Sequential versus Simultaneous Assignment Problems and Two Applications*

Umut Dur† Onur Kesten‡

January 2014

Abstract

We study matching markets from practice, where a set of objects are assigned to a set of agents sequentially in two-rounds. The placement of students to the exam and mainstream public schools in the U.S. and the appointment of teachers to the public schools in Turkey until recently are two examples of such markets. We analyze the mechanisms currently in use in both markets and show that they fail to satisfy desirable fairness and welfare criteria. Moreover, they give participants perverse incentives: misreporting preferences can be beneficial and improved performance on admission test may worsen the assignment of a participant. We characterize the subgame perfect Nash equilibria of the induced preference revelation games in both markets, which motivate us to propose an alternative method, applicable to both markets, through which assignments take place simultaneously in a single round. This may also help explain why the Turkish ministry of education has recently reformed its appointment system.

1 Introduction

Simultaneous allocation systems whereby distribution of all resources takes place in a single round, are widely used for solving static allocation problems. On the other hand, when faced

---

*We thank Battal Doğan, Thayer Morrill, Tayfun Sönmez, and Utku Ünver for helpful discussions. We also gratefully acknowledge various comments we received at University of Texas, Boston College, GAMES 2012, and Koç University Winter Workshop.

†Department of Economics, North Carolina State University, 2801 Founders Drive, Raleigh, NC 27695, USA; e-mail: umutdur@gmail.com.

‡Tepper School of Business, Carnegie Mellon University, 5000 Forbes Ave, Pittsburgh, PA 15213, USA. E-mail: okesten@andrew.cmu.edu.
with subtle allocational constraints, policy makers often resort to sequential allocation systems whereby different sets of resources are distributed in different rounds. In this paper we study sequential allocation systems in the context of indivisible good assignment and assess the advantages and disadvantages of such a system when contrasted with an analogous simultaneous allocation system.

In a sequential assignment problem, a set of agents are to be assigned a set of objects in a sequential fashion in (at least) two-rounds, and each agent is entitled to receive at most one object. More specifically, there is a first round of assignments in which all agents actively participate by reporting their rank-ordered preferences when only a subset of objects is available. The first round is then followed by a second round of assignments in which all remaining objects are assigned to the agents who were unassigned in the first round. We are motivated by two applications of this problem from practice: student placement to exam and regular public schools in the U.S. and the appointment of teachers to state schools in Turkey.

In Boston and New York City, there are two types of public schools: exam and regular (mainstream) schools.\footnote{There are different types of regular schools.} Students who wish to apply to exam schools take a centralized test and are then ranked based on their scores. Meanwhile, regular schools rank students based on certain predetermined criteria, i.e. proximity and sibling status. The admissions for both type of schools are processed separately. In general, the admission decisions to the exam schools are completed well before any students are assigned to the regular schools. In particular, students are assigned to exam schools via a serial dictatorship mechanism, and the unassigned students are then assigned to regular schools via a student-proposing deferred acceptance (DA) mechanism (Gale and Shapley, 1962) in a second round. (See the Appendix for a detailed description of the assignment systems in Boston and New York City.)

In Turkey, the assignment of teachers to teaching positions in state schools takes place via a centralized process overseen by the Turkish Ministry of Education (TMoE). Every year the TMoE offers a standardized test to those university graduates who wish to serve in state-sponsored jobs. Although this test is mostly taken by new university graduates, many who have graduated in the past are also eligible to take it if they wish so.\footnote{Some of them can be currently employed as a teacher.} In a given year, the appointment of teachers to
state schools are based solely on the candidates’ performance on that year’s test. There are two types of teaching positions in each specialization. These are the tenured positions which offer a lifetime employment guarantee and contractual positions which can be taken only for a fixed number of years (typically for only a few years) and the conditions of employments are based on a specific contract mutually agreed upon by the school and the teacher. Although an otherwise identical tenured position is generally preferable to a contractual position, it is also common to observe strong preference for contractual positions in major metropolitan cities such as Istanbul over tenured positions in smaller cities or rural areas.

In a given year the TMoE first announces the list of all available tenured positions in each school and each specialization throughout Turkey. Then each applicant, be it a new graduate or an existing contractual teacher, submits rank-ordered preference lists over the available tenured positions before a certain deadline announced by the ministry. In this first round, existing contractual teachers who are seeking a new position are also restricted to rank-list only tenured positions. Applicants are then assigned to the available positions by a serial dictatorship mechanism induced by the test scores. If a contractual teacher is unassigned in the first round, then she retains her current job assignment. Otherwise, she is appointed to a tenured position and a contractual position at her old school opens up for a new appointment. Typically within a few weeks after the first round, the TMoE announces the list of available contractual positions. And in this second round, only the unassigned new graduates are permitted to apply to these contractual positions. Applicants are again assigned via a serial dictatorship mechanism induced by the test scores. (See Table 1 in the Appendix for some summary statistics of this system in recent years.)

We show that the multiple-round student assignment system in the U.S. and the teacher appointment system in Turkey share a number of serious deficiencies. Among other shortcomings, both systems fail to generate Pareto-efficient or fair assignments, and both systems induce strategic action on the part of applicants while deciding what rank-order lists to submit in each round. We then ask whether such shortcomings can be overcome by alternative systems and turn to investigate general sequential assignment systems when the mechanisms used in each round

---

3Existing teachers employed in tenured positions are not allowed to participate this assignment procedure.
4In other words, any contractual position currently filled by an applicant cannot be rank-listed by any applicants including its current occupant.
satisfy certain properties that are by and large deemed desirable in the matching literature.

We argue that the deficiencies of the systems in the U.S. and Turkey are not specific to these particular contexts. Our analysis indicates that there may indeed be a fundamental problem with achieving distributional and strategic goals via sequential assignment systems that employ mechanisms that satisfy even very mild requirements. Our results are as follows:

If $\Psi = (\varphi^1, \varphi^2)$ is a straightforward sequential assignment system, then it is also wasteful (Theorem 1). That is, any straightforward system necessarily generates inefficient outcomes. If $\Psi = (\varphi^1, \varphi^2)$ is a straightforward sequential assignment system which employs mechanisms non-wasteful mechanisms in both rounds then such a system cannot generate fair outcomes (Theorem 2). Similarly, if $\Psi = (\varphi^1, \varphi^2)$ is a straightforward sequential assignment system which employs non-wasteful mechanisms in both rounds and $\varphi^1$ selects a fair outcome whenever all objects prioritize agents in the same order, then $\Psi$ does not respect improvements in the priority order (Theorem 3). If $\Psi = (\varphi^1, \varphi^2)$ is a sequential assignment system which employs mechanisms $\varphi^1$ and $\varphi^2$ in rounds 1 and 2 such that $\varphi^1$ is individually rational and non-wasteful, and $\varphi^2$ is non-wasteful, then such a system is prone to manipulation (Theorem 4).

We also characterize the subgame perfect Nash equilibria (SPNE) induced by a sequential preference revelation game of a sequential assignment system. We find that when both $\varphi^1$ and $\varphi^2$ are individually rational, non-wasteful and either [population monotonic and non-bossy] or fair, then every SPNE outcome of the preference revelation game associated with system $\Psi$ leads to a non-wasteful and individually rational matching (Theorem 5). On the other hand, when both $\varphi^1$ and $\varphi^2$ are individually rational, non-wasteful, population monotonic, and minimally fair, then every SPNE outcome of the preference revelation game associated with system $\Psi$ leads to a matching that has no priority violations (Theorem 6 and 7). As corollaries of these results, we provide a detailed account on the characteristics of the set of SPNE for each of the two applications that motivate our study.

Our analysis points to clear disadvantages of sequential assignment systems and provides justification for the alternative use of single round assignment systems when possible. This conclusion is also supported by the recent transition of the TMoE from the system analyzed here to a simpler single round simultaneous assignment system. More broadly, these observations

---

5 As far as we are aware, this transition took place without the involvement of any economists in the decision.
motivate us to advocate the use of a suitable adaptation of Gale and Shapley’s celebrated Deferred Acceptance (DA) mechanism to the specific context as a single round assignment system. In particular, in the context of teacher assignment, the dominant strategy outcome of DA Pareto dominates any SPNE of the old assignment system.

1.1 Related Literature

The main characteristic of the type of problems we study here that distinguishes them from the vast majority of the problems considered in the literature is that they involve sequential assignment of indivisible resources. Whereas the set of agents and resources are predetermined and fixed in a standard, single-round simultaneous assignment problem, in a sequential assignment problem agents and resources considered within a round may as well depend on the decisions made in a previous round. Differently put, in a sequential assignment problem, an agent might have the ability to choose which round he is to get which assignment. Yet, the two type of problems still share similar strategic and distributional objectives.

While there is now an extant literature on school choice plans, as far as we are aware, virtually all of these models abstract away from the multi-round nature of these problems and exclusively focus on a single-round simultaneous assignment system. Still, we are not the first to point out the deficiencies of a sequential student assignment system. In their detailed examination of NYC student assignment plan, Abdulkadiroğlu, Pathak and Roth (2009) argue that the current multi-round assignment plan may result in unstable student assignments. They write:

“We would have preferred to integrate these two rounds into one, by having applicants include the specialized schools in their preference lists. (The two-round design creates a possibility of unstable allocations, as when a student gets an offer from a specialized school, but not from a nonspecialized school he prefers that would have had a place for him after the specialized-school students have declined places.)”

While Abdulkadiroğlu, Pathak and Roth (2009) caution against potential stability issues resulting from the sequential system in NYC, in the present paper we identify potential incentive and welfare issues associated with such systems.
On the other hand, the teacher assignment problem (TAP) in Turkey, described above, has features reminiscent of the Student Placement Problem (SPP) due to Balinski and Sönmez (1999) and the House Allocation Problem with Existing Tenants (HAPwET) due to Abdulkadiroğlu and Sönmez (1999). As in the context of SPP, in TAP too, applicants are ranked based on their test scores, and fairness (i.e., favoring applicants with better test scores) is a central goal. And, as in HAPwET, some of the applicants— the contractual teachers— have private endowments—the contractual positions they currently occupy— that may later become available for reassignment to other applicants.

A paper that is also related to ours is Ergin and Sönmez (2006), where the authors characterize the set of NE of the widely-used Boston mechanism and show that this set coincides with the set of stable matchings. We find that while this conclusion need not hold generally for any sequential assignment problem, in the context of TAP (but not in that of SCPwERS) the set of SPNE of the sequential preference revelation game is also equal to the stable set.\(^6\)

The only paper, that we are aware of, to consider sequential assignment is Westkamp (2012), where the author studies the German college admissions system which operates through a combination of Boston and college-proposing deferred acceptance mechanism.\(^7\) Similar to Ergin and Sönmez (2006), Westkamp too characterizes the set of SPNE of this game as being the stable set. While we also provide characterizations of SPNE for both of our applications, we show that the equivalence of SPNE to the stable set in general may not always be guaranteed. Most notably, differently than Westkamp, our focus here is on properties of a general sequential assignment system and on showing that the sources of the deficiencies may be inherently related to the sequential nature of the assignment system. As such, we show that these deficiencies may be impossible to avoid regardless of what specific mechanism is used in any round. Another difference is that, we consider sequential systems composed of strategy-proof mechanisms while Westkamp characterizes a sequential assignment system in which both rounds feature mechanisms that are vulnerable to manipulation. Hence, we show that the strategic vulnerabilities of sequential systems cannot be overcome even though strategy-proofness is guaranteed within each

---

\(^6\) In the context of TAP, stability is characterized by the combination of individual rationality, fairness, and nonwastefulness.

\(^7\) A major reason behind the current two-round German college admissions system is to accommodate affirmative action considerations.
Braun et al. (2011) compare the performance of the sequential German college admissions systems and a modified version of DA mechanism through experiments. The results of the experiments show that the current practice in Germany harms the high-performing students and creates incentives to misreport their preferences. On the other hand, the modified DA mechanism improves the welfare of the high-performing applicants.

The rest of the paper is organized as follows. Section 2 introduces the formal model. Section 3 provides a detailed description of the sequential systems in the U.S. and Turkey. Section 3 presents impossibility results concerning general sequential systems. Section 4 characterizes the SPNE of general sequential systems as well as those of the two motivating applications. Section 5 presents a simple alternative to sequential systems. Section 6 concludes.

2 Model

Let \( I^* = \{i_1, i_2, \ldots, i_n\} \) be the set of all agents, \( S^* = \{s_1, s_2, \ldots, s_m\} \) be the set of all objects, \( q^* = (q_{s_1}, q_{s_2}, \ldots, q_{s_m}) \) be the capacity vector for all objects. Let \( \emptyset \) represent being unassigned option for both agents and objects. Let \( \succ^* = (\succ_{s_1}, \succ_{s_2}, \ldots, \succ_{s_m}) \) denote a priority profile, where \( \succ_{s} \) is the strict priority order for object \( s \) such that \( \emptyset \succ_{s} i \) means that agent \( i \) is not acceptable for object \( s \). We allow for an object to be socially or privately owned. Let \( h^* = (h(i))_{i \in I^*} \) be an ownership profile, where \( h(i) \) is the object for which agent \( i \) has the property right (i.e., her endowment) such that \( h(i) = \emptyset \) means that agent \( i \) has no property right over any object. Each agent \( i \) can own at most one object, i.e., \( |h(i)| \leq 1 \). On the other hand, an object can be owned by more than one agent. In particular, an object \( s \in S \) can be owned by at most \( q_s \) agents. Unless otherwise mentioned an object \( s \) is owned by either 0 or \( q_s \) agents.

Each agent \( i \) has a strict (i.e., complete, transitive, and antisymmetric) preference relation \( P_i \) over \( S \cup \{\emptyset\} \). Let \( R_i \) denote the associated at least as good as relation of agent \( i \). We thus have

\[
\text{s } R_i \text{ s'} \iff \text{s } P_i \text{ s'} \text{ whenever } s \neq s'.
\]

Westkamp conjectures that the incentive issues observed in the current German college admissions system may not be solved by adopting another sequential system.
Let $\succ$ and $P$ be the sets of all possible priority and preference profiles. A **sequential assignment problem**, or a **problem** for short, is a 6-tuple $(I, S, P, q, \succ, h)$ where $I \subseteq I^*$, $S \subseteq S^*$, $P = (P_i)_{i \in I} \in P$ is a preference profile, $q = (q_s)_{s \in S}$, $\succ = (\succ_s)_{s \in S} \in \succ$ and $h = (h(i))_{i \in I}$.

