The Division Problem under Constraints*

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Very Preliminary

Abstract: We characterize axiomatically an extension of the uniform rule proposed to solve the division problem under contraints, which consists of allocating a given amount of an homogeneous and perfectly divisible good among a subset of agents with single-peaked preferences on an exogenously given interval of feasible shares. We show that the extended uniform rule is the unique one satisfying strategy-proofness, equal treatment of equals, bound monotonicity, and admissible contraction.

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1 Introduction

In the division problem an amount of a perfectly divisible good has to be allocated among a set of agents with single-peaked preferences on the set of all positive amounts of the good. An agent has a single-peaked preference if he considers that there is an amount of the good (the peak) strictly preferred to all other amounts and in both sides of the peak the preference is monotonic, decreasing at its right and increasing at its left. A profile is a vector of single-peaked preferences, one for each agent. It would then be desirable that the chosen vector of allotments of the good depended on the profile. But since preferences are idiosyncratic they have to be elicited by a rule selecting, for each profile of single-peaked preferences, a vector of allotments adding up to the total amount of the good. But in general, the sum of the peaks will be either larger or smaller than the total amount to be allocated. Then, a rule has to solve a positive or negative rationing problem, depending on whether the sum of the peaks exceeds or fails short the amount of the good. Rules differ from each other on how this rationing problem is resolved in terms of its induced properties like the strategic incentives faced by agents, efficiency, fairness, monotonicity, consistency, etc.

The literature on the division problem describes many examples of allocation problems that fits well with this general description. For instance, a group of agents participate in an activity that requires a fixed amount of labor (measured in units of time). Agents have a maximal number of units of time to contribute and consider working as being undesirable. Suppose that labor is homogeneous and the wage is fixed. Then, strictly monotonic and quasi-concave preferences on the set of bundles of money and leisure generate single-peaked preferences on the set of potential allotments where the peak is the amount of working time associated to the optimal bundle. Similarly, a group of agents join a partnership to invest in a project (an indivisible bond with a face value, for example) that requires a fixed amount of money (neither more nor less). Their risk attitudes and wealth induce single-peaked preferences on the amount to be invested. Finally, a group of firms with different sizes have to jointly undertake a unique project of a fixed size. Since they may be involved in other projects their preferences are single-peaked on their respective shares of the project. In all these cases, it is required that a rule solves the rationing problem arising from a vector of peaks that do not add up the needed amount. The uniform rule has emerged as a satisfactory way of solving the division problem. It tries to allocate the good as equally as possible keeping the bounds imposed by efficiency. Sprumont (1991) started a long list of axiomatic characterizations of the uniform rule by showing first that it is the unique efficient, strategy-proof and anonymous rule, and second that anonymity
in this characterization can be replaced by envy-freeness.\(^1\)

However, in many applications (like those described above), agents’ allotments may be constrained by lower and upper bounds. For instance, each agent may only be able to contribute to the activity with an amount of labor, or to invest in the project, if his allotment belongs to a given interval. Therefore, in all these cases the division problem is restricted further by feasibility constraints that are described by a family of closed intervals of non-negative feasible allotments, one for each agent. It is then natural to assume that each agent has a closed interval of feasible allotments and his idiosyncratic preferences are single-peaked on this interval. Moreover, we will be interested in situations where agents’ participation is voluntary; namely, all strictly positive feasible allotments have to be considered by any agent as being strictly preferred to receive zero (the allotment associated to the prospect of non-participating in the division problem).

Until recently, a large part of the literature on the division problem has assumed implicitly that all allotments were feasible.\(^2\) In this paper we assume that each agent’s allotment either has to belong to a given feasible interval of allotments or else it has to be equal to zero.\(^3\) Hence, a division problem under constraints is composed by the set of agents, the amount of the good to be allocated among them, the vectors of lower and upper bounds of their feasible intervals, and their single-pealed preferences on their respective feasible intervals. Given a division problem under constraints, it may be the case that there does not exist a vector of feasible allotments, one for each one of the agents, adding up to the total amount to be allocated. Hence, given any division problem under constraints, a rule has two components. First, the choice of an admissible and non-empty subset of agents among whom it is possible to allocate the amount of the good keeping their feasibility constrains; if there is no such subset, then the rule has to choose the zero allotment for all agents. Second, and given this chosen admissible non-empty subset of agents, the rule has to assign

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\(^{2}\)Kibris (2003), Kim (2010), Bergantiños, Massó and Neme (2012a, 2012b), and Manjunath (2012) consider different types of restrictions on the set of feasible and/or acceptable shares. Moulin (1999) and Ehlers (2002a, 2002b) introduce (for technical reasons) maximal capacity constraints in the division problem to deal with the family of fixed path mechanisms. At the end of this Introduction we will briefly describe their contributions and main differences with our approach.

