large literature exists that takes a reduced-form approach to model the prices of commodity derivatives and (re-)insurance, a deeper understanding of their pricing requires a richer model of these economic fundamentals. Indeed, financial engineering methods are usually silent when it comes to quantifying the exact role of economic variables on these contingent claim prices.

Generally speaking, there are two approaches to modelling commodity prices, prior to modelling their contingent claims. The first approach belongs to the literature on finance and financial engineering, which uses models based on diffusion processes, and begins with Black in [7]. The multifactor models of [8] and [9], the CEV model of [12], the mean reverting models of [10] are just a few examples. The second approach belongs to the economics literature and is based on rational commodity storage. In particular, this paper is interested adapting for derivative and insurance pricing the framework found in the Deaton and Laroque models [13]-[15] (the details of a storage model have been laid out in a full chapter in [11]). By making
explicit the price elasticity parameter, this paper also relates to the CEV option pricing model in [16].

This paper therefore aims to merge the two approaches mentioned above by taking an established methodology from the rational expectations theory of storage, and then modifying and applying it to a financial engineering framework. As a result, it will develop a new methodology that quantifies directly the impact of economic variables on commodity (re)insurance and derivative prices. To achieve this requires tackling several mathematical and technical problems, which we solve in this paper. The findings described in this paper contribute to an active literature in the “financialization” of commodities. The methods and results should be useful to academics, traders, and practitioners in finance and financial engineering, as well as those in insurance, reinsurance, and risk management.

2. Representative agent model

Let us consider a representative agent with an isoelastic utility function given by

\[ u(x) = \frac{x^{1-\alpha}}{1-\alpha}, \text{if } \alpha \neq 1 \text{ and } u(x) = \log(x) \text{ otherwise.} \]

In this paper, we focus our attention on a single good, and treat spending on the other good as a residual that can be added to the agent’s utility by using a quasi-linear utility function. Therefore, we consider the agent will be solving the following problem:

\[ \max_{x,m} [ku(x) + m], \quad s.t. \quad px + m = B. \]

where \( m \) is residual income for all other goods, \( k \) is a constant, \( p_x \) is the price of good \( x \) and \( B \) is the budget constraint. It is straightforward to show that the demand is as follows:
\[ x^* = \left( \frac{k \alpha}{(1 - \alpha)p} \right)^{\frac{1}{1 - \alpha}}. \]

If we want to look at the inverse demand it can be shown that

\[ p = \frac{k \alpha}{1 - \alpha} \left( \frac{1}{x^*} \right)^{1 - \alpha}. \]

To simplify the notation, by denoting \( c = \frac{k \alpha}{1 - \alpha} \) and \( \gamma = \alpha - 1 \), one can consider the following inverse demand function

\[ p(x) = cx^\gamma. \]

The form of the demand function for different values of \( c \) and \( \gamma \) are depicted in the following graph:

**Figure 1:** Demand functions for different values of \( c \) and \( \gamma \)

As we are focusing our attention only on one good, we can re-scale the demand unit and then one can consider \( c = 1 \).
3. Model

3.1 The demand and price process

Let us consider a stochastic demand process following geometric Brownian motion (gBm) dynamics as follows:

\[
\frac{dx_t}{x_t} = \mu dt + \sigma dw_t,
\]

where in this case \((w_t)_{0 \leq t \leq T}\) is a standard Brownian motion, \(\mu\) is a number representing the rate of growth in consumption, and \(\sigma\) represents the magnitude of the demand volatility. It is known that the dynamics of the demand process given above can be written as follows:

\[
x_t = x_0 e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma w_t}, \quad t \geq 0.
\]

This model allows for different markets to be studied, as the demand functions are allowed to vary. This means we can study the effect of economic variables as well as speculation on the market demand, and the resulting derivatives and insurance contracts. Considering the iso-elastic demand function, the price dynamics for a demand function \(p\) can be given as follows

\[
p_t = p(x_t) = cx_t^\gamma = p_0 e^{\gamma (\mu - \frac{1}{2} \sigma^2)t + \gamma \sigma w_t}.
\]

If we consider a new Brownian motion \(B_t = -w_t\), one can rewrite the price dynamics as follows

\[
p_t = p_0 e^{\gamma (\mu - \frac{1}{2} \sigma^2)t - \gamma \sigma B_t}.
\]

