Pricing European Style Options

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Abstract

The Black-Scholes option model created a revolution in finance. It was perceived that the model opened up a methodology to price option contracts. The methodology has been problematic as numerous empirical contradictions and anomalies have been noted. Born out of Frequentist decision theory, a key assumption in the formula is that all parameters are known. When viewed as an estimator, however, it is shown that it does not converge to a population parameter. Consequently, a new model is built using Bayesian decision theory rather than Frequentist decision theory. This is done as it assures that the estimator will be both admissible and coherent, something that cannot generally happen with existing methods using Itô calculus or binomial trees. The model proposed is derived in two distinct ways. The first is distribution-free and presumes no first moment. The second follows the work of Harris in deriving the density functions of various asset classes. The former should be robust, the later, powerful.

1 Introduction and Literature

A new model of option pricing is necessary. While the publication of the Black-Scholes option pricing model was a watershed moment in finance, the challenge has been that the model does not empirically work. The original authors report lack of empirical support for their model in the original paper and Yilmaz in his masters thesis catalogs anomalies found in the literature [1, 2]. After showing that mean-variance formulas, in general, do not give rise to estimators that are admissible and consistent with the underlying economic theory, this paper makes a sharp break with the existing literature. Quite a few issues that are currently considered of great import will no longer matter. Things such as the prediction and
estimation of heteroskedasticity do not matter because most of the distributions involved are askedastic.

After a basic introduction to the existing literature, the paper will construct a game-theoretic representation of option pricing for equity securities. The method will be consistent with Bayesian decision theory. The use of Bayesian methods assures that the result will be both admissible and coherent. For robustness, a distribution-free method will be used initially, followed by a method that follows Harris in [3].

Harris in [3] derives the various probability distributions of returns for equity securities and provides a general solution to construct a probability distribution function for returns on capital. In Harris’ construction, separate densities exist for firms that are merging out of existence for cash, for stock, going through bankruptcy or continuing as going concerns. Since some of these base distributions lack a mean, the mixture distribution will lack a mean. This mathematical finding is consistent with the empirical line of work beginning with Mandelbrot’s seminal article in [4].

Specifically, Harris in [3] argues that returns are not data. Rather, prices and volumes are data. If returns for investing are defined as

$$r_t = \frac{p_{t+1}q_{t+1}}{p_tq_t},$$

then it follows that the distribution of $r_t$ depends upon the distributions involved with $\{p_t, q_t, p_{t+1}q_{t+1}\}$ and $r_t$ does not possess its own distribution independent of data based factors.

The specific properties of the problem must be addressed to generate a solution. The first is that most of the underlying distributions that are involved lack a first moment. The result is that sample means and tools based on the method of least squares are unavailable. The second is that since option prices depend upon data, option prices are a statistic. As such it is necessary that the derived statistic be admissible in the Wald sense [5]. Also, the statistic must be coherent in the de Finetti sense since option premiums are a form of gamble [5]. The fourth is that current price relates coherently to future risks through the time value of money. The fifth is that all intervening potential cash flows, such as dividends, are accounted for. The sixth is that all existential risks to the firm are appropriately accounted for.


1.1 Axiomatic Systems

Both the Black-Scholes Option Pricing Model and the Capital Asset Pricing Model, from which it can be derived, are built on Frequentist axioms. This observation provides a lens to understand why the empirical problems in Black-Scholes are so extensive. Problems such as \( x_{t+1} = \beta x_t + \epsilon_{t+1}, \beta > 1 \) are known not to have a Frequentist or Likelihoodist estimator that converges to a population parameter [6]. The difficulty, of course, is that this is no different from starting wealth times a reward plus a shock is equal to future wealth. Models such as the Capital Asset Pricing Model cannot be solved using Frequentist or Maximum Likelihood methods.

Two branches of statistics have a decision theory constructed, Bayesian and Frequentist. Economists normally are trained in Frequentist decision theory. In practice, both Bayesian and Frequentist decision theory is nothing more than standard microeconomics under uncertainty, possibly with some greater emphasis on probability and the secondary properties of the decision.

If admissible solutions existed for this class of problems in Frequentist decision theory, then a discussion would be made as to which of the two models should be used. Frequentist decision theory assures an unbiased estimator, a minimum guarantee against false positives, and the ability to control power. Bayesian methods cannot offer any of these things, although in a few specific cases a Bayesian point estimator will also be an unbiased estimator. Bayesian methods ignore bias as an issue, and it is only discussed with reference to Frequentist questions. Conversely, Bayesian solutions are guaranteed to be admissible, coherent solutions. Frequentist decisions are admissible only to the extent they match a Bayesian solution at every sample, or if they match Bayesian solutions at the limit. Frequentist solutions are not coherent. In the context of option pricing, this would imply that a clever actor or set of actors could game the market maker into taking a sure loss in all states of nature if that market maker chose to use Frequentist tools.

1.1.1 Why Frequentist Estimators are Excluded

Of the distributions in [3], almost none have either a first moment of a statistic sufficient for the parameters. In general, where a sufficient statistic exists, a non-Bayesian solution is admissible [7]. Although a complete article on admissibility and predictions would be preferable, it is possible to show the general reason non-Bayesian estimators are excluded.
Consider first the case of the difference equation

\[ x_{t+1} = \beta x_t + \epsilon_{t+1}, \beta > 1. \] (2)

This equation has been shown by White in [6] to lack a Frequentist or Likelihoodist solution that converges to the population parameter. Nonetheless, distribution-free estimators exist such as Theil’s regression [8] or quantile regression. Assume that the sampling distribution of \( \hat{\beta} - \beta \) is the same density as the posterior density of \( \beta \) using Bayesian methods with the remaining parameters marginalized out and a uniform prior over \( \beta \). Clearly, it would appear that the Bayesian regression would not stochastically dominate the Frequentist and so they should be interchangeable.

That would be true, except that there exists prior information regarding \( \beta \) written right into the equation. Since \( \beta > 1 \) it implies that zero prior mass should exist over the range \((-\infty, 1]\). Now consider a daily rate of increase, one that is quite near unity. Because Frequentist sampling distributions are symmetric here and assume infinite repetition, it must happen that some estimates of \( \hat{\beta} \) will be less than one. Indeed, such things happen in Frequentist estimation commonly for both finance and macroeconomics. This estimator has a zero probability of being true.

Consider the portion of the Frequentist sampling distribution less than or equal to one as being of area \( K \) and greater than one of area \( 1 - K \). By placing zero mass where \( \beta \leq 1 \), the density of the posterior over the allowed region is

\[ \frac{1 - K}{K} \] (3)

times greater than the sampling distribution. Since it is well known that Bayesian estimators can dominate Frequentist, but not the converse is not true, it follows that with zero mass in the disallowed region and the posterior density being intrinsically denser than the sampling distribution, a researcher would be better off with the Bayesian estimator for all allowed possible parameter estimates of \( \beta \).

A similar effect occurs for data drawn from the truncated Cauchy distribution,

\[ \left[ \pi \left( \frac{\sigma}{\sigma^2 + (x - \mu)^2} \right) \right]^{-1}. \] (4)

The maximum likelihood estimator has no known analytic solution, and the number of roots may vastly exceed the number of data points. The minimum variance unbiased estimator is the median, which is systematically to the right of \( \mu \) due to truncation. In addition, the minimum variance unbiased estimator for \( \mu \) is known
to be less efficient than the Bayesian estimator, without taking the bias into account [9].

Finally, Frequentist predictions depend upon the pivotal quantity $\sqrt{n}(\hat{\mu} - \mu)/\sigma$ being representative and it is not due to truncation [10].

1.2 Black-Scholes as an Estimator

The Black-Scholes option pricing model is a touchstone in the discussion of other option pricing models. There are multiple methods of discussing the appropriateness of using Black-Scholes in addition to empirical falsification. The first challenge is to address the proof within its system of axioms for correctness. It would be wonderful to say that is undoubtedly correct or incorrect, but the history of mathematics is one of apparently correct proofs being invalidated through the work of later mathematicians.

It is the consensus that within Frequentist theory that this proof is correct. Indeed, due to its perceived correctness the Kungliga Vetenskapsakademien decided to award a Nobel Prize based on it. What may be less than clear is the impact the axiomatic system has on the usefulness of the proof itself. This paper argues that the proof is hollow and vacuous in practice.

The argument made here is that the Black-Scholes option pricing model does not give rise to a useful estimator even if both the assumptions and the derivation are correct in all aspects. While this may seem surprising, it should not be given the arguments in section 1.1.1. To solve this problem one need only look at two aspects of the original paper.

The first and least problematic issue is the nature of the terminal distribution. Black and Scholes in [1] include the following assumption:

“\text{“The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is log-normal.”}"

Harris in [3] derives the distribution of returns for all asset classes. The only place such a distribution would happen is when a cash-for-stock merger has been announced and fully approved by all parties. Because this is a Frequentist construction and not a Bayesian construction, for the log-normal distribution to be present, the merger must happen with a probability of one.

This observation raises several rather peculiar issues. First, pure cash-for-stock mergers are exceedingly rare among public companies, and so the incidence
of available events would be small. Second, mergers are usually announced no more than six months of the event and mergers that happen in ninety or one hundred and twenty days are very common. The distribution requires that the merger is a certainty and not subject to rejection so the period the options could run may well be less than a month. Even if an estimator is found, it follows that a test would be problematic to perform. The third and greater issue is that the variance is zero. By assumption, the interest rate is known and constant, and there are no transaction costs. As such, this security is in practice no different from a zero coupon bond with certainty of payoff, with no risks to liquidity and no risks to the interest rate. Under the assumptions of Black-Scholes, there is nothing to estimate as the price is merely the discounted present value.

Black and Scholes make a fatal observation for their model regarding giving rise to an estimator. They observe that it is sufficient to accept the Capital Asset Pricing Model (CAPM) as valid and that their model can be derived from the standard $\beta$ equation. This observation begs the estimator question then for the CAPM. Although there are multiple versions of the CAPM, they all share a feature that prevents an estimator from forming.

As with Black-Scholes, the CAPM assumes all parameters are known with probability one. Why this is problematic is easier to see this when written in terms of wealth created by a single asset. As mentioned earlier, it must be the case that $w_{i,t+1} = \mu_i w_{i,t} + \epsilon_{i,t+1}, \mu_i > 1$. Mann and Wald in [11] have shown that the correct estimator is the ordinary least squares estimator for all values of $\mu_i$. White in [6] has shown that the distribution of $\hat{\mu}_i - \mu_i$ is the Cauchy distribution for any distribution of $\epsilon_{i,t+1}$ centered on zero with a finite, positive variance. It is the same as saying that no inference can ever be performed on the estimator and which Sen in [12] reports is perfectly inefficient as an estimator relative to any median based estimator for a center of location. That is to say, its asymptotic precision is zero. As such, the variance for the Black-Scholes model cannot be estimated.