\(^{3}\)We assume that each agent’s lower and upper bounds of the feasible interval of allotments are those strictly positive if the lower bound is strictly positive; that is, it is always possible to assign the zero allotment to all agents, even when zero does not belong to the interval of (strictly positive) feasible allotments.
to each of its members a feasible allotment in such a way that their sum adds up the total amount to be allocated.

Our contribution in this paper is to define an extension of the uniform rule to this class of division problems under constraints and to provide an axiomatic characterization of it by using two classes of desirable properties. The first class is related to the behavior of the rule at a given division problem under constraints. First, *efficiency*. A rule is efficient if it always selects Pareto optimal allocations. Efficiency guarantees that in solving the rationing problem (either positive or negative) no amount of the good is wasted. Second, *equal treatment of equals*. A rule satisfies equal treatment of equals if identical participants receive the same allotment.\(^4\) The second class is related to the restrictions that the properties impose on a rule when comparing its proposal at different division problems with maximal capacity constraints (*i.e.*, when either the set of agents, their upper bounds or the amount to be shared change). First, *strategy-proofness*. A rule is strategy-proof if no agent can profitably alter the rule’s choice by misrepresenting his preferences. Namely, strategy-proofness guarantees that truth-telling is a weakly dominant strategy in the direct revelation game induced by the rule. Second, *bound monotonicity*. It imposes restrictions on how the rule changes when the upper or the lower bound of the interval of feasible allotments of one agent changes. Third, *admissible contraction*. A rule satisfies admissible contraction if whenever (i) the set of admissible coalition in one problem is contained in the set of admissible coalitions in another problem, and (ii) the coalition chosen by the rule second and larger problem is admissible for the first and smaller problem then, the rule has to select the same coalition of agents in the two problems.

The paper contains first a preliminary result, Theorem 1, where we show that in the subclass of division problems under constrains having the property that the full set of agents is admissible (*i.e.*, it is possible to allocate the total amount of the good among all agents respecting all feasible constraints), the feasible uniform rule is the unique rule satisfying efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity. The feasible uniform rule on this subclass of division problems under constraints tries to allocate the good among all agents in the most egalitarian way respecting not only the bounds imposed by efficiency, but also those imposed by the feasibility constraints. Using this characterization, we show in Theorem 2 that the extended uniform rule on the class

\(^4\)As we will see in Section 3, this is a weakening of the usual property that requires that equal agents are treated equally; we show that this version is too strong because now it may be necessary that in order to satisfy the feasibility constraints one agent is excluded from the division and receives the zero allotment while an identical agent is included and receives an strictly positive feasible allotment. Hence, the rule will have to specify how this choice among these two identical agents is done.
of all division problems under constraints selects first, using a monotonic and responsive order on the family of all non-empty and finite subsets of agents, an admissible coalition (if any; otherwise it chooses the zero allotment for all agents) and then it applies the feasible uniform rule to the reduced division problem under constraints obtained by restricting the original problem to this admissible subset of agents.