Fix a problem $(I, S, P, q, \succ, h)$. A matching is a function $\mu : I \rightarrow S \cup \emptyset$ such that the number of agents assigned to an object $s$ does not exceed the total number of the copies of $s$ and each agent can be assigned to at most one object, i.e., $|\mu^{-1}(s)| \leq q_s$ and $|\mu(i)| \leq 1$ for all $s \in S$ and $i \in I$. Let $\mathcal{M}$ be the set of all matchings. A matching $\mu$ is **non-wasteful (NW)** if there exists no agent-object pair $(i, s)$ such that $|\mu^{-1}(s)| < q_s$, $i \succ_s \emptyset$ and $s P_i \mu(i)$. A matching $\mu$ is **individually rational (IR)** if no agent is assigned to an object either she finds worse than being unassigned option or she is unacceptable for. Formally, a matching $\mu$ is individually rational if $\mu(i) \not\succ i$ and $i \succ_{\mu(i)} \emptyset$ for all $i \in I$. A matching $\mu$ **Pareto dominates** another matching $\mu'$ if each agent weakly prefers her assignment in $\mu$ to her assignment in $\mu'$ and at least one agent $i$ strictly prefers her assignment in $\mu$ to her assignment in $\mu'$. A matching $\mu$ is **Pareto efficient** if it is not Pareto dominated by another matching $\mu'$. A matching $\mu$ is **fair** if whenever an agent prefers some other agent’s assignment to her own, then the other agent has a higher priority for that object than herself. Formally, if $\mu$ is fair then for every $i, j \in I$, $\mu(j) P_i \mu(i)$ implies $j \succ_{\mu(j)} i$.

A matching $\mu$ is **stable** if it is non-wasteful, individually rational and fair. A matching $\mu$ is **mutually fair (MF)** if there does not exist an agent-object pair $(i, s)$ such that (1) $i$ ranks $s$ at the top of his preference list (2) $\mu(i) \neq s$ (3) there exists another agent $i'$ with lower priority for $s$ than $i$ and $\mu(i') = s$. Let $r(P_i, s)$ be the rank of $s$ in the preference list $P_i$. A matching $\mu$ **favors higher ranks (FHR)** if $i$ is assigned to a worse object than $s$ then all agents assigned to $s$ rank $s$ at least as high as $i$. Formally, $\mu$ favors higher ranks if whenever there exists an agent-object pair $(i, s)$ such that $s P_i \mu(i)$ then $r(P_i, s) \geq r(P_j, s)$ for all $j \in \mu^{-1}(s)$.

A **mechanism** $\varphi$ is a mapping that associates a matching to a given problem. Denote the outcome selected by mechanism $\varphi$ for problem $(I, S, P, q, \succ, h)$ by $\varphi(I, S, P, q, \succ, h)$ and the match of agent $i \in I$ by $\varphi_i(I, S, P, q, \succ, h)$.

---

9This definition is different from the standard non-wastefulness notion (see Balinski and Sonmez (1999)). Here, we add $i \succ_s \emptyset$ condition. In the standard student placement or school choice problem all agents are acceptable for objects. If a student is unacceptable to a school then the unfilled seats for that school is not wasted. In fact, the student is not assigned to one of schools with available seat for some reason.

10Favoring higher ranks was introduced by Kojima and Unver (2013). Different from them we are not requiring all the seats of school $s$ to be filled.
A mechanism is non-wasteful {(mutually) fair} [individually rational] <favors higher ranks> if its outcome is non-wasteful {(mutually) fair} [individually rational] <able favoring higher ranks> in a given problem.

A mechanism $\varphi$ is weakly non-bossy (WNB) if for any $P = (P_j)_{j \in I}$ and $P_i'$ if $\varphi_i(I, S, P, q, \succ, h) = \varphi_i(I, S, (P_i', P_{-i}), q, \succ, h) = \emptyset$ then $\varphi_i(I, S, P, q, \succ, h) = \varphi_i(I, S, (P_i', P_{-i}), q, \succ, h)$.

A mechanism $\varphi$ is resource monotonic (RM) if $\varphi_i(I, S, P, q, \succ, h) R_i \varphi_i(I, S, P, (q_s', q_{-s}), \succ, h)$ for all $s \in S, q_s' \leq q_s, i \in I$ and $P \in P$.

A mechanism $\varphi$ is (weakly) population monotonic (PM) if $\varphi_j(I, S, (P_i', P_{-i}), q, \succ, h) R_j \varphi_j(I, S, P, (q_s', q_{-s}), \succ, h)$ for all $j \in I \setminus \{i\}$ where $(\varphi_j(P) = \emptyset$ and) $\emptyset P_i'x$ for all $x \in S$.

A mechanism $\varphi$ is monotonic if it is resource and population monotonic.

A mechanism $\varphi$ is independent of irrelevant agents (IIA) if $\varphi_i(I \setminus \{k\}, S, (P_j)_{j \in I \setminus \{k\}}, q, \succ, h) = \varphi_i(I, S, P, q, \succ, h)$ where $P_k : \emptyset P_kx$ for all $x \in S, i \in I \setminus \{k\}$ and $P \in P^I$.

A mechanism $\varphi$ is strategy-proof (SP) if it is always a dominant strategy for each agent to report his preferences truthfully. Formally, for every $i \in I$ and every $P_i'$, and every $P$, we have

$$\varphi_i(I, S, P, q, \succ, h) R_i \varphi_i(I, S, P_i', P_{-i}, q, \succ, h).$$

In Table 1, we summarize the performance of the well-known mechanisms based on the axioms defined.

<table>
<thead>
<tr>
<th></th>
<th>NW</th>
<th>IR</th>
<th>Fair</th>
<th>MF</th>
<th>FHR</th>
<th>WNB</th>
<th>PM</th>
<th>wPM</th>
<th>RM</th>
<th>SP</th>
<th>IIA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Object-proposing DA</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Agent-proposing DA</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Top Trading Cycles</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Boston Mechanism</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Serial Dictatorship</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>NW</th>
<th>IR</th>
<th>Fair</th>
<th>MF</th>
<th>FHR</th>
<th>WNB</th>
<th>PM</th>
<th>wPM</th>
<th>RM</th>
<th>SP</th>
<th>IIA</th>
</tr>
</thead>
</table>

Table 1. Performance of Mechanisms

11 Our population monotonicity definition is similar to that of Kojima and Ünver (2013). Differently from their definition, the assignment of agent $i$ is not removed from the problem in our specification.

12 In the sequel we provide descriptions of the (agent-proposing) DA and the serial dictatorship mechanisms. We refer the reader to the extant literature for the descriptions of the remaining mechanisms.

13 Serial Dictatorship mechanism is defined in an environment where all objects have the same priority order and in this environment the outcome of Serial Dictatorship is fair.
A sequential assignment system, or a system for short, is a pair of mechanisms $\Psi = (\varphi^1, \varphi^2)$ such that\footnote{The notations $q|S^1$ and $\succ |S^1$ respectively denote the restrictions of $q$ and $\succ$ to the set of objects in $S^1$. Here, $P^1 = (P^1_i)_{i \in I}$ is the preference profile over the available objects in step 1. Similarly, $P^2 = (P^2_i)_{i \in I^2}$ is the preference profile over the available objects in step 2.}

1. $\varphi^1$ operates on the restricted problem $(I, S^1, P^1, q|S^1, \succ |S^1, h)$ whose primitives are the set of all agents, a subset of all objects available for assignment in the first round (defined by the application), and the preferences and priorities over available objects, and\footnote{The notations $q|S^1$ and $\succ |S^1$ respectively denote the restrictions of $q$ and $\succ$ to the set of objects in $S^1$. Here, $P^1 = (P^1_i)_{i \in I}$ is the preference profile over the available objects in step 1. Similarly, $P^2 = (P^2_i)_{i \in I^2}$ is the preference profile over the available objects in step 2.}

2. $\varphi^2$ operates on the restricted problem $(I^2, S^2, P^2, q^2, \succ |S^2, h)$ whose primitives are the set of all agents (without property rights) who are unassigned in the first round, the set of all objects available for assignment in the second round (defined by the application), the preferences, and priorities over available objects. More precisely,

\begin{align*}
I^2 &= \{i : \varphi^1_i(I, S^1, P^1, q|S^1, \succ |S^1, h) = \emptyset \text{ and } h(i) = \emptyset\}, \\
S^2 &= S \setminus S^1, \\
q^2_s &= q_s - \{|i : h(i) = s \text{ and } \varphi^1_i(I, S^1, P^1, q|S^1, \succ |S^1, h) = \emptyset\| \forall s \in S^2.
\end{align*}

It is important to note that since the number of available copies of each object in the second round depends on the assignment in the round 1, the problem in the second round (including the participating agents as well as available objects) is “endogenously” determined through the assignments made in the first round. Then, the assignment of agent $i$ for a problem under system $\Psi$ is formally defined as:

$$\psi^1_i(I, S^1, S^2, P^1, P^2, q, \succ, h) = \begin{cases} \\
\varphi^1_i(I, S^1, P^1, q|S^1, \succ |S^1, h) \text{ if } \varphi^1_i(I, S^1, P^1, q|S^1, \succ |S^1, h) \neq \emptyset, \\
h(i) \text{ if } \varphi^1_i(I, S^1, P^1, q|S^1, \succ |S^1, h) = \emptyset \text{ and } h(i) \neq \emptyset, \\
\varphi^2_i(I^2, S^2, P^2, q^2, \succ |S^2, h) \text{ otherwise.} \\
\end{cases}$$

A system is straightforward if no agent ever gains by ranking available objects non-truthfully in any round she participates. Formally, for every $i \in I$, every pair $(P'_i, P''_i)$ and
every $P$, we have

$$
\Psi_i(I, S^1, S^2, P|S^1, P|(S^2, I^2), q, \succ, h) R_i \Psi_i(I, S^1, S^2, (P_i', P_{-i})|S^1, (P_i'', P_{-i})|(S^2, I^{2'}), q, \succ, h)
$$

where $I^{2'}$ is the corresponding sets of schools and agents available in Round 2 for preference profile $(P_i', P_{-i})$.

A system is non-wasteful {fair} [individually rational] if for every initial problem $P$, $\Psi(I, S^1, S^2, P|S^1, P|(S^2, I^2), q, \succ, h)$ is a non-wasteful {fair} [individually rational] matching.\footnote{Note that this definition ignores potential strategic behavior of agents in any of the two rounds. This will cause no loss of generality as our analysis will be focusing on straightforward systems.}

We say that $\tilde{\succ}$ is an improvement in the priorities for agent $i \in I$ if \hbox{1.} $i \succ_s j \implies i \tilde{\succ}_s j$ for all $s \in S$, \hbox{2.} there exists at least one agent $j'$ and school $s'$ such that $j \succ_s i \tilde{\succ}_s j'$, and \hbox{3.} $k \succ_s l \iff k \tilde{\succ}_s l$ for all $s \in S$ and $k, l \in I \setminus \{i\}$. A system $\Psi$ respects improvements in the priorities if $\tilde{\succ}$ is an improvement in the priorities for agent $i \in I$, then for any $i \in I$ we have

$$
\Psi_i(I, S^1, S^2, P|S^1, P|(S^2, I^{2'}), q, \tilde{\succ}, h) R_i \Psi_i(I, S^1, S^2, P|S^1, P|(S^2, I^2), q, \succ, h)
$$

where $I^{2'}$ is the corresponding sets of schools and agents available in Round 2 for priority order $\tilde{\succ}$.

In the rest of the paper, whenever there is no ambiguity, we fix the set of agents, objects, quotas, priority orders, and the ownership profile, and represent the outcome of a system for a given problem by $\Psi(P^1, P^2)$. In order to be consistent with two real-life applications, we will use schools instead of objects in the rest of the paper.

### 3 Two Applications

#### 3.1 School Choice Problem with Exam and Regular Schools (SCPwEXRS)

A school choice problem with exam and regular schools, or a problem for short, consists of\footnote{We are using a similar notation with Balinski and Sonmez (1999).}

1. A set of schools $S = \{s_1, s_2, ..., s_m\}$. $S$ is composed of two disjoint sets, i.e. exam and
regular schools. Let \( S^e \) be the set of exam schools and \( S^r \) be the set of regular schools and \( S = S^e \cup S^r \).

2. A capacity vector \( q = (q_s)_{s \in S} \) where \( q_s \) is the number of available seats in \( s \in S \).

3. A set of students \( I = \{i_1, i_2, ..., i_n\} \).

4. A preference profile \( P = (P_i)_{i \in I} \) where \( P_i \) is the strict preference of \( i \) over \( S \cup \emptyset \).

5. A priority order \( \succ = (\succ_s)_{s \in S} \) where \( \succ_s \) is the strict priority order of students in \( I \) for school \( s \).

6. An ownership profile \( h = (h(i))_{i \in I} \) where \( h(i) \) is the school for which agent \( i \) has the property right.

All the available seats in both type of schools are social endowments. Therefore, \( h(i) = \emptyset \) for all \( i \in I \).

Let \( c(i) \) be the test score of applicant \( i \in I \) and \( c \) be the test score profile of all applicants, \( c = (c(i))_{i \in I} \). In the school choice problem with exam and regular schools only the exam schools rank the students based on their exam score. That is, for each \( s \in S^e \) \( i \succ j \) if and only if \( c(i) > c(j) \). On the other hand, the regular schools use some predetermined exogenous rules (proximity, sibling) while ranking students. All exam schools rank students in the same order. However, the priorities of students for any two different regular school do not have to be the same. Let \( \succ \) be the set of all possible priority profiles in this environment. Then for any \( \succ' \in \succ \), \( \succ' = \succ' \) for all \( s, s' \in S^e \).

