The Object Allocation Problem with Random Priorities

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Abstract

The existing priority-based object allocation literature restricts the objects’ priorities to be deterministic. However, agents might be probabilistically prioritized ex-ante, for instance, through a non-uniform tie breaking rule. This paper generalizes the deterministic setting by allowing priorities to be random. In this probabilistic environment, we first introduce a fairness notion called claimwise stability in the spirit of usual stability of Gale and Shapley (1962). We show that while the well-adopted agent-optimal stable mechanism is claimwise stable, it might be ordinally dominated by another claimwise stable rule. We then introduce a new mechanism called constrained probabilistic serial, which is built on the probabilistic serial mechanism of Bogomolnaia and Moulin (2001). We show that it is claimwise stable and constrained ordinally efficient. Then the paper systematically compares the agent-optimal stable and constrained probabilistic serial mechanisms in terms of strategic and fairness properties. Lastly, we provide an axiomatic characterization of the constrained probabilistic serial mechanism.

JEL classification: C78, D61, D63, D82.

Keywords: constrained probabilistic serial mechanism, agent-optimal stable mechanism, constrained ordinal efficiency, claimwise stability, characterization.

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1 Introduction

In a priority-based object allocation problem, there are sets of agents and indivisible objects that are to be distributed among the agents. Each agent has a strict preference over objects and is entitled to receive one object (including the null object representing being unassigned). Similarly, each object has a priority order (may involve ties) over agents. No monetary transfer is allowed. Student placement in public schools (see Abdulkadiroğlu and Sönmez (2003)) is a well-known real-life example of such allocation problems.

Theoretical advancements in matching theory have proved useful in practical object allocation problems. Especially, many school districts in the US have been successfully redesigned by matching theorists. Student placement, however, is still not problem free. For instance, the following passage is from Boston mayor Menino’s 2012 State of City address:

“Pick any street. A dozen children probably attend a dozen of different schools. Parents might not know each other; children might not play together. They can not carpool, or study for the same tests. We will not have the schools our kids deserve until we build school communities that serve them well”. Menino (2012).

A striking fact supporting what Mayor Menino said above is that 19 kids from the same street in Boston go to 15 different schools (Ebbert and Ulmanu (2011)). As one of the important aspects of fostering community cohesion is to have children from the same community going to the same schools, some have advocated replacing the school choice1 with the neighborhood system.2 However, school choice enables families to express their preferences and honors their choices as much as possible. Therefore, it definitely adds an important value to student placement. Hence, increasing community cohesion is important.

1In the school choice system, students submit a ranked list of schools based on their preferences, and schools have priorities over students. Given these information, a centralized institution determines a student-school matching by performing a certain matching algorithm. In the US, many school districts, including Boston, New York, San Francisco, and Chicago, employ the school choice system.

2Having the same community students going to different schools has other important bad consequences such as higher transportation costs and the longer routes of school buses. Seelye (2012) reports that transportation costs $80.4 million a year in Boston, which is about %9.4 percent of the school system’s operating budget, almost twice the national average.
while keeping the school choice.

The existing school choice literature allows school priorities to be deterministic and weak (i.e., students might be tied). The common practice to deal with ties is to first randomly break them, and then perform the mechanism based on the obtained strict priorities. Fortunately, random tie-breaking gives us a way to lessen the cohesion problem above. The central authority can improve community cohesion in expectation by favoring the students from the same community in the course of random tie-breaking. Namely, through such a non-uniform random tie-breaking, the same community students would have higher chance of ranked higher in the ex-post strict priorities. This in turn increases the probability that the same community students will go to the same school.\textsuperscript{3,4} Breaking the ties non-uniformly, however, means that seemingly equal-priority students in the coarse priorities are not “exactly” tied in the mechanism designer’s eyes.

However, once coarse priorities are taken as a primitive of the problem (as the existing literature does), we can not capture the students’ “exact” positions at schools as explained above. In order to fully capture and incorporate them into the mechanism design, we allow priorities to be random in this paper. Our approach is very important as what mechanism design recommends would drastically change from coarse priorities to random ones. For instance, let us consider two agents $i, j$ and one object $a$ for which both agents are tied.\textsuperscript{5} Then with respect to this coarse priority order, as agents are equal, the “fair” matching is random, giving the object to each agent with probability $1/2$. On the other hand, for some reason, let us consider that either agent, say agent $i$, is favored in the course of random tie-breaking by giving him $2/3$ chance of having higher priority. This means that the probability that agent $i$ has higher priority than agent $j$ is $2/3$ with respect to the ex-ante random priority

\textsuperscript{3}In the last paragraph of this section, we will illustrate this point mathematically as well. We will also give some other real-life applications of non-uniform tie breaking.

\textsuperscript{4}Ashlagi and Shi (2013) also consider the cohesion problem, and they propose another approach to lessen it. Instead of non-uniform random tie breaking, they offer the correlated ex-post implementation of random assignments.

\textsuperscript{5}While we motivate the research in the school choice context, it is not restricted to that. Because of this, we will use agent-object rather than student-school terminology for general descriptions.
order. That is, agents are not exactly tied as opposed to what the above coarse priority says. Then if we take the random priority as a primitive of the problem, as agents are not equal anymore, giving 1/2 of the object to each agent would not be fair.\footnote{We will introduce a fairness notion in the random priority setting. In this particular example, it will give the object to agent $i$ with probability 2/3.}

As random priorities convey different information to the mechanism designer than what coarse priorities do, it is important to study the object allocation problem with random priorities. To this end, the current paper attempts to develop a theory in the random priorities setting, and, to the best of our knowledge, it is the first paper in this direction. We first introduce a fairness notion called “claimwise stability” in the setting, where objects’ priority order profile is a lottery over deterministic and strict priorities and assignments (matchings) are allowed to be random. We can consider the objects as divisible and the entries of an assignment as their assigned shares to agents. Then, for a given problem and matching, we say that agent $i$ has a justified claim against agent $j$ for object $a$ if the assigned share of object $a$ to the latter is greater than the probability that he has higher priority than the former at object $a$\footnote{The probability that agent $j$ has higher priority than agent $i$ at object $a$ is defined as the sum of the probabilities of the deterministic priorities (with respect to the given lottery) under which the former has higher priority than the latter at the object.} plus agent $i$’s total share of his more preferred objects. We say that a matching is claimwise stable if no agent has a justified claim.

What is the intuition behind claimwise stability? There are two main motivating arguments coming from the claim problem (a.k.a. bankruptcy or rationing) and matching literatures. For ease of understanding, let us consider a problem consisting of two agents $i, j$ and one object $a$. We can interpret the probability that agent $i$ has higher priority than agent $j$ as the share of object $a$ for which the former has claim against the latter (similarly, for agent $j$). Then claimwise stability recommends giving each agent as much share as his claim. That is, it advocates distributing shares proportionally to agents’ claims. This kind of claim proportional allocation rules is very common in practice, and it is often considered as a fairness criterion in claim problems (for instance, see Thomson (2003)). On the other
hand, in our general setting with more than one object, agent \( i \) (similarly for agent \( j \)) can satisfy some fraction of his demand with his more preferred objects. In this case, naturally, claimwise stability allows agent \( j \) to acquire more of object \( a \) as much as at most that fraction.

The other motivating argument for claimwise stability is due to the stability of Gale and Shapley (1962) (hereafter, we often refer to stability of Gale and Shapley (1962) as usual stability to avoid confusion between notions). To this end, we first invoke the “consumption process” representation of random assignments by Bogomolnaia and Heo (2012).\(^8\) Any random assignment can be seen as a consumption process where each agent eats objects in decreasing order of his preference at a speed of one over the unit time interval. In the course of consumption process, claimwise stability rules out any time instant at which an agent envies someone else for the object he is eating while the former has higher priority.\(^9\) In other words, it requires usual stability at every point of time in the course of the consumption process. Hence, it can be seen as the natural generalization of usual stability to the current random environment. Claimwise stability indeed collapses to usual stability in the conventional deterministic priority domain for deterministic matchings.\(^10\)

Claimwise stability is the central notion throughout the paper. We then look for mechanisms satisfying it along with other desirable properties. The first natural rule coming to mind is the agent-optimal stable mechanism of Gale and Shapley (1962) (hereafter, \( DA \)) because of its superior properties in the conventional setting and the fact that it is widely used in many real-life problems.\(^11\) While \( DA \) turns out to be claimwise stable, its main drawback is that agents might prefer another claimwise stable assignment regardless of their cardinal utilities (representing their ordinal preferences over objects). That is, formally speaking, \( DA \)

\(^{8}\)This kind of representation was first introduced by Heo (2010) and called “preference-decreasing consumption schedules”.

\(^{9}\)This point is explained in detail in the model section.

\(^{10}\)Its relation to other recent probabilistic fairness notions is discussed in the model section.

\(^{11}\)New York City (which has the largest public school system in the country with over a million students) and Boston (which has over 60,000 students enrolled in the public school system) school districts have been using the agent-optimal stable mechanism (Abdulkadiroğlu et al. (2005) and Abdulkadiroğlu et al. (2005)) to assign students to schools.
might be “ordinally dominated” by another claimwise stable mechanism.

The lack of efficiency of DA leads us to look for other claimwise stable rules. To this end, we introduce “constrained probabilistic serial mechanism” (henceforth, CPS), which is built on the probabilistic serial mechanism (hereafter, PS) of Bogomolnaia and Moulin (2001). They offer PS for the priority-free object allocation problem; hence, it does not take care of claimwise stability constraints. We adopt it to our setting in a way that CPS takes care of these constraints, making it claimwise stable. In contrast to DA, CPS is not ordinally dominated by another claimwise stable mechanism (it might be ordinally dominated by a non-claimwise stable mechanism, yet, it is not a problem specific to CPS as claimwise stability and ordinal efficiency are incompatible). Then the paper systematically compares DA and CPS in terms of strategic and fairness properties. While CPS turns out to be constrained ordinally efficient and performs better than DA on the fairness ground, the converse is true with respect to strategic issues. Lastly, we provide a characterization of CPS: it is the unique non-wasteful, claimwise stable and binding mechanism. Non-wastefulness is a desirable standard efficiency property. Bindingness, on the other hand, requires each agent to continue eating his preferred objects as long as no one else prevents him to do so through claimwise stability constraints.

In the beginning of the paper, we mention a real-life application of non-uniform tie breaking, which suggests taking random priorities, rather than coarse ones, as a primitive of the problem. In what follows, we first mathematically illustrate how non-uniform tie breaking helps us to increase the community cohesion in expectation and give some other real-life applications of non-uniform tie breaking. Let us consider two students $i, j$ from the same community and another one $k$ from a different community. Assume that there are two schools $a, b$ with two available seats at the former and one at the latter, and all students are tied in both schools’ coarse priorities. Let students unanimously prefer school $a$ to school $b$. If the ties are broken uniformly (hence, any deterministic priority order is equally likely), 12

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12It does not matter whether ties are broken independently across schools through multiple tie breaking or dependently through single tie breaking. Please see Footnote 16 for the descriptions of these two common
then each student is matched with school $a$ with probability $2/3$ under $DA$.\footnote{In this random environment, $DA$ refers to the outcome obtained by applying it to every possible deterministic priority realization and summing them by multiplying their corresponding realization probabilities.} This basically implies that the same community students $i, j$ will go to the same school with probability $1/3$ under the ex-post implementation of $DA$. Let us now break the ties non-uniformly so that students $i, j$ are the top two students with probability $2/3$ at school $a$. In this case, they go to the same school with probability $2/3$ under the ex-post implementation of $DA$, increasing the cohesion in expectation.

Another similar documented problem that can be lessened in expectation through non-uniform random tie breaking is having fewer students assigned to their walk-zone schools than what is intended in Boston public schools. Dur et al. (2013) demonstrate the problem even though walk-zone students are given higher priorities in half of the seats in the current system. They offer alternative ways (increasing the walk-zone slots and changing the precedence order with respect to which slots are filled) that prove to be useful in data, yet they might still work in the opposite direction in theory.\footnote{They also consider two lottery numbers instead of single one for breaking the ties in priorities and empirically show that it reduces the problem. However, an important drawback of it is generated efficiency losses (Abdulkadiroglu et al. (2009)).} On the other hand, non-uniform tie breaking favoring walk-zone students would assign more of such students in expectation both in theory and data. Hence, it can be seen as another possible remedy for the problem. We can indeed increase the number of real-life applications of non-uniform tie breaking. Generally speaking, however, non-uniform tie breaking is relevant whenever the social planner is to treat equal-priority agents unequally in expectation because of various constraints or society’s preferences over matchings.

\section{Related Literature}

This paper attempts to develop an object allocation theory with random priorities. The closest work to the current paper is Kesten and Ünver (2012). They introduce two different tie-breaking methods.
ex-ante stability notions for random assignments in the setting where priorities are deterministic (not random, as in the current study) and coarse.\textsuperscript{15} They show that \textit{DA} does not satisfy either of them and introduce two mechanisms satisfying their stability notions along with some other desirable properties. While both the current study and Kesten and "Unver (2012) offer fairness notions for random assignments, the priority structures considered in the papers are different. While we allow priorities to be random, they only consider deterministic priorities that are allowed to involve ties (note that our priorities do not involve ties). Hence, there is no direct logical relation between the concepts. A detailed formal discussion on the relation between the papers is provided in the model section.

As mentioned in the introduction, the non-uniform tie breaking has serious applications. There are papers that have studied other aspects of random tie-breaking. Erdil and Ergin (2008) demonstrate that depending on the way of breaking the ties, \textit{DA} might be dominated by another stable matching. In the same context, Abdulkadiroglu et al. (2009) compare the single tie-breaking and multiple tie-breaking rules\textsuperscript{16} under \textit{DA} in terms of efficiency and incentive properties. Their work reveals that more students receive their top choices under the former (yet, there is no stochastic dominance relation between them); hence, they conclude that the former has better efficiency properties than the latter.