Several papers are closely related to the present one. First, Bergantiños, Massó and Neme (2012a) studies the division problem with maximal capacity constraints under the assumption that the sum of all upper bounds is larger than the total amount of the good that has to be distributed. Second, Kibris (2003) studies the division problem with maximal capacity constraints assuming free-disposability of the good. Then a rule assigns to each division problem with maximal capacity constraints a vector of shares satisfying the constraints and adding up less or equal than the total amount. Kibris (2003) characterizes an extension of the uniform rule to his setting with free-disposability. Third, Bergantiños, Massó and Neme (2012b) considers the division problem when agents’ participation is voluntary. Each agent has an idiosyncratic interval of acceptable shares (which, in contrast with our setting here, is private information) where his preferences are single peaked. Then a rule proposes to each agent either to not participate at all or an acceptable share. Bergantiños, Massó and Neme (2012b) shows that strategy-proofness is too demanding in this setting. Then, they study a subclass of efficient and consistent rules and characterize extensions of the uniform rule that deal explicitly with agents’ voluntary participation. Fourth, Kim (2010) characterizes, in the same setting than Bergantiños, Massó and Neme (2012b) with voluntary participation, a rule (called the generalized uniform rule) by the properties of efficiency, no-envy, separability and weak resource continuity. Fifth, Manjunath (2012) proposes a division problem where each agent’s preferences are characterized by the top share and a minimum share in such a way that the agent is indifferent between any two quantities that are either below the minimum acceptable share or above the top share. Manjunath (2012) first shows that, under different fairness properties, strategy-proofness and efficiency are incompatible and second, he characterizes axiomatically different rules that solve the rationing problem in his setting. Finally, the division problem with maximal capacity constraints is also considered by Moulin (1999).\footnote{In Moulin (1999) the maximal capacity constraints are justified on the basis of technical simplicity in order to define the priority rationing methods by an ordinary path and to define the duality operator that cuts the main proof in half.} He characterizes the class of all fixed path mechanisms as the set of rules satisfying efficiency, strategy-proofness, consistency and resource monotonicity. The constrained uniform rule presented in this paper is the fixed path rationing method of Moulin (1999) using the main diagonal as path. Ehlers
(2002a) presents a shorter proof of the main result in Moulin (1999) and Ehlers (2002b) extends it by showing that, for problems with strictly more than two agents, the class of all fixed path mechanisms coincides with the set of rules satisfying weak one-sided resource monotonicity, strategy-proofness and consistency.

The paper is organized as follows. In Section 2 we describe the model. In Section 3 we define several desirable properties that a rule may satisfy. In Section 4 we define the uniform rule for the subclass of division problems where the grand coalition is admissible. Using the uniform rule on this subclass we define, for all division problems under constraints, the extended uniform rule induced by a monotonic and responsive order on the family of all finite and non-empty subsets of agents and state two axiomatic characterizations. In Theorem 1 we show that (on the subclass of division problems where the grand coalition is admissible) the uniform rule is the unique rule that satisfies efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity. In Theorem 2 we show that a rule (on the class of all division problems under constraints) satisfies efficiency, strategy-proofness, equal treatment of equals, bound monotonicity, and admissible contraction if and only if it coincides with the extended uniform rule induced by a monotonic and responsive order on the family of all non-empty and finite subsets of agents. Section 5 contains some final remarks stating other desirable properties that the extended uniform rule also satisfies and that we use in the proofs of the two theorems, contained in Section 6.

2 Preliminaries

Let $t > 0$ be an amount of an homogeneous and perfectly divisible good. A finite set of agents is considering the possibility of dividing $t$ among a subset of them, to be determined according to their preferences. We will consider situations where the amount of the good $t$ and the finite set of agents may vary. Let $\mathbb{N}$ be the set of positive integers and let $\mathcal{N}$ be the family of all non-empty and finite subsets of $\mathbb{N}$. The set of agents is then $N \in \mathcal{N}$ with cardinality $n$. In contrast with Sprumont (1991), we consider decision problems where the amount received by each agent $i \in N$ is constrained either to belong to a given closed interval $[l_i, u_i] \subset [0, +\infty)$, determined by minimal and maximal exogenous constraints ($l_i$ and $u_i$, respectively), or to be equal to zero. That is, an agent is either excluded from the division (and receives zero) or else his allotment has to be feasible. We are interested in settings where the participation of the agents in the division problem is voluntary in the sense that all feasible allotments are strictly better than receiving zero. Thus, agent $i$’s preferences $\succeq_i$ are defined on the set $\{0\} \cup [l_i, u_i]$, where $[l_i, u_i] \subset [0, +\infty]$ is agent $i$’s interval of feasible allotments. We assume that $\succeq_i$ is a complete, reflexive, and transitive binary
relation on \( \{0\} \cup [l_i, u_i] \). Given \( \succeq_i \), let \( \succ_i \) be the antisymmetric binary relation induced by \( \succeq_i \) (i.e., for all \( x_i, y_i \in \{0\} \cup [l_i, u_i] \), \( x_i \succ_i y_i \) if and only if \( y_i \succeq_i x_i \) does not hold) and let \( \sim_i \) be the indifference relation induced by \( \succeq_i \) (i.e., for all \( x_i, y_i \in \{0\} \cup [l_i, u_i] \), \( x_i \sim_i y_i \) if and only if \( x_i \succeq_i y_i \) and \( y_i \succeq_i x_i \)). We will also assume that \( \succeq_i \) is single-peaked on \([l_i, u_i]\) and we will denote by \( p_i \in [l_i, u_i] \) agent \( i \)'s peak. Formally, agent \( i \)'s preferences \( \succeq_i \) is a complete preorder on the set \( \{0\} \cup [l_i, u_i] \) that satisfies the following additional properties:

(P.1) there exists \( p_i \in [l_i, u_i] \) such that \( p_i \succ_i x_i \) for all \( x_i \in [l_i, u_i] \setminus \{p_i\} \);

(P.2) \( x_i \succ_i y_i \) for any pair of shares \( x_i, y_i \in [l_i, u_i] \) such that either \( y_i < x_i \leq p_i \) or \( p_i \leq x_i < y_i \); and

(P.3) \( x_i \succ_i 0 \) for all \( x_i \in [l_i, u_i] \setminus \{0\} \).

Observe that agent \( i \)'s preferences are defined on \( \{0\} \cup [l_i, u_i] \) and are independent of \( t \). Moreover, we are admitting the possibilities that \( l_i = 0 \) and \( l_i = p_i = u_i \). Conditions (P.1) and (P.2) are the standard single-peaked restrictions on \([l_i, u_i]\) while condition (P.3) conveys the minimal voluntary participation requirement that all strictly positive allotments in the feasible interval are strictly preferred to the zero allotment. A preference \( \succeq_i \) of agent \( i \) is (partly) characterized by the triple \((l_i, p_i, u_i)\). There are many preferences of agent \( i \) with the same \((l_i, p_i, u_i)\); however, they differ only on how two shares on different sides of \( p_i \) are ordered while all of them coincide on the ordering on the shares on each of the sides of \( p_i \). This multiplicity will often be irrelevant. We will assume throughout the paper that for any agent \( i \), the bounds \( l_i \) and \( u_i \) are fixed and exogenously given while the preference \( \succeq_i \) over the interval \([l_i, u_i]\) is idiosyncratic and has to be elicited through a direct revelation mechanism. As we have already discussed in the Introduction, we are interested in division problems where allotments may be restricted by objective feasibility or capacity constraints while every preference \( \succeq_i \) satisfying (P.1), (P.2), and (P.3) is a legitimate one for agent \( i \).

Let \( N \in \mathcal{N} \) be a set of agents. A profile \( \succeq_N = (\succeq_i)_{i \in N} \) is an \( n \)-tuple of preferences satisfying properties (P.1), (P.2) and (P.3) above. Given a profile \( \succeq_N \) and agent \( i \)'s preferences \( \succeq'_i \), we denote by \((\succeq_i', \succeq_N \setminus \{i\})\) the profile where \( \succeq_i \) has been replaced by \( \succeq'_i \) and all other agents have the same preferences. When no confusion arises we denote the profile \( \succeq_N \) by \( \succeq \).

A division problem under constraints (a problem for short) is a 5-tuple \((N, t, l, u, \succeq)\) where \( N \in \mathcal{N} \) is the finite set of agents, \( t \) is the amount of the good to be divided, \( l = (l_i)_{i \in N} \)

\footnote{See Bergantiños, Massó, and Neme (2012) for an analysis of efficient and consistent rules in the division problem when the interval \([l_i, u_i]\) is the set of idiosyncratic acceptable allotments for agent \( i \) and participation is voluntary.}
is the vector of lower constraints, \( u = (u_i)_{i \in N} \) is the vector of upper constraints, and \( \succeq \) is a profile. Although the vector of lower and upper constraints are part of the definition of the profile \( \succeq \), for convenience, we explicitly include them in the description of a problem. Let \( \mathcal{P} \) be the set of all problems. A problem where all agents have single-peaked preferences on \([0, +\infty)\) is known as the division problem; i.e., for all \( i \in N \), \( l_i = 0 \), \( u_i = +\infty \), and (P.1) and (P.2) hold.\(^7\)