The error in thinking is that if the terminal distribution is log-normal given perfect information on the parameters, then it must also be log-normal when the parameters are not known. In the general case, Harris in [3] shows that under the Markowitzian assumptions, that the distribution of returns must be the Cauchy distribution when the assumption of perfect knowledge is relaxed. Since no Frequentist solution exists, the question would be whether or not a Bayesian solution exists. Harris in [3] observed that White’s proof in [6] has a Bayesian interpretation. White had multiplied the likelihood by the square root of the Fisher information regarding the equation. As the square root of Fisher information is the
Jeffreys prior, it follows that the same equation can be turned around and used with Bayesian methods. As a consequence, Bayesian financial models, both in ratio form and in time series form exist. These models are not supportive of the mean-variance models as they do not have a mean. Dropping the assumption that the parameters are known brings an end to mean-variance models and the related empirical tools.

1.3 Binomial Option Pricing

The binomial option pricing model makes perfect sense and also can only be used to price a put contract. That is because returns for going concerns will be drawn from a distribution without a mean.

For a firm that will continue to exist after the expiry, a sale price for the shares will exist. For purposes of argument, this price will be denoted $p_T$. The purchase price will be denoted $p_t$. Since a return is a ratio, $\frac{p_T}{p_t} - 1$, it follows that returns follow a ratio distribution.

Since equity securities are traded in a double auction, the winner’s curse does not obtain. As such, the rational behavior for traders is to bid their expected price. With the Markowitzian assumption of many buyers and sellers, the limit book should converge to the normal distribution. Consequently, for securities not systematically away from the equilibrium price, the distribution of returns must converge to the Cauchy distribution. This fact is altered by the limitation of liability and the fact that the budget constraint is finite.

Because the Cauchy distribution lacks a mean, any section of returns that are not doubly bounded will result in divergence, and no binomial price can be set.

Binomial pricing can exist for puts, and this fact will form the basis of this model.

1.4 Pricing European Style Options

To price options, it is necessary to frame the model inside some greater framework. The primary logical framework is the study of financial asset pricing.

The study of financial asset pricing is relatively short. The first such academic study is by Bachelier in his doctoral dissertation, defended in 1900. Leonard Jimmie Savage mailed postcards out to leading economists and mathematicians to read Bachelier’s dissertation. His work was derived from work on rentes and was preceded empirically by Regnault [13]. This set off a flurry of work on Brownian motion and also option pricing. Work by Osborne extended the Brownian motion
argument which was followed on by Samuelson, Boness, and Sprenkle [14, 15, 16, 17]. The approach taken here is to fundamentally rework the math. Because of this, it will resemble the older approach in some ways, but not in others.

Like Bachelier, the study is using the limiting distribution of returns, but instead of the normal distribution, the limiting distribution is the Cauchy distribution for going concerns and other distributions for firms undergoing mergers [3]. Like Boness, this work assumes the non-negativity of prices, but unlike Boness who used the log-normal distribution to evade the issue this work uses the Cauchy distribution truncated at zero for bankruptcy [14]. Like Sprenkle who discounted at the expected rate of return for the stock price, this work considers the terminal value of the security, but discounts at the subjective opportunity cost of funds [17]. Like Black and Scholes, this work considers the no-arbitrage equilibrium but does not use Brownian motion to calculate the option price [1]. Like the mean-variance models, the game in which the price is constructed provides actors with complete historical information, and there is no informational asymmetry. Unlike these articles, this study includes liquidity costs, merger risk and the probability of bankruptcy.

Because European options have a definable terminal value without the possibility of early exercise, the simplest solution is to look at the distribution of possible terminal values. For put prices, the values are bounded at zero and \( k \), the strike price. An expected value, therefore, exists regardless of the distribution chosen. Ignoring time values and liquidity costs, the most straightforward mechanism to value a put contract is to integrate over the set of possible final values. Everything else becomes setting values to present value and adjusting for the costs of liquidity. With call options, there can be no expected value as the future value diverges without bounds. However, it is possible to price calls through equilibrium pricing.

Certain things then become necessary to price European style option contracts. The first of these is an interest rate. In the real world, there is a broad range of available interest rates. In practice, there are offers that are available nationally, such as those from purchasing a bond over an organized national exchange, and there are those that are only available in specific localities or to certain people, such as from credit unions.

In this model, there are two rates, a deposit rate, and a loan rate. The existence of the spread is not for realism. It is for coherence under de Finetti’s Coherence Principle. From the Coherence Principle, it is possible to derive Kolmogorov’s axioms of probability as theorems instead [5]. If there is a deposit and loan rate spread, then the most fundamental laws of probability are assured to hold.

Because a sufficient statistic does not exist for the truncated Cauchy distribu-
tion, non-Bayesian methods must lose information to be used [18]. It makes them inadmissible, and so non-Bayesian methods generally cannot be utilized in this paradigm pushing decision makers into other axiom systems such as de Finetti’s or Savage’s.

The second necessary component is the bankruptcy rate. Without it, the math diverges.

The third required aspect is the bid-ask spread. Like the deposit-loan spread required for coherence, the bid-ask spread is needed for coherence. Coherence permits the existence of call contracts even though no expected value could exist.

The fourth required tool for pricing options is initial endowments. Actors with different endowments may face different decisions. An actor indebted by $100,000 is in a different position than one flush with cash.

There is a fifth factor not required for pricing, except of course in the real world, but which is quite useful. That is to include a probability of merger.

Mergers are often bankruptcy alternatives, and a firm that is merging is actually selling out its underlying physical capital to cover liabilities in a non-statutory liquidation. It permits teasing out the distribution of going concerns with firms that will be absorbed by another firm.

While not required for pricing, this model is shown to be independent of the actual distribution involved. A distribution-free alternative is provided. Allowing the solution to be independent of a particular distribution allows for a robustness that could never happen in Black-Scholes or similar models.

Black-Scholes is very dependent on the distribution involved. While the model in this paper uses the distributions in [3], using the distribution-free alternative offers a robustness check against any errors in thinking during model building. Simple Bayesian model selection would show that the robust alternative was better or worse than the parametric model. That cannot be done with Black-Scholes as the solution depends entirely upon the distribution involved. Further, because some distributions lack important properties, such as a mean, the distribution-free solution can be constructed without requiring one exist.

Even if Harris in [3] is incorrect, this model of option pricing is adequately robust to survive without it.

1.5 Bayesian Likelihood Functions

The difference between a Frequentist probability density function and a Bayesian likelihood function is what variables are random. For the Frequentist, the data is
random. For the Bayesian the parameters are random. So while

\[ f(x|\mu) = \frac{1}{\pi} \frac{1}{1 + (x - \mu)^2} \forall x \in \chi, \tag{5} \]

where \( \chi \) is the sample space is a perfectly good density function, it becomes a likelihood function when it is transformed into

\[ f(x|\mu) = \frac{1}{\pi} \frac{1}{1 + (x - \mu)^2} \forall \mu \in \Theta, \tag{6} \]

where \( \Theta \) is the parameter space.

Randomness is interpreted in Bayesian thinking as being uncertainty rather than chance and due to incomplete information. The quantification of the certainty, or, alternatively, the uncertainty in the sample is through the likelihood function.

### 1.5.1 Distribution-Free Likelihood

For the distribution-free likelihood, a histogram-like density is constructed. It can be a true histogram, or it can be smoothed as with kernel smoothing. Each bar of the histogram has an uncertain height. It is this height which is the parameter. Each height is estimated from a likelihood function. The multinomial is appropriate, but the normal approximation also would work.

### 1.5.2 Harris’ Distributions

Harris in [3] notes that returns are not data, but rather statistics. Prices and volumes are data, but as a statistic is any function of data, it must be the case that returns are statistics. The solution to solving the distribution of returns is to solve the distribution of prices.

Returns are the ratio of the selling price divided by the buying price minus one. The exception to this is cash-for-stock mergers where the final price is fixed in cash. According to Harris, the distribution of those mergers should be log-normal. Likewise, according to Harris, under bankruptcy, it is not prices that go to zero but rather the number of shares. The Bankruptcy Court vacates the ownership claims of the shareholders. The price remains undefined.

For going concerns, Harris argues that the distribution is approximately a truncated Cauchy distribution [3]. For mergers, Harris argues that it is a transformation of the Cauchy distribution. The argument is that firms that are targets for
buyout must be systematically well below their equilibrium price. This difference can be seen in how the distribution would be derived in polar coordinates.

For going concerns, the errors are centered around the equilibrium. The cumulative density function is found by rotating an angle $d\theta$ around the joint density function for the buy and sell price and integrating over both the angle and the radius of errors relative to the point of equilibrium. The cumulative density is then differentiated to determine the density function of returns.

For stock-for-stock mergers, the errors are centered around the current actual price relative to the equilibrium current price and the future price in equilibrium. However, the angle and radius of integration are still anchored at the equilibrium. Centering the density over the current price and the equilibrium final price, but integrating around the joint equilibrium price point creates a skewed version of the Cauchy distribution.

2 The Game

This game proposes a simple option pricing model, but it is not the model of option pricing. Fortunately, through the use of Bayes factors, it should be possible in relatively short order to find nearly optimal solutions for pricing by changing the information inputs and the structure of the models. The available dataset covers a sufficiently long time that definitive models should be possible within a reasonable amount of time, obsolescing this article in short order.

Actors in the game engage in several actions before deciding on portfolio allocations. They set a prior distribution, estimate parameters using a cost function, construct a predictive distribution and use that distribution to engage in decisions regarding securities’ positions.

The game exists as a limited game inside a set of games, one game for each possible security and at one subsection of time in an infinitely repeated game.

2.1 Notation

Depending on the circumstances, most of the appropriate notation is suppressed. This notation will vary slightly from section to section and will be annotated in the text. For example, the bid price for a call option would be $\psi z(n)$, while the asked price would be $z(n)\psi$, with $z(n)$ being the mark up or mark down for liquidity costs. While this is relatively mild, the complete representation of a call as a
function of the variables that map onto it would be:

$$\psi = \psi(p_t, k, t, T, \Delta t, i_D | \Delta t, i_L | \Delta t, \delta | [t, T], n, B, M, \lambda)$$  \hspace{1cm} (7)

Likewise the full notation for a put would be:

$$\phi = \phi(p_t, k, t, T, \Delta t, i_D | \Delta t, i_L | \Delta t, \delta | [t, T], n, B, M, \lambda)$$  \hspace{1cm} (8)

In addition, there are multiple interest rates in the game. For some calculations, a deposit or a loan rate is specifically important and would be denoted $i_D$ or $i_L$. In some places the interest rate is denoted $i_X$ where $X \in \{D, L\}$.

### 2.2 Actors

The game has three types of actors; they are nature, market makers, and participants. At time zero, nature makes an initial move for each actor endowing them with resources. Participants are endowed with $\bar{m}_i$ of liquid wealth, such that $-\infty < \bar{m}_i < \infty$ and $\bar{m}_i \in \mathbb{R}, \forall i \in I$. $I$ is the index of participants. Also, nature endows market makers and participants with $\omega_i$ in risky wealth. Resources are endowed so that $\sum_{i \in I} \bar{m} = 0$ and $\sum_{i \in I} \omega_i = \omega_{\text{universe}}$. Market makers are assumed to be near their equilibrium balance sheets. Likewise, participants are near or at their equilibrium balance sheets. Nature sets a historical dividend policy for the firm.

In this game, market makers are monopolists, while option writers are serial monopolists. A serial monopolist, as used here, implies that only one potential writer appears at a time, but that many may appear across time.