Another important related paper is Bogomolnaia and Moulin (2001). They introduce the \textit{PS} mechanism for the priority-free object allocation problem and show that it is ordinally efficient while the well-known random priority mechanism\textsuperscript{17} is not. In the current

\textsuperscript{15}They say that a random assignment causes \textit{ex-ante schoolwise justified envy} if student \textit{i} has higher priority than student \textit{j} at school \textit{c}, yet the former is assigned to a less preferred school with positive probability, while the latter is assigned to school \textit{c} with positive probability. They call a random assignment \textit{ex-ante stable} if it eliminates \textit{ex-ante schoolwise justified envy} (this is the notion they offer for students belonging to different priority classes). Moreover, they say that a random assignment causes \textit{ex-ante schoolwise discrimination} if a student is assigned to a school with a higher probability than another equal-priority student, while the latter is assigned to a less preferred school with positive probability. They refer to a random assignment as \textit{strongly ex-ante stable} if it eliminates both \textit{ex-ante schoolwise justified envy} and \textit{ex-ante schoolwise discrimination}.

\textsuperscript{16}In the single tie-breaking rule, each student is randomly given a single number, and it is used to break the ties in every school. On the other hand, in multiple tie-breaking, each student is randomly given a number at each school to break the ties.

\textsuperscript{17}Under the random priority mechanism, an ordering of agents is chosen uniformly; then agents choose their remaining favorite object one at a time following the order.
paper, we adopt \(PS\) to the priority-based object allocation setting for the first time. Similar to Bogomolnaia and Moulin (2001), we show that our mechanism is constrained ordinally efficient, whereas well-known \(DA\) is not. This shows that the appealing properties of \(DA\) do not carry over to the current random environment. Similar points are also shared in some other papers in different settings, including Erdil and Ergin (2008) and Kesten and Ünver (2012). In an incomplete information setting regarding the preferences of students, Featherstone and Niederle (2008) reveal that truth-telling can be an equilibrium under the Boston mechanism, and it can first-order stochastically dominate \(DA\), both in theory and in the laboratory. Similarly, in an incomplete information setting with common ordinal preferences, Abdulkadiroglu et al. (2011) demonstrate that every student at least weakly prefers any symmetric equilibrium of the Boston mechanism to the dominant strategy equilibrium of \(DA\) in the priority-free setting. Similar results are obtained in Troyan (2012) and Miralles (2008).

The \(PS\) mechanism has received much attention in the literature. Partly due to its superior properties, it has been extended to various richer domains. Katta and Sethuraman (2006) generalize \(PS\) to the weak preference domain. Heo (2012) considers the more general multi-unit demand case and extends \(PS\) to that setting. Yilmaz (2010, 2009) lets agents have private endowments in the strict and weak preferences settings and generalizes \(PS\) to those domains respectively. Kojima (2009) extends it to the multi-quota setting. Aliogullari et al. (2013) modify \(PS\) to increase the expected number of agents matched with their preferred objects; and a recent paper by Budish et al. (2013) introduce a variant of \(PS\) accommodating various real-life constraints. Besides this line of research, there has also been a recent surge in \(PS\)’s characterizations: Bogomolnaia and Heo (2012) and Hashimoto et al. (2012) independently characterize the \(PS\) mechanism.
3 Model and Results

There are finite sets of agents $N$ and objects $O$ that are to be distributed among agents. Each agent $i \in N$ has a preference relation $R_i$, which is a complete, transitive, and antisymmetric binary relation over $O$. We write $aP_ib$ when $aR_ib$ and $a \neq b$. Let $\mathcal{R}$ and $q_a$ be the set of preference relations and the number of copies of object $a$, respectively. For ease of analysis, we assume that $q_a = 1$ for each $a \in O$ and $|N| = |O|$.$^{18}$

In the conventional priority-based object assignment model, each object $a$ is endowed with deterministic priority order $\succ_a$, which is a complete, strict, and transitive binary relation over $N$. Let $\succ = (\succ_a)_{a \in O}$ and $\zeta$ be the priority order profile and the set of such profiles, respectively.

This paper departs from the above conventional setting and allows priorities to be random. Formally, we write $\Delta$ for the priority order profile, which is a probability distribution (lottery) over $\zeta$. We write $\Delta_a$ for the priority order of object $a$, which is the marginal probability distribution of the priority order of object $a$ under $\Delta$. There is no restriction on $\Delta$; hence, objects’ priorities may be independent as well as correlated.$^{19}$ Let $\Delta(\succ)$ be the probability of $\succ = (\succ_a)_{a \in O}$ under $\Delta$. We write $supp(\Delta) = \{\succ \in \zeta : \Delta(\succ) > 0\}$ for the support of $\Delta$. We define $Pr_\Delta(i \succ_a j) = \sum_{\succ \in \zeta : i \succ_a j} \Delta(\succ)$. In words, it is the probability that agent $i$ has higher priority than agent $j$ at object $a$. In the rest of the paper, we fix the set of agents and objects and simply write $(R, \Delta)$ for the generic problem.

A matching $\sigma = [\sigma_{i,a}]_{i \in N, a \in O}$ is a matrix such that for all $i \in N$ and $a \in O$, (i) $0 \leq \sigma_{i,a} \leq 1$, (ii) $\sum_{a \in O} \sigma_{i,a} = 1$, and (iii) $\sum_{i \in N} \sigma_{i,a} = 1$. Here, $\sigma_{i,a}$ represents the probability that agent $i$ is matched with object $a$. Let $\sigma_i$ and $\sigma^a$ denote the random assignments of agent $i$ and object $a$ at $\sigma$, respectively. A matching $\sigma$ is deterministic if $\sigma_{i,a} \in \{0, 1\}$ for all $i \in N$ and $a \in O$. Let $\mathcal{X}$ be the set of all matchings. We write $\mathcal{M}$ for the proper subset of $\mathcal{X}$.

$^{18}$All the results carry over to the more general cases of multi-unit quota, the presence of the null object, and $|N| \neq |O|$. We will consider extensions later in the paper.

$^{19}$This generality is important in both theory and practice. Single and multiple tie breaking rules, which are well-studied and used ones, yield independent and correlated priorities, respectively (Abdulkadiroglu et al. (2009)).
consisting of only deterministic matchings.

A probability distribution $\lambda$ over $\mathcal{M}$ is called lottery. Formally, $\lambda = (\lambda_{\mu})_{\mu \in \mathcal{M}}$ is such that, for all $\mu \in \mathcal{M}$, $0 \leq \lambda_{\mu} \leq 1$ and $\sum_{\mu \in \mathcal{M}} \lambda_{\mu} = 1$. We write $\sigma^\lambda$ for the matching induced by $\lambda$, i.e., $\sigma^\lambda_{i,a} = \sum_{\mu \in \mathcal{M}}: \mu_i = a \lambda_{\mu}$.

**Fact 1** (Birkhoff-Von Neumann). *Any matching can be induced by a lottery $\lambda$ over $\mathcal{M}$.*

Because of the above well-known fact, in the rest of the paper, we consider matchings instead of lotteries. Given an agent $i$ and object $a$, let $SU(R_i, a) = \{c \in O : c \succ_i a\}$ (the strict upper contour set of agent $i$ at object $a$), and $U(R_i, a) = SU(R_i, a) \cup \{a\}$ (the upper contour set of agent $i$ at object $a$).

Given two assignments $\sigma_i$ and $\sigma'_i$ for agent $i$, the former *ordinally dominates* the latter if $\sum_{c \in U(R_i, a)} \sigma_{i,c} \geq \sum_{c \in U(R_i, a)} \sigma'_{i,c}$ for each $a \in O$, with holding strictly for at least one object. That is, $\sigma_i$ first-order stochastically dominates $\sigma'_i$ with respect to $R_i$. A matching $\sigma$ ordinally dominates $\sigma'$ if, for all $i \in N$, either $\sigma_i$ ordinally dominates $\sigma'_i$ or $\sigma_i = \sigma'_i$, with the former holding for at least one agent. A matching is *ordinally efficient* if it is not ordinally dominated by another matching.

A deterministic matching $\mu$ is *stable* at $(R, \succ)$ if there exists no pair of agents $i, j$ such that $\mu_j \succ_i \mu_i$ and $i \succ_{\mu_j} j$. A matching $\sigma$ is *ex-post stable* at $(R, \succ)$ if it can be induced by a lottery over stable matchings. Below introduces the key notion of the paper.

**Definition 1.**

(i) *Given a problem* $(R, \Delta)$, *we say that agent* $i$ *has a justified claim against agent* $j$ *for object* $a$ *at matching* $\sigma$ *if* $\sigma_{j,a} > Pr_{\Delta}(j \succ_a i) + \sum_{c \in SU(R_i, a)} \sigma_{i,c}$.

(ii) *A matching* $\sigma$ *is claimwise stable if no agent has a justified claim against someone else for an object.*

We can motivate the claimwise stability notion through the proportional distribution criterion in the claim problem (a.k.a., bankruptcy or rationing) literature and the usual stability of Gale and Shapley (1962). For any given pair of agents $i, j$ and object $a$, we can
interpret \( Pr\Delta(j \succ_a i) \) as the claim of agent \( j \) on object \( a \) against agent \( i \). If, for a moment, we assume that object \( a \) is the only object, then claimwise stability recommends distributing assignment probabilities proportionally to agents’ claims. This kind of assignment rule is commonly used and often considered as a fairness criterion in claim problems (for instance, see Thomson (2003) and Moulin (2002)). On the other hand, in the case of multiple objects, claimwise stability naturally lets agent \( j \) obtain more of object \( a \) as much as at most \( \sum_{c \in SU(R_i,a)} \sigma_{i,c} \) (the fraction of agent \( i \)’s demand satisfied with his more preferred objects). Hence, claimwise stability can be seen as a natural counterpart of the proportional allocation principle in the current object allocation model.

For exploring the connection between the claimwise and usual stability notions, we first invoke the consumption process representation of random assignments by Bogomolnaia and Heo (2012). Any random assignment can be seen as an outcome of the consumption process where, over the unit time interval, agents continuously acquire probability shares of objects in decreasing order of their preferences at a speed of one. The assigned probabilities under a given random assignment represent the time-shares in which agents eat corresponding objects. Let us think of a pair of agents \( i,j \), object \( a \), and matching \( \sigma \). We can consider \( Pr\Delta(j \succ_a i) \) as the time-share in which agent \( j \) has higher priority than agent \( i \) for object \( a \). Similarly, let \( \sum_{c \in SU(R_i,a)} \sigma_{i,c} \) represent the time-share in which agent \( i \) eats his more preferred objects to object \( a \) at \( \sigma \). Then if \( \sigma_{j,a} > Pr\Delta(j \succ_a i) + \sum_{c \in SU(R_i,a)} \sigma_{i,c} \), it implies that there is a time interval during which agent \( j \) eats object \( a \), while agent \( i \) consumes his less preferred objects even though he has higher priority for object \( a \). Therefore, claimwise stability rules out any time instant at which an agent envies someone else for the object he is eating while the former has higher priority. In other words, it requires the usual stability at every point of time in the course of the consumption process. The figure below depicts it.
Figure 1: Agents eat objects over the above unit time interval. The orange region (R.1) represents the time-share in which agent $j$ has higher priority than agent $i$ for object $a$ (the length of it is $Pr_{\Delta}(j \succ_a i)$). Similarly, agent $i$ is prioritized in the blue region (R.3). On the other hand, the green region (R.2) shows the time-share in which agent $i$ eats his more preferred objects. Therefore, if $Pr_{\Delta}(j \succ a) + \sum_{c \in SU(R_i,a)} \sigma_{i,c}$ (the sum of orange (R.1) and green regions (R.2)), then it implies that agent $j$ consumes object $a$ for some period of time falling into the red region (R.4). This, however, means that, for some period of time, agent $j$ eats object $a$, while agent $i$ consumes a less preferred object even though he has higher priority. In other words, the usual stability is violated within that time interval (note that this is independent of the regions’ locations on the unit interval above).

It is easy to see that in the conventional deterministic priority domain, stability and claimwise stability coincide with each other for deterministic matchings. However, there are non-deterministic claimwise stable matchings as well (for instance, it is easy to see that any lottery over stable matchings is claimwise stable). Hence, a natural question is whether non-deterministic claimwise stable matchings implement stable matchings ex-post; in other words, are they ex-post stable? The answer turns out be affirmative as shown below.

**Proposition 1.** Claimwise stable matchings are ex-post stable.

*Proof.* See Appendix. }

Kesten and Ünver (2012) introduce two stability notions for random assignments when objects have deterministic and coarse priorities. Let us write $i \sim_a j$ for agents $i, j$ belonging to the same priority class for object $a$. Then they say that matching $\sigma$ induces ex-ante schoolwise justified envy of agent $i$ toward agent $j$ if $i \succ_a j$ and $\sigma_{i,b} > 0$ for some object $b$ such that $aP_i b$ and $\sigma_{j,a} > 0$. They call matching $\sigma$ ex-ante stable if it does not cause any ex-ante schoolwise justified envy. Moreover, they say that matching $\sigma$ induces ex-ante schoolwise discrimination between students $i, j$ if $i \sim_a j$ and $\sigma_{i,b} > 0$ for some object $b$, where $aP_i b$ and $\sigma_{i,a} < \sigma_{j,a}$. Matching $\sigma$ is called strongly ex-ante stable if it eliminates both ex-ante
schoolwise justified envy and ex-ante schoolwise discrimination. As the priority structures in Kesten and Ünver (2012) and the current paper are different, there is no general logical relation between the stability notions. However, as agents are either prioritized or tied in their setting, the counterpart of it in the current environment might be the subdomain where, for any object, the probability that any agent has higher priority than someone else is in \{0, 1/2, 1\}. We, then, can define a natural isomorphism between this special domain and their model as follows: For any pair of agents \(i, j\) and object \(a\), \(i \sim_a j \iff Pr_\Delta(i >_a j) = 1/2\), and \(i >_a j \iff Pr_\Delta(i > j) = 1\). Then given \((R, \Delta)\), any strongly ex-ante stable matching in the isomorphic deterministic priority profile is claimwise stable at \((R, \Delta)\), yet the converse is not true.\(^{21}\)

A mechanism \(\psi\) is a function assigning a matching for every problem. A mechanism \(\psi\) is claimwise stable if \(\psi(R, \Delta)\) is claimwise stable at any problem \((R, \Delta)\). Mechanism \(\psi\) is ordinarily efficient if \(\psi(R, \Delta)\) is not ordinarily dominated at any \((R, \Delta)\).