The set of feasible allocations of problem \((N, t, l, u, \succeq)\) is

\[
FA(N, t, l, u, \succeq) = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i \in (0,t) \text{ and, for each } i \in N, \; x_i \in \{0\} \cup [l_i, u_i] \right\}.
\]

This set is never empty since the allocation \((0, \ldots, 0) \in \mathbb{R}^N_+\) is always feasible. Moreover, there are problems for which \((0, \ldots, 0)\) is the unique feasible allocation; for instance the problem \((N, \succeq, t, l, u)\) where \( N = \{1, 2\} \), \( t = 10 \), and \( \succeq_1 \) and \( \succeq_2 \) are characterized by \((l_1, p_1, u_1) = (l_2, p_2, u_2) = (1, 2, 3)\).

A coalition \( S \subseteq N \) is admissible (at problem \((N, t, l, u, \succeq)\)) if either \( S \) is empty or it is feasible to divide \( t \) among all agents in \( S \); namely, coalition \( S \neq \emptyset \) is admissible if there exists \( x = (x_i)_{i \in S} \in \mathbb{R}^S_+ \) such that \( \sum_{i \in S} x_i = t \) and \( l_i \leq x_i \leq u_i \) for all \( i \in S \). Hence, \( S \neq \emptyset \) is admissible if and only if \( \sum_{i \in S} l_i \leq t \leq \sum_{i \in S} u_i \). We denote by \( A(N, t, l, u, \succeq) \) the set of all admissible coalitions at problem \((N, \succeq, t, l, u)\). The set \( A(N, \succeq, t, l, u) \) is non-empty because it always contains the empty coalition.

A rule \( f \) assigns to each problem in \( \mathcal{P} \) a feasible allocation; that is, \( f(N, \succeq, t, l, u) \in FA(N, t, l, u, \succeq) \) for all \((N, t, l, u, \succeq) \in \mathcal{P} \). Hence, a rule \( f \) can be seen as a systematic way of assigning to each problem \((N, t, l, u, \succeq) \in \mathcal{P} \) two different but related aspects of the solution of the problem. First, the admissible coalition \( S \in A(N, \succeq, t, l, u) \). If \( S \neq \emptyset \) we denote it by

\[
c^f(N, t, l, u, \succeq) = \{ i \in N \mid f_i(N, t, l, u, \succeq) \in [l_i, u_i] \}
\]

and call its members participants. Often, and when no confusion arises because the problem \((N, t, l, u, \succeq)\) will be obvious from the context we write \( c^f \) instead of \( c^f(N, t, l, u, \succeq) \). Obviously, if \( i \notin c^f(N, \succeq, t, l, u) \) then \( f_i(N, \succeq, t, l, u) = 0 \). Second, how the amount \( t \) is divided among the participants in \( c^f(N, t, l, u, \succeq) \); i.e., if \( c^f \neq \emptyset \)

\[
\sum_{i \in c^f} f_i(N, t, l, u, \succeq) = t.
\]

\(^7\)See Thomson (1994) and Ching and Serizawa (1998) for earlier axiomatic analysis of the division problem with variable population and/or variable amount of the good. Observe that in a division problem agent \( i \) may strictly prefer 0 to some strictly positive amount. Thus, if we add to the division problem the additional requirement that all allotments are acceptable and \( l_i = 0 \) then, if \( u_i = 0 \), \( x_i \succ_i 0 \) for all \( x_i \in (0, u_i] \) must hold (that is, (P.3) holds).
We will see later that to identify rules satisfying appealing properties we may have some freedom when choosing one among all admissible coalitions while the properties will determine a unique way of dividing the amount of the good among the participants.

3 Properties of Rules

In this section we define several properties that a rule may satisfy. The first three are basic and standard properties already used in many axiomatic analysis of the division problem.

A rule is efficient if it always selects a Pareto optimal allocation.