### 2.3 Actions

Securities are in decision-theoretic terms, lotteries. Buying $n$ shares of an asset has a different probability distribution than buying $n$ call contracts when seen in reward space. As such, they are different lotteries with different density functions. Some combinations of lotteries map to the same probability distribution as another lottery. Buying an equity security is the same as buying a call option and selling a put contract in the sense that they have the same density function. The game is restricted to decisions to be executed at time $t$. Although actors have access to a wide range of risky assets, all assets except the one of interest are collectively held as an individual’s portfolio designated $\omega_i$. That portfolio contains all other risky wealth. This serves two functions.
First, instead of having the notation $p_t$ each firm would have to be identified, and so there would end up with some notation such as $p_{if}^t$ for each firm $1 \ldots f$ in $F$. That would add no information. Second, the purpose of this article is to formulate a model of option pricing based on a single security. While it is true that there may be demand for a set of options from many different firms, this paper does not discuss portfolio effects.

Actors have several possible actions, denoted $a$ or sometimes $a'$ and so forth, they can take on several possible lotteries. The actions are:

- Place market order to:
  - Buy
    * shares in the underlying security
    * contracts in a European style option on the underlying security
      (also called going long)
  - Sell
    * shares in the underlying security
    * contracts in a European style option on the underlying security
      (also called writing)

- Place limit order to:
  - Buy
    * shares in the underlying security
    * contracts in a European style option on the underlying security
  - Sell
    * shares in the underlying security
    * contracts in a European style option on the underlying security

- Place multiple orders in a convex combination of the above order types
- Do nothing

The set of all possible actions over all possible permitted combinations of securities is denoted $\mathcal{A}$.

Some actions are functionally excluded by dominance. For example, buying a call and selling a put while short selling the underlying security would be equivalent to “do nothing,” except that there would be transaction costs. Hence, do
nothing dominates doing something, where that something is more costly than a simpler solution. Although the move is not excluded in the game by rule, it is dominated and so including “Do Nothing” permits a pragmatic rationality-based bounding for the set $\mathcal{A}$.

For most practical purposes, limit orders are the same as “do nothing.” If a security is trading at $10 per share and a buy limit order for $15 is placed, this is no different from placing a market order and so will be treated as a market order. However, a security trading at $10 per share when a buy limit order of $5 is issued will result in no action. As only one static moment is under consideration here, though it is a generic moment, limit orders will either map to a market order or orders to do nothing.

There are five possible simple lotteries. They are:

- Equity security
- Call option
- Put option
- Deposit contract
- Loan contract

Compound lotteries are made up of several simple lotteries.

The outcome of the deposit and loan contracts are known with certainty and so result in the same outcome in all states of nature, except in the case of the bankruptcy of the obligor. By assumption, all obligations are paid by someone.

If a buy order would exhaust and exceed all possible liquid assets, then a loan would automatically be granted. Likewise, any net cash generated by transactions would automatically create a deposit contract to the extent the cash exceeded any outstanding debt.

2.4 Assumptions

There are six explicit prices in the game, four of which are exogenous. The exogenous prices are $p_t$ and $p_T$ which are the current and future price of some security; $i_D$ and $i_L$, which are the interest rates on deposits and loans respectively at time $t$. The endogenous prices are $\phi$ and $\psi$, which are the current prices for put and call options if liquidity costs are ignored. There is also an implicit price $\lambda$, which is used to price the cost of liquidity. The cost of liquidity is treated as exogenous.
Essentially this reflects that the price of liquidity is set in the broad market and is not firm-specific, though it may vary by asset subclass or on other variables.

For participants, all moves happen at time $t$. The consequences of these actions are paid out at time $T$. The interval $T - t = \Delta t$. For the market makers, it is possible for actions to be taken in continuous time over the interval $[t, T]$ and implicitly there will be other participants available as counter-parties over the interval.

In addition to the current exogenous market prices, there are three other variables which affect the price of option contracts. They are the strike price, denoted $k$, the number of contracts or shares purchased denoted $n$, and $\delta$, which is the future value of dividends over the interval $(t, T)$ marked up to future value at the deposit rate $i_D$. As is standard in economic notation, a variable noted with a $*$ is at the equilibrium value.

Although securities are priced at $p_t, p_T, \psi$, and $\phi$, they are marked up or down to the bid and ask price by requiring a liquidity premium. The liquidity premium, from Abbott[19], is being modeled as a function $z(n)$ such that:

$$z(n) = e^{n \lambda}$$

In addition to price variability, securities are subject to existential risks. In particular, the probability of bankruptcy or merger whose probability is denoted $B$ and $M$.

Both $\psi$ and $\phi$ are unknown functions of the above variables.

Interest rates and the future value of dividends are, of course, dependent on the interval of time over which they are to happen. Likewise, to simplify notation

$$i_D = i_D|\Delta t, t$$

and

$$i_L = i_L|\Delta t, t$$

and

$$\delta = \delta|[t, T]$$

Market makers and participants that write options are assumed to be profit maximizers. Participants that are option buyers are assumed to be either profit maximizers or utility maximizers. Utility maximizers are assumed to have strictly concave utility with heterogeneous preferences.

Participant actors have heterogeneous endowments. A consequence of this is that different actors face different interest rates. A result of this is that they have different reservation prices.
All participants are assumed to have access to sufficient credit facilities that they could engage in any transaction up to and past the point of profitability.

Participants in the game have perfect knowledge of all relevant historical data, and the data set is vast. Actors base their actions on the predictive distribution created from the data from the beginning of the data set to time $t - 1$. As the set is very large, it is assumed that the differences in parameter estimates are less than the number of significant digits and so in a discrete space are equal. Implicitly this presumes no actor holds a degenerate prior. The Bayesian predictive distribution for as yet unseen random variable $\tilde{x}$ is:

$$
\Pr(\tilde{x}|X) \equiv \int_{\theta \in \Theta} \Pr(\tilde{x}|\theta) \Pr(\theta|X) d\theta, \quad (13)
$$

where $X$ is the sample data and $\theta$ is the vector of parameters, which are in parameter set $\Theta$. Because of the integration, the predictive distribution does not depend upon $\theta$.

2.5 Nature of the Contracts

2.5.1 Underlying Equity Security

The firm issues a finite number of shares. At time $T$ there are three possible states of nature for the firm. It can have continued as a going concern, it could have been merged out of existence, or it could have become bankrupt. If it merged, then it could have been merged for cash or shares in the acquiring firm.

The firm has a board of directors. This board can change its dividend policy from its historical dividend policy. This is important because it impacts option price valuation.

Option prices must converge to a single point value. A range of potential option prices is not useful. For the short put, the solution is simple. The expectation is found, and this implies that the cost function involved is quadratic loss. To understand why, if the problem had been stated in the form of another loss function, such as the linear absolute loss function, then the solution would have been something other than the expectation. In the case of absolute linear loss, the solution would be the anticipated median. For some dividends, such as those for preferred stock, normal errors are a reasonable approximation for large samples, and the expected dividend then is also a reasonable point estimator. This is because the only decision for the Board of Directors to make is whether or not to pay the dividend.
Each dividend is a Bernoulli trial, though for a large set the normal approximation is adequate.

The same would be true for dividends of the form

\[ \delta_{\Delta t} = \beta \delta_0 \Delta t + \epsilon \]  

where this breaks down would be for dividends that grow exponentially.

When estimated by regression, the result should be a Cauchy distributed posterior once nuisance parameters have been marginalized out.[3] When projected into a predictive distribution, though, truncation moves the center of location from the median to the mode as the median is shifted by the absence of the possibility of negative dividends.

Construction of a point estimator now depends upon two possible cases. In the first case, the actual loss function for a particular actor is known then that is the loss function that must be minimized. In that case, the loss created by choosing an incorrect parameter estimate is explicitly known to the actor, and it is illogical to use any other. The second case is more interesting in that it may allow a generalization of stochastic calculus beyond what is currently in use.

The second case requires some description before assuming it into existence. Decision theory in all its forms presumes that there is a utility in finding the true value of a parameter. Indeed, this is a strange assumption unless population parameters serve as the maximization of a society-wide social utility function. In essence, parameters are revealed as preferred locations for prices to converge to. Since preferences are biological in origin, this implies a biological component to returns[20, 21].

Now it is necessary to consider the nature of the potential distributions involved. For the normal distribution, the center of location is simultaneously the mean, median and mode, yet when estimation is performed, it is rarely anything other than the sample mean, in the absence of prior knowledge. This is because the sample mean is the admissible estimator. Implicitly then, the social cost function for this problem is quadratic loss if the problem is solved in a market. Any other loss function would imply a different estimator and all other estimators carry a higher Bayes risk. If society is loss averse, or at least the marginal actor who sets prices is loss averse, then risk minimization implies society uses quadratic loss in this case.

Since dividends cannot be negative, a truncated Cauchy distribution for dividends implies the mode is the only potential estimator. Since the mode is the
solution to an all-or-nothing loss function, this implies the social utility is

$$L(\theta, \text{mode}(x)) = \begin{cases} 0 & : \theta \in \text{mode}(x) \pm \epsilon \\ c & : \theta \notin \text{mode}(x) \pm \epsilon, \end{cases} \quad (15)$$

where $\epsilon$ is an acceptable region of practical equivalence and $c$ is the cost of choosing the wrong estimate, and $\text{mode}(x)$ is the estimator of the mode.

### 2.5.2 Option Contracts

Participants can either write or buy any finite number of European style put or call options. A European style equity option is an option on an underlying security that can only be exercised at maturity. In this case, maturity is at time $T$. In addition to, or in lieu of a portfolio of option contracts, they could purchase or sell any finite quantity of the underlying security.

A put contract grants the buyer the right, but not the obligation to require the writer to buy a security at a predetermined price denoted $k$ and called the *strike price*. A call contract grants the buyer the right, but not the obligation to require the writer to sell a security at a previously chosen strike price, denoted $k$.

Unlike equity securities or bonds, which are fixed in quantity over the short run, option contracts can exist without limit. Financial intermediaries create financial contracts, and these contracts are flexible in quantity. This mechanism is similar to the manner in which banks create money.

When a new bank forms, the equity is loaned out. The proceeds of these loans result in deposits to the banks. This money is then loaned out again, creating new deposits in the process, until some contractual or regulatory limit is reached.

Similarly, market makers in this game insure the market against adverse movements by writing options and by making a market for those parties that wish to absorb those risks. This assures the market that participants can become underwriters of those risks. In a sense, the market maker acts as a Lloyd’s association does in insurance when combined with a reinsurer to cover the risk of failure by the contract writers. For purposes of the game, only market makers can create derivative securities.

Like a bank certificate of deposit, no option obligation exists until a market maker agrees to open an account and create one. Likewise, a risky option position cannot be closed early without a counter-party willing to absorb the risk. By insuring the primary markets against certain types of risks, it makes it possible to increase the size of the primary markets by permitting the sale of risks considered unacceptable to one party to be sold to another party.
3 The Profit Function for Short Puts

3.1 The Critical Importance of the Short Put

The capacity to solve any element of the system revolves around the short put. It is the only portion of the system where an expected profit exists. As such, puts are necessary for financial stability. Although the European style short put seems like an esoteric concept, it is, in fact, a simpler mathematical construction than a bank deposit. It is entirely possible bank deposits exist in this sea of instability because of the nature of the put contract.