Given a problem \((R, \Delta)\) where \(\text{supp}(\Delta) = \bigcup_{k=1}^n \{>^k\}\), a natural solution \(\psi(R, \Delta)\) might be of the following form: \(\psi(R, \Delta) = \sum_{k=1}^n \Delta(>^k)\psi^k(R, >^k)\), where each \(\psi^k(R, >^k)\) is a stable matching at \((R, >^k)\). Indeed, this form defines a class of rules we call “separable mechanisms”. Below, we outline a well-known member of this class.

### 3.1 The Agent-Optimal Stable Mechanism

In this section, we outline the deferred acceptance algorithm (Gale and Shapley (1962)) producing the agent-optimal stable matching in the conventional domain \(\mathcal{R}^{[N]} \times \zeta\).

**Step 1.** Each agent applies to his first choice object. Each object that receives one or more offers holds the best acceptable offer and rejects the rest.

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\(^{20}\)Neither our model nor theirs implies the other. Priorities are random here, while they are deterministic involving indifference classes in Kesten and Ünver (2012).

\(^{21}\)Any lottery over stable matchings is claimwise stable, yet it is not even necessarily ex-ante stable. On the other hand, as ex-ante stability does not say anything for equal-priority students, it might not be claimwise stable whenever the priorities involve indifference classes.
In general,

**Step t.** Each agent who was rejected in step \( (t - 1) \) applies to his best acceptable choice in the set of objects to which he did not apply before. Each object holds the best acceptable offer among the set of offers held at step \( (t - 1) \) and the offers it receives in this step and rejects the rest.

The algorithm terminates when no agent applies to an object, and the tentatively held offers at the termination step are realized as assignments.

The deferred acceptance algorithm, denoted by \( DA \) in the paper, is defined for strict priority orders. In other words, it is not well defined for coarse priorities. To overcome this caveat, in many real-life problems, ties are first broken randomly; then \( DA \) is applied to the obtained strict priority order. Therefore, in our setting, \( DA \) refers to the outcome obtained by applying it to each possible realization of the random priority profile and summing them by multiplying their corresponding realization probabilities. Formally, given a problem \( (R, \Delta) \),

\[
DA(R, \Delta)_{i,a} = \sum_{\succ \in \text{supp}(\Delta)} \Delta(\succ) DA(R, \succ)_{i,a}
\]

for each \( i \in N \) and \( a \in O \).

Due to Gale and Shapley (1962), we know that \( DA \) is stable in the conventional setting. The following proposition shows that separable mechanisms, in particular \( DA \), are claimwise stable.

**Proposition 2.** Separable mechanisms, in particular \( DA \), are claimwise stable.

*Proof.* Let \( \psi \) be a separable mechanism and consider a problem \( (R, \Delta) \) where \( \text{supp}(\Delta) = \bigcup_{k=1}^{n} \{\succ^k\} \). For ease of notation, we suppress the dependency of mechanisms' outcomes on the problem instance \( (R, \Delta) \). As \( \psi \) is separable, we can write \( \psi = \sum_{k=1}^{n} \Delta(\succ^k) \psi^k \) where each \( \psi^k \) is a deterministic stable matching with respect to priority order \( \succ^k \). This implies that, at each matching \( \psi^k \), either an agent is the top priority one in his matched object or all other agents having higher priority than him are matched with their preferred objects. This basically implies that, for any agents \( i, j \) and object \( a \),

\[
\psi_{i,a} \leq Pr_{\Delta}(i \succ_a j) + \sum_{c \in SU(R_j,a)} \sigma_{j,c},
\]

showing that \( \psi \) is claimwise stable.
A natural desideratum is to have a mechanism that is claimwise stable and ordinally efficient. It is well known that the usual stability and efficiency are incompatible even in the conventional deterministic setting (Balinski and Sönmez (1999)). On the other hand, due to Proposition 1, we know that any claimwise stable matching can be written as a lottery over stable matchings. These results imply that claimwise stability and ordinal efficiency are incompatible.

**Corollary 1.** Suppose $|N| \geq 3$. Then there is no claimwise stable mechanism that is ordinally efficient.

Given the above negative result, we look for mechanisms that are claimwise stable and not ordinally dominated by another claimwise stable rule. We say that a matching $\sigma$ is **constrained ordinally efficient** if no other claimwise stable matching ordinally dominates it. Mechanism $\psi$ is constrained ordinally efficient if $\psi(R, \Delta)$ is constrained ordinally efficient at every problem $(R, \Delta)$.

It is well known that $DA$ dominates (in terms of efficiency) any other stable rule in the conventional deterministic domain (Gale and Shapley (1962)). In sharp contrast to this fact, the following result demonstrates that it is not even constrained ordinally efficient in the current random environment.

**Proposition 3.** Suppose $|N| \geq 4$. Then $DA$ is not constrained ordinally efficient.

**Proof.** Let $N = \{i, j, k, z\}$, $O = \{a, b, c, d\}$, and the preference profile of agents be as follows:

\[
R_i : a, b, c, d; R_j : d, b, a, c; R_k : b, a, c, d; R_z : d, c, a, b. \quad \quad \Box
\]

---

22Recall our initial supposition that $|N| = |O|$.

23Hence, it also dominates any other separable rule in our environment.

24Similarly, in the deterministic and coarse priority domains, Erdil and Ergin (2008) show that applying $DA$ after breaking the ties is dominated by another stable matching. This result does not imply Proposition 3 here. The reason is that any Pareto superior stable matching to $DA$ in their setting is not necessarily claimwise stable in the corresponding random priority profile. On the other hand, as $DA$ does not satisfy the stability notions of Kesten and Üner (2012), they do not look at its efficiency properties.

25The earlier an object comes, the more it is preferred. For instance, agent $i$ prefers object $a$ to object $b$, and so on and so forth.
The priority order profile of objects $\Delta$ is such that $supp(\Delta)$ consists of the following four deterministic profiles:

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Let the probability of each of the above deterministic profiles in $supp(\Delta)$ be $1/4$. Then, the $DA$ outcome is given below:

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<td>$i$</td>
<td>$3/4$</td>
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<td>$j$</td>
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<td>$k$</td>
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Now, consider the following matching $\sigma$:

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<td>$z$</td>
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</table>

The reader can easily verify that $\sigma$ is claimwise stable and ordinally dominates $DA(R, \Delta)$. Therefore, $DA$ is not constrained ordinally efficient.

---

Note that even though we do not rule out correlated priorities, they are independent in this example.
**Remark 1.** DA dominates any other stable mechanism in the conventional deterministic domain. In the current model, nevertheless, it turns out to be not even constrained ordinally efficient under a natural generalization of stability. What is the intuition behind this sharp difference? DA outcome at any deterministic priority order in the support of $\Delta$ is independent of those at other deterministic priority orders. However, claimwise stability constraints relax as agents acquire shares of their preferred objects (because of the second term on the right-hand side of the claimwise stability constraint; see Definition 1). The lack of this updating process in the course of DA makes it not even constrained ordinally efficient.

In the rest of the paper, we introduce a new mechanism built on the PS mechanism of Bogomolnaia and Moulin (2001), which will turn out to be both claimwise stable and constrained ordinally efficient. We then compare DA with our new mechanism in terms of strategic and fairness properties.

First, we describe the PS mechanism of Bogomolnaia and Moulin (2001). Over the unit time interval, each agent continuously eats available objects at a speed of one in the decreasing order of his preference. The eaten shares by an agent by time $t = 1$ turn out to be his PS assignment. Bogomolnaia and Moulin (2001) show that, in contrast to the well-known random priority mechanism, PS is ordinally efficient and satisfies some other desirable properties. Since then, the PS mechanism has received much attention in the literature.

The PS mechanism has been introduced for priority-free object allocation problems. It does not take care of claimwise stability constraints; hence, it is not claimwise stable.\(^{27}\) Below, we modify PS to accommodate claimwise stability constraints.

\(^{27}\)For instance, $N = \{i, j\}$ and $O = \{a, b\}$ with $R_i = R_j : a, b$. Let the priorities be deterministic such that agent $i$ has higher priority than agent $j$ for both objects. Then, $PS_{i,a} = PS_{i,b} = PS_{j,a} = PS_{j,b} = 1/2$, which is not claimwise stable.
3.2 Constrained Probabilistic Serial Mechanism

Similar to PS, in the course of CPS, agents continuously acquire shares of objects at a speed of one in decreasing order of their preferences. The only difference between the mechanisms is the rule governing when agents stop eating objects. In the course of CPS, an agent stops eating an object when either relevant claimwise stability constraints start binding or the object is totally exhausted (whichever occurs first). The algorithm moves to a new step whenever an agent stops eating his current object and terminates when all objects are eaten away.

Before giving the formal definition of the algorithm, we illustrate how it works on a simple example. Let \( N = \{i, j, k\} \), \( O = \{a, b, c\} \), and \( R_i : a, b, c; R_j : b, a, c; \) and \( R_k : b, c, a \). Let \( \text{supp}(\Delta) \) consist of the following deterministic priority orders:

\[
\begin{array}{ccc|ccc}
\succ & \succ' \\
 a & b & c & a & b & c \\
 j & i & j & i & i & k \\
 i & k & i & j & j & i \\
 k & j & k & k & k & j \\
\end{array}
\]

Let \( \Delta(\succ) = 3/4 \) and \( \Delta(\succ') = 1/4 \).

**Step 1.** Each agent first attempts to eat his favorite object. Agents \( j, k \) both desire to eat object \( b \), and as \( Pr_\Delta(k \succ_b j) = 3/4 \), agent \( j \) is allowed to eat at most \( 1/4 \) of object \( b \). As agent \( i \) is the only one attempting to eat object \( a \), no constraint is imposed on him in the current round. This step, hence, terminates at \( t^1 = 1/4 \). By the end of it, agents have eaten \( 1/4 \) of their respective objects, and agent \( j \) stops eating object \( b \).

**Step 2.** Agent \( j \) now attempts to eat his second-choice object \( a \). Given that \( Pr_\Delta(j \succ_a i) = 3/4 \) and agent \( j \) has already eaten \( 1/4 \) of his more preferred object in the previous step, agent \( i \) is allowed to eat at most \( 1/2 \) of object \( a \) (by claimwise stability definition). As he has already eaten \( 1/4 \) of object \( a \), this step terminates at \( t^2 = 1/2 \), and agent \( i \) stops eating object \( a \).
**Step 3.** Agent $i$ now attempts to eat object $b$. Even though $Pr_\Delta(i \succ_b k) = 1$, as agent $i$ eats $1/2$ of object $a$, agent $k$ is allowed to eat at most $1/2$ of object $b$. On the other hand, since he has already eaten $1/2$ of object $b$, this step terminates by $t^3 = 1/2$, and agent $k$ stops eating object $b$. No agent indeed eats any positive amount within this step.

**Step 4.** Agent $k$ now attempts to eat object $c$. As $1/4$ of objects $a$ and $b$ are left from the previous rounds, this step terminates at time $t^4 = 3/4$. By the end of this step, agents have eaten an additional $1/4$ of their respective objects, and objects $a$ and $b$ are totally exhausted.

**Step 5.** All agents attempt to eat object $c$, and they all continue to eat until it is totally exhausted.\(^{28}\) This step, hence, terminates at $t^5 = 1$, and by the end of it, all objects have been eaten away. Below is the CPS outcome:

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<tr>
<td>$k$</td>
<td>$0$</td>
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Now, we give the formal definition of the algorithm. Each agent $i$ is endowed with a unit eating speed function $w_i$, i.e., for any $t \in [0, 1]$, $w_i(t) = 1$, where $w_i(t)$ is the speed at which agent $i$ consumes an object at time $t$.

Given a matching $\sigma$, $A = (A_i)_{i \in N}$ where $A_i \subseteq O$, a priority order profile $\Delta$, below we define $\Theta^a_i$ for each agent $i \in N$ and each object $a \in O$:

$$\Theta^a_i(\sigma, \Delta, A) = \min \{Pr_\Delta(i \succ_a j) \sum_{c \in SU(R_ja)} \sigma_{j,c} - \sigma_{i,a} : \forall j \in N \setminus \{i\} \text{ such that } aR_j b \forall b \in A_j\}.\quad (\Theta^a_i(\sigma, \Delta, A) = 1 \text{ if the above set is empty}).$$

\(^{28}\)This does not contradict claimwise stability as each agent has already eaten at least $1/2$ of his more preferred object.
\[ \Theta^a_i \] will keep track of how much more agent \( i \) can eat object \( a \) without violating claimwise stability constraints within each step of the algorithm.