The current system used in Boston is a serial dictatorship followed by deferred acceptance mechanism (SD-DA) and works as follows:

**Round 1:**

- Only exam schools, \( S^e \), are available for assignment in this round and all students can participate in. Students submit their preferences over the set \( S^e \) and \( \emptyset \). Let \( P^1 = (P^1_i)_{i \in I} \) be the list of submitted preference in round 1. Therefore, the problem considered in round 1 is \((I, S^e, P^1, q|S^e, \succ|S^e, h)\).

---

18 Alternatively, one can define the problem as 5-tuple by excluding \( h \).

19 Here the test score profile is an exogenous rule which is used to determine the priority order.
• Serial dictatorship mechanism is applied to the problem \( (I, S^e, P^1, q|S^e, \succ |S^e, h) \): The agent with the highest score is assigned to his top choice in the list he submitted, the next agent is assigned to his top choice among the remaining schools, and so on.

• Let \( \mu_1 \) denote the assignment in round 1.

**Round 2:**

• The problem considered in round 2 is \( (I^2, S^2, P^2, q^2, \succ |(S^2, I^2), h) \). \( I^2, S^2 \) and \( q^2 \) are calculated as described in Section 2. Note that \( S^2 = S^r \), \( q^2 = (q_s)_{s \in S^r} \) and all the unassigned students in the first round participate in.

• Student proposing deferred acceptance mechanism is used in the placement process:
  
  – Each agent \( i \in I^2 \) applies to the top ranked school in \( P^2_i \). Each school \( s \in S^2 \) tentatively accepts all best offers up to its quota \( q^2_s \) according to its priority order. The rest are rejected.
  
  – In general; each agent \( i \in I^2 \) applies to the highest ranked school in \( P^2_i \) which has not rejected him yet. Each school, which holds tentatively accepted offers or receives new offers in this round, tentatively accepts all best acceptable offers, among the new and previously held ones, up to its quota according to its priority order. The rest are rejected.
  
  – The algorithm terminates when there are no more rejections

• Let \( \mu_2 \) be the final assignment in round 2.

The placement of agent \( i \in I \) induced by the SD-DA is:

\[
\mu(i) = \begin{cases} 
\mu_1(i) & \text{if } \mu_1(i) \neq \emptyset, \\
\mu_2(i) & \text{otherwise}.
\end{cases}
\]

### 3.2 Teacher Assignment Problem (TAP)

A teacher assignment problem, or a problem for short, consists of
1. A set of schools $S = \{s_1, s_2, ..., s_m\}$. $S$ is composed of two disjoint subsets, i.e. contractual and tenured schools. Let $S^c$ be the set of contractual schools and $S^t$ be the set of tenured schools and $S = S^c \cup S^t$. 

2. A capacity vector $q = (q_s)_{s \in S}$ where $q_s$ is the number of available seats in $s \in S$. 

3. A set of students $I = \{i_1, i_2, ..., i_n\}$. $I$ is composed of two disjoint subsets, i.e. existing teachers and new graduates. Let $I^e$ be the set of existing teachers and $I^n$ be the set of new graduates and $I = I^e \cup I^n$. 

4. A preference profile $P = (P_i)_{i \in I}$ where $P_i$ is the strict preference of $i$ over $S \cup \emptyset$. 

5. A priority order $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is the strict priority order of applicants in $I$ for school $s$. 

6. An ownership profile $h = (h(i))_{i \in I}$ where $h(i)$ is the school for which agent $i$ has the property right. 

   Each $i \in I^e$ has property right over a school in $S^c$, $\sum_{i \in I^e} 1(h(i) = s) = q_s$ for all $s \in S^c$. For each new graduate $i \in I^n$ $h(i) = \emptyset$. The number of available seats in $s \in S^c$ is $q_s = |h^{-1}(s)|$. All the available seats in tenured schools are social endowments. 

   The strict priority order of applicants in $I$ for each school $s$, $\succ_s$, is determined according to the centralized test score of each agent and the property rights. Each tenured school $s \in S^t$ ranks applicants based on their test score: $i \succ_s j$ if and only if $c(i) > c(j)$. Each existing teacher currently working in a contractual school $s \in S^c$ is given right to keep his position unless he is assigned to a better school. That is, each contractual school $s \in S^c$ ranks its current teachers at the top of its priority order. Each contractual school $s \in S^c$ considers each existing teacher working in another contractual school as unacceptable. All the new graduates are ranked based on their test score. The priority order of each contractual school $s \in S^c$ as: 

   - For all $i, j \in I$ such that $h(i) = s$, $h(j) \neq s$ then $i \succ_s j$ 
   - For all $i, j \in I$ such that $h(i) = h(j)$, $i \succ_s j$ if and only if $c(i) > c(j)$ 

   All existing teachers are assumed to prefer their current position to $\emptyset$. 

14
• For each \( s \in S^c \) and all \( i \in I^c \) such that \( h(i) \neq s \) then \( \emptyset \succ_s i \)

• For each \( s \in S^c \) and all \( i \in I^n \) and \( j \in I^c \) such that \( h(j) \neq s, i \succ_s j \).

It is worth to mention that if there exist two new graduates \( i, j \in I^n \) and two schools \( s, s' \in S \) such that \( j \succ_s j \) and \( i \succ_{s'} j \) then \( \succ \) cannot be a possible priority profile in this environment, i.e. \( \succ \not\in \succ \).

As one can notice, in the TAP the priority order can be constructed by using the test scores and the ownership profile. Alternatively, we can define the TAP as \((I, S, P, q, c, h)\). To be consistent with the general framework we define problem as \((I, S, P, q, \succ, h)\).

The system that was in use in Turkey until very recently is the **two-round serial dictatorship mechanism** (TSSD) and works as follows:

**Round 1:**

• Only tenured schools, \( S^t \), are available for assignment in this round and all teachers can participate in. Teachers submit their preferences over the set \( S^t \) and \( \emptyset \). Let \( P^1 = (P^1_i)_{i \in I} \) be the list of submitted preference in round 1. Therefore, the problem considered in round 1 is \((I, S^t, P^1, q|S^t, \succ |S^t, h)\).\(^{21}\)

• Serial dictatorship mechanism is applied to the problem \((I, S^t, P^1, q|S^t, \succ |S^t, h)\): The agent with the highest score is assigned to his top choice in the list he submitted, the next agent is assigned to his top choice among the remaining schools, and so on.

• Let \( \mu_1 \) denote the assignment in round 1.

**Round 2:**

• The problem considered in round 2 is \((I^2, S^2, P^2, q^2, \succ |(S^2, I^2), h)\). \( I^2 \), \( S^2 \) and \( q^2 \) are calculated as described in Section 2. Note that \( S^2 = S^c \) and only the unassigned new graduates participate in.\(^{22}\)

\(^{21}\)Since only the tenured schools are considered in this round priority order for all available schools will be the same and it is equivalent to the order of test scores.

\(^{22}\)Since only the new graduates can participate in this round each school \( s \in S^2 \) ranks the agents in \( I^2 \) based on test scores.
Serial dictatorship mechanism is used in the assignment process: The agent with the highest
test score is assigned to his top choice in the list he submitted. The number of available
seats in that school is updated and if it falls to zero that school is removed. The agent with
the second highest test score is assigned to his top choice among the remaining schools,
and so on.

Let \( \mu_2 \) be the final assignment in round 2.

The placement of agent \( i \in I \) induced by the TSSD is:

\[
\mu(i) = \begin{cases} 
\mu_1(i) & \text{if} \; \mu_1(i) \neq \emptyset \\
h(i) & \text{if} \; \mu_1(i) = \emptyset \; \text{and} \; h(i) \neq \emptyset \\
\mu_2(i) & \text{otherwise.}
\end{cases}
\]

4 Deficiencies of General Sequential Systems

We next show that sequential systems, regardless of the specific mechanisms used in each round,
may be inherently flawed. To this end, we offer some impossibility results. In Theorem 1, we
show that any straightforward system is wasteful and therefore not inefficient.

**Theorem 1** There does not exist a straightforward and non-wasteful system.

**Proof.** We prove by contradiction. Suppose \( \Psi(\varphi^1, \varphi^2) \) is straightforward and non-wasteful. We
consider two different cases. In the first case, all schools are social endowment. In the second
case, we allow some schools to be owned by agents.

**Case 1:** There are three schools \( S = \{s_1, s_2, s_3\} \) with one available seat and two agents
\( I = \{i_1, i_2\} \). Let \( S^1 = \{s_2, s_3\} \) and \( h(i_1) = h(i_2) = \emptyset \). The priority structure of each school is
the same and given as \( i_1 \succ_s i_2 \succ_s \emptyset \) for all \( s \in S \). Let the true preferences be as follows:

\[
s_2P_{i_1}s_3P_{i_1}s_1P_{i_1}\emptyset
\]

\[
s_1P_{i_2}s_2P_{i_2}s_3P_{i_2}\emptyset
\]
There is only one non-wasteful matching $\mu_1 = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_2 & i_1 & \emptyset \end{pmatrix}$. In the first round, we have the following problem $(I, S^1, P^1, q|S^1, \succ |S^1, h)$ where $s_2 P_{i_1}^1 s_3 P_{i_1}^1 \emptyset$ and $s_2 P_{i_2}^1 s_3 P_{i_1}^1 \emptyset$. The matching selected in round 1 is $\varphi_{i_1}^1(I, S^1, P^1, q|S^1, \succ |S^1, h) = s_2$, $\varphi_{i_2}^1(I, S^1, P^1, q|S^1, \succ |S^1, h) = \emptyset$.

Now consider the following preference profile:

$\begin{pmatrix} s_2 \overline{P}_{i_1} s_3 \overline{P}_{i_1} s_1 P_{i_1} \emptyset \\ s_2 \overline{P}_{i_2} s_3 \overline{P}_{i_2} s_1 P_{i_2} \emptyset \end{pmatrix}$

There are two non-wasteful matchings $\mu_2 = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & i_1 & i_2 \end{pmatrix}$ and $\mu_3 = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & i_2 & i_1 \end{pmatrix}$. In the first round, we have the following problem $(I, S^1, \overline{P}^1, q|S^1, \succ |S^1, h)$ where $s_2 \overline{P}_{i_1}^1 s_3 \overline{P}_{i_1}^1 \emptyset$ and $s_2 \overline{P}_{i_2}^1 s_3 \overline{P}_{i_2}^1 \emptyset$. Note that $(I, S^1, \overline{P}^1, q|S^1, \succ |S^1, h) = (I, S^1, P^1, q|S^1, \succ |S^1, h)$ but $\varphi^1(I, S^1, \overline{P}^1, q|S^1, \succ |S^1, h) \neq \varphi^1(I, S^1, P^1, q|S^1, \succ |S^1, h)$.

Case 2: Consider the same example. We only change the example by giving the property rights of school $s_1$ to $i_1$. That is, $h(i_1) = s_1$ and $h(i_2) = \emptyset$. The non-wasteful matchings for both preference profiles will be the exact matching that we get in Case 1. Therefore, we have contradiction in this case too.

Since Pareto efficiency implies non-wastefulness, Theorem 1 has an immediate corollary.

**Corollary 1** There does not exist a straightforward and Pareto efficient system.

Note that a system which always select a null matching in which all agents are unassigned is fair and respects improvements. In Theorem 2 we show that when the mechanisms used in both rounds are non-wasteful then there does not exist any straightforward system which is fair. Similarly, in Theorem 3, we show that when the mechanisms used in both rounds are non-wasteful and the mechanism used in the first round is fair then any straightforward system fails to respect improvements in the priority order.

**Theorem 2** Let $\Psi = (\varphi^1, \varphi^2)$ be a system. If $\varphi^1$ and $\varphi^2$ are non-wasteful and $\Psi$ is straightforward, then $\Psi$ cannot be fair.
Proof. As in the proof of Theorem 1, we consider two different cases. In the first case, all schools are social endowment. In the second case, we allow some schools to be owned by agents.

Case 1: There are three schools $S = \{s_1, s_2, s_3\}$ with one available seat and three agents $I = \{i_1, i_2, i_3\}$. Let $S^1 = \{s_2, s_3\}$, and $h(i_1) = h(i_2) = h(i_3) = \emptyset$. The priority structure of each school in $S$ is the same and given as $i_1 \succ_s i_2 \succ_s i_3 \succ_s \emptyset$ for all $s \in S$. Let the true preferences be as follows:

$$s_2 P_{i_1} s_3 P_{i_1} s_1 P_{i_1} \emptyset$$
$$s_1 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset$$
$$s_1 P_{i_3} s_2 P_{i_3} s_3 P_{i_3} \emptyset$$

When all agents act truthfully any fair system composed of two non-wasteful mechanisms will select: $\mu_1 = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_2 & i_1 & i_3 \end{pmatrix}$. In the first round, we have the following problem $(I, S^1, P^1, q|S^1, \succ |S^1, h)$ where $s_2 P_{i_1} s_3 P_{i_1} \emptyset$, $s_2 P_{i_2} s_3 P_{i_2} \emptyset$ and $s_2 P_{i_3} s_3 P_{i_3} \emptyset$. The matching selected in round 1 is $\varphi^1_{i_1}(I, S^1, P^1, q|S^1, \succ |S^1, h) = s_2$, $\varphi^1_{i_2}(I, S^1, P^1, q|S^1, \succ |S^1, h) = \emptyset$ and $\varphi^1_{i_3}(I, S^1, P^1, q|S^1, \succ |S^1, h) = s_3$.

Now consider the following preference profile:

$$s_2 \overline{P}_{i_1} s_3 \overline{P}_{i_1} s_1 \overline{P}_{i_1} \emptyset$$
$$s_2 \overline{P}_{i_2} s_3 \overline{P}_{i_2} s_1 \overline{P}_{i_2} \emptyset$$
$$s_1 \overline{P}_{i_3} s_2 \overline{P}_{i_3} s_3 \overline{P}_{i_3} \emptyset$$

When all agents act truthfully any fair system composed of two non-wasteful mechanisms will select: $\mu_2 = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_3 & i_1 & i_2 \end{pmatrix}$. In the first round, we have the following problem $(I, S^1, \overline{P}^1, q|S^1, \succ |S^1, h)$ where $s_2 \overline{P}_{i_1} s_3 \overline{P}_{i_1} \emptyset$, $s_2 \overline{P}_{i_2} s_3 \overline{P}_{i_2} \emptyset$ and $s_2 \overline{P}_{i_3} s_3 \overline{P}_{i_3} \emptyset$. Note that $(I, S^1, \overline{P}^1, q|S^1, \succ |S^1, h) = (I, S^1, P^1, q|S^1, \succ |S^1, h)$ but $\varphi^1(I, S^1, P^1, q|S^1, \succ |S^1, h) \neq \varphi^1(I, S^1, P^1, q|S^1, \succ |S^1, h)$.