This change is necessary because \(\gamma < 0\), and as a result \(-\gamma \sigma > 0\). Using the Ito formula for the function \(f(x, t) = ce^{\gamma (\mu - \frac{1}{2} \sigma^2)t - \gamma \sigma x}\) gives
\[
\frac{\partial f}{\partial t} = \gamma \left( \mu - \frac{1}{2} \sigma^2 \right) f, \quad \frac{\partial f}{\partial x} = (\gamma \sigma) f, \quad \frac{\partial^2 f}{\partial x^2} = \gamma^2 \sigma^2 f, 
\]

resulting in

\[
dp_t = \gamma \left( \mu + \frac{1}{2} (\gamma - 1) \sigma^2 \right) p_t dt - \gamma \sigma p_t dB_t 
\]

For simplicity, we can write the stochastic differential equation (SDE) of the price process as follows

\[
\frac{dp_t}{p_t} = \nu dt + \eta dB_t, p_0 > 0
\]

where \( \nu = \gamma \left( \mu + \frac{1}{2} (\gamma - 1) \sigma^2 \right) \) and \( \eta = -\gamma \sigma \).

It is worth considering some conditions under which the model makes greater economic sense. The first condition is that the drift term of the price, i.e., \( \nu \), must be non-negative. Since \( \gamma \leq 0 \) then this is equivalent to checking that

\[
\mu + \frac{1}{2} (\gamma - 1) \sigma^2 \leq 0.
\]

However, on the other hand, the market price of risk needs to be non-negative to make sure that market participants will participate. For that reason, it is necessary to check whether \( \nu - r > 0 \). This condition will certainly yield the previous one. The two conditions make economic sense, but in general they are not necessary to obtain solutions.

### 3.2 Loss distribution

Let us consider a time horizon \( T \) at which we want to introduce a loss variable and make an insurance contract to hedge against the risk of the losses. In this paper, we consider the following non-negative random variable as the loss
\[ L = (p_0 - e^{-rT}p_T)_+. \]

We wish to find out the distribution of the loss variable \( L \). First, it is not difficult to see that for 
\( x < 0 \), we have \( F_L(x) = 0 \), for \( x = 0 \) we have \( F_L(0) = P(p_0 \leq e^{-rT}p_T) = N \left( \frac{\gamma(\mu - \frac{1}{2}\sigma^2)T - rT}{-\gamma\sigma\sqrt{T}} \right) \)

and for \( x > p_0 \), we have \( F_L(x) = 1 \). Now let us consider \( p_0 \geq x > 0 \). In this case we have
\[
F_L(x) = 1 - P(L > x) = 1 - P((p_0 - e^{-rT}p_T)_+ > x) = 1 - P(p_0 - e^{-rT}p_T > x)
\]
\[
= 1 - P \left( p_0 \left( 1 - e^{-rT}e^{\gamma(\mu - \frac{1}{2}\sigma^2)T - \gamma\sigma B_T} \right) > x \right)
\]
\[
= 1 - P \left( e^{rT \left( 1 - \frac{x}{p_0} \right)} > e^{\gamma(\mu - \frac{1}{2}\sigma^2)T - \gamma\sigma B_T} \right)
\]
\[
= 1 - P \left( \frac{\log e^{rT \left( 1 - \frac{x}{p_0} \right)} - \gamma \left( \mu - \frac{1}{2}\sigma^2 \right) T}{-\gamma\sigma\sqrt{T}} > B_1 \right)
\]
\[
= 1 - N \left( \frac{rT + \log \left( 1 - \frac{x}{p_0} \right) - \gamma \left( \mu - \frac{1}{2}\sigma^2 \right) T}{-\gamma\sigma\sqrt{T}} \right)
\]
\[
= N \left( \frac{\gamma \left( \mu - \frac{1}{2}\sigma^2 \right) T - rT - \log \left( 1 - \frac{x}{p_0} \right)}{-\gamma\sigma\sqrt{T}} \right).
\]

In sum, we get
\[
F_L(x) = \begin{cases} 
N \left( \frac{\gamma \left( \mu - \frac{1}{2}\sigma^2 \right) T - rT - \log \left( 1 - \frac{x}{p_0} \right)}{-\gamma\sigma\sqrt{T}} \right), & 0 \leq x < p_0, \\
0, & x < 0, \\
1, & x \geq p_0
\end{cases}
\]