We define \( M(a, A, N) = \{ i \in N : aR_i b \forall b \in A_i \} \). Let \( \text{top}(i, A_i) \) be the favorite object of agent \( i \) in \( A_i \). Let \( A^0 = (A_i^0)_{i \in N} \) where \( A_i^0 = O \) for each \( i \in N \), \( c^0 = 0 \), and \( \sigma^0 = [0] \) (matrix of zeros). Suppose that \( A^{s-1}, c^{s-1}, \) and \( \sigma^{s-1} \) are already defined. Then, for any \( a \in \bigcup_{i \in N} A_i^{s-1} \):

\[
y^s(a) = \min \{ y : \sum_{i \in M(a, A^{s-1}, N)} \int_y \int_{c^{s-1}} w_i(t)dt + \sum_{i \in N} \sigma_i^{s-1} = 1 \}
\]

\( (y^s(a) = +\infty, \text{if } M(a, A^{s-1}, N) = \emptyset) \).

Define now,

\[
y^s = \min \{ y^s(a) : \forall a \in \bigcup_{i \in N} A_i^{s-1} \} \quad \theta^s = \min \{ \Theta_i^{\text{top}(i, A_i^{s-1})}(\sigma^{s-1}, \Delta, A_i^{s-1}) : \forall i \in N \}
\]

\[ c^s = \min \{ y^s, \theta^s \} \]

\[ E = \{ a \in \bigcup_{i \in N} A_i^{s-1} : c^s = y^s(a) \}. \]

\[ A_i^s = \begin{cases} A_i^{s-1} \setminus E \cup \text{top}(i, A_i^{s-1}) & \text{if } c^s = \Theta_i^{\text{top}(i, A_i^{s-1})}(\sigma^{s-1}, \Delta, A_i^{s-1}) \\
A_i^{s-1} \setminus E & \text{otherwise} \end{cases} \]

For each \( i \in N \):

\[
\sigma_i^{s,a} = \begin{cases} \sigma_i^{s-1} + \int_{c^{s-1}}^c w_i(t)dt & \text{if } a = \text{top}(i, A_i^{s-1}) \\
\sigma_i^{s-1} & \text{otherwise} \end{cases}
\]

As the algorithm moves to a next step when an agent stops eating an object and everything (i.e., both agents and objects) is finite, the process terminates in finite steps. The generated assignment \( \sigma^s \) in step \( s \), in which all the objects are exhausted realizes as the outcome. We write \( CPS(R, \Delta) \) for the final outcome. We call the mechanism giving \( CPS(R, \Delta) \) for every problem \((R, \Delta) \) “Constrained Probabilistic Serial” mechanism and denote it by \( CPS \).

Budish et al. (2013) introduce the generalized probabilistic serial mechanism accommodating multi-unit allocations and various real-life constraints such as group-specific quotas in the school
choice setting or curriculum constraints in course allocations. Their setting is priority-free as that of Bogomolnaia and Moulin (2001), and their constraints basically specify the least and highest amounts of an object that can be assigned to a group of agents collectively. The generalized probabilistic serial mechanism allows agents to eat objects as long as such constraints permit. While our above mechanism and theirs are constructed in a methodologically similar way, they are independent of each other as the constraint structures each of them accommodates are different.

In the rest of the paper, we compare DA and CPS in terms of efficiency, strategic and fairness properties. The following lemma, which will be used in the proof of Theorem 1, is of interest on its own.

**Lemma 1.** \(CPS(R,\succ) = DA(R,\succ)\) at every \((R,\succ) \in \mathcal{R}^{|N|} \times \zeta\).

**Proof.** See Appendix.

In words, it says that CPS coincides with DA in the conventional deterministic priorities domain.

**Theorem 1.** CPS is claimwise stable and constrained ordinally efficient.

**Proof.** See Appendix.

With the above result, we obtain the first sharp difference between CPS and DA: while CPS is constrained ordinally efficient, DA is not.

### 3.3 Strategic Properties

**Definition 2.** A mechanism \(\psi\) is strategy-proof if, at every problem instance \((R,\Delta)\) and for any \(i \in N\) and \(R'_i \in \mathcal{R}\), either \(\psi_i(R,\Delta)\) ordinally dominates \(\psi_i(R'_i, R_{-i}, \Delta)\) or \(\psi_i(R,\Delta) = \psi_i(R'_i, R_{-i}, \Delta)\).\(^{29}\)

In words, under a strategy-proof mechanism, no agent can ever benefit by misreporting his preference regardless of his cardinal utilities. While Bogomolnaia and Moulin (2001) show that PS is not strategy-proof, they demonstrate that its weaker version below is satisfied.

\(^{29}\)\(R_{-i}\) stands for the preference profile of agents except that of agent \(i\).
Definition 3. A mechanism $\psi$ is weakly strategy-proof if, at every problem instance $(R, \Delta)$ and for any $i \in N$ and $R'_i \in R$, $\psi_i(R, \Delta)$ is not ordinally dominated by $\psi_i(R'_i, R_{-i}, \Delta)$.

In contrast to strategy-proofness, the weaker version allows beneficial misreporting for some (but not all) cardinal utilities representing the agents’ ordinal preferences.

Due to Dubins and Freedman (1981) and Roth (1982), we know that DA is strategy-proof in the conventional deterministic setting.\(^{30}\) Given this result, it is straightforward to see that DA continues to be strategy-proof in the current richer environment.

Proposition 4. DA is strategy-proof.

In spite of its robust strategic properties, the main disadvantage of DA is the lack of constrained efficiency. On the other hand, the result below reveals that claimwise stability and constrained ordinal efficiency are incompatible even with weak strategy-proofness. Hence, rather than being a problem of DA, there is a general tension between these three properties. For the proof of the following result, we follow the same analytical approach as other well-known similar impossibility results in the literature. Namely, we provide a problem instance $(R, \Delta)$ at which no claimwise stable and constrained ordinally efficient rule is weakly strategy-proof.\(^{31}\)

Theorem 2. Suppose $|N| \geq 4$. Then there is no claimwise stable mechanism that is constrained ordinally efficient and weakly strategy-proof.

Proof. See Appendix.

What is the intuition behind the above impossibility result? In order for a mechanism to achieve constrained ordinal efficiency, it has to update claimwise stability constraints. Yet, as they depend on the preferences of agents, that necessary updating process for achieving constrained efficiency enables agents to manipulate these constraints in favor of themselves through misreporting their preferences.

As CPS is claimwise stable and constrained ordinally efficient, Theorem 2 immediately implies that CPS is not weakly strategy-proof.

\(^{30}\)Yet this result does not carry over to the two-sided matching context (Roth (1982)).

\(^{31}\)In Footnote 36, we also mention a general form of $\Delta$ for which the impossibility result holds.
Corollary 2. Suppose $|N| \geq 4$. Then CPS is not weakly strategy-proof.

3.4 Further Fairness Properties

We show that both CPS and DA are claimwise stable, which is a desirable property from the fairness point of view. In this section, we investigate further desirable fairness properties of the mechanisms.

Definition 4.

(i) A mechanism $\psi$ is envy-free if, at every problem instance $(R, \Delta)$ and for every pair of agents $i, j$, either $\psi_i(R, \Delta)$ ordinally dominates $\psi_j(R, \Delta)$ with respect to $R_i$ or $\psi_i(R, \Delta) = \psi_j(R, \Delta)$.

(ii) A mechanism $\psi$ satisfies equal-treatment of equals if, at every problem instance $(R, \Delta)$ and for every pair of agents $i, j$ such that $R_i = R_j$, $\psi_i(R, \Delta) = \psi_j(R, \Delta)$.

First, it is easy to see that both CPS and DA do not satisfy either of the above properties. This negative result is indeed very expected as agents are prioritized in our setting. In the following definition, we modify the above properties in a natural way to adopt them to our setting.

Definition 5.

(I) A mechanism $\psi$ satisfies limited envy-freeness if, for any pair of agents $i, j$, $\psi_i(R, \Delta)$ is not ordinally dominated by $\psi_j(R, \Delta)$ with respect to $R_i$ at every problem $(R, \Delta)$ such that (i) $\Pr_{\Delta}(i \succ a j) \geq 1/2$ and (ii) $\Pr_{\Delta}(i \succ a k) \leq \Pr_{\Delta}(j \succ a k)$ for each $a \in O$ and $k \in N$.

(II) A mechanism $\psi$ satisfies limited equal-treatment of equals if, for any pair of agents $i, j$, $\psi_i(R, \Delta) = \psi_j(R, \Delta)$ at every problem $(R, \Delta)$ such that (i) $R_i = R_j$, (ii) $\Pr_{\Delta}(i \succ a j) = 1/2$ and (iii) $\Pr_{\Delta}(i \succ a k) = \Pr_{\Delta}(j \succ a k)$ for each $a \in O$ and $k \in N$.

Note that the above notions are far weaker than the previous ones. They not only depend on the priorities but also require no agent’s assignment to be ordinally dominated by someone else’s assignment.

Proposition 5. Suppose $|N| \geq 3$. Then

(i) neither CPS nor DA satisfies limited envy-freeness;

(ii) while CPS satisfies limited equal-treatment of equals, DA does not.
Proof.

(i) First, we show that CPS does not satisfy limited envy-freeness. Let $N = \{i, j, k\}$ and $O = \{a, b, c\}$. The preference profile of agents is as follow:

$$
R_i : a, b, c; R_j : b, a, c; R_k : a, c, b.
$$

Let $Pr_{\Delta}(s \succ^c s') = 1/2$ for any $s, s' \in N$, and the priorities of the other objects be as follows:

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Let $\Delta(\succ) = 1/8$, $\Delta(\succ') = 3/8$, and $\Delta(\succ'') = 1/2$. As we can see, $Pr_{\Delta}(i \succ_d j) = 1/2$, and $Pr_{\Delta}(i \succ_d k) = Pr_{\Delta}(j \succ_d k)$ for all $d \in O$. Therefore, the conditions of limited envy-freeness hold.

Then, $CPS(R, \Delta)$ is given below:

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<td>$i$</td>
<td>$1/8$</td>
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<tr>
<td>$j$</td>
<td>$1/8$</td>
<td>$9/16$</td>
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<tr>
<td>$k$</td>
<td>$3/4$</td>
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As we can see from above, $CPS_j(R, \Delta)$ ordinally dominates $CPS_i(R, \Delta)$ with respect to $R_i$. Therefore, CPS does not satisfy limited envy-freeness.

Next, we provide a problem instance at which DA does not satisfy both limited envy-freeness and limited equal-treatment of equals. Let $N = \{i, j, k\}$ and $O = \{a, b, c\}$. Assume that $R_i = R_j : a, b, c$; and $R_k : b, a, c$. The priority profiles in $supp(\Delta)$ are as follows:

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Therefore, CPS does not satisfy limited envy-freeness.
Let $\Delta(\succ) = \Delta(\succ') = 1/2$. Note that agents $i$ and $j$ satisfy the assumptions of limited envy-freeness and limited equal-treatment of equals. Then, $DA_{i,a}(R, \Delta) = DA_{i,b}(R, \Delta) = 1/2$ and $DA_{j,a} = DA_{j,c} = 1/2$. One can easily verify that $DA_i$ ordinally dominates $DA_j$, thereby, $DA$ satisfies neither limited envy-freeness nor limited equal-treatment of equals.

(ii) As we have already shown that $DA$ does not satisfy limited equal-treatment of equals, in this part, we only need to demonstrate that $CPS$ does satisfy it. Given a problem instance $(R, \Delta)$, assume that there are agents $i, j$ such that (i) $R_i = R_j$, (ii) $Pr_{\Delta}(i \succ_j a) = 1/2$ and (iii) $Pr_{\Delta}(i \succ_k a) = Pr_{\Delta}(j \succ_k a)$ for each $a \in O$ and $k \in N$.

Let us enumerate the objects in $O = \{a_1, a_2, ..., a_k\}$ such that $a_1 P_i a_2$, .. and so on. Then, by the construction of $CPS$, both agents $i$ and $j$ first attempt to eat object $a_1$. Given that they are endowed with the unit eating speed functions and our supposition, it is clear that $CPS(R, \Delta)_{i,a_1} = CPS(R, \Delta)_{j,a_2}$. This means that they attempt to eat their second favorite object at the same time. Then, by the same logic, we have $CPS(R, \Delta)_{i,a_2} = CPS(R, \Delta)_{j,a_2}$. The same argument for all other remaining objects proves that $CPS_i(R, \Delta) = CPS_j(R, \Delta)$.

\[\square\]

### 3.5 A Characterization of CPS

In this section, we provide an axiomatic characterization of $CPS$ in a more general setting than what we started with. Here, we assume the existence of the null object, denoted by $\emptyset$, representing being unassigned. Agents’ preferences are now over $O \cup \emptyset$, and the null object is not scarce, i.e., $|q_\emptyset| = \infty$. The extension of $CPS$ to the presence of the null object is straightforward.

As we already mentioned before, there has been a recent surge in the characterization of the celebrated $PS$ mechanism of Bogomolnaia and Moulin (2001): Bogomolnaia and Heo (2012) and Hashimoto et al. (2012) independently provide different characterizations of $PS$. In particular, Bogomolnaia and Heo (2012) introduce the consumption process representation of random assignments (described before in the paper), which has proven very useful in the characterization of $PS$.

In obtaining our axiomatization, we invoke the consumption process representation as well. Even though we describe it earlier in the paper, for the sake of completeness, we first outline it
here again. Each matching can be considered as a consumption process where, over the unit time interval, agents continuously acquire shares of objects in decreasing order of their preferences at a speed of one. In the course of this process, agents’ switching times between objects generate all different matchings. For instance, in PS, agents continue to consume their preferred objects until they are exhausted. On the other hand, in CPS, they do so until either the corresponding claimwise stability constraints bind or the objects are totally exhausted (whichever occurs first).