Case 2: Consider the same example. We only change the example by giving the property rights of school $s_1$ to $i_1$ and keep everything else the same. That is, $h(i_1) = s_1$. Given the
problems observed in the first rounds are the same as in the Case 1 we will observe the same problem in the second round as in the Case 1. Hence, the same matchings will be selected in both preference profile as in the Case 1 and we can show the contradiction. ■

**Theorem 3** Let \( \Psi = (\varphi^1, \varphi^2) \) be a straightforward system. If \( \varphi^1 \) is non-wasteful and fair and \( \varphi^2 \) is non-wasteful, then \( \Psi \) does not respect improvements in the priority order.

**Proof.** We prove by contradiction. Suppose \( \Psi(\varphi^1, \varphi^2) \) is straightforward, \( \varphi^1 \) is non-wasteful and fair and \( \varphi^2 \) is non-wasteful and \( \Psi \) respects improvements. We consider two different cases.

**Case 1:** There are three schools \( S = \{s_1, s_2, s_3\} \) with one available seat and three agents \( I = \{i_1, i_2, i_3\} \). Let \( S^1 = \{s_2, s_3\} \), and \( h(i_1) = h(i_2) = h(i_3) = \emptyset \). The priority structure of each school in \( S \) is the same and given as \( i_1 \succ_s i_2 \succ_s i_3 \succ_s \emptyset \) for all \( s \in S \). Let the true preferences be as follows:

\[
\begin{align*}
s_2 & P_{i_1} s_3 P_{i_1} s_1 P_{i_1} \emptyset \\
s_1 & P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset \\
s_1 & P_{i_3} s_2 P_{i_3} s_3 P_{i_3} \emptyset
\end{align*}
\]

When all agents act truthfully any straightforward system with mechanisms as described in the statement will select: \( \mu_1 = \left( \begin{array}{ccc} s_1 & s_2 & s_3 \\ i_3 & i_1 & i_2 \end{array} \right) \).

Now consider the following priority order: \( i_1 \succ_s' i_3 \succ_s' i_2 \succ_s' \emptyset \) for all \( s \in S \). When all agents act truthfully any straightforward system with mechanisms as described in the statement will select: \( \mu_2 = \left( \begin{array}{ccc} s_1 & s_2 & s_3 \\ i_2 & i_1 & i_3 \end{array} \right) \). Since, \( \mu_1(i_3)P_{i_3}\mu_2(i_3) \Psi \) does not respect the improvement of \( i_3 \).

**Case 2:** Consider the same example. We only change the example by giving the property rights of school \( s_1 \) to \( i_1 \) and keep everything else the same. That is, \( h(i_1) = s_1 \). Given the problems observed in the first rounds are the same as in the Case 1 we will observe the same problem in the second round as in the Case 1. Hence, the same matchings will be selected in both preference profile as in the Case 1 and we can show the contradiction. ■
In Theorem 4, we show that if a non-wasteful mechanism is used in both rounds and the mechanism used in the first round is individually rational, then that system is vulnerable to manipulation and fails to be Pareto efficient, or even non-wasteful.

**Theorem 4** Let $\Psi = (\varphi^1, \varphi^2)$ be a system. If $\varphi^1$ is non-wasteful and individually rational and $\varphi^2$ is non-wasteful then $\Psi$ fails to be straightforward.

**Proof.** We consider two different cases. In the first case, all schools are social endowment. In the second case, we allow some schools to be owned by agents.

**Case 1:** There are three schools $S = \{s_1, s_2, s_3\}$ with one available seat and two agent $I = \{i_1, i_2\}$. Let $S^1 = \{s_2, s_3\}$, and $h(i_1) = h(i_2) = \emptyset$. The priority structure of each school is the same and given as $i_1 >_s i_2 >_s \emptyset$ for all $s \in S$. Let the true preferences be as follows:

$$s_2P_{i_1}s_3P_{i_1}s_1P_{i_1}\emptyset$$

$$s_1P_{i_2}s_2P_{i_2}s_3P_{i_2}\emptyset$$

In the first round, if both agents act truthfully and submit their ranking lists by keeping the relative order of available schools and $\emptyset$ in $P$ then there are two non-wasteful and individually rational matchings: $\mu_1^1 = \begin{pmatrix} s_2 & s_3 \\ i_1 & i_2 \end{pmatrix}$ and $\mu_1^2 = \begin{pmatrix} s_2 & s_3 \\ i_2 & i_1 \end{pmatrix}$. No matter which one of these two matchings is selected in the first round none of the agents can participate the second round and $s_1$ is available in the second round. Therefore, the unique non-wasteful matching selected in the second period is $\mu_2^1 = \mu_2^2 = \begin{pmatrix} s_1 \\ \emptyset \end{pmatrix}$. That is, any system satisfying conditions mentioned in the statement of Theorem 4 assigns $i_2$ to either $s_2$ or $s_3$. Let $\tilde{\Psi}$ be a system selecting $(\mu_1^1, \mu_1^1)$ and $\overline{\Psi}$ be a system selecting $(\mu_1^2, \mu_2^2)$. That is, the outcome of $\tilde{\Psi}$ is $\mu^1 = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & i_1 & i_2 \end{pmatrix}$ and the outcome selected by $\overline{\Psi}$ is $\mu^2 = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & i_2 & i_1 \end{pmatrix}$.

Suppose $i_2$ submits the following preference list in the first round: $\emptyset P_{i_2}s_2P_{i_2}s_3$. There is a unique individually rational and non-wasteful matching in round 1: $\mu_1^1 = \begin{pmatrix} s_2 & s_3 \\ i_1 & \emptyset \end{pmatrix}$ and both
\( \Psi \) and \( \Psi \) select \( \mu' \). Based on the matching selected in the first round \( i_2 \) can participate second round and \( s_1 \) is available in the second round. When \( i_2 \) submits \( P''_{i_2} \) : \( s_1P''_{i_2}\emptyset \) then there is a unique non-wasteful matching: \( \mu'_2 = \left( \frac{s_1}{i_2} \right) \). That is, \( (P'_{i_1}, P''_{i_1}) \) pair is a profitable deviation for \( i_2 \) under \( \Psi \) and \( \Psi \). Note that \( P''_2 \) is \( i_1 \)'s true relative order over the available schools in the second round.

**Case 2:** Consider the same example. We only change the example by giving the property rights of school \( s_1 \) to \( i_1 \). That is, \( h(i_1) = s_1 \) and \( h(i_2) = \emptyset \). Since the set of available schools in round 1 is not changed then in the first round any non-wasteful matching \( i_1 \) will be assigned to another school and he will give up his property rights for \( s_1 \). Therefore, \( s_1 \) will be socially endowed in the second period as in the Case 1. One can follow the steps in Case 1 and show that the impossibility result is robust to the ownership structure, i.e. whether all schools are socially endowed or not.

Recall that when defining a system we impose restrictions. For instance, the unfilled seats of \( s \in S^1 \) are not available in round 2 or if an agent with endowment is not assigned to a school in \( S^1 \) cannot participate the second round. It is worth to note that these restrictions do not play any role in obtaining the above results. Indeed, Theorem 2, 3 and 4 hold in the absence of these restrictions.

**Remark 1** Theorem 2, 3 and 4 hold in the following cases:

1) The schools in \( S^1 \) with unfilled seats are available in the second round.
2) Unassigned agents with endowment are allowed to participate the second round.
3) Certain portion of the schools are available and not all students are allowed to participate in the first round.

If we use a non-wasteful mechanism in the first round then Theorem 1 holds in the absence of these restrictions.

**Remark 2** When a non-wasteful mechanism is used in the first round Theorem 1 holds in the following cases:

1) The schools in \( S^1 \) with unfilled seats are available in the second round.
2) Unassigned agents with endowment are allowed to participate the second round.

3) Certain portion of the schools are available and not all students are allowed to participate in the first round.

In TAP and SCPwEXRS different combinations of DA and SD mechanisms are used. Both DA and SD are non-wasteful and individually rational. Hence, Theorem 4 has the following immediate corollary for the two applications we have considered.

**Corollary 2** SD-DA used in SCPwEXRS and TSSD used in TAP are vulnerable to manipulation

We focus on the case where agents report their roundwise true preferences over the available schools. In Proposition 1 we show the deficiencies of both systems in practice.

**Proposition 1** SD-DA used in SCPwEXRS and TSSD used in TAP are wasteful, not fair and do not respect improvement in the priority order (test scores).

**Proof.** Proof follows from the proofs of Theorem 1, 2 and 3. ■

In the TAP, the sequence in which assignment is done cannot be changed since in order to have an available contractual position first a contractual teacher should be assigned to another school. On the other hand, in the school choice system the order can be changed by first assigning students to the regular schools and then assign the remaining students to the exam schools. Then, one can wonder whether the deficiencies of the sequential system used in school choice system are a consequence of considering exam schools before the regular school. It is easy to show that if we first consider the regular schools then the exam schools then we face the same deficiencies. Under this case the mechanism used to place students will be deferred acceptance followed by serial dictatorship mechanism (DA-SD).

**Corollary 3** DA-SD in SCPwEXRS is manipulable, wasteful, not fair, leads to avoidable welfare loss and does not respect improvements in the priority order.

In the Appendix A we present two examples to illustrate how SD-DA and TSSD mechanisms fail to satisfy the desired properties.
5 Equilibrium Analysis of the Preference Revelation Games

In Section 4, we have shown that the systems in Turkey and the U.S. are not straightforward. We further argue that it may not be difficult for the agents to identify strategies to manipulate these systems. In this section we first investigate possible ways of manipulation under the sequential systems. Then, we turn to analyzing the properties of the preference revelation games associated with the current systems. Since both systems are composed of two rounds we consider the subgame perfect Nash equilibrium (SPNE) as the main solution concept. We analyze the games under complete information of payoffs, available strategies and priorities. Agents are assumed to play simultaneously and the outcome of the first round is publicly announced.

We first show that, in the general setting, if an agent can gain from misreporting, then this implies that she will also be weakly better off by declaring all available schools unacceptable in the first round, and reporting her true relative-preferences over the available schools in the second round.

Denote €’s true relative preference over the available schools in round $t \in \{1, 2\}$ including $\emptyset$ with $P^t_i$.

**Proposition 2** Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that both $\varphi^1$ and $\varphi^2$ are strategy-proof, individually rational and $\varphi^1$ is non-wasteful and weak population monotonic. If there exists a preference pair $(Q^1_i, Q^2_i)$ such that $\Psi_i((Q^1_i, P^1_{-i}), (Q^2_i, P^2_{-i})) \neq P_i \Psi_i(P^1, P^2)$ then $\Psi_i((\tilde{Q}^1_i, P^1_{-i}), (Q^2_i, P^2_{-i}))$ where $\tilde{Q}^1_i = \emptyset \tilde{Q}^1_i x$ for all $x \in S^1$ and $\tilde{Q}^2_i = P^2_i$.

**Proof.** Suppose $\Psi_i((Q^1_i, P^1_{-i}), (Q^2_i, P^2_{-i})) P_i \Psi_i(P^1, P^2)$ and $\Psi_i((Q^1_i, P^1_{-i}), (Q^2_i, P^2_{-i})) P_i \Psi_i((\tilde{Q}^1_i, P^1_{-i}), (\tilde{Q}^2_i, P^2_{-i}))$. Due to individual rationality $\Psi_i(P^1, P^2) R_i \emptyset$. Hence, $\Psi_i((Q^1_i, P^1_{-i}), (Q^2_i, P^2_{-i})) P_i \emptyset$ and $\Psi_i((Q^1_i, P^1_{-i}), (Q^2_i, P^2_{-i})) \in S$. We consider two possible cases.

**Case 1:** $\Psi_i((Q^1_i, P^1_{-i}), (Q^2_i, P^2_{-i})) \in S^1$. In this case, $\varphi^1_i(Q^1_i, P^1_{-i}) = \Psi_i((Q^1_i, P^1_{-i}), (Q^2_i, P^2_{-i}))$ and $i$ does not participate in the second round. Since $\varphi^1$ is strategy-proof and $\varphi^1_i(Q^1_i, P^1_{-i}) P_i \emptyset$, $\varphi^1_i(P^1) R_i \varphi^1_i(Q^1_i, P^1_{-i}) P_i \emptyset$. Hence, $\Psi_i(P^1, P^2) = \varphi^1_i(P^1) R_i \Psi_i((Q^1_i, P^1_{-i}), (Q^2_i, P^2_{-i}))$. This is a contradiction.