**Efficiency** (ef) A rule \( f \) is efficient if for each problem \((N, t, l, u) \in \mathcal{P}\) there is no feasible allocation \((y_i)_{i \in N} \in \mathcal{FA}(N, t, l, u)\) with the property that \(y_i \succeq_i f_i(N, t, l, u)\) for all \(i \in N\) and \(y_j \succ_j f_j(N, t, l, u)\) for some \(j \in N\).

**Remark 1** Let \( f \) be an efficient rule and let \((N, t, l, u) \in \mathcal{P}\) be a problem. Then, \(\sum_{i \in c^f} y_i \geq i\) implies that \(f_i(N, t, l, u) \geq p_i\) for all \(i \in c^f\) and \(\sum_{i \in c^f} y_i < i\) implies that \(f_i(N, t, l, u) \leq p_i\) for all \(i \in c^f\).

Rules require each agent to report a single-peaked preference on \(\{0\} \cup [l_i, u_i]\). A rule is strategy-proof if it is always in the best interest of agents to reveal their preferences truthfully; namely, truth-telling as a weakly dominant strategy in the direct revelation game induced by the rule.

**Strategy-proofness** (sp) A rule \( f \) is strategy-proof if for each problem \((N, t, l, u) \in \mathcal{P}\), agent \(i \in N\), and single-peaked preference \(\succeq'_i\) on \(\{0\} \cup [l_i, u_i]\),

\[
 f_i(N, \succeq_N, t, l, u) \succeq_i f_i(N, (\succeq'_i, \succeq_{N \setminus \{i\}}), t, l, u).
\]

Given a problem \((N, \succeq_N, t, l, u)\) we say that agent \(i \in N\) manipulates \( f \) at profile \(\succeq_N\) via \(\succeq'_i\) if \(f_i(N, (\succeq'_i, \succeq_{N \setminus \{i\}}), t, l, u) \succ_i f_i(N, \succeq_N, t, l, u)\).

A rule satisfies strong equal treatment of equals if identical agents receive the same allotment.

**Strong equal treatment of equals** (sete) A rule \( f \) satisfies strong equal treatment of equals if for every problem \((N, t, l, u) \in \mathcal{P}\) such that there are agents \(i, j \in N\), \(i \neq j\), and \(\succeq_i = \succeq_j\) then, \(f_i(N, t, l, u) = f_j(N, t, l, u)\).\(^8\)

Strong equal treatment of agents is incompatible with efficiency. To see that, consider any problem \((N, t, l, u)\) where \(N = \{1, 2, 3\}, t = 10, (l_i, p_i, u_i) = (4, 5, 10)\) for \(i = 1, 2, 3\),

\(^8\)Remember that \(\succeq_i = \succeq_j\) implies \([l_i, u_i] = [l_j, u_j]\).
and \(\succeq_1 \equiv \succeq_2 \equiv \succeq_3\). Since the allotment \((10/3, 10/3, 10/3) \notin FA(N, t, l, u, \succeq)\) any \(f\) satisfying strong equal treatment of equals has the property that \(c^f = \emptyset\) and \(f_i(N, t, l, u, \succeq) = 0\) for all \(i = 1, 2, 3\). However, \((0, 5, 5)\) Pareto dominates \((0, 0, 0)\). Thus, under constraints, efficiency and strong equal treatment of equals are incompatible. For this reason, we restrict our attention to the weaker notion of the property requiring that only equal \textit{participants} must be treated equally. The example above suggests that a rule satisfying equal treatment of equal (participants) will have to use some criteria to select among the three allotments \((0, 5, 5)\), \((5, 0, 5)\), and \((5, 5, 0)\) (and corresponding set of participants); but we will deal with that later.

A rule satisfies equal treatment of equals if identical participants receive the same allotment.

**Equal treatment of equals (ete)** A rule \(f\) satisfies \textit{equal treatment of equals} if for every problem \((N, t, l, u, \succeq) \in \mathcal{P}\) such that there are agents \(i, j \in N, i \neq j, \succeq_i = \succeq_j\), and \(i, j \in c^f(N, t, l, u, \succeq)\) then, \(f_i(N, t, l, u, \succeq) = f_j(N, t, l, u, \succeq)\).