Before moving to the axioms, we define the steps of the consumption process and put them in order. In the course of the consumption process, agents first attempt to eat their best objects, and it constitutes the initial step. Then whenever an agent stops consuming his current object and attempts to eat his best remaining object, the process moves to the next step and so on and so forth. One caveat in the ordering of steps is that there might be a multiple of them happening at the same time. Indeed, an agent might apply to different objects successively in decreasing order of his preference at the same time instant.

In order to overcome the above problem, we add another time dimension to the consumption process. It is ticking away with steps as opposed to the usual time in the unit interval [0, 1], which ticks away as agents acquire the probability shares of objects. Formally, for a given mechanism ψ and a problem instance (R, Δ), let S be the set of steps occurring in the course of the consumption process ψ(R, Δ). Below, we suppress the dependency of S and ψ on the problem instance. Let us consider the set of discrete artificial time instances T = {1, 2, ..., n}, where |T| is equal to the number of steps. We then assign an artificial time index to each step. Formally, let ι : S → T be a bijective function, where ι(s) is the artificial time index assigned to step s. As agents consume objects in decreasing order of their preferences, we assume that ι(s) < ι(s′) for any steps s, s′ in which the same agent attempts to eat different objects and prefers the one corresponding to step s. Apart from that, there is no other restriction on ι. Moreover, we write s(t) for the usual time instance by which step s starts. We then define an order ≺ on S as follows:

For any given s, s′ ∈ S, s ≺ s′ if either s(t) < s′(t) or s(t) = s′(t) & ι(s) < ι(s′).

In words, if s ≺ s′, then we say that step s occurs before step s′ in the course of ψ. By introducing ι, we order the steps happening at the same time and let them happen successively
according to the order $\prec$. Note that the outcome of a consumption process is independent of the ordering of steps happening at the same time. It is easy to see that $\prec$ is a strict, complete, and transitive binary relation on $S$.

For a given matching $\sigma$, agent $i$, and object $a$, Hashimoto et al. (2012) define $F(R_i, a, \sigma_i) = \sum_{c \in U(R_i, a)} \sigma_{i,c}$. They call it “agent $i$’s surplus at object $a$ under $\sigma_i$”. Then they introduce the following axiom.

**Ordinal Fairness** (Hashimoto et al. (2012)): A mechanism $\psi$ is ordinally fair, if, at every problem instance $R$,\(^{32}\) there are no pair of agents $i, j$ and object $a$ such that $F(R_i, a, \psi_i) < F(R_j, a, \psi_j)$ and $\psi_{j,a} > 0$.

A mechanism $\psi$ is *non-wasteful* if, for any problem $(R, \Delta)$ and any agent $i$, $aP_i b$ for some $b$ and $\psi(R, \Delta)_{i,b} > 0$, then $\sum_{j \in N} \psi(R, \Delta)_{j,a} = 1$.

Hashimoto et al. (2012) show that $PS$ is the only mechanism that is non-wasteful and ordinally fair. In our setting, on the other hand, it is easy to see that claimwise stability and ordinal fairness are incompatible. This is very expected as agents are not equal in the current environment (they are prioritized), and ordinal fairness is a kind of equality property. In what follows, we adopt ordinal fairness to our priority-based setting.

**Definition 6.** Mechanism $\psi$ is binding if, at any problem $(R, \Delta)$ and for any pair of agents $i, j$ and object $a$, $F(R_i, a, \psi_i) < F(R_j, a, \psi_j)$ and $\psi_{j,a} > 0$, then there exists an agent $k$ such that the following hold:

(i) $\psi(R, \Delta)_{i,a} = Pr_\Delta(i \succ a k) + \sum_{c \in SU(R_k, a)} \psi(R, \Delta)_{k,c}$.

(ii) The steps in the consumption process $\psi(R, \Delta)$ can be ordered in a way that if $s$ and $s'$ be the steps in which agent $k$ applies to object $a$ and agent $i$ stops eating it, respectively, then $s \prec s'$.

For the justification of the above property, we first observe that, under any consumption process $\psi$, $F(R_i, a, \psi_i) < F(R_j, a, \psi_j)$ and $\psi_{j,a} > 0$ imply that agent $i$ stops eating object $a$ before it is totally exhausted. Under a binding mechanism, it happens only if some agent $k$ (agents $k$ and $j$ might be the same) applies to object $a$ in a step before agent $i$ stops eating it, and the claimwise stability constraint imposed on the latter by the former binds. That is, the above property requires

\(^{32}\)No priority in their setting.
any agent to continue eating his preferred object until someone else applies to it and prevents him from eating it through claimwise stability constraints. Therefore, we can say that agents are doing their bests in eating their preferred objects under binding mechanisms.

In order to understand the role of ordering of steps in the above axiom, consider a problem instance consisting of $N = \{i, j\}$ and $O = \{a, b\}$. The preference profile of agents is as follows: $R_i : a, b$; and $R_j : b, a$. The priority orders of objects are deterministic such that $j \succ_a i$ and $i \succ_b j$. Then consider the matching $\sigma$ under which $\sigma_{i, b} = 1$ and $\sigma_{j, a} = 1$. That is, agent $i$ is assigned to object $b$, and agent $j$ is assigned to object $a$. It is easy to see that $\sigma$ is claimwise stable. However, it is not binding. In order to see that, first observe $F(R_i, a, \sigma_i) < F(R_j, a, \sigma_j)$ and $\sigma_{j, a} > 0$; and, similarly, $F(R_j, b, \sigma_j) < F(R_i, b, \sigma_i)$ and $\sigma_{i, b} > 0$. Then in order for matching $\sigma$ to be binding, the two conditions in Definition 6 have to hold. First, it is easy to verify that claimwise stability constraints bind at matching $\sigma$. We, hence, need to be able to order the steps in the consumption process $\sigma$ in the above aforementioned way. Let $s_1$ and $s_2$ be the steps in which agents $i$ and $j$ stop eating their best objects and apply to their second best alternatives, respectively. Hence, it has to be that $s_1 \prec s_2$, yet, at the same time, $s_2 \prec s_1$. This contradicts the definition of $\prec$; therefore, $\sigma$ is not binding. This, in words, shows that agents are not doing their best in eating their preferred objects, i.e., either of them voluntarily stops eating his best object.

Before the characterization result, the following lemma, which will be used in the proof of Theorem 3, is of interest on its own.

**Lemma 2.** A constrained ordinally efficient matching is non-wasteful.

**Proof.** See Appendix.

**Theorem 3.** A mechanism $\psi$ is non-wasteful, claimwise stable, and binding if and only if it is CPS.

**Proof.** See Appendix.

**Remark 2.** If $|N| \geq |O|$ (this nests the absence of the null object case), we do not need non-wastefulness, and the characterization is given by claimwise stability and bindingness.

In the following examples, we show the independence of the axioms.
Example 1: Due to Hashimoto et al. (2012), we know that $PS$ is non-wasteful, ordinally fair, hence binding. Yet it is not claimwise stable whenever $|N| \geq 2$.

Example 2: The mechanism leaving every agent unassigned is claimwise stable and binding, yet wasteful.

Example 3: If $|N| \geq 2$ and $|O| \geq 2$, then a non-wasteful and claimwise stable mechanism is not necessarily binding. To see this, consider the example given right before Theorem 3 above. The matching instance given there is non-wasteful and claimwise stable. Yet it is not binding.

Remark 3. Erdil (2011) shows that $DA$ is wasteful. Moreover, it is not binding as well. This can be seen via the $DA$ outcome given in Proposition 3. There $F(R_j, b, DA_j) < F(R_i, b, DA_i)$, where $DA_i, b > 0$. It is easy to verify that the first condition in the axiom does not hold.

4 Extensions

4.1 Different Number of Objects and Agents and the Presence of the Null Object

We originally assume that $|N| = |O|$. Yet if $|N| > |O|$, then nothing will change in our analysis above; all the results carry over. On the other hand, if $|N| < |O|$ (note that this case nests the presence of the null object setting), then wastefulness will be an issue. That is, there might be claimwise stable matchings that are wasteful. For instance, the matching at which every agent is unassigned is claimwise stable. Another direct example is $DA$ that is claimwise stable, yet wasteful (Erdil (2011)). On the other hand, from the characterization part, we know that $CPS$ is non-wasteful in this more general case (we show it under the presence of the null object, yet it easily holds in the case of $|N| < |O|$ without the null object). As one could argue that wasteful objects might be interpreted as justified claims against the social planner, we might require non-wastefulness from claimwise stable matchings. In this case, $DA$ would not be even claimwise stable, while all the results regarding $CPS$ would carry over.
4.2 Multi-Quota Case

In our analysis, we assume that there is only one copy (unit quota) from each object (except the null object). Yet we pointed out that it is just a simplification assumption; our results hold for the multi-quota case as well.

In the multi-quota case, the claimwise stability definition is problematic. For example, consider a problem consisting of \( N = \{i, j\} \) and \( O = \{a\} \), where \( q_a = 2 \). Assume that \( R_i = R_j : a, \emptyset \); and \( Pr_\Delta(i \gg_a j) = 1 \). Then under any claimwise stable matching \( \sigma: \sigma_{j, a} = 0 \) even if \( \sigma_{i, a} = 1 \). This obviously does not make sense as agent \( i \) is already assigned to his favorite object with certainty; therefore, giving the other copy to agent \( j \) would not be unfair to agent \( i \) in any sense.

In order to overcome the above caveat, we consider each copy of an object as a different object endowed with the originally given priority order. For instance, if we have two copies of object \( a \), then we consider the second copy of it as a different object, denoted by \( a' \), and its priority order is the same as that of object \( a \). The next question in this construction is how we define preferences of agents over the artificially created set of objects. Given two artificial objects \( a' \) and \( b' \) and the original preference relation \( R_i \) of agent \( i \), we define \( R'_i \) as follows: \( a'R'_ib' \) iff \( aRibi \). We can define \( R'_i \) over the copies of the same objects in any way. Now, through this artificial construction, we can transform every multi-unit quota problem to a unit quota one and solve the latter problem instead of the former. Below, we see how our construction solves the problem in the above given instance.

Now, let us consider the problem as if there are two objects, say \( a \) and \( a' \), with \( q_a = q_{a'} = 1 \). For the preference profile of agents, construct new preferences \( R'_i = R'_j : a, a', \emptyset \). Moreover, \( Pr_\Delta(i \gg_{a'} j) = Pr_\Delta(i \gg_a j) = 1 \). Then the random assignment \( \sigma \), where \( \sigma_{i, a} = 1 \) and \( \sigma_{j, a'} = 1 \), is claimwise stable.\(^{33}\)

Given that we transform multi-quota problems to unit-quota ones, all of the results also apply to the multi-quota setting as well.

\(^{33}\)It is the efficient claimwise stable matching.
5 Conclusion and Future Research

This paper studies the object allocation problem with random priorities for the first time in the literature. We first introduce a fairness notion called claimwise stability. While the well-known DA mechanism turns out to be claimwise stable, it is dominated by another claimwise stable rule. Given this important shortcoming of DA, we adopt the PS mechanism of Bogomolnaia and Moulin (2001) to the current setting and introduce the constrained probabilistic serial mechanism. It is both claimwise stable and constrained ordinally efficient. Then the paper compares the agent-optimal stable and constrained probabilistic serial mechanisms in terms of strategic and fairness properties. Lastly, we provide a characterization of CPS. The following table summarizes our comparison between CPS and DA.

<table>
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<th>DA</th>
<th>CPS</th>
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<td>✓</td>
</tr>
<tr>
<td>Constrained Ordinal Efficiency</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>Non-wastefulness</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>Strategy-proofness</td>
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<td>×</td>
</tr>
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<td>Limited Envy-freeness</td>
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</tr>
<tr>
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<td>✓</td>
</tr>
<tr>
<td>Bindingness</td>
<td>×</td>
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</table>

While CPS achieves constrained ordinal efficiency and exhibits better fairness properties than DA, the converse is true with respect to strategic issues. CPS is built on PS, and there are other variants of PS performing well in different settings. This paper strengthens the idea that adopting PS to other environments might be a fruitful future research direction.

Appendix

Proof of Proposition 1. Let σ be a claimwise stable matching at problem (R, >). If it is deterministic, then it is easy to see that it is stable. Let us assume that it is random. Consider an agent i
and object $a$ such that $\sigma_{i,a} > 0$. Then, as $\sigma$ is claimwise stable, any agent $j$ who has higher priority than agent $i$ at object $a$ receives at least $\sigma_{i,a}$ share of his strictly preferred objects. This implies that matching $\sigma$ can be induced by a lottery over deterministic matchings under which whenever agent $i$ is matched with object $a$, then agent $j$ is matched with one of his preferred objects. As this is true for all agents and objects, we can write $\sigma$ as a lottery over stable matchings showing that it is ex-post stable.

Proof of Lemma 1. In the course of CPS, agents first apply to their respective favorite objects. All applying agents are rejected except the one with the highest priority. Recall that, according to the definition of CPS, no agent starts eating an object until there is no rejected one. Next, the rejected agents in the previous round apply to their second best objects. Similar to above, all agents are rejected except the one with the highest priority among the tentatively accepted one in the previous round and the ones who applied to the object in this step, and so on. When there is no remaining rejected agent, they consume their tentatively assigned objects (note that the assignment is deterministic). This wordy explanation of CPS easily shows that $CPS(R,>) = DA(R,>)$ for every problem $(R,>) \in \mathcal{R}^{|N|} \times \zeta$.