**Case 2:** $\Psi_i((Q^1_i, P^1_{-i}), (Q^2_i, P^2_{-i})) \in S^2$. In this case, $\varphi^2_i(Q^2_i, P^2_{-i}) = \Psi_i((Q^1_i, P^1_{-i}), (Q^2_i, P^2_{-i}))$ and $\varphi^1_i(Q^1_i, P^1_{-i}) = \emptyset$. Due to individual rationality, $i$ is unassigned when he submits $\tilde{Q}^1_i = \emptyset \tilde{Q}^1_i x$ for all $x \in S^1$.
If \( x \in S^1 \), then \( \varphi_j^1(Q^1_i, P^1_{-i}) \in S^1 \) due to individual rationality \((\varphi_j^1(Q^1_i, P^1_{-i}) \in S^1)\) and population monotonicity \((\varphi_j^1(Q^1_i, P^1_{-i}) \in S^1)\). Moreover, if \( \varphi_j^1(Q^1_i, P^1_{-i}) \neq \varphi_j^1(\tilde{Q}^1_i, P^1_{-i}) \), then \( \varphi_j^1(\tilde{Q}^1_i, P^1_{-i}) \) should have filled all its available seats in matching \( \varphi^1(Q^1_i, P^1_{-i}) \). Otherwise non-wastefulness of \( \varphi^1 \) is violated. That is, only the agents who are assigned to a school in \( \varphi^1(Q^1_i, P^1_{-i}) \) become better off in \( \varphi^1(\tilde{Q}^1_i, P^1_{-i}) \). Hence, the same set of students are assigned to a school in \( S^1 \) and each school fills the same number of seats in \( \varphi^1(Q^1_i, P^1_{-i}) \) and \( \varphi^1(\tilde{Q}^1_i, P^1_{-i}) \). In other words, the same set of agents will participate the second round and the quotas of each school in \( S^2 \) will be the same when \( i \) submits \( Q^1_i \) and \( \tilde{Q}^1_i \). Due to the strategy-proofness \( i \) cannot get a better school than \( \varphi^2_i(Q^2_i = P^2_i, P^2_{-i}) \) in \( \varphi^2(Q^2_i, P^2_{-i}) \). Therefore, \( \Psi_i((\tilde{Q}^1_i, P^1_{-i}), (\tilde{Q}^2_i, P^2_{-i})) \in \Psi_i((Q^1_i, P^1_{-i}), (Q^2_i, P^2_{-i})) \). This contradicts is a contradiction.

Note that the mechanisms used in the first and the second rounds of TSSD, SD-DA and DA-SD satisfy the conditions mentioned in Proposition 2. Therefore, for a given problem if an agent has a way of manipulating these sequential assignment systems, then she can equivalently extract all the benefits of that manipulation by ranking \( \emptyset \) as the first choice in the first round and acting truthfully in the second round.

**Corollary 4** If an agent benefits from misreporting under TSSD in TAP or SD-DA (DA-SD) in SCPwEXRS then ranking \( \emptyset \) as the first choice in the first round and acting truthfully in the second round extracts all benefits from manipulation.

In Section 4 we have shown that TSSD is not straightforward. However, Proposition 2 shows that not all of the applicants can benefit from misreporting their preferences. In particular, existing teachers cannot benefit from misreporting. Because when an existing tenant ranks \( \emptyset \) as her first choice, she gets her current position, which is the worst outcome she receives under truth-telling.

**Corollary 5** Under TSSD, existing teachers cannot benefit from misreporting.

In Proposition 3, we show that if a system satisfies the conditions in Proposition 3, then the only way to manipulate that system is to truncate the relative preference over the available schools in Round 1.
Proposition 3 Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that both $\varphi^1$ and $\varphi^2$ are strategy-proof and $\varphi^1$ is individually rational, non-wasteful and weak population monotonic. If whenever there exists a preference pair $(Q^1, Q^2)$ such that $\Psi_i((Q^1_i, P^1_i), (Q^2_i, P^2_i))P_i\Psi_i(P^1, P^2)$ then there exists a school $s \in S^1$ where $\emptyset Q^1_s$ and $sP^1 \emptyset$.

Proof. By contradiction, we show that there does not exist a profitable deviation in which all the acceptable schools under the true preference are ranked above being unassigned option in the reported preference list of the first round. We consider two cases: $\varphi^1_i(P^1) = \emptyset$ and $\varphi^1_i(P^1) \neq \emptyset$.

Case 1: Due to strategy-proofness $\varphi^1_i(Q^1_i, P^1_i) = \emptyset$. If $h(i) \neq \emptyset$ then $\Psi_i((Q^1_i, P^1_i), (Q^2_i, P^2_i)) = \Psi_i(P^1, P^2)$. If $h(i) = \emptyset$ then consider the preference profile $\tilde{Q}^1_i : \emptyset Q^1_i s$ for all $s \in S^1$. Due to non-wastefulness and population monotonicity, the same set of students are assigned to schools in $S^1$ and each school fills the same number of seats under $\varphi^1(\tilde{Q}^1_i, P^1_i)$, $\varphi^1(P^1)$ and $\varphi^1(Q^1_i, P^1_i)$. Then the same set of agents will participate in the second round and the quotas of each school in $S^2$ will be the same when $i$ submits $P^1_i, Q^1_i$ and $\tilde{Q}^1_i$. Since $\varphi^2$ is strategy-proof, $i$ cannot be assigned to a better school than $\Psi_i(P^1, P^2) = \varphi^2(P^2)$.

Case 2: Due to strategy-proofness if $\varphi^1_i(Q^1_i, P^1_i) = \emptyset$ then $\Psi_i((Q^1_i, P^1_i), (Q^2_i, P^2_i))$ then $\varphi^1_i(P^1) = \Psi_i(P^1, P^2) R_i \Psi_i((Q^1_i, P^1_i), (Q^2_i, P^2_i))$. That is, there does not exist a profitable deviation in which $i$ is assigned a school in $S^1$. On the other hand $\varphi^1_i(P^1_i) = \emptyset$ cannot be true due to the strategy-proofness of $\varphi^1$, i.e. $P^1_i$ would be a profitable deviation for someone whose real preference is $Q^1_i$ and that agent will get $\varphi^1_i(P^1)$. Note that $\varphi^1_i(P^1)P^1_i \emptyset$ since $\varphi^1$ is individually rational. By our construction $\varphi^1_i(P^1)Q^1_i \emptyset$. Hence, an agent with preference profile $Q^1_i$ can be better off by submitting $P^1$. 

Remark 3 Proposition 2 and 3 hold in the following cases:

1) The schools in $S^1$ with unfilled seats are available in the second round.

2) Unassigned agents with endowment are allowed to participate the second round.

Both DA and SD mechanisms are strategy-proof, non-wasteful, population monotonic and individually rational. Therefore, all three systems, TSSD, DA-SD and SD-DA, satisfy the conditions mentioned in Proposition 3. Hence, truncating the preference list over the schools in round 1 is the only way to manipulate TSSD in TAP or, SD-DA (DA-SD) in SCPwEXRS.
Corollary 6 If an agent benefits from misreporting under TSSD in TAP or SD-DA (DA-SD) in SCPwEXRS, then that agent truncates his preference list of acceptable schools in round 1 by excluding at least one acceptable school.

Now we are ready to start our equilibrium analysis. In the following theorems we provide general results on the equilibrium analysis of systems.

Theorem 5 Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that

- $\varphi^1$ is individually rational, non-wasteful and either fair or [population monotonic and weak non-bossy].
- $\varphi^2$ is individually rational, non-wasteful and either fair or [monotonic, independent of irrelevant agents and weak non-bossy].

Every SPNE outcome of the preference revelation game associated with $\Psi$ leads to a non-wasteful and individually rational matching.

Proof. Let $Q = (Q^1_i, Q^2_i)_{i \in I}$ be an SPNE profile and $\mu$ be the associated equilibrium outcome. First note that for any $i \in I$ we cannot have $\emptyset \succ_{\mu(i)} i$ because both $\varphi^1$ and $\varphi^2$ are individually rational. If $\mu$ is individually irrational then there exists $i \in I$ such that $\emptyset P_i \mu(i)$. If $\emptyset P_i \mu(i)$ then submitting $P'_i = \emptyset P'_i x$ for all $x \in S_i$ in round $t \in \{1, 2\}$ is a profitable deviation for agent $i$. Therefore, $Q$ cannot be SPNE profile which is a contradiction.

Suppose $\mu$ is wasteful. Then, there exists $i \in I$ such that $s P_i \mu(i), i \succ s \emptyset$ and $|\mu^{-1}(s)| < q_s$. We consider two cases and we show that if $\mu$ is wasteful then $i$ can profitably deviate.

Case 1: Suppose $s \in S^1$. Since $\varphi^1$ is non-wasteful, $\emptyset Q^1_i s$. Consider following preference profile $P'_i = s P'_i \emptyset P'_i x$ for all $x \in S^1 \{s\}$. Denote $\varphi^1(P'_i, Q^1_i)$ by $\mu_1$. Due to the individual rationality either $\mu_1(i) = s$ or $\mu_1(i) = \emptyset$. We consider two subcases:

$\varphi^1$ is individually rational, non-wasteful and fair. By rural hospital theorem (Roth, 1986) the same set of students are assigned to schools and each school fills the same number of seats in all fair, non-wasteful and individually rational matchings. Than consider the outcome of sequential DA mechanism (McVitie and Wilson, 1971) where student $i$ applies after all students
are tentatively assigned. Since DA is population monotonic\textsuperscript{23} the number of students tentatively assigned to $s$ before $i$’s turn is less than $q_s$. When it is $i$’s turn he will be assigned to $s$. Hence, $\mu_1(i) \in S^1$ and this school is $s$ which is the only acceptable one in $P_i'$.

$\varphi^1$ is individually rational, non-wasteful, population monotonic and weak non-bossy: If $\mu_1(i) = s$ then $(P'_i, Q^2_i)$ is a profitable deviation for $i$. If $\mu_1(i) = \emptyset$ then $|\mu_1^{-1}(s)| = q_s$. Otherwise, $\mu_1$ is wasteful. Let $\bar{I} = \{j \in I | \mu(j) \neq \mu_1(j) = s\}$. Since $|\mu_1^{-1}(s)| = q_s$ and $|\mu^{-1}(s)| < q_s$, $\bar{I} \neq \emptyset$. For all $j \in \bar{I}$ we have $\mu(j) Q^1_j s Q^1_j 0$. Otherwise, $\varphi^1$ cannot be non-wasteful or individually rational. Now consider problem $(\mathcal{P}_i, Q^1_{i-})$ where $0 \in \mathcal{P}_i x$ for all $x \in S^1$. Due to the non-bossiness and individual rationality $\varphi^1(\mathcal{P}_i, Q^1_{i-}) = \mu_1$. When we consider problems $(\mathcal{P}_i, Q^1_{i-})$ and $(Q^1_i, Q^1_{i-})$ due to population monotonicity all students should weakly prefer $\mu_1$ to $\mu$. However, agents in $\bar{I}$ prefer $\mu$ to $\mu_1$. This violates population monotonicity.

**Case 2:** Suppose $s \in S^2$. If $\mu(i) \in S^2 \cup \emptyset$ then by using the same steps in Case 1 one can show that $(Q^1_i, P''_i)$ is a profitable deviation for $i$ where $P''_i = s P''_i 0 P''_i x$ for all $x \in S^2 \setminus \{s\}$. If $\mu(i) \in S^1$ then we show that $(\mathcal{P}_i, P''_i)$ is a profitable deviation for $i$. Since $\varphi^1$ is individually rational and fair and/or population monotonic, if $\varphi^1_j(Q^1_i) \in S^1$ then $\varphi^1_j(\mathcal{P}_i, Q^1_{i-}) \in S^1$. Therefore, in the second round the set of agents is a subset of $I^2 \cup \{i\}$ and set of available seats weakly increases compared to the case in which $i$ plays $Q^1_i$. Let $I'_2$, $\mu_2$ and $\bar{q}^2$ be the set of agents, matching selected and quota vector in round 2 when $i$ submits $(\mathcal{P}_i, P''_i)$, respectively. Due to individual rationality, $\mu_2(i)$ is either $s$ or $\emptyset$. We show that $\mu_2(i)$ cannot be $\emptyset$. Suppose $\mu_2(i) = \emptyset$. Let $\bar{I}'_2 = \{j \in I'_2 | \mu(j) \neq \mu_2(j) = s\}$. Since $\varphi^2$ is non-wasteful, $|\mu_2^{-1}(s)| = q_s$, $\bar{I}'_2 \neq \emptyset$ and $\mu(j) Q^2_j \mu_2(j)$ for all $j \in \bar{I}'_2$. We consider two subcases:

$\varphi^2$ is individually rational, non-wasteful and fair: The students who participate in round 2 except $i$ cannot fill all the available seats of $s$. This follows from the same argument in Case 1.

$\varphi^2$ is individually rational, non-wasteful, non-bossy, monotonic and independent of irrelevant agent. Due to non-bosiness and individual rationality $\mu_2$ will be selected when $i$ submits $\mathcal{P}_i^2 = 0 P^2_i x$ for all $x \in S^2$. Let $\mu'_2$ be the outcome of $\varphi^2$ when we only consider agents in $I'_2 \setminus \{i\}$ keeping everything else the same. Due to monotonicity and independence of irrelevant

\textsuperscript{23}In fact, when the number of agents decreases all the remaining agents are assigned to weakly better schools by DA.
agent, $\mu_2'(j) = \mu_2(j)Q_2^j\mu(j)$ for all $j \in I_2 \setminus \{i\}$. This contradicts with the fact that for all $k \in \bar{I}_2$, which is a subset of $I_2 \setminus \{i\}$, $\mu(k)Q_2^k\mu(k)$. ■

Although many well-known mechanisms satisfy the conditions in Theorem 5, the celebrated top trading cycles (TTC) is not one of them. Indeed, we may have a wasteful equilibrium if we use the TTC mechanism in the second round. We illustrate this point in the following example.\(^{24}\)

**Example 1** Let $S = \{s_1, s_2, s_3, s_4\}$, $S^1 = \{s_1\}$, $q = (1, 1, 1, 1)$ $I = \{i_1, i_2, i_3, i_4\}$ and $h(i_1) = h(i_2) = h(i_3) = h(i_4) = \emptyset$. Priorities and preferences are given as

<table>
<thead>
<tr>
<th>$\succ s_1$</th>
<th>$\succ s_2$</th>
<th>$\succ s_3$</th>
<th>$\succ s_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_1$</td>
<td>$i_2$</td>
<td>$i_3$</td>
<td>$i_3$</td>
</tr>
<tr>
<td>$i_2$</td>
<td>$i_4$</td>
<td>$i_2$</td>
<td>$i_2$</td>
</tr>
<tr>
<td>$i_3$</td>
<td>$i_3$</td>
<td>$i_1$</td>
<td>$i_1$</td>
</tr>
<tr>
<td>$i_4$</td>
<td>$i_1$</td>
<td>$i_4$</td>
<td>$i_4$</td>
</tr>
<tr>
<td>$P_{i_1}$</td>
<td>$P_{i_2}$</td>
<td>$P_{i_3}$</td>
<td>$P_{i_4}$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$s_1$</td>
<td>$s_2$</td>
<td>$s_2$</td>
</tr>
<tr>
<td>$s_1$</td>
<td>$s_3$</td>
<td>$s_4$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

Let $\Psi = (\varphi^1, \varphi^2)$ be a system where $\varphi^1$ is a serially fair mechanism and $\varphi^2$ is TTC. Consider the following strategy profile:

- In the first round students submit their true preferences over $S^1 \cup \emptyset$: $s_1P_{i_1}^1\emptyset$, $s_1P_{i_2}^1\emptyset$, $\emptyset P_{i_3}^1 s_1$ and $\emptyset P_{i_4}^1 s_1$.