The graph of \( F_L(x) \) is depicted as follows:
Figure 2: Graph of the distribution function $F_x(L)$ describing the loss function

3.3 Premium

Now we can use risk-neutral valuation principles to price any contract $H = h(p_T)$ by using the risk-free probability measure $Q$ as follows

$$\text{Price} = e^{-rT}E_Q(h(p_T)) = e^{-rT}E\left(\frac{dQ}{dP} h(p_T)\right),$$

where

$$\frac{dQ}{dP} = e^{\left(\frac{m^2}{2\eta^2\pi} - \frac{m}{\eta}\right)} \left(\frac{e^{-rT}p_T}{p_0}\right)^\frac{m}{\eta^2},$$

and $m = \nu - r$. From this, we can show that

$$\pi(L) = E\left(\frac{dQ}{dP} h(P_T)\right) = \int_0^1 \text{VaR}_t \left(h(p_T)\right) d\Gamma(t),$$

where $\Gamma(t) = N\left(N^{-1}(t) - \frac{|m|\sqrt{T}}{\eta}\right)$. 
4. Designing optimal insurance and option pricing

4.1 Moral hazard issues

In this section we consider how to use the above model for purposes of pricing optimal insurance contracts as well as options. In the case of designing an optimal insurance contract we should consider moral hazard and see if we can design a contract that rules out it. The literature on actuarial mathematics deals with this problem in the following manner by considering that both insurer and insuree should feel the losses.

We assume that contract $X$ should be such that both $X$ and $(L - X)$, where $L = (p_0 - e^{-rT}p_T)_+$ is the loss, increase when $L$ increases. Therefore, we consider contracts $X = k(L)$, where $k$ belongs to the following set

$$C = \{k: R_+ \rightarrow R_+ | k(x) and x - k(x) are non-decreasing in x\}.$$ 

4.2 Optimal solution

Next, we set up an optimal insurance problem and try to find an optimal solution. For that we assume that the insuree is a risk-averse agent whose risk is measured according to a distortion risk measure $\rho$ on the set of non-negative random variables defined as follows

$$\rho(X) = \int_0^1 \text{VaR}_t(X) d\Pi(t).$$

Here $\Pi: [0,1] \rightarrow [0,1]$ is a non-decreasing function so that $\Pi(0) = 0$ and $\Pi(1) = 1$. This family of risk measures includes very important examples, e.g., Value at Risk with:
\[ \Pi(t) = 1_{[\alpha,1]} \]

or Conditional Value at Risk with:

\[ \Pi(t) = \frac{t - \alpha}{1 - \alpha} 1_{[\alpha,1]} . \]

The insuree’s global loss is the part of the loss that is not covered by insurer, added up to the amount that is paid for premium, i.e.,

\[ \text{Global loss} = L - X + \pi(X). \]

Since distortion risk measures are cash-invariant, the risk of the global loss is \( \rho(L - X) + \pi(X) \).

In order to study insurance premiums, we consider an optimal insurance design problem as proposed in [5] and [6] (or similarly with a budget constraint in [4]):

\[ \min_{k \in C} \rho(L - k(L)) + \delta \pi(k(L)), \]

for a risk loading factor \( \delta \geq 1 \) that is used by the insurance company. Using the marginal indemnification function method (MIF) introduced in [5] and developed in [3], [4] and [6], this problem can be re-written as follows

\[ \min_{0 \leq k' \leq 1} \int_0^1 \left( \delta \left( 1 - \Gamma(F_L(t)) \right) - \left( 1 - \Pi(F_L(t)) \right) \right) k'(t) dt, \]

where \( k' \) is the derivative of \( k \). The optimal solution is then given by \( X = k(L) \), where

\[ k'(t) = \begin{cases} 1, & 1 - \Pi(F_L(t)) > \delta \left( 1 - \Gamma(F_L(t)) \right) \\ 0, & 1 - \Pi(F_L(t)) \leq \delta \left( 1 - \Gamma(F_L(t)) \right). \end{cases} \]
4.3 Evolution of the contracts with $\gamma$

In this section, we assume that there are values $a, b \in (0,1)$ so that $1 - \Pi(x) > \delta (1 - \Gamma(x))$ on $(a, b)$ and $1 - \Pi(x) < \delta (1 - \Gamma(x))$ on $(0, a) \cup (b, 1)$. This assumption holds for many interesting cases including $\rho = \text{VaR}$ and CVaR.