Proof of Theorem 1. We first need to show that, for any problem $(R,\Delta)$, $CPS(R,\Delta)$ is indeed a matching, i.e., $\sum_{i \in N} CPS(R,\Delta)_{i,a} = 1$ and $\sum_{a \in O} CPS(R,\Delta)_{i,a} = 1$ for all $a \in O$ and $i \in N$. For ease of notation, let $\sigma = CPS(R,\Delta)$. By the definition of CPS, agent $i$ continues to eat until $\sum_{a \in O} \sigma_{i,a} = 1$. On the other hand, as $|N| = |O|$ and $q_a = 1$ for every $a \in O$, it also implies that $\sum_{i \in N} \sigma_{i,a} = 1$ for every $a \in O$.

Now, we claim that $\sigma$ is claimwise stable. Assume for a contradiction that there exist agents $i, j$ and an object $a$ such that $\sigma_{i,a} > Pr_\Delta(i >_a j) + \sum_{c \in SU(R_j,a)} \sigma_{j,c}$. Let $s_0$ and $s_1$ denote the steps by which agent $i$ starts to consume object $a$ and stops eating it, respectively. Then, by our supposition, $Pr_\Delta(i >_a j) + \sum_{c \in SU(R_j,a)} \sigma_{j,c}^s - \sigma_{i,a}^s < 0$ (this is due to our supposition and the fact that $\sum_{c \in SU(R_j,a)} \sigma_{j,c}^s \leq \sum_{c \in SU(R_j,a)} \sigma_{j,c}$). Then, the construction of CPS implies that there exists a step $\tilde{s} < s_1$ such that $Pr_\Delta(i >_a j) + \sum_{c \in SU(R_j,a)} \sigma_{j,c}^{\tilde{s}} - \sigma_{i,a}^{\tilde{s}} = 0$. By the definition of $\Theta_i^s$, it
implies that agent $i$ can eat object $a$ till at most step $\tilde{s}$ in the course of CPS, which contradicts our supposition. Hence, CPS is claimwise stable.

Next, we claim that CPS is constrained ordinally efficient. Assume for a contradiction that there exists a claimwise stable matching $\phi$ ordinally dominating $\sigma$ (recall that $\sigma = CPS(R, \Delta)$). We first introduce some notations: Given a matching $\psi$, we define $F(R_i, a, \psi_i) = \sum c \in U(R_i, a) \psi_{i,c}$.

Let $\pi_1 = \min\{F(R_i, a, \phi_i) : \text{for all } (i, a) \in N \times O\}$, $\pi_k = \min\{F(R_i, a, \phi_i) : F(R_i, a, \phi_i) > \pi_{k-1} \text{ for all } (i, a) \in N \times O\}$. We also write $\Pi = \{F(R_i, a, \phi_i) : \text{for all } (i, a) \in N \times O\}$.

In what follows, we will show that $F(R_i, a, \phi_i) = F(R_i, a, \sigma_i)$ for all agent $i$ and object $a$, which would mean that $\phi = \psi$, and this will contradict our starting supposition that $\phi$ ordinally dominates $\sigma$. Below, we inductively prove that $F(R_i, a, \phi_i) = F(R_i, a, \sigma_i)$ for all agent-object pair $(i, a)$ such that $F(R_i, a, \psi_i) = \pi_k$. First, for $k = 1$, pick an agent-object pair $(i, a)$ such that $F(R_i, a, \phi_i) = \pi_1$.

Below are the two possible cases.

**Case 1** Let $F(R_i, a, \phi_i) = 0$. Then, as $\phi$ ordinally dominates $\sigma$, we have $F(R_i, c, \sigma_i) = 0$ for all $c \in U(R_i, a)$. In particularly, we have $F(R_i, a, \sigma_i) = 0$.

**Case 2** Let $F(R_i, a, \phi_i) > 0$. First, we prove our claim when object $a$ is the favorite object of agent $i$. By the definition of CPS, all agents first attempt to eat their favorite object until either claimwise stability constraints bind or the object is totally exhausted (whichever occurs first). If we write $a(j)$ for the favorite object of agent $j \in N$, then $F(R_j, a(j), \phi_j) \geq \pi_1$ (by the definition of $\pi_1$). Given that $\phi$ is claimwise stable, each agent $i \in N$ can eat his favorite object at least until time $t' = \pi_1$ in the course of CPS. This along with the fact that $F(R_i, a, \sigma_i) \leq F(R_i, a, \phi_i) = \pi_1$ shows that $F(R_i, a, \sigma_i) = \pi_1$.

Now, let us assume that $a$ is not the favorite object of agent $i$. Let us say that it is his second best object (the other cases would follow from the same argument). Then, from the above analysis, we know that $\sigma_{i,a(i)} = \phi_{i,a(i)}$, where $a(i)$ is the favorite object of agent $i$. This along with our supposition that $\phi$ ordinally dominates $\sigma$ implies that $F(R_i, a, \sigma_i) = F(R_i, a, \phi_i) = \pi_1$.

Let us assume that $F(R_i, a, \phi_i) = F(R_i, a, \sigma_i)$ for all agent-object pair $(i, a)$ such that $F(R_i, a, \phi_i) \leq \pi_{k-1}$.

Let us pick a pair $(i, a)$ such that $F(R_i, a, \phi_i) = \pi_k$. We want to show that $F(R_i, a, \sigma_i) = F(R_i, a, \phi_i)$. First, we assume that $\phi_{i,a} > 0$. This implies that $F(R_i, b, \phi_i) \leq \pi_{k-1}$ where $b$ is the
object just preferred to object $a$ by agent $i$. Let us write $F(R_i, b, \phi_i) = t$. By the definition of $CPS$, agent $i$ attempts to eat object $a$ by time $t$ (note that, by induction hypothesis, $F(R_i, b, \sigma_i) = t$). In what follows, we first show that agent $i$ does not stop eating object $a$ before time $t' = \pi_{k-1}$ in the course of $CPS$ (Note that, by construction, $t \leq \pi_{k-1}$. On the other hand, if $t = \pi_{k-1}$, then no need to show this part).

Let $N(a) = \{ j \in N : \phi_{j,a} > 0 \& F(R_j, a, \phi_j) \leq t' \}$. Then, as $\sum_{j \in N} \phi_{j,a} = 1$, in particularly, we have $\sum_{j \in N(a)} \phi_{j,a} \leq 1$. This along with the induction hypothesis implies that object $a$ is not exhausted before time $t'$ in the course of $CPS$.

Therefore, the only reason that might stop agent $i$ from eating object $a$ before time $t'$ in $CPS$ is claimwise stability constraints. Assume that he stops eating object $a$ before $t'$, i.e., claimwise stability constraints imposed on him for object $a$ bind before $t'$. This implies that there exists an agent $k$ such that $\sigma_{i,a} = Pr\Delta(i > a k) + \sum_{c \in SU(R_k,a)} \sigma_{k,c}$. By our supposition, $\sigma_{i,a} < t' - t$, where $t' = \pi_{k-1}$. This implies that $\sum_{c \in SU(R_k,a)} \sigma_{k,c} < \pi_{k-1}$. Then, we have two cases to consider.

**Case (i).** If $\sum_{c \in SU(R_k,a)} \phi_{k,c} \leq \pi_{k-1}$, then, by induction hypothesis, we have $\sum_{c \in SU(R_k,a)} \sigma_{k,c} = \sum_{c \in SU(R_k,a)} \phi_{k,c}$. In this case, however, as $\phi_{i,a} > \sigma_{i,a}$, $\phi$ could not have been claimwise stable, which would have contradicted our very first assumption.

**Case (ii).** Let us consider the case where $\sum_{c \in SU(R_k,a)} \phi_{k,c} > \pi_{k-1}$. Recall that $F(R_i, b, \phi_i) = F(R_i, b, \sigma_i) = t < \pi_k$, where $b$ is the object just preferred to object $a$ by agent $i$. Let $\pi_{k'} \in \Pi$ be such that it just comes after $t$. That is, $t < \pi_{k'}$ and $\pi_{k'} \leq \pi_h$ for all $\pi_h \in \Pi$ such that $\pi_h > t$.\(^{34}\) We first show that agent $i$ does not stop eating object $a$ in the course of $CPS$ before time $t' = \pi_{k'}$. The proof is very similar to the proof of the base step of the induction. In the course of $CPS$, an agent $j$ who stopped eating his current object by time $t$ will apply to his next best available object, say $z$. Note that if $\phi_{j,z} = 0$, then $F(R_j, z, \sigma_j) = F(R_j, z, \phi_j) = t$ (by induction hypothesis). Hence, for such an agent $j$, let us assume that $\phi_{j,z} > 0$. On the other hand, for all other agents, i.e, ones who did not stop eating their objects by $t$ will continue to eat their current objects. Then, in order to make analogy with the base step, we can consider the objects agents are eating by time $t$ as their best objects. Then, given that $\phi$ is claimwise stable, agents will continue to eat their best objects.

\(^{34}\)Note that $\pi_{k'}$ might be equal to $\pi_{k-1}$. 35
at least till $t' = \pi_{k'}$ (as claimwise stability constraints allow,\textsuperscript{35} and $\pi_{k'}$ comes just after $t$ by our construction). Hence, agent $i$ continues to eat object $a$ at least until $t' = \pi_{k'}$ in the course of $CPS$.

Now, let $\pi_{k''} \in \Pi$ come just after $\pi_{k'}$ (if $\pi_{k'} = \pi_{k-1}$, then no need to show this part). That is, $\pi_{k'} < \pi_{k''}$ and $\pi_{k''} \leq \pi_h$ for all $\pi_h \in \Pi$ such that $\pi_{k'} < \pi_h$. In this case, let $t = \pi_{k'}$ and $t' = \pi_{k''}$.

Then, by the symmetric argument as above, we can easily show that agent $i$ continues to eat object $a$ between $\pi_{k'}$ and $\pi_{k''}$ in the course of $CPS$. Then, if we continue in the same manner till time $t' = \pi_{k-1}$, we can easily prove that agent $i$ does not stop eating object $a$ at least till $t' = \pi_{k-1}$ in the course of $CPS$.

We now claim that agent $i$ continues to eat object $a$ between times $t' = \pi_{k-1}$ and $t'' = \pi_k$ as well. Notice that, by construction, $\pi_k$ is the element of $\Pi$ which is just greater than $\pi_{k-1}$. Then, this part also directly follows from the same argument as above. Hence, $F(R_i, a, \sigma_i) \geq \pi_k$. Given that $\pi_k = F(R_i, a, \phi_i) \geq F(R_i, a, \sigma_i)$, we have $F(R_i, a, \sigma_i) = \pi_k$.

In the above analysis, we assume that $\phi_{i,a} > 0$. Now, suppose that $\phi_{i,a} = 0$. As $F(R_i, a, \phi_i) = \pi_k$ and $\phi_{i,a} = 0$, there exists an object $b$ such that $F(R_i, b, \phi_i) = \pi_k$ and $\phi_{i,b} > 0$ (note that it implies that $bP_i a$). From above, we have $F(R_i, b, \sigma_i) = F(R_i, b, \phi_i) = \pi_k$. Then, by our supposition, $\pi_k = F(R_i, a, \phi_i) \geq F(R_i, a, \sigma_i)$. This along with $F(R_i, b, \sigma_i) = \pi_k \leq F(R_i, a, \sigma_i)$ demonstrates that $F(R_i, a, \sigma_i) = \pi_k$, which finishes the proof.

\[
\square
\]

Proof of Theorem 2. Consider a problem instance consisting of $N = \{i, j, k, z\}$ and $O = \{a, b, c, d\}$. The preference profile of agents is as follows:

$$R_i : b, a, c, d; \ R_j : b, a, d, c; \ R_k : b, d, a, c; \ R_z : c, b, d, a.$$  

Let the random priority profile $\Delta = (\Delta_a)_{a \in O}$ be such that (i) the priorities of objects $a$, $c$, and $d$ are deterministic such that agents $j$, $i$, and $k$ are the top priority students at objects $a$, $c$, and $d,$ respectively and (ii) the support of $\Delta_b$ consists of the following deterministic priorities $\succ_b$ and $\succ'_b$ with $\Delta(\succ_b) = 1/6$ and $\Delta(\succ'_b) = 5/6. \text{\textsuperscript{36}}$

\textsuperscript{35}As $\phi$ is claimwise stable.  
\textsuperscript{36}It is easy to follow from the proof that we indeed only need (i) agents $j$, $i$, and $k$ are the top priority students at objects $a$, $c$, and $d$, respectively, and (ii) $0 < \Delta(\succ_b) \leq 1/6$. That is, if such a structure exists in $\Delta$, then the impossibility result holds.  

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Let $\psi$ be a constrained ordinally efficient and claimwise stable mechanism.

Then, we claim that, under the true preference profile $R$, $\psi(R, \Delta)_{i,a} = 0$, $\psi(R, \Delta)_{i,b} = 1/6$ and $\psi(R, \Delta)_{i,c} = 5/6$. For ease of notation, we just write $\psi$ to denote the outcome.

First, given that object $b$ is the best alternative of agent $i$ and $Pr_{\Delta}(j \succ b i) = 1$, due to claimwise stability, we have $\psi(R, \Delta)_{j,b} = 0$. Then, given this along with object $a$ being the second choice of agent $j$, by claimwise stability, we have $\psi_{i,a} = 0$ (as $Pr_{\Delta}(j \succ a i) = 1$). Indeed, $\psi_{j,a} = 1$ as he is the highest priority agent with certainty at object $a$ and $\psi_{j,b} = 0$. On the other hand, $\psi_{i,b} \leq 1/6$ due the claimwise stability of $\psi$ and the facts that $Pr_{\Delta}(k \succ b i) = 5/6$ and object $b$ is the favorite one of agent $k$.