- Students participating in the second round submit true preference over the available schools.

Under this preference profile $\Psi$ will select the following matching: $\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & \emptyset \\ i_1 & i_3 & i_2 & \emptyset & i_4 \end{pmatrix}$. One can verify that any student cannot get a better allocation when he deviates and all other students keep their strategy, $(P^1, P^2)$ is SPNE. In $\mu$ the seat in $s_4$ is wasted. ■

Using Example 1, we can further show that any equilibrium of $\Psi = (\varphi^1, \varphi^2)$ all SPNE outcome of the preference revelation game associated with $\Psi$ is wasteful, where $\varphi^1$ is a fair mechanism and $\varphi^2$ is TTC.

**Theorem 6** Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that

\(^{24}\)Once again, we do not describe this mechanism for brevity.
• $\varphi^1$ is individually rational, non-wasteful, mutually fair and able favoring higher ranks, or

• $\varphi^1$ is individually rational, non-wasteful, population monotonic, weakly non-bossy and fair.

Every SPNE outcome of the preference revelation game associated with $\Psi$ leads to a matching $\mu$ in which there does not exist $(i,j)$ pair such that $\mu(j) \in S^1$, $\mu(j) P_i \mu(i)$ and $i \succ_{\mu(j)} j$.

**Proof.** Let $Q = (Q^1_i, Q^2_i)_{i \in I}$ be an SPNE profile and $\mu$ be the associated equilibrium outcome. First note that $\mu(k) R_k h(k)$ for all $k \in I$. Otherwise, $Q$ cannot be SPNE (Theorem 5). Suppose there exist two agents $i, j \in I$ such that $\mu(j) \in S^1$, $\mu(j) P_i \mu(i)$ and $i \succ_{\mu(j)} j$. Let $\mu(j) = s$, $I' = \{i' \in I | s P_{e} \mu(i') \text{ and } i' \succ_s j\}$ and $\hat{i} \in I'$ be the student who has the highest priority for $s$ among the ones in $I'$. We claim that submitting $Q' : s Q' 0 Q' x$ for all $x \in S^1 \setminus \{s\}$ in Round 1 is a profitable deviation for $\hat{i}$.

Let $\varphi^1(Q^1) = \mu_1$, $\varphi^1(Q', Q^1_{-\hat{i}}) = \bar{\mu}_1$ and $\bar{I}_1 = \{k \in I | \mu_1(k) \neq \bar{\mu}_1(k) = s\}$. Since $\varphi^1$ is individually rational then either $\bar{\mu}_1(\hat{i}) = s$ or $\bar{\mu}_1(\hat{i}) = \emptyset$. If $\bar{\mu}_1(\hat{i}) = s$ then $Q'$ is a profitable deviation for $\hat{i}$. If $\bar{\mu}_1(\hat{i}) = \emptyset$ then $|\bar{\mu}_1(\hat{i})| = q_s$, $k \succ_s \hat{i}$ for all $k \in \bar{\mu}_1(\hat{i})$, $\bar{\mu}_1(j) \neq s$ and $\bar{I}_1 \neq \emptyset$. Otherwise mutual fairness\textsuperscript{25} and/or non-wastefulness of $\varphi^1$ would be violated. Suppose $\bar{\mu}_1(\hat{i}) = \emptyset$.

$\varphi^1$ favors higher ranks and mutually fair: Since $\varphi^1$ favors higher ranks, $sQ^1_k x$ for all $x \in S^1 \cup \emptyset$ and $k \in \bar{I}_1$. Then, $\mu_1$ cannot be mutually fair.

$\varphi^1$ is fair, weak non-bossy and population monotonic: Consider the profile $(Q'', Q^1_{-\hat{i}})$ where $Q'' : \emptyset Q'' x$ for all $x \in S^1$. Due to weak non-bossiness and individual rationality $\varphi^1(Q'', Q^1_{-\hat{i}}) = \bar{\mu}_1$. Due to population monotonicity and individual rationality $\bar{\mu}_1(l) = s Q^1_l \mu_1(l)$ for all $l \in \bar{I}_1$. Since $l \succ_s \hat{i} \succ_s j$ and $s Q^1_l \mu_1(l)$ for all $l \in \bar{I}_1$, fairness is violated in matching $\mu_1$. ■

Theorem 5 and 6 can also be interpreted as giving the conditions for a mechanism to induce NE outcomes that are fair, individually rational, and non-wasteful. Consequently, this also implies the result of Ergin and Sönmez (2006) who show that every NE outcome of the preference revelation game induced by the Boston mechanism is fair, individually rational, and non-wasteful. It is easy to check that the Boston mechanism is indeed serially fair, individually rational, non-wasteful, population monotonic, and weakly non-bossy.

\textsuperscript{25}Recall that mutual fairness is a weaker condition than fairness.
Theorem 7 Let \( \Psi = (\varphi^1, \varphi^2) \) be a system such that \( \varphi^1 \) is individually rational, population monotonic and

- \( \varphi^2 \) is individually rational, non-wasteful, monotonic, independent of irrelevant agents, weak non-bossy and fair, or
- \( \varphi^2 \) is individually rational, non-wasteful, mutually fair and able favoring higher ranks.

Every SPNE outcome of the preference revelation game associated with \( \Psi \) leads to a matching \( \mu \) in which there does not exist \((i, j)\) pair such that \( \mu(j) \in S^2 \), \( \mu(j)P_i\mu(i) \) and \( i \succ_{\mu(j)} j \).

Proof. Let \( Q = (Q^1_i, Q^2_i)_{i \in I} \) be an SPNE profile and \( \mu \) be the associated equilibrium outcome. First note that \( \mu(k)R_kh(k) \) for all \( k \in I \). Otherwise, \( Q \) cannot be SPNE (Theorem 5). Suppose there exist two agents \( i, j \in I \) such that \( \mu(j)P_i\mu(i) \), \( \mu(j) \in S^2 \) and \( i \succ_{\mu(j)} j \). Let \( \mu(j) = s \). There are two cases: (1) \( i \) does not participate in the second round because either \( \mu(i) \in S^1 \) or \( h(i) \neq \emptyset \) (2) \( i \) participates the second round.

Suppose that \( i \) participates in the second round. Without loss of generality denote the student with the highest priority for \( s \) among the ones who prefer \( s \) to his assignment and participate in the second round with \( i \). Note that when nobody deviates from his strategy in the first round the set of schools and agents in the second round will not change. Therefore, we can prove that \( i \) can benefit from deviating to \( Q': sQ'Q'x \) for all \( x \in S^2 \setminus \{s\} \) by following the same steps in the proof of Theorem 6.

Now suppose that there does not exist an agent \( i' \) such that (1) \( i' \in I^2 \), (2) \( sP_{i'}\mu(i') \) and (3) \( i' \succ_{s} j \). Then \( \mu(i) \in S^1 \). We claim that submitting \( (\bar{Q}, Q') \) is a profitable deviation for \( i \) where \( \bar{Q} : \emptyset Qx \) for all \( x \in S^1 \). Due to individual rationality \( \varphi^1_1(\bar{Q}, Q') = \emptyset \). Due to the population monotonicity and individual rationality if \( \varphi^1_j(\bar{Q}, Q^1_{-i}) \in S^1 \) then \( \varphi^1_j(Q^1) \in S^1 \). Therefore, in the second round the set of agents is a subset of \( I^2 \cup \{i\} \) and set of available seats weakly increases compared to the case in which \( i \) plays \( Q^1_i \). Let \( I_2', \bar{P}_2 \) and \( \bar{q}^2 \) be the set of agents, matching selected and quota vector in round 2 and when \( i \) submits \( (\bar{Q}, Q') \), respectively. Due to individual rationality, \( \bar{P}_2(i) \) is either \( s \) or \( \emptyset \). Suppose \( \bar{P}_2(i) = \emptyset \). Let \( \bar{I}_2 = \{ j \in I_2' | \mu(j) \neq \bar{P}_2(j) = s \} \). Since \( \varphi^2 \) is non-wasteful and mutually fair, \( |\bar{P}_2^{-1}(s)| = q_s, \bar{I}_2 \neq \emptyset, k \succ_{s} i \) for all \( k \in \bar{P}_2^{-1}(s), \bar{P}_2(j) \neq s \).

We continue with \( \bar{P}_2(i) = \emptyset \).
\( \varphi^2 \) favors higher ranks and mutually fair: Since \( \varphi^2 \) favors higher ranks, \( sQ_k^2 x \) for all \( x \in S^2 \cup \emptyset \) and \( k \in \tilde{I}_2 \). Then, \( \mu_2 \) cannot be mutually fair.

\( \varphi^2 \) is fair, weak non-bossy, monotonic, independent of irrelevant agents: Due to weak non-bosiness and individual rationality \( \mu_2 \) will be selected when \( i \) submits \( \tilde{P}_i^2 = \emptyset \tilde{P}_i^2 x \) for all \( x \in S^2 \). Let \( \mu'_2 \) be the outcome of \( \varphi^2 \) when we only consider agents in \( I_2' \setminus \{i\} \) keeping everything else the same. Due to independent of irrelevant agent, \( \mu'_2(l) = \mu_2(l) \) for all \( l \in I_2' \setminus \{i\} \). Due to monotonicity, \( \mu(l)Q_i^2 \mu'_2(l) = \mu_2(l) \) cannot be true for any \( l \in I_2' \setminus \{i\} \). This contradicts with the fact that \( \varphi^2 \) is fair.

5.1 Subgame Perfect Nash Equilibria of SD-DA in SCPwEXRS

In this subsection, we analyze the SPNE of the current sequential assignment system used in SCPwEXRS. Recall that in SCPwEXRS serial dictatorship mechanism is applied in the first round and deferred acceptance mechanism is applied in the second round. Serial dictatorship mechanism is individually rational, non-wasteful, population monotonic, non-bossy and strategy-proof. Moreover, in the SCPwEXRS it selects a fair (mutually fair) outcome when only the exam schools are available. Deferred acceptance mechanism is individually rational, non-wasteful, population monotonic, strategy-proof and fair (mutually fair). By following Theorem 5 one can see that every SPNE outcome of SD-DA leads to a non-wasteful and individually rational matching under the true preferences.

**Corollary 7** Every SPNE outcome of the preference revelation game associated with SD-DA leads to a non-wasteful and individually rational matching under agents’ true preferences.

**Proof.** Follows from Theorem 5. ■

In the SCPwEXRS, not all SPNE of the preference revelation game associated with SD-DA leads to a fair matching under agents’ true preferences. However, SPNE of the preference revelation game associated with SD-DA leads to a matching where the priorities of the exam schools are respected under agents’ true preferences.

**Corollary 8** Every SPNE outcome of the preference revelation game associated with SD-DA leads to a matching \( \mu \) in which there does not exist \( (i, j) \) pair such that \( \mu(j) \in S^e \), \( \mu(j)P_i \mu(i) \) and \( i \succ_{\mu(j)} j \).
Proof. Follows from Theorem 6.

Recall that only the students who have not been assigned to an exam school participate in the second round. Since it is the last round and we are using a strategy-proof mechanism to assign the participants to the available schools agents cannot benefit from misreporting. That is, it is weakly dominant strategy for all students to submit true preference over the available schools in round 2. Without loss of generality in the rest of this subsection we assume that students act truthfully in the second round of SD-DA.

Let \((S, I, P, P_S, q)\) be the associated college admission problem of the school choice problem with exam and regular schools, \((S, I, P, \succ, q, h)\) where for each school \(s\) \(iP_s j\) if and only if \(i \succ_s j\). In particular, the unique difference between college admission problem and school choice problem is that in the college admission problem schools are active and have preferences over students, \(P_S = (P_s)_{s \in S}\), whereas in the school choice problem schools are passive and considered as schools to be consumed. A matching \(\mu\) in school choice problem is individually rational, non-wasteful and fair if and only if it is stable for its associated college admission problem (Balinski and Sönmez, 1999). Moreover, for each college admission problem there exists a unique stable matching which is preferred to any other stable matching by all students. This matching is called student optimal stable matching. Similarly, student optimal stable matching is individually rational, non-wasteful, fair and preferred to any other individually rational, non-wasteful and fair matching by all students. In the following proposition, we show that in any school choice problem with exam and regular schools there exists at least one SPNE outcome of preference revelation game associated with SD-DA mechanism which is (weakly) preferred to any individually rational, non-wasteful and fair matching by all students.

Proposition 4 In any SCPwEXRS, there always exists at least one SPNE outcome of preference revelation game associated with SD-DA mechanism which (weakly) Pareto dominates any individually rational, non-wasteful and fair matchings

Proof. We show the existence of a SPNE outcome which (weakly) Pareto dominates the student optimal stable matching\(^{26}\). Denote the student optimal stable matching with \(\mu\). Then consider the following strategy profile \(\bar{P} = (\bar{P}_i^1, \bar{P}_i^2)_{i \in I}\):

\(^{26}\)Here, we mean the most preferred individually rational, fair and non-wasteful matching by all agents.
• Student $i$ submits $\tilde{P}_i^1 : \mu(i)\tilde{P}_i^1 x \tilde{P}_i^1 \emptyset$ for all $x \in S^e \setminus \{\mu(i)\}$ in the first round if $\mu(i) \in S^e$.

• Student $i$ submits $\tilde{P}_i^1 : \emptyset \tilde{P}_i^1 x$ for all $x \in S^e$ in the first round if $\mu(i) \in S^r$.