![Graph](image)

**Figure 3**: Representing the optimal solution: interesting cases

The existence of the optimal solution and its form depends on $F_L(0)$. However, we know that

$$F_L(0) = N \left( \frac{\gamma \left( \mu - \frac{1}{2} \sigma^2 \right) T - rT}{-\gamma \sigma \sqrt{T}} \right) = N \left( \frac{\left( \mu - \frac{1}{2} \sigma^2 \right) \sqrt{T}}{-\sigma} - \frac{r \sqrt{T}}{|\gamma| \sigma} \right).$$

Therefore, increasing the absolute value of $\gamma$ will decrease the value of $F_L(0)$. The optimal solution in this case either: i) does not exist, ii) is a stop loss policy, or iii) is a two-layer policy. This result can be shown as in the following figure by depicting $F_L$ for different values of $\gamma$. 

11
We have three cases:

1. If $F_L(0) > b$, or \( \frac{r\sqrt{T}}{\sigma\left(N^{-1}(b)\frac{\left(\frac{\mu - \frac{1}{2}\sigma^2}{\tau}\right)}{\frac{1}{2}}\right)} < \gamma \), then $k' = 0$ and there is no contract.

2. If $a < F_L(0) < b$, or \( \frac{r\sqrt{T}}{\sigma\left(N^{-1}(a)\frac{\left(\frac{\mu - \frac{1}{2}\sigma^2}{\tau}\right)}{\frac{1}{2}}\right)} < \gamma < \frac{r\sqrt{T}}{\sigma\left(N^{-1}(b)\frac{\left(\frac{\mu - \frac{1}{2}\sigma^2}{\tau}\right)}{\frac{1}{2}}\right)} \), then there is a stop loss policy with retention level that solves

\[
N\left(\gamma \left(\mu - \frac{1}{2}\sigma^2\right)T - rT - \log\left(1 - \frac{b^*}{p_0}\right)\right) = b.
\]

This results in

\[
b^* = p_0 \left(1 - \exp\left(\gamma \left(\mu - \frac{1}{2}\sigma^2\right)T - rT + \gamma\sigma\sqrt{T}N^{-1}(b)\right)\right).
\]
3. Finally, if, \( F_L(0) < a \) or \( \gamma < \frac{r \sqrt{T}}{\sigma N^{-1}(a) - \frac{(\mu - \frac{1}{2} \sigma^2) r}{\sigma}} \), then the contract is a two-layer policy with upper retention level \( b^* \) given above and lower retention level \( a^* \) given as

\[
a^* = p_0 \left( 1 - \exp \left( \gamma \left( \mu - \frac{1}{2} \sigma^2 \right) T - r T + \gamma \sqrt{T} N^{-1}(a) \right) \right).
\]

\[\text{Figure 5: Cases 1 through 3 in terms of values of } a \text{ and } b\]

\[\text{4.4 Example: VaR}\]

Based on general case that we discussed above, let \( a \) be the solution to \( \delta \left( 1 - \Gamma(a) \right) = 1 \) or \( a = \Gamma^{-1} \left( 1 - \frac{1}{\delta} \right) \). This means

\[
\delta \left( 1 - N \left( N^{-1}(a) - \frac{|m| \sqrt{T}}{\eta} \right) \right) = 1
\]

or
\[ a = N \left( \frac{|m|\sqrt{T}}{\eta} + N^{-1} \left( 1 - \frac{1}{\delta} \right) \right). \]

It is also clear that in this case \( b = \alpha \).

There are three cases:

1. If \( N \left( \frac{\gamma (\mu - \frac{1}{2} \sigma^2) T - rT}{-\gamma \sigma \sqrt{T}} \right) \geq \alpha \). This is equivalent to \( \gamma \left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} N^{-1}(\alpha) \geq rT \).

   Since \( \gamma \leq 0 \), therefore, \( \left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} N^{-1}(\alpha) \leq 0 \), and as a result we have to check if

   \[ \gamma \leq \frac{rT}{\left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} N^{-1}(\alpha)}. \]

   In this case, \( k' = 0 \) and there is no insurance contract.