Now, assume for a contradiction that $\psi_{i,b} < 1/6$. As $Pr_{\Delta}(i \succ c z) = 1$, it implies that $\psi_{z,c} < 1/6$. On the other hand, since agent $z$ is the top agent at the priority order of object $b$ and $\psi$ is claimwise stable, we have $\psi_{k,b} < 1/6$. These along with $\psi_{j,b} = 0$ implies that $\psi_{z,b} > 0$. On the other hand, as $\psi_{i,a} = 0$, $\psi_{i,b} < 1/6$, and agent $i$ is the top priority one in object $c$, we have $\psi_{i,c} > 0$. This allocation, however, can not be constrained ordinally efficient as agent $i$ can receive arbitrarily small amount of object $b$ from agent $z$ in return of the same amount of object $c$. As $\psi_{i,b} < 1/6$, this trade would not violate claimwise stability (it would have violated if $\psi_{i,b} = 1/6$ due to agent $k$), and it would improve the welfare of agents $i$ and $z$. Hence, $\psi_{i,b} = 1/6$.

By claimwise stability, $\psi_{i,b} = 1/6$ and $\psi_{i,a} = 0$ imply that $\psi_{z,c} = 1/6$. We now claim that $\psi_{i,d} = 0$. If it were not be the case, then we would have had $\psi_{k,c} > 0$. In this case, yet, agent $i$ could give positive amount of object $d$ to agent $k$ in return of the same amount of object $c$. While this trade would improve their welfare, it would not violate claimwise stability. This contradicts the constrained ordinal efficiency of $\psi$, hence, $\psi_{i,d} = 0$. Therefore, $\psi_{i,b} = 1/6$ and $\psi_{i,c} = 5/6$.

Next, consider the false preference relation $R'_i : a, b, c, d$. Let us write $\psi'$ for the outcome at $(R'_i, R_{-i}, \Delta)$. Then, we claim that $\psi'_{i,a} = 1/6$, $\psi'_{i,b} = 1/6$, and $\psi'_{i,c} = 2/3$. Due to claimwise stability
of $\psi$ and $Pr_\Delta(k \succ_i b) = 5/6$ along with the fact that object $b$ is the favorite one of agent $k$, we have $\psi'_{i,b} \leq 1/6$. In this case, $\psi'_{i,a}$ might be positive if agent $j$ obtains object $b$ with some positive probability (this might be possible here as $b$ is not the top object of agent $i$ with respect to $R'_i$, and $Pr_\Delta(j \succ_k b) = 1/6$). Moreover, as $Pr_\Delta(k \succ_j b) = 5/6$, we have $\psi'_{j,b} \leq 1/6$. This implies that $\psi'_{i,a} \leq 1/6$.

Now, assume for a contradiction that $\psi'_{i,a} < 1/6$ and $\psi'_{i,b} = 1/6$. Due to the claimwise stability of $\psi$, they imply that $\psi'_{z,c} \geq 1/6$. On the other hand, due to claimwise stability and $\psi'_{i,a} < 1/6$, we have $\psi'_{j,b} < 1/6$ (note that $Pr_\Delta(i \succ_j b) = 1$). By claimwise stability, we have $\psi'_{j,a} > 0$. These altogether show that agent $i$ can get an arbitrarily small amount of object $a$ from agent $j$ in return of the same amount of object $b$, which would improve their welfare. This would not violate claimwise stability, as (i) $\psi'_{j,b} < 1/6$, (ii) $\psi'_{z,c} \geq 1/6$, and (iii) $Pr_\Delta(j \succ_k b) = 1/6$. This, however, contradicts our supposition.

Next, assume that $\psi'_{i,a} = 1/6$ and $\psi'_{i,b} < 1/6$. As $\psi'_{i,a} = 1/6$, we have $\psi'_{j,b} = 1/6$. Then, by claimwise stability of $\psi$, we have $\psi'_{z,c} < 1/3$. This shows that $\psi'_{k,b} < 1/3$. Hence, $\psi'_{j,b} > 0$. On the other hand, as $\psi'_{i,a} + \psi'_{i,b} < 1/3$, by claimwise stability of $\psi$, $\psi'_{t,c} < 1/3$ for each $l \in \{j, k, z\}$. Hence, $\psi'_{t,c} > 0$. In this case, agent $i$ can receive an arbitrarily small amount of object $b$ from agent $z$ in return of the same amount of object $c$, which would improve the welfare of agents $i$ and $z$. Moreover, this would not violate claimwise stability, as, by our supposition, $\psi'_{i,b} < 1/6$.

The last case is $\psi'_{i,a} < 1/6$ and $\psi'_{i,b} < 1/6$. By claimwise stability, $\psi'_{z,c} < 1/3$. Then, by the same reasoning as above, $\psi'_{t,c} > 0$. On the other hand, as $Pr_\Delta(i \succ_j b) = 1$, we have $\psi'_{j,b} < 1/6$. Similarly, as $Pr_\Delta(z \succ_k b) = 1$ and $\psi'_{z,c} < 1/3$, we have $\psi'_{k,b} < 1/3$. Then, these inequalities imply that $\psi'_{z,b} > 0$. In this case, however, agent $i$ can receive arbitrarily small amount of object $b$ from agent $z$ in return of the same object $c$. This trade would improve the welfare of both agents and not violate claimwise stability as $\psi'_{i,b} < 1/6$.

We therefore show that $\psi'_{i,a} = \psi'_{i,b} = 1/6$. Now, we claim that $\psi'_{i,d} = 0$. Assume for a contradiction that $\psi'_{i,d} > 0$. As $\psi'_{z,c} \leq 1/3$, it implies that either (or both) $\psi'_{j,c} > 0$ or $\psi'_{k,c} > 0$. If $\psi'_{k,c} > 0$, then agent $i$ can give some amount of object $d$ to agent $k$ in return of the same amount of object $c$, and this would not violate claimwise stability while it would make them better off, contradicting the constrained ordinal efficiency of $\psi$ (recall that agents $i$ and $k$ are the top priority objects).
ones at objects \( c \) and \( d \), respectively). On the other hand, if \( \psi'_{j,c} > 0 \) and \( \psi'_{k,c} = 0 \), then we have two cases to consider. First, if \( \psi'_{j,d} < \psi'_{k,b} \), then agent \( i \) can give arbitrarily small amount of object \( d \) to agent \( j \) in return of the same amount of object \( c \). This would make them better off while not violating claimwise stability. On the other hand, if \( \psi'_{j,d} = \psi'_{k,b} \) (it can not be greater due to the claimwise stability), then we have \( \psi'_{j,a} > 0 \) (note that by supposition, \( \psi'_{i,d} > 0 \) and \( \psi'_{k,c} = 0 \)). First, if \( \psi'_{k,b} = \psi'_{j,d} = 0 \), then we have \( \psi'_{i,d} = 1 \) as he is the top priority agent in object \( d \). This case contradicts our starting supposition that \( \psi'_{i,d} > 0 \). Let us assume \( \psi'_{k,b} = \psi'_{j,d} > 0 \). In this case, agent \( j \) can give some positive amount of object \( d \) to agent \( k \) in return of the same amount of object \( a \). While this trade is compatible with claimwise stability, it would improve their welfare. This, however, contradicts the constrained ordinal efficiency of \( \psi' \), showing that \( \psi'_{i,d} = 0 \). Therefore, \( \psi'_{i,a} = \psi'_{j,b} = 1/6 \), and \( \psi'_{i,c} = 2/3 \). It is easy to verify that \( \psi'_{i} \) ordinally dominates \( \psi_{i} \) with respect to \( R_{i} \), which finishes the proof.

\[ \square \]

**Proof of Lemma 2.** For a given problem \((R, \Delta)\), assume that matching \( \sigma \) is constrained ordinally efficient, i.e., it is not ordinally dominated by another claimwise stable matching. We claim that \( \sigma \) is non-wasteful.

Assume for a contradiction that there exists an agent \( i \) such that \( aP_{i}b \) with \( \sigma_{i,b} > 0 \) and \( \sum_{j \in N} \sigma_{j,a} < 1 \). Then, by constrained ordinal efficiency, we have \( \sigma_{i,a} = Pr_{\Delta}(i \succ_{a} j) + \sum_{c \in SU(R_{j,a})} \sigma_{j,c} \) for some agent \( j \) (since, otherwise, we can give arbitrarily small amount of object \( a \) to agent \( i \), which would improve the welfare of agent \( i \) without violating claimwise stability).\(^{37}\) As \( Pr_{\Delta}(j \succ_{a} i) \leq 1 \), it shows that \( \sigma_{j,a} \geq \sum_{c \in SU(R_{j,a})} \sigma_{j,c} \). This along with our initial supposition \( \sum_{j \in N} \sigma_{j,a} < 1 \) shows that \( \sigma_{j,a} < 1 - \sum_{c \in SU(R_{j,a})} \sigma_{j,c} \). Then, this implies that there exists an object \( d \) such that \( aP_{j}d \) and \( \sigma_{j,d} > 0 \). By the same argument above, it implies that there exists an agent \( k \) such that \( \sigma_{j,a} = Pr_{\Delta}(j \succ_{a} k) + \sum_{c \in SU(R_{k,a})} \sigma_{k,c} \). First, we claim that \( k \neq i \). Assume for a contradiction that \( k = i \). Then, we know that \( \sigma_{i,a} \geq 1 - Pr_{\Delta}(j \succ_{a} i) \) and \( \sigma_{j,a} \geq 1 - Pr_{\Delta}(i \succ_{a} j) \). These altogether imply that \( \sigma_{i,a} + \sigma_{j,a} \geq 2 - Pr_{\Delta}(i \succ_{a} j) - Pr_{\Delta}(j \succ_{a} i) \). From here, since \( Pr_{\Delta}(i \succ_{a} j) = 1 - Pr_{\Delta}(j \succ_{a} i) \), we obtain \( \sigma_{i,a} + \sigma_{j,a} \geq 1 \), which contradicts our very first supposition, hence, \( k \neq i \).

\(^{37}\)Since there is no constraint in the allocation of the null object if \( \emptyset P_{i}b \) where \( \sigma_{i,b} > 0 \), then this would directly contradict the constrained ordinal efficiency of \( \sigma \).
Next, by the same argument as the one above, we have an object \( z \) such that \( aP_kz \) and \( \sigma_{k,z} > 0 \). This in turn implies that there exists an agent \( h \) such that \( \sigma_{k,a} = Pr_\Delta(k \succ a h) + \sum_{c \in SU(R_{h,a})} \sigma_{h,c} \). Then, by the symmetric argument, we can easily show that \( h \neq j \). Moreover, we want to show that \( h \neq i \) as well.

Assume for a contradiction that \( h = i \). We already have \( \sigma_{i,a} \geq 1 - Pr_\Delta(j \succ a i) \) and \( \sigma_{j,a} \geq 1 - Pr_\Delta(i \succ a k) \). The last finding also implies that \( \sigma_{k,a} \geq 1 - Pr_\Delta(i \succ a k) \). Now, we can decompose \( Pr_\Delta(i \succ a k) \) as: \( Pr_\Delta(i \succ a k) = \sum_{\succ \in supp(\Delta): j \succ a i \succ a k} \Delta(\succ) + \sum_{\succ \in supp(\Delta): i \succ a k \succ a j} \Delta(\succ) + \sum_{\succ \in supp(\Delta): i \succ a j \succ a k} \Delta(\succ) \). The sum of the last two terms is less than or equal to \( Pr_\Delta(i \succ a j) \). On the other hand, we have \( \sigma_{i,a} \geq Pr_\Delta(a \succ i) \). The first term, moreover, is less than or equal to \( Pr_\Delta(j \succ a k) \). Similar to above, we also have \( \sigma_{j,a} \geq Pr_\Delta(j \succ a k) \). These findings, therefore, show that \( Pr_\Delta(i \succ a k) \leq \sigma_{i,a} + \sigma_{j,a} \). Hence, \( \sigma_{k,a} \geq 1 - \sigma_{i,a} - \sigma_{j,a} \). This implies that \( \sigma_{i,a} + \sigma_{j,a} + \sigma_{k,a} \geq 1 \), which contradicts our very first supposition. Hence, \( h \in N \setminus \{i,j,k\} \).

Then, if we continue as before, we obtain an object \( u \) such that \( aP_hu \) and \( \sigma_{h,u} > 0 \). This implies that there exists an agent \( l \) such that \( \sigma_{h,a} = Pr_\Delta(h \succ a l) + \sum_{c \in SU(R_{l,a})} \sigma_{l,c} \). Then, by following the same steps above, we can easily show that \( l \in N \setminus \{i,j,k,h\} \).

From our above analysis, we observe that agent \( i \) can not eat object \( a \) more than \( \sigma_{i,a} \) since it would otherwise violate claimwise stability constraint imposed on agent \( i \) due to agent \( j \). We can illustrate this relation by drawing an arrow coming from agent \( i \) and going to agent \( j \). If we do the same thing to all other agents as well, we would obtain the following figure:

\[
i \to j \to k \to h \to l.
\]

In the above analysis, we prove that each agent is different in the above sequence. Moreover, continuing in the same manner as above would give us an sequence of agents in the above sense where each of them is different. This, however, is impossible as the set of agents is finite. Hence, \( \sigma \) is non-wasteful.

\[\square\]

Proof of Theorem 3. “If” Part: We have already showed that CPS is claimwise stable and constraint ordinally efficient (we proved it in the absence of the null object, yet, the results easily
carry over to the presence of the null object case). Hence, by Lemma 2, it is non-wasteful. Therefore, it is enough to demonstrate that \( CPS \) is binding. To this end, given any problem instance \((R, \Delta)\), pick an agent-object pair \((i, a)\). In the course of the consumption process \( CPS \), if he stops eating object \( a \) by the time it is totally exhausted, then there is no agent \( k \) such that \( F(R_i, a, CPS_i) < F(R_k, a, CPS_k) \) and \( CPS_{k,a} > 0 \). On the other hand, if it is not the case, then, by the definition of \( CPS \), there has to be an agent \( k \) who has applied to object \( a \) before agent \( i \) stops eating it (note that this is due the definition of \( \Theta \)) and \( CPS_{i,a} = Pr_\Delta(i \succ k) + \sum_{c \in SU(R_k, a)} CPS_{k,c} \). Therefore, \( CPS \) is binding.