• Student $i$ submits his true preferences over the regular schools and $\emptyset$ in the second round whenever he is active, i.e. $\tilde{P}_i^2 = P_i|(S^r \cup \emptyset)$.

Denote the outcome of SD-DA mechanism under this preference profile by $\nu$. We first show that $\nu(i)R_i \mu(i)$ for all $i \in I$. Under preference profile $\tilde{P}$ only students in $\mu^{-1}(s)$ apply to each $s \in S^e$. Hence, $\nu(i) = \mu(i)$ for all $i \in \bigcup_s \mu^{-1}(s)$. Consider problem $(S^r, I^2, (q_s)_{s \in S^r}, (\tilde{P}_i^2)_{i \in I^2}, \succ^r)$ where $I^2 = \bigcup_{s \in S^r \cup \emptyset} \mu^{-1}(s)$ and $\tilde{P}_j^2 = P_j|(S^r \cup \emptyset)$. Define $\nu' : I^2 \rightarrow S^r \cup \emptyset$ and $\mu' : I^2 \rightarrow S^r \cup \emptyset$ such that $\nu'(i) = \nu(i)$ and $\mu'(i) = \mu(i)$ for all $i \in I^2$. One can verify that $\mu'$ and $\nu'$ are individually rational, fair and non-wasteful for the problem $(S^r, I^2, (q_s)_{s \in S^r}, (\tilde{P}_i^2)_{i \in I^2}, \succ^r)$. Moreover, $\nu'$ is the student optimal individually rational, fair and non-wasteful. Hence, $\nu(i)R_i \mu(i)$ for all $i \in I^2$.

In Example 2 we show that in some problems there exists a student who strictly prefers $\nu$ to $\mu$.

To show that $\tilde{P}$ is SPNE, we first look at the subgames in the second round. Each subgame can be considered as an independent school choice problem. Truth-telling is weakly dominant strategy under DA mechanism. Therefore, submitting true preferences in the second round is a NE in each subgame.

Now we analyze the strategies in the first round. First consider $i \in I^2$. Since $\nu(i)R_i \mu(i)$, $\nu^{-1}(s) = \mu^{-1}(s)$ for all $s \in S^e$ and $\mu$ is fair and non-wasteful all the seats of the exam schools that $i$ prefers to $\nu(i)$ are filled by students with better exam score in $\nu$. Therefore, all exam schools that $i$ prefers to $\nu(i)$ fill their seats before $i$’s turn and $i$ cannot be assigned to a better exam school no matter what he submits.

Now consider student $j$ who is assigned to an exam school. Since $\nu^{-1}(s) = \mu^{-1}(s)$ for all $s \in S^e$ and $\mu$ is fair and non-wasteful all the seats of the exam schools that $i$ prefers to $\nu(i)$ are filled by students with better exam score. We should also check whether he can be assigned to a more preferred regular school. If $j$ deviates and participates in the second round then we should consider the subgame where $I^2 \cup j$ are active. Without loss of generality we change the preference profile of $j$ by placing $\emptyset$ just before $\mu(j)$ and represent it with $P_j'$. Let $\tilde{P}_j' = P_j'|(S^r \cup \emptyset)$. It is easy to see that if $j$ can be assigned to a better school than $\mu(j)$ in $(S^r, I^2 \cup j, (q_s)_{s \in S^r}, (\tilde{P}_i^2)_{i \in I^2 \cup j}, \succ^r)$ then he will be assigned to the same school in $(S^r, I^2 \cup j, (q_s)_{s \in S^r}, (\tilde{P}_j', \tilde{P}_i^2)_{i \in I^2}, \succ^r)$. Define
a new matching \( \mu'' : \bigcup_{s \in S^r \cup \emptyset} I^2 \to S^r \cup \emptyset \) such that \( \mu''(i) = \mu(i) \) for all \( i \in I^2 \setminus j \) and \( \mu''(j) = \emptyset \). Then it is easy to see that \( \mu'' \) is individually rational, fair and non-wasteful in problem \((S^r, I^2 \cup j, (q_s)_{s \in S^r}, (\tilde{P}^s_j, (\tilde{P}^2_i)_{i \in I^2}), \succ_s^r)\). As a consequence of the rural hospital theorem (Roth, 1986) in all stable matchings the set of students assigned to a real school will be the same. Therefore, DA mechanism will not assign \( j \) to a better school then \( \mu(j) \) if he deviates and participates the second round. 

We illustrate the result of Proposition 4 in the following example.

**Example 2** Let \( S^e = \{s_1\} \), \( S^r = \{s_2, s_3\} \), \( q = (1, 1, 1) \) and \( I = \{i_1, i_2, i_3\} \). Priorities and preferences are given as

\[
\begin{array}{ccc}
\succ_{s_1} & \succ_{s_2} & \succ_{s_3} \\
i_1 & i_2 & i_3 \\
i_2 & i_1 & i_2 \\
i_3 & i_3 & i_1 \\
\end{array}
\]

\[
\begin{array}{ccc}
P_{i_1} & P_{i_2} & P_{i_3} \\
s_2 & s_3 & s_2 \\
s_1 & s_2 & s_3 \\
\emptyset & \emptyset & \emptyset \\
s_3 & s_1 & s_1 \\
\end{array}
\]

We can find the student optimal individually rational, fair and non-wasteful matching by applying DA mechanism to the associated school choice problem. The rounds of the DA mechanism is:

<table>
<thead>
<tr>
<th>Round 1</th>
<th>Round 2</th>
<th>Round 3</th>
<th>Round 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>( i_1^* ), ( i_3^* ), ( i_2^* )</td>
<td>( i_1^* ), ( i_2^* ), ( i_3^* )</td>
<td>( i_1^* ), ( i_2^* ), ( i_3^* )</td>
</tr>
</tbody>
</table>

In every round the students with star are tentatively held by schools. The final outcome of DA mechanism is \( \mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & i_3 \end{pmatrix} \). Consider the following strategy profile:

- **In the first round students submit following profiles:** \( s_1 \tilde{P}^1_{i_1} \emptyset, \emptyset \tilde{P}^1_{i_2} s_1, \emptyset \tilde{P}^1_{i_3} s_1 \).

- **Students participating in the second round submit true preference over the regular schools.**
We can verify that the strategy profile is SPNE by checking the proof of Proposition 4. The outcome of this strategy profile is: 
$$\mu' = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & i_3 \end{pmatrix}$$
and it Pareto dominates the student optimal individually rational, fair and non-wasteful matching 
$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_3 & i_2 \end{pmatrix}$$.

As a consequence of Proposition 4 and Example 2 we cannot say that every SPNE outcome of the preference revelation game associated with SD-DA leads to a non-wasteful, individually rational and fair matching under agents’ true preferences. On the other hand, we can relate every non-wasteful, individually rational and fair matching under agents’ true preferences to an SPNE outcome of the preference revelation game associated with SD-DA.

**Theorem 8** Every non-wasteful, individually rational and fair matching under agents’ true preferences is led by a SPNE outcome of the preference revelation game associated with SD-DA.

**Proof.** We refer to the proof of Theorem 9. One can easily modify the proof of Theorem 9 and follow the same rounds. ■

### 5.2 Subgame Perfect Nash Equilibria of TSSD

Recall that in TAP, SD mechanism is applied in both rounds. SD mechanism is individually rational, non-wasteful, population monotonic, non-bossy and strategy-proof. Moreover, it selects a fair (mutually fair) outcome in a TAP when the available schools are not owned by participating agents in Round 1 and 2. By following Theorem 5, 6 and 7 one can see that every SPNE outcome of TSSD leads to a non-wasteful, individually rational and fair matching under the true preferences.

**Corollary 9** Every SPNE outcome of the preference revelation game associated with TSSD leads to a non-wasteful, individually rational and fair matching under agents’ true preferences.

**Proof.** Follows from Theorem 5, Theorem 6 and Theorem 7. ■

We illustrate this result in the following example.
Example 3 Let $S = \{a, b, c\}$, $q = (1,1,1)$, $I_e = \{e\}$ and $I^n = \{t_1, t_2\}$. Teacher $e$ is currently working in school $a$ and the other two schools are tenured positions. The ranking based on the test score is given by: $c(t_1) > c(t_2) > c(e)$. The true preferences over schools and utilities of teachers are given as:

<table>
<thead>
<tr>
<th></th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$a$</td>
<td>$a$</td>
<td>3</td>
</tr>
<tr>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
<td>2</td>
</tr>
<tr>
<td>$a$</td>
<td>$c$</td>
<td>$c$</td>
<td>1</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>0</td>
</tr>
</tbody>
</table>

Since a strategy-proof mechanism is used in the last round, teachers can only benefit from a deviation in the first round. In round 1 strategies are: $bc\emptyset$, $cb\emptyset$, $b\emptyset c$, $c\emptyset b$, $\emptyset bc$ and $\emptyset cb$. We read $bc\emptyset$ as $b$ is ranked over $c$ and $c$ is ranked over $\emptyset$. Due to individual rationality $\emptyset bc$ and $\emptyset cb$ give the same outcome and we represent both strategies by $\emptyset$. The payoff tables are given below. Here, $t_2$ is the matrix player, $t_1$ is the column player and $e$ is the row player.

<table>
<thead>
<tr>
<th>$bc\emptyset$</th>
<th>$cb\emptyset$</th>
<th>$b\emptyset c$</th>
<th>$c\emptyset b$</th>
<th>$\emptyset$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$bc\emptyset$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
<td>$1,2,1$</td>
<td>$3,3,2$</td>
</tr>
<tr>
<td>$cb\emptyset$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
<td>$1,2,1$</td>
<td>$3,3,2$</td>
</tr>
<tr>
<td>$b\emptyset c$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
<td>$1,2,1$</td>
<td>$1,0,2$</td>
</tr>
<tr>
<td>$c\emptyset b$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
<td>$1,2,1$</td>
<td>$3,3,2$</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
<td>$1,2,1$</td>
<td>$1,0,2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$cb\emptyset$</th>
<th>$bc\emptyset$</th>
<th>$b\emptyset c$</th>
<th>$c\emptyset b$</th>
<th>$\emptyset$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$bc\emptyset$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
</tr>
<tr>
<td>$cb\emptyset$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
</tr>
<tr>
<td>$b\emptyset c$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
</tr>
<tr>
<td>$c\emptyset b$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>$1,2,1$</td>
<td>$1,1,2$</td>
<td>$1,2,1$</td>
<td>$1,0,1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$b\emptyset c$</th>
<th>$c\emptyset b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$bc\emptyset$</td>
<td>$1,2,1$</td>
</tr>
<tr>
<td>$cb\emptyset$</td>
<td>$1,2,1$</td>
</tr>
<tr>
<td>$b\emptyset c$</td>
<td>$1,2,0$</td>
</tr>
<tr>
<td>$c\emptyset b$</td>
<td>$1,2,3$</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>$1,2,0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$c\emptyset b$</th>
<th>$b\emptyset c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$bc\emptyset$</td>
<td>$1,2,1$</td>
</tr>
<tr>
<td>$cb\emptyset$</td>
<td>$1,2,1$</td>
</tr>
<tr>
<td>$b\emptyset c$</td>
<td>$1,2,1$</td>
</tr>
<tr>
<td>$c\emptyset b$</td>
<td>$1,2,1$</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>$1,2,1$</td>
</tr>
</tbody>
</table>

36
The bold payoffs represent the NE outcomes. (1,2,1) corresponds to the payoff of the school optimal fair, non-wasteful and individually rational matching and (3,3,2) corresponds to the payoff of the agent optimal fair, non-wasteful and individually rational matching. Moreover student optimal and college optimal matchings are the only two matchings which are fair, non-wasteful and individually rational.

In the following theorem, we show that in TAP every non-wasteful, individually rational and fair matching under agents’ true preferences can be related to an SPNE outcome of TSSD.

**Theorem 9** Every non-wasteful, individually rational and fair matching under agents’ true preferences is led by a SPNE outcome of the preference revelation game associated with TSSD.

**Proof.** Let μ be a non-wasteful, individually rational and fair matching under agents’ true preferences. Then consider the following strategy profile \( Q = (Q^1_i, Q^2_i)_{i \in I} \) where \( Q^t_i \) is the submitted preferences in round \( t \in \{1, 2\} \) such that

- if \( \mu(i) \in S^t \) then \( \mu(i)Q^1_i xQ^1_i \emptyset \) for all \( x \in S^t \setminus \{\mu(i)\} \),
- if \( \mu(i) \in S^c \) and \( h(i) = \emptyset \) then \( \emptyset Q^1_i x \) for all \( x \in S^t \) and \( \mu(i)Q^2_i xQ^2_i \emptyset \) for all \( x \in S^c \setminus \{\mu(i)\} \),
- if \( \mu(i) = h(i) \neq \emptyset \) then \( \emptyset Q^1_i x \) for all \( x \in S^t \),
- if \( \mu(i) = h(i) = \emptyset \) then \( \emptyset Q^1_i x \) for all \( x \in S^t \) and \( \emptyset Q^2_i x \) for all \( x \in S^t \).

The outcome of this strategy profile is \( \mu \). Due to individual rationality, if \( \mu(i) \in S \) then \( i \) cannot be better off by submitting a preference profile which makes him unassigned.
Consider the second round. Agent $i$ participates in the second round if $h(i) = \emptyset$ and $\forall Q_i^t x$ for all $x \in S^t$. Suppose there exists a teacher, $j$, among the ones participating round 2 who can get $sP_j \mu(j)$ by deviating from his strategies in $Q$ where $s \in S^c = S^2$. Since $\mu$ is non-wasteful $|\mu^{-1}(s)| = q_s$. Moreover, due to the fairness any student in $\mu^{-1}(s)$ either has higher test score or is an existing teacher in $s$. Therefore, all seats of $s$ are filled before $j$’s turn in round 2 and she cannot get that school no matter what she submits. Therefore, in any subgame in round 2 a Nash equilibrium is selected under preference profile $Q^2$.