2. If \( a \leq N \left( \frac{\gamma (\mu - \frac{1}{2} \sigma^2) T - rT}{-\gamma \sigma \sqrt{T}} \right) < \alpha \).

   This is equivalent to \( N \left( \frac{|m|\sqrt{T}}{\eta} + N^{-1} \left( 1 - \frac{1}{\delta} \right) \right) \leq N \left( \frac{\gamma (\mu - \frac{1}{2} \sigma^2) T - rT}{-\gamma \sigma \sqrt{T}} \right) < \alpha \).

   First, let us look to the right inequality.

   a. If \( N \left( \frac{\gamma (\mu - \frac{1}{2} \sigma^2) T - rT}{-\gamma \sigma \sqrt{T}} \right) < \alpha \) and \( \left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} N^{-1}(\alpha) \leq 0 \)

      we get \( \gamma > \frac{rT}{\left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} N^{-1}(\alpha)} \).

   b. If \( N \left( \frac{\gamma (\mu - \frac{1}{2} \sigma^2) T - rT}{-\gamma \sigma \sqrt{T}} \right) < \alpha \) and \( \left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} N^{-1}(\alpha) > 0 \)

      we get \( \gamma < \frac{rT}{\left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} N^{-1}(\alpha)} \).
Second, the left inequality results in

\[
\frac{\left(\gamma \left(\mu + \frac{1}{2} (\gamma - 1) \sigma^2\right)\right) \sqrt{T}}{-\gamma \sigma} - \frac{r \sqrt{T}}{-\gamma \sigma} + \frac{N^{-1} \left(1 - \frac{1}{\delta}\right)}{-\gamma \sigma} \leq \frac{\gamma \left(\mu - \frac{1}{2} \sigma^2\right) T - r T}{-\gamma \sigma \sqrt{T}}
\]

\[
= \frac{\gamma \left(\mu - \frac{1}{2} \sigma^2\right) \sqrt{T}}{-\gamma \sigma} + \frac{-r \sqrt{T}}{-\gamma \sigma},
\]

resulting in,

\[
\frac{2N^{-1} \left(1 - \frac{1}{\delta}\right)}{\sigma \sqrt{T}} \leq \gamma.
\]

In this case, the contract is a stop loss policy with retention level \( b^* \) that solves

\[
N \left(\frac{\gamma \left(\mu - \frac{1}{2} \sigma^2\right) T - r T - \log(1 - \frac{b^*}{p_0})}{-\gamma \sigma \sqrt{T}}\right) = \alpha.
\]

If we solve for \( b^* \) we find

\[
b^* = p_0 \left(1 - \exp\left(\left(\frac{\gamma \left(\mu - \frac{1}{2} \sigma^2\right) T + \sigma \sqrt{T} N^{-1}(\alpha) - r T}{-\gamma \sigma \sqrt{T}}\right)\right)\right)
\]

3. If \( N \left(\frac{\gamma \left(\mu - \frac{1}{2} \sigma^2\right) T - r T}{-\gamma \sigma \sqrt{T}}\right) < \alpha \) or \( \frac{2N^{-1} \left(1 - \frac{1}{\delta}\right)}{\sigma \sqrt{T}} > \gamma \).

In this case, the contract is a two-layer contract with lower and upper retention levels

\( a^*, b^* \), where \( b^* \) is as in case 2 and that \( a^* \) solves \( N \left(\frac{\gamma \left(\mu - \frac{1}{2} \sigma^2\right) T - r T - \log(1 - \frac{a^*}{p_0})}{-\gamma \sigma \sqrt{T}}\right) = a \),

which similarly gives
\[ a^* = p_0 \left( 1 - \exp \left( \gamma \left( \mu - \frac{1}{2} \sigma^2 \right) T - rT + \gamma \sigma \sqrt{T} N^{-1}(\alpha) \right) \right). \]

Note however that, \( N^{-1}(\alpha) = N^{-1} \left( N \left( \frac{\left| m \right| \sqrt{T}}{\eta} + N^{-1} \left( 1 - \frac{1}{\delta} \right) \right) \right) = \frac{\left| m \right| \sqrt{T}}{\eta} + N^{-1} \left( 1 - \frac{1}{\delta} \right). \)