To understand the relative simplicity of an equity put contract when compared to a bank deposit or a bank loan, it is important to think about what a bank deposit grants the actor. Depositors receive a debt obligation and a long put that they did not pay a premium for. If interest rates increase enough, then a depositor will remove their deposit and redeposit the money at the higher rate. If interest rates fall, then the depositor can continue to receive the higher rate.

Banks create a bond, often at a fixed rate of interest, and it includes a long American style put. Implicitly the bank would loan the customer the money for the premium through a discounted deposit rate and possibly a penalty for early withdrawal. For a time deposit, the depositor receives a rate $i_D$ from time $t$ to time $T$ unless a higher rate appears. Then the depositor is free to choose the higher of the two rates. In the absence of a penalty for early withdrawal, the depositor is guaranteed the supremum of the available rates over some period of time.

Deposits are far more complex, mathematically, than a European style put on an equity security. The obligation is only at the end of the period with a defined premium to open the contract. The writer has no automatic future obligation to the buyer, such as the obligation to write another put. A bank, on the other hand, as a common carrier, must agree to accept a rate marked up deposit from the same customer who canceled the prior agreement.

This relatively simple contract is the building block of all American style options, European style call options and standard banking products.

3.2 The Profit Function

The profit functions\(^1\) for participants are such that revenues are marked down by an exponentially growing cost of liquidity in $n$, and costs are marked up by an

\(^1\)Economists usually define the profit function in terms of a maximization. That assumption is relaxed temporarily.
exponentially growing cost of liquidity in \( n \). The price of a long options position is the anticipated cost plus profit. Positions are entered into to maximize utility by insuring against risk rather than generating a profit for the buyer. By assumption, at the margin, actors are risk averse with an increasing utility of wealth, and so for purposes of this game, there are no speculators. The presence of speculators in the market can have an impact to be discussed later in section 7.

As market makers could be the permanent holders of all short positions, it is the market maker’s self-interest to assure not only that sufficient premiums are being collected, but also that the option writers are profitable in the long run.

Two of the three possible single contract short positions can have no expected profitability as the expectation diverges and therefore does not exist. Expected gains and losses for short selling a stock or call option cannot be defined. As it happens, this is not a difficulty in equilibrium.

As will be shown later, the call price, given a large cash endowment, in equilibrium is
\[
\psi = \phi + pt \left( \frac{1}{1+iP} - \frac{k+\delta}{1+iP} z(n) \right).
\]
Every variable on the right-hand side is well defined except \( \phi \) and \( \delta \). Setting aside considerations of \( \delta \) for a moment, the question becomes “is there an optimal put premium given an obligation by the market maker to write \( n \) put contracts without a volume limitation?”

The goal is to set a price such that the volume sold produces maximal profitability. The profit function for \( n \) short put contracts is:
\[
\Delta \Pi_P^c(n) = n \left( \frac{\max \left( 0, k - p_T \right)}{z(n)} \right)
\]
In this equation \( x \in \{D, L\} \). As the goal is to get participants to voluntarily choose optimal volume, it must first be determined what is an optimal volume.

At contract termination, there are three possible existential states that the function \( \max \left( 0, k - \frac{p_T}{z(n)} \right) \) could finish in. They are

1. The firm is bankrupt
2. The firm has been merged out of existence
3. The firm is a going concern

In the bankrupt state, the writer pays \( k \) since \( p_T = 0 \). This happens with probability \( B \). \( B \) is understood in a Bayesian sense as \( B|t \), where \( t \) is information. This is also true for the other existential states.
The expected profit, given bankruptcy, is:

\[ E(\Delta \Pi_S^P(n)|Bankruptcy) = \left( \frac{n}{z(n)} \phi (1 + i_X) - nk \right) E(B) \]  

(17)

Without bankruptcy, it is not possible to calculate option premiums as the integrals would hopelessly diverge in all existential states.

Mergers are often a bankruptcy substitute, as such, it is reasonable to believe that post-merger returns are drawn from a different distribution than for firms which are a going concern. In the Going Concern state of nature, returns are approximated by the Cauchy distribution.

Real world data is skewed. It is reasonable to believe this is due to the intertemporal budget constraint. The denominator of a return, if short-selling is ignored, is determined by the purchasing actor’s budget constraint. Because observed transactions must have happened, purchases do not impact skew. On the other hand, for a return to exist then a sale must have happened. However, some actors attempting to sell assets find that the market fails and the trade does not close.

This skew is created by an owner placing a security in the market for sale with one of two possible results. Either a trade happens, or it does not. The probability of a trade happening if the closing price is zero is one hundred percent. The probability of a trade occurring if the closing price approaches infinity is zero percent. Holding the purchase price constant implies a declining probability of observing the numerator at any price to the right of any other price. This implies that an observed return is a return, given a transaction happens, multiplied by the probability of a transaction happening.

The intertemporal budget constraint is ignored here because liquidity is explicitly modeled. If it were not, then it is advised one adopt the model by Abbott or a similar model such as appears in [19]. As a result, the data can be treated as truncated, but symmetric since the budget constraint is subsumed by the cost of liquidity and the willingness of the market maker to engage in all finite transactions for a sufficient price.

If \( G \) is thought of as the probability of a firm continuing as a going concern and \( M \) the probability of merger, then there are two ways to handle the relationship between \( G, M, \) and \( B \). They are:

1. \( G + M + B = 1 \)

2. Or by having both of the following conditions be true:
(a) \( G|\text{not bankrupt} + M|\text{not bankrupt} = 1 \)

(b) \( (G|\text{not bankrupt} + M|\text{not bankrupt})(1 - B) + B = 1 \)

The method used is the multinomial choice of \( G + M + B = 1 \).

The profit function in the merged state is a compound distribution. It includes mergers for cash and mergers for stock. As it is not partitioned into multiple distributions, a non-analytic solution is being offered here. The method of histograms permits an approximate solution for this problem [22]. Fundamentally, predicted values are based on the probability that the final price, \( p_T \) will be inside a particular partition.

Noting that in equation 16 that a loss would happen anywhere \( p_T < kz(n) \), implies that out of the money options, that is those greater than the strike price should be exercised due to liquidity costs. One partition, should then cover the no loss region of \( p_T \geq kz(n) \). The number of partitions should be optimally chosen to minimize information loss. The optimal number of partitions is assumed to be \( S + 1 \), where \( S \in \mathbb{Z}^+ \).

The probability of being in a given slice, given a merger will happen in the contract period, is unknown, but is estimated here using the multivariate normal of dimension \( S + 1 \). Each partition is mutually exclusive and does not covary with others. The beliefs regarding the probability of being in a particular in-the-money slice, \( s \in \{1 \ldots S\} \), is:

\[
\Pr \left( \frac{s-1}{S} kz(n) \leq p_t < \frac{s}{S} kz(n) | \text{Merged} \right) \sim \mathcal{N}(\mu_s, \sigma_s^2)M
\]

and for the out of the money slice:

\[
\Pr (z(n)p_t \geq k | \text{Merged}) \sim \mathcal{N}(\mu_{S+1}, \sigma_{S+1}^2)M
\]

A multinomial distribution could have been used as well.

As partitions are, by this construction, of equal width, this creates an expected profit function of:

\[
E \left( \Delta \Pi_S^P(n) | \text{Merged} \right) = E(M) \frac{n}{z(n)} \phi(1 + iX)
- nkE(M) \left\{ (1 - \mu_{S+1}) - \sum_{s=1}^{S} \mu_s \left( \frac{s - 1}{S} \right) \right\}
\]

\[
(20)
\]
The second \( z(n) \) from equation 16 vanishes because \( z(n) \) scales the partitions so that each partition is \( \frac{kz(n)}{S} \) wide, but the midpoint is discounted by \( \frac{1}{z(n)} \). Likewise, as \( k \) scales the width of the histogram and appears as the paid out strike price, \( k \) gets pulled out to the side.

In the going concern state of nature, the expected profit function is:

\[
E(\Delta \Pi_S^n)_{\text{Going Concern}} = E(G) \left[ \frac{n}{z(n)} \phi(1+i_x) - 0 - \frac{2n}{\pi + 2 \tan^{-1}\left(\frac{\mu_G}{\sigma_G}\right)} \int_0^{kz(n)} \left( k - \frac{p_T}{z(n)} \right) \frac{\sigma_G}{\sigma_G^2 + (p_T - \mu_G)^2} dp_T \right] \tag{21}
\]

For simplicity, it was assumed the contract was an all or nothing contract. It will marginally overstate costs if this is not true for small values of \( n \). The center of location is \( \mu_G \) and the parameter of spread is \( \sigma_G \). Note however that the current spot price is information and that \( \mu_G \) and \( \sigma_G \) are notationally shortened from \( \mu_G|_{p_t} \) and \( \mu_G|_{p_t} \). It should also be noted that additional information beyond the strike price could be included such as accountancy data or dividend payments.

Because the distribution is truncated at zero due to the existence of bankruptcy, the coefficient of integration is \( \frac{2}{\pi + 2 \tan^{-1}\left(\frac{\mu_G}{\sigma_G}\right)} \) instead of \( \pi^{-1} \).

Prior to evaluating the parametric form for the going concern, it may be valuable to consider using the distribution free form, as in equation 20, for the going concern. Then the joint profit function for all three states of nature, in approximation, becomes:

\[
E(\Delta \Pi_S^P(n)) = \frac{n}{z(n)} \phi(1+i_x) - nk \left[ E(B) + E(M) \left\{ (1 - \mu_{S+1}^M) - \sum_{s=1}^{S} \mu_s^M \frac{(s - \frac{1}{2})}{S} \right\} + E(G) \left\{ (1 - \mu_{S+1}^G) - \sum_{s=1}^{S} \mu_s^G \frac{(s - \frac{1}{2})}{S} \right\} \right] \tag{22}
\]
Setting a function $\Lambda(k)$ as the net unrecoverable loss function, as:

$$\Lambda(k) = k \left[ E(B) + E(M) \left\{ (1 - \mu^M_{s+1}) - \sum_{s=1}^{S} \mu^M_s \left( \frac{s - \frac{1}{2}}{S} \right) \right\} + E(G) \left\{ (1 - \mu^G_{s+1}) - \sum_{s=1}^{S} \mu^G_s \left( \frac{s - \frac{1}{2}}{S} \right) \right\} \right],$$

(23)

the problem simplifies to the more visually tractable:

$$E(\Delta \Pi^P_s(n)|n; p_t; k) = \frac{n}{z(n)} \phi(1 + iX) - n\Lambda(k)$$

(24)

Since $p_t$ and $k$ are exogenous, this permits a solution for $\phi$ given an optimal value for $n$.