**“Only If” Part:** Let us pick a mechanism \( \psi \) which is non-wasteful, claimwise stable, and binding. We want to show that \( \psi(R, \Delta) = CPS(R, \Delta) \) for every problem \((R, \Delta)\).

Now, we use the same construction as in the proof of Theorem 1. For the sake of completeness, we repeat it here. First, given a matching \( \sigma \), agent \( i \) with preference \( R_i \) and object \( a \), we define \( F(R_i, a, \sigma_i) = \sum_{c \in U(R_i, a)} \sigma_{i,c} \). For ease of notation, we suppress the dependency of mechanisms \( \psi \) and \( CPS \) on the problem instance \((R, \Delta)\) and just write \( \psi \) and \( CPS \) to denote their outcomes at \((R, \Delta)\). Let \( \pi_1 = \min\{F(R_i, a, \psi_i) : \) for all \((i, a) \in N \times O\}, \pi_k = \min\{F(R_i, a, \psi_i) : F(R_i, a, \psi_i) > \pi_{k-1} \) for all \((i, a) \in N \times O\}. \) We also write \( \Pi = \{F(R_i, a, \psi_i) : \) for all \((i, a) \in N \times O\}. \)

Now, we claim that \( F(R_i, a, \psi_i) = F(R_i, a, CPS_i) \) for all agent \( i \) and object \( a \), which would imply that \( \psi = CPS \). To this end, we inductively prove that \( F(R_i, a, \psi_i) = F(R_i, a, CPS_i) \) for all agent-object pair \((i, a)\) such that \( F(R_i, a, \psi_i) = \pi_k \).

Let \( k = 1 \) and \( F(R_i, a, \psi_i) = \pi_1 \). Assume that \( a \) is the best object of agent \( i \). Then, by claimwise stability of \( \psi \) and the construction of \( CPS \), it is easy to see that \( CPS_{i,a} \geq \pi_1 \). Hence, if \( \pi_1 = 1 \), then we have \( F(R_i, a, \psi_i) = F(R_i, a, CPS_i) \). Assume that \( \pi_1 < 1 \). Then, we have two cases to consider.

**Case 1.** There exists no agent \( h \) such that \( F(R_i, a, \psi_i) < F(R_h, a, \psi_h) \) and \( \psi_{h,a} > 0 \). By non-wastefulness, it implies that object \( a \) is totally exhausted by time \( \pi_1 \) in the course of \( \psi \). This can be easily seen once we consider \( \psi \) as a consumption process where agents continuously acquire objects in decreasing order of their preferences. Given this along with \( F(R_j, b, \psi_j) \leq F(R_j, b, CPS_j) \) for all agent-object pair \((j, b)\) such that \( F(R_j, b, \psi_j) = \pi_1 \), we have \( F(R_i, a, CPS_i) = F(R_i, a, \psi_i) = \pi_1 \).

**Case 2.** Let us assume that there exists an agent \( h \) such that \( F(R_i, a, \psi_i) < F(R_h, a, \psi_h) \) and
\( \psi_{h,a} > 0 \). This means that agent \( a \) stops eating object \( a \) before it is totally exhausted. Then, as \( \psi \) is binding, there exists an agent \( k \) who applies to object \( a \) in a step before the one in which agent \( i \) stops eating object \( a \). If we write \( s(k) \) and \( s(i) \) to respectively denote those steps, then we have \( s(k) \prec s(i) \). Moreover, we also have \( \psi_{i,a} = Pr_{\Delta}(i \succ a)k + \sum_{c \in SU(R_{k,a})} \psi_{k,c} \).

First, if \( a \) is the best object of agent \( k \), then the above binding claimwise stability constraint implies that \( Pr_{\Delta}(k \succ a \ i) = 1 - \pi_1 \). Since, in this case, \( \sum_{c \in SU(R_{k,a})} \phi_{k,c} = 0 \) regardless of mechanism \( \phi \), and, under any claimwise stable mechanism \( \phi \), we have \( \phi_{i,a} \leq \pi_1 \). This along with \( F(R_i, a, CPS_i) \geq \pi_1 \) implies that \( F(R_i, a, CPS_i) = \pi_1 \).

Consider the case where \( SU(R_{k,a}) \neq \emptyset \). In this case, the above binding claimwise stability constraint implies that \( Pr_{\Delta}(k \succ a \ i) = 1 \) and \( \sum_{c \in SU(R_{k,a})} \psi_{k,c} = \pi_1 \) (due to the fact that \( \sum_{c \in SU(R_{k,a})} \psi_{k,c} \geq \pi_1 \)). Let object \( b \) be the just preferred object to object \( a \) by agent \( k \). Then, we have \( F(R_k, b, \psi_k) = \pi_1 \). This case, therefore, gives another agent-object pair \((k, b)\) such that \( F(R_k, b, \psi_k) = \pi_1 \), which is similar to our starting pair \((i, a)\).

Next, we repeat the above analysis for agent \( k \). If there is no object \( c \in U(R_k, b) \) such that Case 2 applies to it, then, by the same arguments, we have \( F(R_k, b, CPS_k) = F(R_k, b, \psi_k) = \pi_1 \). This in turn implies that \( CPS_{i,a} = \pi_1 \) due to claimwise stability, hence, \( F(R_i, a, CPS_i) = \pi_1 \).

Let us assume that there exists an object \( c \in U(R_k, b) \) such that Case 2 applies to it. This implies that there exists an agent \( j \) such that he applies to object \( c \) in a step before the one in which agent \( k \) stops eating it in the course of consumption process \( \psi \). Let \( s(j) \) denote that step. Note that \( s(j) \prec s(k) \prec s(i) \) (recall that step \( s(k) \) is the one in which agent \( k \) applies to object \( a \), which is dispreferred to object \( c \) by him). Moreover, we have \( \psi_{k,c} = Pr_{\Delta}(k \succ c \ j) + \sum_{d \in SU(R_{j,c})} \psi_{j,d} \).

If we continue our analysis by applying the above steps to agent \( j \), we will either directly show that \( F(R_k, b, CPS_k) = \pi_1 \) or find another agent \( h \) such that, for some object \( d \in U(R_j, c) \), \( \psi_{j,d} = Pr_{\Delta}(j \succ d \ h) + \sum_{\psi_{h,e} \in SU(R_{h,d})} \psi_{h,e} \). In the former, \( F(R_k, b, CPS_k) = \pi_1 \) implies that \( F(R_i, a, CPS_i) = \pi_1 \) (by claimwise stability of \( CPS \)). On the other hand, in the latter, we obtain agent \( h \) such that \( \psi_{j,d} = Pr_{\Delta}(j \succ d \ h) + \sum_{\psi_{h,e} \in SU(R_{h,d})} \psi_{h,e} \). Moreover, as \( \psi \) is binding, step \( s(h) \) in which agent \( h \) applies to object \( d \) occurs before step \( s(j) \). That is, we have \( s(h) \prec s(j) \prec s(k) \prec s(i) \).

If we continue in the same manner as above, then there are two possible cases. Under the first case, we find an agent through the above steps such that Case 2 does not apply to him for any
corresponding object. In this case, similar to above, \( F(R_i, a, CPS_i) = \pi_1 \) would follow from the claimwise stability of \( CPS \). Under the other case, we find a sequence of ordered steps as above. Yet, since everything is finite, this would give us a cycle in the ordering of steps. This, on the other hand, would contradict the transitivity of \( \prec \). Therefore, this case is impossible, showing that there has to be an agent-object pair falling into the first case, proving that \( F(R_i, a, CPS_i) = \pi_1 \).

For the induction hypothesis, assume that \( F(R_i, a, \psi_i) = F(R_i, a, CPS_i) \) for all \( F(R_i, a, \psi_i) = \pi_k \) where \( k < k' \). Let \( F(R_i, a, \psi_i) = \pi_{k'} \). We want to show that \( F(R_i, a, CPS_i) = \pi_{k'} \).

First, our induction hypothesis along with the definition of \( CPS \) and the claimwise stability of \( \psi \) implies that \( F(R_i, a, CPS_i) \geq \pi_{k'} \).

Similar to above, if there is no agent \( k \) such that \( F(R_i, a, \psi_i) < F(R_k, a, \psi_k) \) and \( \psi_{k,a} > 0 \), then this implies that agent \( i \) continues to consume object \( a \) until it is totally exhausted in the course of consumption process \( \psi \). In this case, by the previous arguments, \( F(R_i, a, \psi_i) = F(R_i, a, CPS_i) = \pi_{k'} \), which finishes the proof.

For the other case, let us assume that, for some agent \( k \), \( F(R_i, a, \psi_i) < F(R_k, a, \psi_k) \) and \( \psi_{k,a} > 0 \). As \( \psi \) is binding, there has to be an agent \( j \) such that \( \psi_{i,a} = Pr_\Delta(i \succ_a j) + \sum_{c \in SU(R_j, a)} \psi_{j,c} \).

Moreover, agent \( j \) applies to object \( a \) in a step before the one in which agent \( i \) stops eating object \( a \).

Assume that \( \psi_{i,a} > 0 \). Let object \( b \) be the object just preferred to object \( a \) by agent \( j \) (if object \( a \) is the best alternative of agent \( j \), then the proof follows from the claimwise stability of \( CPS \) and the above binding claimwise stability constraint). As agent \( j \) applies to object \( a \) before agent \( i \) stops eating it, we have \( F(R_j, b, \psi_j) \leq \pi_{k'} \). If it holds strictly, then, by the induction hypothesis, we have \( F(R_j, b, \psi_j) = F(R_j, b, CPS_j) < \pi_{k'} \). Then, by the claimwise stability of \( CPS \), we have \( CPS_{i,a} = \psi_{i,a} \). This along with induction hypothesis and \( \psi_{i,a} > 0 \) shows that \( F(R_i, a, \psi_i) = F(R_i, a, CPS_i) = \pi_{k'} \). For the other case, assume that \( F(R_j, b, \psi_j) = \pi_{k'} \). In this case, similar to previous arguments, we apply the same steps to agent \( j \). In the course of this iterative process, there are two cases to consider. In the first case, we find an agent \( h \) and the corresponding object \( c \) such that \( F(R_h, c, \psi_h) < \pi_{k'} \) (in this case, object \( c \) is the one just preferred to object \( b \) by agent \( h \)). In this case, as the same as above, the proof would follow from the claimwise stability of \( CPS \) and the induction hypothesis. On the other hand, the other case would give us a
cycle in the ordering of steps as before, which would constitute a contradiction. This shows that we would find an agent-object pair as in the former (case), showing that \( CPS_{i,a} = \psi_{i,a} \). This along with the induction hypothesis and \( \psi_{i,a} > 0 \) show that \( F(R_i, a, \psi_i) = F(R_i, a, CPS_i) = \pi_{k'} \).

If \( \psi_{i,a} = 0 \), then let object \( b \) be such that \( F(R_i, b, \psi_i) = F(R_i, b, CPS_i) \). This implies that \( \psi_{i,c} = 0 \) for all \( c \in U(R_i, a) \setminus U(R_i, b) \). Our binding claimwise stability constraint \( \psi_{i,a} = Pr_{\Delta} (i \succ_j a) + \sum_{c \in SU(R_j, a)} \psi_{j,c} \) along with \( \psi_{i,a} = 0 \) implies that \( Pr_{\Delta} (j \succ_i a) = 1 \) and \( \sum_{c \in SU(R_j, a)} \psi_{j,c} = 0 \).

By the first step of induction, we know that \( \sum_{c \in SU(R_j, a)} CPS_{j,c} = 0 \) as well. Hence, by claimwise stability of CPS, we have \( CPS_{i,a} = 0 \). The same arguments can be directly applied to objects \( c \in U(R_i, a) \setminus U(R_i, b) \) to demonstrate that \( CPS_{i,c} = 0 \) for each of such object \( c \) (recall that, for any \( c \in U(R_i, a) \setminus U(R_i, b) \), \( \psi_{i,c} = 0 \)). Therefore, \( F(R_i, a, \psi_i) = F(R_i, a, CPS_i) = \pi_{k'} \).

In above, we assume that object \( a \) is the best alternative of agent \( i \). Let us suppose that it is his second best object. Let object \( b \) is his first choice. Then, we have \( F(R_i, b, \psi_i) = F(R_i, a, \psi_i) = \pi_1 \). Moreover, by our above analysis, we have \( F(R_i, b, \psi_i) = F(R_i, b, CPS_i) = \pi_1 \). Given this, the same right above argument would prove that \( \psi_{i,a} = CPS_{i,a} = 0 \) implying \( F(R_i, a, \psi_i) = F(R_i, a, CPS_i) \).

The other cases follow from the same arguments. This finishes the proof.

\[ \square \]

**Acknowledgment**

I am grateful to Fuhito Kojima for the insightful comments and suggestions. I owe special thanks to Ahmet Alkan, Azar Abizade, Umut Mert Dur, and Yusufcan Masatlioglu for their thorough comments. I thank Tayfun Sönmez, Utku Ünver, Yeon-Koo Che, William Thomson, Özgür Kıbrıs, and Mehmet Barlo for their valuable suggestions.

**References**