Therefore,
\[ a^* = p_0 \left( 1 - \exp \left( \gamma \left( \mu - \frac{1}{2} \sigma^2 \right) T - rT + \gamma \sigma \sqrt{T} \left( \frac{\left( \gamma \left( \mu + \frac{1}{2} (\gamma - 1) \sigma^2 \right) - r \right) \sqrt{T}}{-\gamma \sigma} + N^{-1} \left( 1 - \frac{1}{\delta} \right) \right) \right) \right) \]
\[ = p_0 \left( 1 - \exp \left( \gamma \left( \mu - \frac{1}{2} \sigma^2 \right) T - rT \pm \gamma \left( \mu - \frac{1}{2} \sigma^2 + \frac{1}{2} \gamma \sigma^2 \right) T + rT + \gamma \sigma \sqrt{T} N^{-1} \left( 1 - \frac{1}{\delta} \right) \right) \right) \]

So finally, we get
\[ a^* = p_0 \left( 1 - \exp \left( -\frac{1}{2} \gamma^2 \sigma^2 T + \gamma \sigma \sqrt{T} N^{-1} \left( 1 - \frac{1}{\delta} \right) \right) \right). \]

### 4.5 Calibration and simulation for VaR

Let us use the following calibration for the parameters:

\( T = p_0 = 1, \sigma = 0.1, \mu = 0.01, r = 0.05, \alpha = 0.95, \delta = 1.1 \) and let us consider \( \gamma \in [-50,0] \).

First, since \( \left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} N^{-1}(\alpha) = \left( 0.01 - \frac{1}{2} (0.1)^2 + 0.1(1.645) \right) = 0.1659 > 0 \), case 1 and case 2, a) does not happen.

Second, since \( \frac{2N^{-1}(1-\frac{1}{\delta})}{\sigma \sqrt{T}} = \frac{2(-1.335)}{0.1} = -26.7 \), based on case 2, b) for \(-26.7 \leq \gamma \leq 0\) the lower retention level is 0 and the upper retention level is given by \( b^* \). Finally, for \(-50 \leq \gamma \leq -26.7\) the upper and lower retention levels are given by \( b^* \) and \( a^* \), respectively,
Figure 6: Calibration solutions graphed against $\gamma$

If we want to make sure that the market price of risk is non-negative is given as follows:
Therefore, we need to find the roots of this inequality with respect to $\gamma$

$$\gamma_{1,2} = -\left(\mu - \frac{1}{2} \sigma^2\right) \pm \sqrt{\left(\mu - \frac{1}{2} \sigma^2\right)^2 + 2r\sigma^2}$$

$$= -\left(0.01 - \frac{1}{2}(0.1)^2\right) \pm \sqrt{\left(0.01 - \frac{1}{2}(0.1)^2\right)^2 + 2(0.05)(0.1)^2} \frac{(0.1)^2}{(0.1)^2}$$

$$= -(0.5) \pm 100\sqrt{0.001025}$$

which yields the solution: $-0.5 \pm 3.2 = \{-3.7, 2.8\}$.

This shows that a more sophisticated answer is for $\gamma \in [-50, -3.7]$.

<Additional tests and empirical evidence forthcoming in the next draft>

5. Conclusion

The financialization of commodities has brought considerable renewed interest in finance and risk management research to this asset class. Black’s model [7] remains the standard for pricing commodity derivatives, and most models are reduced-form in the style of [8]. But to gain a deeper understanding of these markets, both for exchange-traded derivatives as well as insurance instruments, it is important to explicitly model the economic variables that determine the stochastic price process. To obtain explicit solutions to this problem, however, is not trivial.
This paper develops a more structural framework to price commodity derivatives and insurance. The contingent claim methodology that is proposed here is based on standard risk-neutral valuation arguments, but inspired by the rational storage models of Deaton-Larocque [13-15]. It can be then applied to exchange-traded, OTC derivative, or insurance instruments.

In this paper, we show how to obtain commodity-specific pricing solutions in terms of the deeper economic parameters such as the price elasticity of demand for a given commodity, as well as the loss function that best describes the trader or hedger (e.g. value-at-risk, or conditional value-at-risk). We also consider the role played by the risk specification among a class of distortion risk measures. Results are presented for risk management applications, where optimal contract types are obtained in terms of the parameter space. In some cases, no contract is offered. These findings should be useful for academics and practitioners in commodity finance, derivatives, risk management and insurance.

References