We now introduce the property of bound monotonicity, which imposes restrictions on how the rule changes when the upper or lower bounds of the interval of feasible allotments of one agent changes. Take a problem \((N, t, l, u, \succeq)\) where the upper bound of agent \(k\) decreases to \(u'_k < u_k\) without changing his preferences (i.e., \(\succeq'_k\) coincides with \(\succeq_k\) on \([l_k, u'_k])\). A natural notion of bound monotonicity would say the following. First, if \(f_k(N, t, l, u, \succeq) \leq u'_k\) then, \(f(N, t, l, (u'_k, u_{-k}), (\succeq'_k, \succeq_{-k})) = f(N, t, l, u, \succeq)\). Second, if \(u'_k < f_k(N, t, l, u, \succeq)\) then, \(f_k(N, t, l, (u'_k, u_{-k}), (\succeq'_k, \succeq_{-k})) = u'_k\) and \(f_i(N, t, l, (u'_k, u_{-k}), (\succeq'_k, \succeq_{-k})) \geq f_i(N, t, l, u, \succeq)\) for all \(i \in N \setminus \{k\}\), to compensate the unavoidable change of agent \(k\)’s allotment by changing all agents’ allotments in the same and opposite direction. However, the condition \(f_k(N, t, l, (u'_k, u_{-k}), (\succeq'_k, \succeq_{-k})) = u'_k\) is incompatible with efficiency. Consider any problem \((N, t, l, u, \succeq)\) where \(N = \{1, 2\}, t = 10, \text{ and } [l_1, u_1] = [l_2, u_2] = [7, 11]\). By efficiency, \(f(N, t, l, u, \succeq) \neq (0, 0)\). Hence, by feasibility, \(f(N, t, l, u, \succeq) \in \{(10, 0), (0, 10)\}\). To see that \(f\) can not satisfy the above notion of bound monotonicity, suppose that \(f(N, t, l, u, \succeq) = (10, 0)\) (the analysis of the other case proceeds similarly). Let \((N, t, l, (u'_1, u_2), (\succeq'_1, \succeq_2))\) be a new problem that coincides with the former one except that now the upper bound of agent 1 is \(u'_1 = 9\) and \(\succeq'_1\) coincides with \(\succeq_1\) on \([7, 9]\). Since \(u'_1 = 9 < f_1(N, t, l, u, \succeq) = 10\), the second condition on the proposed bound monotonicity property above would require that \(f_1(N, t, l, (u'_1, u_2), (\succeq'_1, \succeq_2)) = 9 = u'_1\). But this would contradict the feasibility of \(f\) since \(FA(N, t, l, (u'_1, u_2), (\succeq'_1, \succeq_2)) = \{(0, 0), (0, 10)\}\). By efficiency, the new allotment should be \(f(N, t, l, (u'_1, u_2), (\succeq'_1, \succeq_2)) = (0, 10)\). The incompatibility arises because \(c^f(N, t, l, u, \succeq) = \{1\}\) but \(\{1\}\) is not an admissible coalition in the new
problem \((N, t, l, (u'_1, u_2), (\succeq'_1, \succeq_2))\). In order to restore the compatibility between a bound monotonicity principle and efficiency we weaken the former notion by requiring that either \(f(N, t, l, u, \succeq) = f(N, t, l, (u'_1, u_2), (\succeq'_1, \succeq_2))\) if \(f_k(N, t, l, u, \succeq) \leq u'_k\) or \(f_k(N, t, l, (u'_1, u_2), (\succeq'_1, \succeq_2)) = u'_k\) and \(f_i(N, t, l, u', \succeq') \geq f_i(N, t, l, u, \succeq)\) for all \(i \in N \setminus \{k\}\) if \(u'_k < f_k(N, t, l, u, \succeq)\) only when \(c^f(N, t, l, u, \succeq) \in A(N, t, l, (u'_1, u_2), (\succeq'_1, \succeq_2))\). Since \(A(N, t, l, (u'_1, u_2), (\succeq'_1, \succeq_2)) \subset A(N, t, l, u, \succeq)\) always holds, we also require that \(c^f(N, t, l, (u'_1, u_2), (\succeq'_1, \succeq_2)) = c^f(N, t, l, u, \succeq)\). Similarly for changes in the lower bound of an agent. We now define the property of bound monotonicity formally.