First order conditions for this form are:

$$\frac{dE(\Delta \Pi^P_s(n))}{dn} = \frac{\phi(1 + iX)}{z(n)} - \frac{\phi(1 + iX)\lambda n}{z(n)} - \Lambda(k) \equiv 0$$

(25)

With a little manipulation, the equation can be brought into product-log form and $n^*$ can be arrived at thus:

$$n^* = \frac{\left( 1 - W \left( \frac{e^{\Lambda(k)} \phi(1+iX)}{\phi(1+iX)} \right) \right)}{\lambda},$$

(26)

where $W(y)$ solves:

$$y = W(y) \exp(W(y))$$

(27)

Unfortunately, $\phi$ has yet to be solved for. Still, it illustrates the important inverse relationship between optimal volume and $\lambda$. Since volume is observed in the market while $\phi$ can only be infered, $\phi$ is solved for as:

$$\phi = \frac{\Lambda(k)z(n^*)}{(1+iX)(1-n^*\lambda)}$$

(28)

For completeness, second order conditions support a maximum when:

$$n\lambda < 2$$

(29)
The profit function to write a short put, by substitution, is:

\[ \sup \Delta \Pi_P^S(n) = \frac{n^* \Lambda(k)}{1 - \lambda n^*} - n^* \max \left(0, k - \frac{p_T}{z(n^*)} \right), \]  

(30)

the expectation for which is:

\[ E(\sup \Delta \Pi_P^S(n)) = \frac{n^* \Lambda(k)}{1 - \lambda n^*} - n^* \Lambda(k) \]  

(31)

Although this is an approximation, it has a nice form. Writers receive a percentage markup over costs. In this model, \( \phi(n) \), is a function of \( n^* \) and not \( n \). The alternative would be for \( \phi(n) \) to vary directly with \( n \). There is an important conceptual difference that goes to the core of banking. If \( \phi(n) \) is a constant, then all mark-ups and mark-downs are taken by the market maker. The market maker’s role is to absorb the volume. It is a liquidity cost and not a size effect. It represents timing and the ability to maintain stable supply and demand curves. Size effects are absorbed by the liquidity function.

On the other hand, \( \phi(n) \) represents a rotation of the short run supply curve to an upward sloping short-run curve. This is different from a simple repricing of the underlying security, from \( p_t \) to \( p_t' \). It would imply the existence of a market participant with pricing power. In equilibrium, this should not occur but could be imagined in a world where a principal market maker failed, and another actor was willing to underwrite the contracts. It could also be the case where a market maker wanted to exit a line of business and sell its book of business to another actor.

For completeness, this state is provided. Equation 16 becomes:

\[ \Delta \Pi_P^S(n) = \frac{n}{z(n)} \phi(n)(1 + iX) - n \max \left(0, k - \frac{p_T}{z(n)} \right) \]  

(32)

Taking expectations and setting the first derivative to zero generates a differential equation for \( \phi(n) \) whose solution is:

\[ \phi(n) = \frac{\Lambda(k)z(n)}{1 + iX} + \frac{cz(n)}{n}, \]  

(33)

where \( c \in \mathbb{R}^{++} \).

The expected profit function is:

\[ E(\sup \Delta \Pi_P^S(n)) = c(1 + iX) \]  

(34)

The writer is recapturing all costs and collecting a flat fee, independent of \( n, \lambda, \) and \( k \). For that to be the case, the market maker has to transfer capital and liquidity profits to remove the risk from its books.
3.2.1 Parametric Form for Going Concern

The parametric form of the going concern profit function shown in equation 35 is far less tame. Indeed, a simple visual inspection would cause anyone to doubt that the first and second derivative would go anywhere simple or useful. Far more important, the pattern of \( \frac{n}{z(n)}a - nb \), is hopelessly broken here. Although it is the parametric form of the close, distribution-free approximation and at its core, the same pattern must hold, it is not obvious how that would be arrived at.

The simple pattern where \( z(n) \) and \( k \) scale the cost function is obscured by the nature of the integration. The fundamental lessons are the same, except that there isn’t an observed analytic solution for the second derivative test. However, since the distinction between the non-parametric form and the parametric form is the same as increasing the number of partitions to infinity at the limit, taking the problem from an approximation to an exact form, the presence of a maximum will not be impacted by the number of partitions as the number doesn’t impact the results.

\[
E(\Delta \Pi^G_S(n) | \text{Going Concern}) = E(G) \left[ \frac{n}{z(n)} \phi(1 + i_x) - 0 ight. \\
- \frac{2n(k - z(n)\mu)\tan^{-1}\left(\frac{k - z(n)\mu}{\sigma z(n)}\right)}{\pi + 2\tan^{-1}\left(\frac{\mu}{\sigma}\right)} \\
- \frac{2n(k - z(n)\mu)\tan^{-1}\left(\frac{\mu}{\sigma}\right)}{\pi + 2\tan^{-1}\left(\frac{\mu}{\sigma}\right)} \\
+ n z(n) \sigma \left\{ \log(\mu^2 + \sigma^2) \\
- \log \left( \frac{k^2}{z(2n)} - \frac{2k\mu}{z(n)} + \mu^2 + \sigma^2 \right) \right\} \right]
\]  

(35)

The primary issue that makes the problem difficult is that it is rational to exercise out-of-the-money option contracts when adjusted for the cost of liquidity, they are pragmatically in the money. A person holding an option for 10,000 shares of ABCorp with a strike at 50 when the current price is 49.75 would not exercise under the Black-Scholes model but must exercise here if the shares are still desired if the market order to buy the shares on the open market would drive it over $50.
3.3  Parametric Form for Cash-for-Stock Merger

Per Harris in [3], the parametric form in the case of a cash for stock merger is

$$E(\Delta \Pi_S^P(n) | \text{Cash for Stock Merger}) = E(M_C) n \left[ \frac{\phi(1+i_x)}{z(n)} - \int_0^k \left( k - \frac{p_T}{z(n)} \right) \frac{1}{p_T \sqrt{2\pi \sigma_{M_C}^2}} \exp \left( - \frac{(\log(p_T) - \mu_{M_C})^2}{2\sigma_{M_C}^2} \right) dp_T \right].$$

(36)

It varies from the going concern and cash-for-stock merger state in that there is no asset to dispose of. As a result, the upper limit of integration is $k$ instead of $kz(n)$.

This reduces to

$$E(\Delta \Pi_S^P(n)) = E(M_C) n \left[ \frac{\phi(1+i_x)}{z(n)} - \frac{kz(n)(\text{erf}(k\log - \mu_{M_C}))}{\sqrt{2\sigma_{M_C}}} - \frac{\mu_{M_C} + \frac{\sigma_{M_C}^2}{2} \left( 1 - \text{erf}(\mu_{M_C} + \sigma_{M_C} - k\log) \right)}{\sqrt{2\sigma_{M_C}}} + kz(n) \right],$$

(37)

where erf is the error function, the cumulative density of the normal distribution.

3.4  Parametric Form for Stock-for-Stock Mergers

The assumption here is that acquiring firms attempt to purchase target firms that are trading below fair market value. Further, the assumption here is that it is systematically trading below fair market value by an amount $\alpha_t$. The assumption is that the shares of the acquiring firm will trade at or around its equilibrium value at the closing date of the contract.

This, combined with the assumption of many potential buyers or sellers and trades happening in a double auction, should result in $(p_t, p_T)$ to be normally distributed around $(p_t^* + \alpha_t, p_T^*)$, $\alpha_t < 0$. Using Curtiss’ method in [23] to derive the distribution of returns and future prices relative to purchase price, the distribution
of \( p_T \) is
\[
g(p_T) = \sigma_T^2 e^{-\frac{\alpha^2}{2\sigma_t^2}} \left( \sqrt{\pi} \alpha \left( \text{erf} \left( \frac{\alpha}{\sqrt{2}\sigma_t \sqrt{\frac{(p_T - \mu_{MS})^2}{\sigma_{T+1}^2}} + \frac{1}{\sigma_t}} \right) + 1 \right) \right) \exp \left( \frac{\alpha^2 \sigma_{T+1}^2}{2\sigma_t^4 (p_T - \mu_{MS})^2 + 2\sigma_{T+1}^2 \sigma_t^2} \right) + \sqrt{2} \sigma_t^2 \sqrt{\frac{(p_T - \mu_{MS})^2}{\sigma_{T+1}^2} + \frac{1}{\sigma_t^2}} \right)
\]

\[2\sqrt{2} \pi \sqrt{\sigma_t^2 \sigma_{T+1}^2} \sqrt{\frac{(p_T - \mu_{MS})^2}{\sigma_{T+1}^2} + \frac{1}{\sigma_t^2}} (\sigma_t^2 (p_T - \mu_{MS})^2 + \sigma_{T+1}^2)\]

(38)

A slight deviation from Curtiss was made because the use of Curtiss as written would permit infinitely negative prices. The integral was constrained to be greater than or equal to zero. The author has been unable to construct an expectation or an expected loss over the range bounded by zero and the strike price marked up for liquidity costs. As a result, numerical methods will have to be used until a future author solves for the expectation. Consequently, the formula for expected profit given a stock-for-stock merger is

\[
E(\Delta\Pi^P_S(n)|\text{Stock-for-Stock Merger}) = E(M_S) \left[ \frac{\phi(1 + i_x)}{z(n)} - \int_0^{kz(n)} \left( k - \frac{p_T}{z(n)} \right) g(p_T) dp_T \right].
\]

(39)

3.5 Brief Discussion

A brief discussion is in order as this mechanism for pricing puts is slightly different from what would be expected in a mean-variance framework. Note, for example, that there are no dividends in the formula for a put contract. This does not mean dividend payments do not impact the price of puts.

A simple example would be a firm paying liquidating dividends over a period of years. An option on the current price would almost certainly be guaranteed to be \textit{in the money} far enough into the future. That would differ from a firm paying dividends from profits or no dividend at all.

How should one incorporate such a dividend? It should be in the likelihood function, subject to any prior information about dividend payments on prices. It should appear in \( \text{Pr}(p_T|\delta) \) and not as a correcting factor outside the expected cost function. It is inherently true that dividends are uncertain. A board of directors, as in any legislature, is subject to time inconsistency. That is, the games are subgame imperfect. This forces inductive reasoning to properly estimate the role of either announced or historical dividends on future prices.
If prices were not conditioned on information that included dividends, then the effect of dividends is disbursed into the general uncertainty of future prices. In essence, ignoring dividends increases uncertainty, but that does not inherently mean the gain in information is worth the computational costs. It merely means the effects of dividends become hidden in the uncertainty about price changes.

Another missing element is the relation between the current spot price and the strike price. This missing information is captured in $\mu_G, \mu_M, \sigma_G, \sigma_M$ as they are really $\mu_X|p_t$ and $\sigma_X|p_t$. So the strike price vanishes into the posterior via the likelihood function as well.

It is reasonably certain, but not perfectly certain, that a stock currently priced at $50 per share with a put option with a strike price at $100 per share with one year to run in the contract will expire in the money in the absence of information that would cause one to believe that the current price is far from the equilibrium price.

One other slight difference from mean-variance finance is that calls are priced actuarially in mean-variance finance while puts acquire their value through put-call parity. This is reversed as calls have no expected value.

4 No Arbitrage Equilibrium

Most models impose an absence of opportunity to enter into an arbitrage position. It can be justified under a number of possible assumptions or as a consequence of rationality concepts. However, one of the simplest is de Finetti’s coherence principle [5]. de Finetti set about an axiomatization of probability theory in 1937 built around the concept of gambling. Shortcomings in this approach are noted in Shimony, Janes, and Nau in [7, 24, 25]. de Finetti’s Coherence Principle can be stated as:

**Assumption 1** A bookmaker’s betting odds are coherent if a client cannot place a bet or a combination of bets such that no matter what outcome occurs, the bookmaker will lose money.