Now consider the first round. Suppose there exists a contractual teacher $j$ who can get $sP_j \mu(j)$ by deviating from his strategies in $Q$. First note that $s \notin S^c$. Because the system does not allow an agent with ownership to participate in the second round. Then, $s \in S^t$. Since $\mu$ is non-wasteful and fair all seats of $s$ are filled before $j$’s turn and she cannot get that school no matter what she submits. Now we show that a new graduate cannot be better off by deviating. A new graduate cannot increase the number of available seats when each contractual teacher $k \in I^c$ submits $Q_k^1$. That is, the number of available seats in each round cannot be affected by the deviation of a new graduate. Due to fairness and non-wastefulness any school that a new graduate prefers to her assignment are filled with either teachers with higher test score or with the existing teachers. Therefore, no matter what a new graduate submits she cannot get a better school than her assignment in $\mu$.

6 A Simpler Alternative System: Simultaneous Assignment via DA

Theorem 4 and Theorem ?? show that the main reason behind the deficiencies observed in the current systems may simply be due to the fact that assignments are done in sequential fashion. These impossibilities motivate us to advocate one-round assignment systems as alternative systems to sequential assignment whenever it is feasible to do so.

In the assignment systems discussed in this paper, one of the important concerns is assigning the agents to the school without violating the predetermined priorities. Additionally, decreasing the level of gaming and thereby encouraging agents to report their true preferences over the schools is another practical concern. Fortunately, an easy solution is available. By simply
reducing the two rounds into a single round and applying the agent-proposing deferred acceptance algorithm (DA), one can readily address the concerns mentioned above.

For a given assignment problem \((S, I, q, P, \succ, h)\) we can find the outcome of the DA mechanism as follows:

**Step 1**: Each agent applies to her top choice school. Each school, which receives an offer, tentatively accepts all best acceptable offers up to its quota according to its priority ordering. Any unacceptable offer or any offer not honored due to the quota constraint is rejected. If an agent applies to the being unassigned option, then she is permanently assigned to it.

In general,

**Step k**: Each agent who does not have a tentatively accepted offer from the previous round makes an offer to the best school, which has not rejected him yet. Each school, which holds tentatively accepted offers or receives new offers in this round, tentatively accepts all best acceptable offers, among the new and previously held ones, up to its quota according to its priority ordering. Any unacceptable offer or any offer not honored due to the quota constraints is rejected permanently. If an agent applies to the being unassigned option, then she is permanently assigned to it.

The algorithm terminates when no agent is rejected any more. For any problem, DA mechanism selects a fair, non-wasteful and individually rational outcome, i.e., it is stable. Moreover, DA mechanism is immune to preference manipulation and respects the improvements in the test scores (priorities). A natural question to ask is whether or not there is another alternative which satisfies all these desirable features. The following result based on Alcalde and Barbera (1994) and Balinski and Sönmez (1999) gives a negative answer to this question and makes the case for DA as a remedy to the deficiencies of the systems used in the two applications we discussed.27

**Theorem 10** DA is the unique mechanism which is

- fair, individually rational, non-wasteful, strategy-proof, or
- fair, individually rational, non-wasteful, respects improvements in priorities.

27Of course, it is well-known that the outcome of DA is not necessarily Pareto efficient but it Pareto dominates any other stable matching for a given problem.
7 Conclusion

Although strategic and distributional objectives in standard (one-round) assignment problems and sequential assignment problems are quite similar, we have shown that the latter type of problems may be fundamentally different and more challenging when compared with the former type. We have shown that under sequential systems, most desirable properties are lost even though they may be satisfied roundwise. Most remarkably, sequential systems are strategically vulnerable (even if they are strategy-proof roundwise) and force participants to make hard judgment calls about how to rank-list the available options in each round. As a result these systems may lead to inefficient and even wasteful assignments. This suggests that even though sequential systems may arguably be easier to implement in practice (e.g., in the context of school choice), such convenience may come at an important cost. The alternative use of one-round systems, such as the DA, may help avoid these costs when doing so is feasible.

The recent transition in Turkey to one such system\textsuperscript{28} may also provide support to our conclusions. In our July 2012 meeting with the former Minister of Education, Ömer Dinçer, we have explained our concerns about the existing practice. In that meeting Mr. Dinçer stated that he and his planning team were well aware of the challenges posed by the existing system and hinted at potential reforms to follow. Later in that year, the Turkish ministry of education announced that they will discontinue hiring for the contractual positions and that all the contractual teachers will be converted to tenured teachers.

References


\textsuperscript{28}In the recently adopted new system in Turkey, all teachers are assigned via a serial dictatorship. The practice of hiring contractual teachers has been discontinued in recent years.


[6] Sebastian Braun, Nadja Dwenger, Dorothea Kübler and Alexander Westkamp, "Implementing Quotas in University Admissions: An Experimental Analysis", working paper.


A Appendix

A.1 Assignment systems in Boston and NYC

In Boston there are three exam schools\(^\text{29}\) which enroll around 25\% of the seventh grade students.\(^\text{30}\) In a given year, sixth grade students take the centralized exam before December and apply to one of these schools in the following year. A ranking of students are then obtained based on a combination of the exam scores and GPAs from the previous year. The assignment of the students to the exam schools are determined via the serial dictatorship mechanism induced by this ranking. Admitted students receive their acceptance letters from the exam schools by mid-March\(^\text{31}\), and the assignment for the regular schools are determined via DA.

In New York City there are nine exam schools.\(^\text{32}\) The assignments to the exam and regular schools are also implemented sequentially although students submit their preferences over both types of schools at the same time. Every year between 25,000 and 30,000 student take the Specialized High School Admission Test (SHSAT) which is then used to determine the assignments to the exam schools which enroll only about 5,000 annually. Students who take this test submit two different rank-order-lists to the central authority. In the first one they rank-list only the exam schools whereas in the second one they rank-list only the regular schools which do not require any test score. The admission decisions for the specialized high schools are determined based on the scores on SHSAT, while the admissions for the regular schools follow the outcome of DA. Both decisions are concurrently determined. The central authority aims to make placements to the specialized high schools first. Therefore, initially only those students who have been admitted to both an exam and a regular school are informed, and they are asked to make a choice.

\(^{29}\)These schools are Boston Latin Academy, Boston Latin School and the John D. O’Bryant School of Mathematics and Science.

\(^{30}\)In 2012-2013 school year 836 of 3,795 seventh grade students have enrolled to exam schools.

\(^{31}\)Sixth graders can also apply to be transferred to another regular school after mid-March.

\(^{32}\)These schools are Bronx High School of Science, Brooklyn Latin School, Brooklyn Technical High School, High School for Math, Science and Engineering at City College, High School of American Studies at Lehman College, Queens High School for the Sciences at York College, Staten Island Technical High School and Stuyvesant High School.
between the two schools they are admitted to. Subsequently, students who are not assigned in this round are considered and they are assigned to the regular schools once again via DA.

### A.2 Turkish Assignment System

The number of teachers assigned to tenured and contractual positions in 2009 and 2010 is presented in Table 2. For instance in December 2009, 8,850 tenured positions were filled by applicants in the first round. 6,323 of these applicants were existing teachers working in contractual positions. These contractual positions which became available as a consequence of assignments of existing teachers to the tenured positions were filled in the same month.

<table>
<thead>
<tr>
<th>Time of the Assignment</th>
<th>Type of the Positions</th>
<th>Number of Positions Filled</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 2009</td>
<td>Tenured</td>
<td>8,285</td>
</tr>
<tr>
<td>March 2009</td>
<td>Contractual</td>
<td>6,323</td>
</tr>
<tr>
<td>December 2009</td>
<td>Tenured</td>
<td>8,850</td>
</tr>
<tr>
<td>December 2009</td>
<td>Contractual</td>
<td>6,323</td>
</tr>
<tr>
<td>June 2010</td>
<td>Tenured</td>
<td>10,000</td>
</tr>
<tr>
<td>July 2010</td>
<td>Contractual</td>
<td>9,000</td>
</tr>
<tr>
<td>December 2010</td>
<td>Tenured</td>
<td>30,000</td>
</tr>
<tr>
<td>December 2010</td>
<td>Contractual</td>
<td>6,843</td>
</tr>
</tbody>
</table>

Table 2. Number of Teachers Assigned to Tenured and Contractual Positions (2009-2010)

### A.3 Examples

In the following two examples, we illustrate how SD-DA and TSSD mechanisms fail to satisfy the desired properties.

**Example 4** Let $S = \{s_1, s_2, s_3, s_4\}$, $S^c = \{s_3, s_4\}$, $S^r = \{s_1, s_2\}$, $I = \{i_1, i_2, i_3, i_4\}$ and $h(i_1) = h(i_2) = h(i_3) = h(i_4) = \emptyset$. All schools have one available seat, $q_s = 1$ for all $s \in S$. Let true
preferences and test scores be as follows:

\[ s_2 P_{i_1} s_3 P_{i_1} s_1 P_{i_1} s_4 P_{i_1} \emptyset \quad c(i_1) = 90 \]
\[ s_1 P_{i_2} s_4 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset \quad c(i_2) = 88 \]
\[ s_3 P_{i_3} s_1 P_{i_3} s_2 P_{i_3} s_4 P_{i_3} \emptyset \quad c(i_3) = 85 \]
\[ s_4 P_{i_4} s_2 P_{i_4} s_1 P_{i_4} s_3 P_{i_4} \emptyset \quad c(i_4) = 70 \]

The set of available schools in round 1 is \( S^1 = \{s_3, s_4\} \). The outcome selected in round 1 when all the agents act truthfully is \( \mu_1(i_1) = s_3, \mu_1(i_2) = s_4, \mu_1(i_3) = \emptyset \) and \( \mu_1(i_4) = \emptyset \). In round 2 the set of the available schools and set of the applicants allowed to participate are: \( S^2 = \{s_1, s_2\} \) and \( I^2 = \{i_3, i_4\} \). The outcome selected in round 2 when all the agents act truthfully is \( \mu_2(i_3) = s_1 \) and \( \mu_2(i_4) = s_2 \). The final outcome of the SD-DA mechanism is \( \mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{pmatrix} \).

**SD-DA is not Pareto efficient:** There exists another matching \( \mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_2 & i_1 & i_3 & i_4 \end{pmatrix} \) that Pareto dominates the outcome of the SD-DA mechanism, \( \mu \). It is worth to mention that \( \mu' \) is a fair matching. That is, the outcome of the current mechanism is Pareto dominated by a fair matching.

**SD-DA is not Strategy-proof:** If \( i_2 \) ranks \( s_4 \) below \( \emptyset \) in his list in round 1 then the final outcome will be \( \mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_2 & i_4 & i_1 & i_3 \end{pmatrix} \) and \( i_2 \) will be strictly better-off.

**SD-DA does not respect improvements:** If we take \( c'(i_2) = 75 \) then the outcome will be

\[ \mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_2 & i_4 & i_1 & i_3 \end{pmatrix} \quad \text{and} \quad \mu'(i_2)P_{i_2} \mu(i_2). \]

That is, when \( i_2 \) gets higher score he is assigned to a less preferred school.

**SD-DA is not fair:** \( \mu(i_4)P_{i_1} \mu(i_1) \) and \( i_1 \) has higher priority for \( \mu(i_4) = s_2 \).

**SD-DA is wasteful:** Consider the same example with only one agent, \( I = \{i_1\} \). SD-DA mechanism assigns \( i_1 \) to \( s_3 \). But \( i_1 \) prefers \( s_2 \) to its match \( s_3 \) and \( s_2 \) has an empty under the outcome of SD-DA.

Consider the same example with the following modification, \( S^r = \{s_3, s_4\} \) and \( S^e = \{s_1, s_2\} \). All the other things are kept the same. Then it is easy to see that DA-SD mechanism suffers
from the same deficiencies as the SD-DA mechanism. ■

**Example 5** Consider Example 4 with the following modifications: \( h(i_1) = s_1, h(i_2) = s_2, h(i_3) = h(i_4) = \emptyset \). Take the same test scores for students \( i_1, i_3 \) and \( i_4 \). Only change the test score of \( i_2 \) to \( c(i_2) = 80 \). Then TSSD mechanism selects the following matching: \( \mu = \left( \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \\ i_4 & i_2 & i_1 & i_3 \end{array} \right) \).

**TSSD is not Pareto efficient:** There exists another matching \( \mu' = \left( \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{array} \right) \) that Pareto dominates the outcome of the TSSD mechanism \( \mu \). It is worth to mention that \( \mu' \) is a fair matching. That is, the current mechanism is Pareto dominated by a fair matching.

**TSSD is not Strategy-proof:** If \( i_3 \) ranks \( s_4 \) below \( \emptyset \) in the submitted preferences in round 1 then the final outcome will be \( \mu' = \left( \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{array} \right) \) and \( i_3 \) will be strictly better-off. Moreover none of the agents will be hurt.

**TSSD does not respect improvements:** If we take \( c'(i_3) = 75 \) then the outcome will be \( \mu' = \left( \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{array} \right) \) and \( \mu'(i_3)P_{i_3}\mu(i_3) \). That is, when \( i_3 \) gets higher score he is assigned to a less preferred school.

**TSSD is not fair:** \( \mu(i_4)P_{i_4}\mu(i_3) \) and \( i_3 \) has higher priority than \( i_4 \) for \( \mu(i_4) = s_1 \).

**TSSD is wasteful:** Consider the same example with only two agents, \( I = \{i_1, i_3\} \) and \( S = \{s_1, s_3, s_4\} \) where \( h(i_1) = s_1 \) and \( h(i_3) = \emptyset \). The preference of agents are

\[
\begin{align*}
\text{s}_3P_{i_1}s_1P_{i_1}s_4 & \quad c(i_1) = 90 \\
\text{s}_3P_{i_3}s_1P_{i_3}s_4 & \quad c(i_3) = 85
\end{align*}
\]

The matching selected by the TSSD is \( \mu'' = \left( \begin{array}{ccc} s_1 & s_3 & s_4 \\ \emptyset & i_1 & i_3 \end{array} \right) \). But \( i_3 \) prefers \( s_1 \) to its match \( s_4 \) and \( s_1 \) has an empty seat under \( \mu'' \). ■

45