An open question to this assumption, of course, is “do rational actors have to use coherent probabilities?” Although this issue is covered by Ramsey and Savage, it can simply be excluded here by the assumption that the market maker is a profit maximizer and could just choose not to engage in transactions that result in a sure loss, when the alternative was a zero change in profits [5, 26]. The practical implication is that there is a well-bounded set of possible prices.
The binding rule for the bookmaker is that the bookmaker will accept any and all gambles as long as they are finite in number at the posted price.

The rule when combined with the Coherence Principle create a set of binding conditions on the bookmaker that assure an equilibrium price. It should be noted that the absence of arbitrage opportunities is a stronger requirement than the Coherence Principle. The Coherence Principle only prohibits a bookie or market maker from being gamed to the point of a sure loss; it does not prohibit explicitly the market maker or bookie from behaving as a con man and setting up a sure loss for a naïve participant.

5 Call Options Under Various Initial Endowments

As different possible actors could approach the market maker with different reservation prices, different possible no arbitrage equilibrium prices exist. It is important not to read an equilibrium price as the equilibrium price but rather as the equilibrium price conditional upon a state of nature. Each possible state of nature is dependent upon the subjective conditions of the actors approaching the market maker to enter into positions.

5.1 Large Cash Endowment

In this state of nature, it is assumed that $\bar{m} \gg 0$ and that after the position is entered into that sufficient cash exists to maintain the position without borrowing. The profit function, should no action be taken to enter into a position, is:

$$\Pi_{\bar{m} \gg 0} = i_D \bar{m}$$ (40)

The profit functions in table 1 represent a change in profit from the do nothing choice. As such, they are prefixed with a $\Delta$ to make that clear. The profit functions of this state of nature are in table 1. Some elements of these profit functions should be made explicit.

The profit is the net profit at the terminal date of the contract. These are not present values, but nominal future profits. For the long position, although no cost earns interest, the cash used would otherwise have been on deposit and so the change in profit includes the lost interest. The profit function for the position itself is:

$$\Pi_C^L(n) = n \max \left( 0, \frac{p_T}{z(n)} - k \right) - nz(n) \psi$$ (41)
<table>
<thead>
<tr>
<th>Type of Position</th>
<th>Formula for Profit Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long Call</td>
<td>( \Delta \Pi_L^C(n) = n \max(0, \frac{p_T}{z(n)} - k) - nz(n)\psi(1 + i_D) )</td>
</tr>
<tr>
<td>Long Put</td>
<td>( \Delta \Pi_L^P(n) = n \max(0, k - z(n)p_T) - nz(n)\phi(1 + i_D) )</td>
</tr>
<tr>
<td>Long Equity Position</td>
<td>( \Delta \Pi_L^S(n) = \frac{n}{z(n)}p_T - nz(n)p_t(1 + i_D) + n\delta )</td>
</tr>
<tr>
<td>Short Call</td>
<td>( \Delta \Pi_S^C(n) = \frac{n}{z(n)}\psi(1 + i_D) - n \max(0, z(n)p_T - k) )</td>
</tr>
<tr>
<td>Short Put</td>
<td>( \Delta \Pi_S^P(n) = \frac{n}{z(n)}\phi(1 + i_D) - n \max(0, k - \frac{p_T}{z(n)}) )</td>
</tr>
<tr>
<td>Short Equity Position</td>
<td>( \Delta \Pi_S^S(n) = \frac{n}{z(n)}p_t(1 - i_L) - nz(n)p_T - n\delta )</td>
</tr>
</tbody>
</table>

Table 1: Profit Functions Given Sufficiently Large Cash Endowments

Also of note is the interest adjustment to the short equity position, which is \( 1 - i_L \).

There are multiple ways in which a broker-dealer can manage both option and short equity positions. For purposes of this game, interest is paid on the initial balance. In the United States, interest is not paid on the cash received from the short sale. It is held as collateral and used by the broker-dealer until repaid, giving the dealer an additional reward in the form of an interest-free loan.

5.1.1 Derivatives of Potential Arbitrage Positions

The derivatives for the arbitrage position call = put + equity are shown below. In the special case where the final price, \( p_T \), is less than the strike price, \( k \), it is not necessary to separately close the long position while simultaneously purchasing a short position as implied by the separate equations. The joint position becomes:

\[
\Delta \Pi_L^{P+S}(n) = nk - nz(n)\phi(1 + i_D) - nz(n)p_t(1 + i_D) + n\delta
\]  

(42)

The put owner delivers the shares, already purchased at time zero, and delivers them for \( k \) as per the contract.

The table of derivatives for the change in profit functions of the two positions with respect to \( p_T \) is as follows:

Relatively simple math will show that as the only differences between the cases are the interest rate, the derivatives will be the same for all possible outcomes and all states of nature, given either a long or short position. As such, for brevity, the derivatives with respect to \( p_T \) are not shown for the other endowed states.

It follows that if the change in profitability of calls equals the change in profitability of puts and equity positions, then the no arbitrage requirement is met.
<table>
<thead>
<tr>
<th>Type</th>
<th>Price</th>
<th>Derivative of Call</th>
<th>Derivative of Put Plus Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long</td>
<td>$p_T &lt; k$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$p_T = k$</td>
<td>undefined</td>
<td>undefined</td>
</tr>
<tr>
<td></td>
<td>$p_T &gt; k$</td>
<td>$\frac{n}{z(n)}$</td>
<td>$\frac{n}{z(n)}$</td>
</tr>
<tr>
<td>Short</td>
<td>$p_T &lt; k$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$p_T = k$</td>
<td>undefined</td>
<td>undefined</td>
</tr>
<tr>
<td></td>
<td>$p_T &gt; k$</td>
<td>$-nz(n)$</td>
<td>$-nz(n)$</td>
</tr>
</tbody>
</table>

Table 2: Derivatives

5.1.2 Equilibrium Pricing In Long Positions

For each position, one should note that there are three cases. As liquidity costs exist, it could matter if the interval of the bid and the ask prices cover the strike price. As mentioned earlier, consider an option to buy 10,000 shares of stock at $10 per share. Assume the current prices is $10.50 per share to sell 100 shares. Exercising enough contracts to close out 100 shares would cost $1,000 and generate revenue of $1050.00. Assume that selling 10,000 shares would result in a net price of $9.50 per share. The exercising party would pay $100,000 but only receive $95,000 in revenue. It would be unwise to exercise very many of those contracts.

**CASE I:** $\frac{p_T}{z(n)} \geq k$

The change in the profit functions of the call versus put plus equity position must equal, as such:

$$\frac{p_T}{z(n)} - k - z(n)\psi(1 + i_D) = -z(n)\phi(1 + i_D) + \frac{p_T}{z(n)}$$

$$-z(n)p_t(1 + i_D) + \delta$$

(43)

This reduces down to:

$$\psi = \phi + p_t - \frac{k + \delta}{z(n)(1 + i_D)}$$

(44)

To extract a little more intuition out of the equilibrium, consider the case of the *at the money* call price. In that circumstance the strike price is also the same as the stock price so $k = p_t$. In that case the formula becomes:

$$\psi = \phi + p_t \frac{z(n)(1 + i_D) - 1}{z(n)(1 + i_D)} - \frac{\delta}{z(n)(1 + i_D)}$$

(45)
For a long position the interpretation is that the price of a call option is equal to the price of a put option plus the carrying cost of buying the initial shares, marked down to present value and adjusted for liquidity costs minus the present value of dividends missed by holding the call position, again adjusted for liquidity costs.

Two other features are important here. First, the formula is independent of any value of $p_T$, and so no uncertainty is present. Second, although $\delta$ does not bear the usual notation for an estimator as would be the case if it were represented as $\hat{\delta}$; this is done for convenience as dividends, like liquidity costs, appear everywhere.

The proper estimator of dividends appears to be the mode of the predictive distribution. That is due to the truncation of the distributions due to the limitation of liability.

**CASE II:** $k \geq p_T$

The equation for this case is:

$$-z(n)\psi(1 + i_D) = k - z(n)\phi(1 + i_D) - z(n)p_t(1 + i_D) + \delta$$  \hspace{1cm} (46)

This reduces down to:

$$\psi = \phi + p_t - \frac{k + \delta}{z(n)(1 + i_D)}$$  \hspace{1cm} (47)

**CASE III:** $kz(n) > p_T > k$

This case has two subcases, one in which the contracts are exercised as an *all or nothing* execution and those that permit *partial execution* of the total position.

**CASE IIIa: All or Nothing Execution**

If the call option required all-or-nothing execution, then the contract would not be executed. This is the above example of having a nominally quoted price greater than $k$, but when marked down for volume results in a net price below the strike. For a simple formal proof, let $p_T = kz(n) - \xi z(n), \xi > 0$, then it follows that the profit function for a call option is $-z(n)\psi(1 + i_D)$ due to the fact that the value of the contract would be max($0, -\xi) = 0$.

This leads to a somewhat surprising result in academic models that ignore liquidity, that is that the put option should be exercised even though the contract is out of the money.

Again, subject to the overall restriction, let $p_T = kz(n) - \xi z(n), \xi > 0$. It follows that the value of the equivalent position is:

$$k - \frac{kz(n) - \xi z(n)}{z(n)} - z(n)p_t(1 + i_D) + \delta,$$  \hspace{1cm} (48)
as this is greater than the non-exercised profit by an amount $\xi$, the option must be exercised for maximal profitability.

The equilibrium is the same as for the low price equilibrium, which is:

$$\psi = \phi + p_t - \frac{k + \delta}{z(n)(1 + i_D)} \quad (49)$$

**CASE IIIB: Partial Execution**

In the case where some, but not all contracts could be executed profitably, it is assumed there exists a quantity $n'$ such that $0 < n' < n$ and that the execution of $n'$ contracts is profit maximizing.

The terminal profit function, $\Delta \Pi_L^C$, becomes:

$$\Delta \Pi_L^C = n' \left[ \left( \frac{p_T}{z(n')} - k \right) - z(n') \psi(1 + i_D) \right] - (n - n')z(n)\psi(1 + i_D) \quad (50)$$

This reduces to:

$$\Delta \Pi_L^C = n' \left( \frac{p_T}{z(n')} - k \right) - nz(n)\psi(1 + i_D) \quad (51)$$

The terminal profit function, $\Delta \Pi_L^{P+S}(n)$ becomes:

$$\Delta \Pi_L^{P+S}(n) = n' \left[ \left( \frac{p_T}{z(n')} - z(n) p_t (1 + i_D) + \delta \right) - z(n) \phi(1 + i_D) \right]$$

$$+ (n - n') [k - z(n) \phi(1 + i_D) - z(n) p_t (1 + i_D) + \delta] \quad (52)$$

In equilibrium, this reduces to:

$$-nz(n)\psi(1 + i_D) = nk - nz(n) p_t (1 + i_D) + n\delta - nz(n) \phi(1 + i_D) \quad (53)$$

Which is:

$$\psi = \phi + p_t - \frac{k + \delta}{z(n)(1 + i_D)} \quad (54)$$

For all long positions, where the endowment of cash is positive and sufficient to cover the cost of the positions, the equilibrium condition is:

$$\psi = \phi + p_t - \frac{k + \delta}{z(n)(1 + i_D)} \quad (55)$$

For subsequent cases, calculations of Case III are omitted as it is simply a variation of coefficients from the above case due to different interest rates.
5.1.3 Equilibrium In Short Positions

Although the profit equations for long and short option positions are the additive inverse of each other, this is not true for the long and short equity position. This difference results in a bid-ask spread even without liquidity costs, such as where $\lambda = 0$. The difference between the gross amount of a call price the market would be willing to pay, $\psi_L$, is greater than the insurer requires, $\psi_S$. This violates the law of one price but assures coherence.

A proper understanding of this difference is that if this condition is the equilibrium condition, then neither long nor short participants can form an arbitrage position against the market maker if the market maker keeps the spread between the prices.

CASE I: $z(n)p_T \geq k$

There are slight mathematical differences created by going from long to short, but it is the short case which is critical from a policy-making perspective. It is the option writer that needs sufficient reserves to support the system. As the buyer has no method to inspect the writers, indeed, in the American over-the-counter market exercise by long holders is exercised by random assignment, it is dependent upon the market makers to set an adequate reserve and collateral requirements.

The equilibrium condition is:

$$\psi \frac{1 + i_D}{z(n)} + k - z(n)p_T = \phi \frac{1 + i_D}{z(n)} + p_t \frac{1 - i_L}{z(n)} - z(n)p_T - \delta$$  \hspace{1cm} (56)

This resolves to:

$$\psi = \phi + p_t \frac{1 - i_L}{1 + i_D} - \frac{k + \delta}{1 + i_D}z(n)$$  \hspace{1cm} (57)

CASE II&III For economy of space, the calculations are omitted for the other cases as they also resolve to equation 57.

5.2 No Endowment

In this state of nature, it is assumed that $\bar{m} = 0$. Further it is assumed that the participants have sufficient access to credit as to be able to make purchases at an interest cost of $i_L$. The profit function, should no action be taken to enter into a position, is:

$$\Pi_{\bar{m}=0} = 0$$  \hspace{1cm} (58)

The profit functions of this state of nature are in table 3.
<table>
<thead>
<tr>
<th>Type of Position</th>
<th>Formula for Profit Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long Call</td>
<td>( \Delta \Pi_L^C(n) = n \max(0, \frac{p_T}{z(n)} - k) - nz(n)\psi(1 + i_L) )</td>
</tr>
<tr>
<td>Long Put</td>
<td>( \Delta \Pi_L^P(n) = n \max(0, k - z(n)p_T) - nz(n)\phi(1 + i_L) )</td>
</tr>
<tr>
<td>Long Equity Position</td>
<td>( \Delta \Pi_L^S(n) = \frac{n}{z(n)}p_T - nz(n)p_T(1 + i_L) + n\delta \frac{1 + i_L}{1 + i_D} )</td>
</tr>
<tr>
<td>Short Call</td>
<td>( \Delta \Pi_S^C(n) = \frac{n}{z(n)}\psi(1 + i_D) - n\max(0, z(n)p_T - k) )</td>
</tr>
<tr>
<td>Short Put</td>
<td>( \Delta \Pi_S^P(n) = \frac{n}{z(n)}\phi(1 + i_D) - n\max(0, k - \frac{p_T}{z(n)}) )</td>
</tr>
<tr>
<td>Short Equity Position</td>
<td>( \Delta \Pi_S^S(n) = \frac{n}{z(n)}p_T(1 - i_L) - nz(n)p_T - n\delta \frac{1 + i_L}{1 + i_D} )</td>
</tr>
</tbody>
</table>

Table 3: Profit Functions Given No Cash Endowment

There are slight differences in this state of nature from the large endowment state. In order to enter into a long position, the participant has to borrow funds and so costs are marked up by the interest rate. This is opposite the short side where any money received goes to a deposit account. There is also a difference in both equity positions.

In both equity positions, \( \delta \) is a future value. In a long equity position, the receipt of dividends would pay down the debt from the purchase and so must be discounted back to present value so it can be valued at the loan rate of interest. In the short equity position passed dividends are no longer paid from an endowment of cash. As such, passed dividends must be paid from borrowings at the loan rate. As \( \delta \) is defined with reference to the deposit rate, it must first be discounted back to the present value to be costed out at the commercial loan rate.

Basic algebra confirms that the results will be of the same form, but with different coefficients. For the long position, all cases result in the equilibrium formula:

\[
\psi = \phi + p_T - \frac{k}{z(n)(1 + i_L)} - \frac{\delta}{z(n)(1 + i_D)}
\]

For the short position, all cases result in the formula:

\[
\psi = \phi + p_T \frac{1 - i_L}{1 + i_D} - \frac{k(1 + i_D) + \delta(1 + i_L)}{(1 + i_D)^2}z(n)
\]

5.3 Large Endowment of Debt

In this state of nature, it is assumed that \( \bar{m} \ll 0 \). It is assumed that the participants have sufficient access to credit facilities as to be able to make purchases at an interest cost of \( i_L \). It is further assumed that no cash revenue is sufficient to entirely
pay down the debt to a positive cash position. The profit function, should no action be taken to enter into a position, is:

\[
\Pi_{\bar{m} < 0} = i_L \bar{m}
\]  

(61)

The profit functions of this state of nature are in table 4.

<table>
<thead>
<tr>
<th>Type of Position</th>
<th>Formula for Profit Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long Call</td>
<td>[ \Delta \Pi^C_L(n) = n \max(0, \frac{pt}{z(n)} - k) - nz(n) \psi(1 + i_L) ]</td>
</tr>
<tr>
<td>Long Put</td>
<td>[ \Delta \Pi^P_L(n) = n \max(0, k - z(n)p_T) - nz(n) \phi(1 + i_L) ]</td>
</tr>
<tr>
<td>Long Equity Position</td>
<td>[ \Delta \Pi^S_L(n) = \frac{n}{z(n)} p_T - nz(n)p_t(1 + i_L) + n \delta \frac{1+i_L}{1+i_D} ]</td>
</tr>
<tr>
<td>Short Call</td>
<td>[ \Delta \Pi^C_S(n) = n \max(0, z(n)p_T - k) ]</td>
</tr>
<tr>
<td>Short Put</td>
<td>[ \Delta \Pi^P_S(n) = \frac{n}{z(n)} \phi(1 + i_L) - n \max(0, k - \frac{pt}{z(n)}) ]</td>
</tr>
<tr>
<td>Short Equity Position</td>
<td>[ \Delta \Pi^S_S(n) = \frac{n}{z(n)} p_t(1 - i_L) - nz(n)p_T - n \delta \frac{1+i_L}{1+i_D} ]</td>
</tr>
</tbody>
</table>

Table 4: Profit Functions Given a Sufficiently Large Initial Indebtedness

This state of nature is distinguished by all transactions either paying down debt or increasing debt. As such, the deposit rate only appears in the discounting of dividends. The equilibrium equation for the long position in this state is:

\[
\psi = \phi + p_t - \frac{k}{z(n)(1 + i_L)} - \frac{\delta}{z(n)(1 + i_D)}
\]  

(62)

The equation for the short position is:

\[
\psi = \phi + p_t \frac{1 - i_L}{1 + i_L} - \frac{k}{z(n)} \frac{k}{1 + i_L} - \frac{\delta}{1 + i_D}
\]  

(63)

This state of nature is important as it describes the state of nature for the hedge fund industry.

5.4 Small Endowment of Cash

In the case where the participant begins with a small endowment of cash but wishes to make purchases that require the acquisition of debt for profit maximization, it is best to think in terms of the marginal transaction.

This transaction is a single transaction and so is at a single price. The value of the position has to be sufficient to make it worth going into debt to accomplish
the purchase. That permits two ways to think about the problem. One would be to blend the profit function between the deposit and the loan rates. The other would be to concern oneself only with the marginal transaction.

The blended method, while a correct profit function, doesn’t represent the last dollar spent. The marginal long position is the same as the wholly indebted state, while the short position is like the large endowment of cash state.

At the margin, this is no different than the no endowment state, that is \( m = 0 \). As such, the equilibrium conditions are the same as for that state.

5.5 Small Endowment of Debt

In this case, the participant begins with a small endowment of debt. Long purchases result in greater debt, but short positions more than pay off the debt. In that case, the participant is deciding that the overall position is no longer valuable enough to warrant carrying debt to maintain it.

As in the small endowment of cash state, at the margin, the result is the same as the no endowment state, that is \( m = 0 \).

6 Dominant Pricing

The equilibrium prices for the various endowments for call options are shown in table 5.

<table>
<thead>
<tr>
<th>Endowment</th>
<th>Short Formula</th>
<th>Long Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{m} \gg 0 )</td>
<td>( \psi = \phi + p_t \frac{1-i_L}{1+i_L} - \frac{k+\delta}{1+i_L} z(n) )</td>
<td>( \psi = \phi + p_t - \frac{k+\delta}{z(n)(1+i_L)} )</td>
</tr>
<tr>
<td>( \bar{m} \approx 0 )</td>
<td>( \psi = \phi + p_t \frac{1-i_L}{1+i_L} - \frac{k}{(1+i_L)^2} + \phi z(n) )</td>
<td>( \psi = \phi + p_t - \frac{k}{z(n)(1+i_L)} - \frac{\delta}{z(n)(1+i_L)} )</td>
</tr>
<tr>
<td>( \bar{m} \ll 0 )</td>
<td>( \psi = \phi + p_t \frac{1-i_L}{1+i_L} - z(n) \frac{k}{1+i_L} - \frac{\delta}{1+i_L} )</td>
<td>( \psi = \phi + p_t - \frac{k}{z(n)(1+i_L)} - \frac{\delta}{z(n)(1+i_L)} )</td>
</tr>
</tbody>
</table>

Table 5: Endowment Specific Equilibrium Pricing for Call Options

6.1 Put Contracts

The formula for \( \phi \) is simpler and less diverse than the call option. As those who have an endowment of debt have a higher discount rate, then they are willing to accept a lower price. As such, the lowest price wins the contract on the short side. The simple interpretation is that those who are willing to accept significant
amounts of debt on their balance sheets are willing to accept smaller premiums to pay for risks.

For those endowed with cash to write a contract at the higher price either implies that those participants endowed with debt have reached their equilibrium balance sheet and/or credit restrictions prevent further underwriting.

This implies that hedge funds and financial institutions should dominate the market for writing put contracts.

The long put is priced by symmetry as \( z(n) \phi \). As the price in the normal state is low, all types of endowments should be willing to buy long put contracts.

### 6.2 Long Call Contracts

For those willing to carry debt to buy long call contracts, the reservation price is greater than the reservation price for those unwilling to carry debt. That, of course, makes sense. Someone carrying debt would find insurance to be of greater value. A leveraged loss is magnified by the proportion the balance sheet is leveraged.

### 6.3 Short Call Contracts

The reservation price for \( \phi \) for those not heavily indebted is greater than for those who are heavily indebted. Since coherence would require participants willing to enter into either side of the position, either call or put plus a loan, the presence of heavily indebted parties willing to underwrite contracts should preclude the other two sets of endowment pricing from becoming operative. This is due to the unwillingness to compete with indebted parties to underwrite puts.

If the case exists where heavily indebted parties are no longer willing to underwrite call or put contracts, then those near zero would offer lower prices than those heavily endowed with cash. Only in the case where parties are unwilling to incur increased debt to underwrite contracts, like those near zero are required to do when shorting stocks and shorting dividends, will those heavily endowed with cash be able to get their reservation price. Of course the market maker, as an indebted party, would lose money on each transferred risk. So either this state does not exist, or it appears in the run state. Cash-rich investors appear when there is blood in the street or when the government bails out the system.
7 Speculators

Allowing risk loving actors into the game is not disruptive on the long side as a risk-loving actor would be willing to pay a higher premium than a risk-averse person for the right to purchase a gamble. It is disruptive on the short side of the market.

Risk-loving individuals will pay a premium for each gamble, in effect guaranteeing that their wealth will go to zero given enough time. This implies that risk-loving individuals, given enough time, will default on option contracts at high rates.

In markets with nationally insured banks, a market maker would love to accept the higher than normal option prices from risk loving buyers and sell options at unusually large discounts to risk loving sellers. Until the market craters, the makers will make unusually wide profits. The collapse of the writers could result in the collapse of the market makers if dividends have been paid, but this no longer is the case under too big to fail doctrines. In that case, the shareholders of the market makers keep the unusually wide profits but get recapitalized to do it again from risk-averse taxpayers.

Two remedies to this are to require minimum regulatory option premiums with prudential regulations similar in form to that found in the insurance and reinsurance industry or to have a no-bailout provision in the constitutional law. The challenge of the no-bailout provision is that it could be an incredibly costly solution.

8 Alternative Contractual Forms

European style options are not the only possible form. Other option styles include American, Asian and Look-back options. This list is, of course, neither exhaustive nor covers exotic options.

8.1 Asian Style Options

Asian style options are the simplest option to discuss. An Asian style option is based on the average price obtained over a period. Because the sample average has an identical distribution to the population distribution of data, the pricing mechanism is identical to the pricing of European style options.
8.2 American Style Options

It is not clear that a continuous time format would work for American options in many circumstances. It may not be true that equity prices are scale invariant. It may be the case that some circumstances, such as the January effect or witching days may impact both the distributions involved, but also the propensity to require delivery.

An iterative solution is available for American style options. Since an American style option can be thought of as a series of European options, it could be priced as a European style option given that it has not been exercised times the probability of exercise on that date. This requires an additional random variable, which is the probability of exercise on a given date. The measurement of intra-day exercise probably would not improve pricing.

One consequence of this is that the time derivative is probably meaningless as it is unlikely this methodology would permit a sufficiently smooth path. Witching days and jumps such as the January effect could alone cause a rough path.

That is an empirical question for future researchers.

8.3 Look-back Options

Lookback options are options that are based on the optimal price over a sequence of time. The distribution should be derived as in [3] as the ratio of a normal distribution in the denominator and the Gumbel distribution in the numerator.

This distribution, most likely, should be handled non-parametrically but is otherwise priced as a European style option contract.

9 The Budget Constraint

Because the market maker is willing to buy all finite orders for a sufficient markup, it is not necessary to model a budget constraint. The skew inherent in returns due to the budget constraint is captured in Abbott’s [19] construction of the liquidity model.

10 Limitations

The distribution-free model implicitly assumes that the scale parameters are constant. If they were not constant, then the width of the partitions should have a
function that varied their widths. Additionally, there is an implicit assumption that preferences are long-run constant at the margin. If this were not true, then the center of location could move. It is also assumed that the underlying risk and return properties of the firm are constant or that the compensation for changes happens in the dividend. Finally, the distributions are subject to discontinuous change in projection if the model is not scale invariant.

It is quite probable that models are not scale invariant. Between financial announcements, the only change in the underlying would come from changes to liquidity costs, quantity supplied and quantity demanded. As such, the information drivers between announcements should be liquidity based. As underlying financial or dividend information changes, it should trump the incidental liquidity based risks. Over time, larger macroeconomic events may overtake the predictive distribution in things such as the January effect and the business cycle.

11 Conclusion

As monopolies, leveraged institutions will set the price. Although this model would allow for an array of prices, it should be leveraged firms that are the price setters except in a period of disintermediation.

This model is rather simple. Integrate over the area at risk, find the expectation, subtract expected value from the strike price. Discount that value to present value, account for dividends, and set the price of calls as an equilibrium with puts. In concept, none of this is difficult. The implementation is challenging, but the concepts are simple.

Although the Black-Scholes model would be valid if all parameters were known, this is not the case. Humans are not imprinted with parameter information as soon as they consider making a purchase or sale. Although the Black-Scholes model is solved without a limitation of liability, the integrals will not converge in the Bayesian model without it.

This inclusion forces an additional parameter, the probability of bankruptcy, into the model.

Unlike the Black-Scholes model, it is impossible to avoid a discussion of dividends. All firms have a positive probability of paying a dividend over any fixed period of time. Depending on the cost function used, the statistic used for dividends will not generally be zero. An exception is the all-or-nothing cost function. If the modal probability is a payment of zero, then the statistic would be zero.

The Black-Scholes model prices the expected value of a call contract and put
contracts are valued with reference to calls. This is not possible in the Bayesian construction. Put contracts are priced actuarially, and calls cannot be. They are priced as a function of puts.

The Black-Scholes model assumes risk neutrality. Under a Cauchy distribution, this is impossible as the integrals would diverge. Implicitly, risk aversion is present.

Because the Black-Scholes equation is an integration over paths in continuous time, it is necessary to explicitly calculate the impact of time. This is lost in this formulation because it is only necessary to know the terminal distribution. The path does not matter. Indeed, some care will be needed in discussing paths as the Cauchy distribution can be thought of as a projection of the normal distribution from complex space. As such, it may not have a differentiable path.

As important, the center of location does not vanish from the equations. In the Black-Scholes model, the center of location vanishes in the derivation. It cannot vanish here.

Finally, most of the “Greeks” do not mathematically survive. Delta, the partial derivative of price with respect to the underlying price depends upon the state of nature and whether calls or puts are being discussed.

Vega, the partial derivative of the option price with respect to volatility is without mathematical meaning. It is the nature of the Cauchy distribution that realized volatility is a random number even in a Frequentist construction. While the scale parameter of the Cauchy distribution may be a function of information, the derivative would be with respect to the information and not volatility. Indeed, while the scale parameter in reward space measures dispersion, it is a measure of heteroskedasticity in the price space. Vega conceptualized as the change in the option price with respect to the variance is undefined.

As mentioned above, theta, the partial derivative with respect to time may encounter paths too rough to differentiate over. That may or may not be true. Investigations of phenomenon such as the January effect are required to determine adequate smoothness.

Rho presents a special case here. The derivative with respect to the interest rate is quite a credible concept, but there are now two interest rates to deal with. The concern should be for each separate rate, but also the joint relationship.

Lambda is a special challenge because the price is absorbed into the likelihood function causing price elasticity to become a direct function of the sample itself.

A further challenge comes in pricing interest rate options, commodities and more exotic items such as weather derivatives. Not touched were the prices for futures and forwards contracts.
Certain other factors have been excluded from this model that should be explored. The possibility of information asymmetry, moral hazard inside a broker-dealer, the bankruptcy of a writer or a market maker, reputation effects, or effects created by using a repeated game format such as strategic thinking has not been considered here. Separate research on each of these possible changes in assumptions is advised.

This article lays out broad principles that can be used to develop alternative models. As Bayesian inference and the various models of Bayesian decision theory contain well-understood mathematics, a good foundation exists to move forward. It is important to remember, that this is not the model of option pricing, but rather a foot forward to discussing it.

A Appendix–A Conjecture

Because random paths are continuous everywhere but not smooth enough anywhere to perform differentiation, financial calculus has relied on Itô and Stratonovich integrals to create a smooth path of expectations. This is not available for financial models. As shown above, these models will not produce admissible estimators, though a rigorous and more complete discussion is still required.

If \( X \) is a realization of random variables indexed by time so \( X = \{ x_0, x_1 \ldots x_T \} \), then the problem for the economist is to create a useful smooth function whose values depend upon \( X \). Preferably this function would use expectations when most appropriate and not when it is not appropriate. In an ideal construction, the predictions in this alternative method would match those of an Itô method or method constructed on Stratonovich integrals when these methods would also be appropriate.

If we assume that the model, \( f(\theta | X) \) is the true model and either that the predictive path is scale invariant or the location of the discontinuities are understood, then we can define a Bayesian predictive distribution as in equation 13 over the time interval \( t = (\tau, T] \). If we define \( g(\tilde{x}_t) \) as a sequence of predictive mass functions defined at all points in time in \( t \) with the parameters being a smooth function of time, then a smooth path can be formed over the set of predictions.

**Definition 1** The function \( A(g(\tilde{x}_t)) \), known as the anticipation operator, anticipates a future value of \( \tilde{x}_t \) by inverting a function \( h \) such that \( h \) is the admissible estimator under the density \( f(\theta(t)) \) for the parameter \( \theta(t) \) and \( m \) is an implicit cost function that if minimized over the parameter space would produce estimator
h. $A(g(\tilde{x}_t)) = m(\tilde{x}_t)$ where $m(\tilde{x}) = \hat{x}_t$ minimizes the loss everywhere by choosing $\hat{x}_t, \forall t \in (\tau, T]$ such that it is the loss minimizing set of points indexed by the interval $(\tau, T]$.

Note that the anticipation operator does not need to be unique. There may be more than one decision rule that is admissible. For a projection of normal distributions, the loss-minimizing cost function is quadratic loss. For the Cauchy distribution it is the linear absolute loss function, and for the truncated Cauchy distribution it is the all-or-nothing loss function. Because it is assumed that the projected parameters are a smooth function of time, it should follow that the projections are a smooth function of time. The path itself should be a smooth function of time.

To simplify the lives of economists, some special cases can be defined thusly.

**Definition 2** The expectations function, $E(\tilde{x}_t) = A(g(\tilde{x}_t)) = \hat{x}_t$, when the cost function $(\hat{\theta} - \theta)^2$ would produce an estimator for $\hat{\theta}$ that is admissible over $f(\theta)$ and is applied over $\tilde{x}_t$ so that $\hat{x}_t$ minimizes the loss over the prediction at each point in time $t$.

**Definition 3** The ordinarily function, $O(\tilde{x}_t) = A(g(\tilde{x}_t))$, when the cost function $|\hat{\theta} - \theta|$ would produce an estimator for $\hat{\theta}$ that is admissible over $f(\theta)$ and is applied over $\tilde{x}_t$ so that $\hat{x}_t$ minimizes the loss over the prediction at each point in time $t$.

**Definition 4** The usually function, $U(\tilde{x}_t) = A(g(\tilde{x}_t))$, when the cost is zero when $\theta - \varepsilon < \hat{\theta} < \theta + \varepsilon$ and the cost is $b > 0$ otherwise would produce an estimator for $\hat{\theta}$ that is admissible over $f(\theta)$ and is applied over $\tilde{x}_t$ so that $\hat{x}_t$ minimizes the loss over the prediction at each point in time $t$.

The three operators correspond to the mean, median and mode of the prediction at each point in time and the differentiable path is the line connecting the chosen points $\hat{x}_t, \forall t \in (\tau, T]$.

**References**