**Bound monotonicity (bm)** A rule \(f\) satisfies **bound monotonicity** if two conditions hold.

**(bm.1)** Let \((N, t, l, u, \succeq), (N, \succeq', t, l, u') \in \mathcal{P}\) be two problems such that for some agent \(k \in N\), \(u'_k < u_k(\leq t?)\), \(\succeq'_k\) coincides with \(\succeq_k\) on \([l_k, u'_k]\), \(u_i = u'_i\) and \(\succeq_i = \succeq'_i\) for all \(i \in N \setminus \{k\}\), and \(c^f(N, t, l, u, \succeq) \in A(N, t, l', u, \succeq')\). Then, \(c^f(N, t, l, u', \succeq') \geq c^f(N, t, l, u, \succeq)\) and

\[
f_i(N, t, l, u', \succeq') \geq \min \{f_i(N, t, l, u, \succeq), u'_i\} \quad \text{for each } i \in N. \tag{1}
\]

**(bm.2)** Let \((N, t, l, u, \succeq), (N, t', l', u', \succeq') \in \mathcal{P}\) be two problems such that for some agent \(k \in N\), \(l_k < l'_k\), and \(\succeq'_k\) coincides with \(\succeq_k\) on \([l'_k, u_k]\), \(l_i = l'_i\) and \(\succeq_i = \succeq'_i\) for all \(i \in N \setminus \{k\}\), and \(c^f(N, t, l, u, \succeq) \in A(N, t, l', u', \succeq')\). Then, \(c^f(N, t, l', u', \succeq') = c^f(N, t, l, u, \succeq)\) and

\[
f_i(N, t, l', u, \succeq') \leq \max \{f_i(N, t, l, u, \succeq), l'_i\} \quad \text{for each } i \in N. \tag{2}
\]

To clarify the definition assume that \(f_k(N, t, l, u, \succeq) \leq u'_k < u_k\) and \(c^f(N, t, l, u, \succeq) \in A(N, t, l, u', \succeq')\). Then, \(c^f(N, t, l, u', \succeq') = c^f(N, t, l, u, \succeq)\) and (1) hold, which imply that \(f(N, t, l, u, \succeq) = f(N, t, l, u', \succeq')\). Now suppose that \(u'_k < f_k(N, t, l, u, \succeq)\) and \(c^f(N, t, l, u, \succeq) \in A(N, t, l, u', \succeq')\). Then, \(c^f(N, t, l, u', \succeq') = c^f(N, t, l, u, \succeq)\) and (1) hold, which imply that \(f_k(N, t, l, u', \succeq') = u'_k\) and \(f_i(N, t, l, u', \succeq') \geq f_i(N, t, l, u, \succeq)\) for all \(i \in N \setminus \{k\}\). Thus, condition (1) can be rewritten as

\[
f_i(N, t, l, u', \succeq') \geq f_i(N, t, l, u, \succeq) \quad \text{for all } i \in N \setminus \{k\}\] and

\[
f_k(N, t, l, u', \succeq') \geq \min \{f_k(N, t, l, u, \succeq), u'_k\}.
\]

Similarly, condition (2) can be rewritten as

\[
f_i(N, t, l', u, \succeq') \leq f_i(N, t, l, u, \succeq) \quad \text{for all } i \in N \setminus \{k\}\] and

\[
f_k(N, t, l', u, \succeq') \leq \max \{f_k(N, t, l, u, \succeq), l'_k\}.
\]

A rule satisfies admissible contraction if the following requirement holds. Consider two problems where the set of admissible coalitions of the first one is contained in the set of
admissible coalitions of the second one. Assume that the coalition chosen by the rule in the second problem is admissible for the first one. Then, the rule chooses the same coalition of participants in the two problems.

**Admissible Contraction (ac)** A rule $f$ satisfies admissible contraction if for any two problems $(N, t, l, u, \succeq) , (N', t', l', u', \succeq') \in \mathcal{P}$ such that $c^f (N, t, l, u, \succeq) \in A (N', t', l', u', \succeq') \subset A (N, t, l, u, \succeq)$ then,

$$c^f (N', t', l', u', \succeq') = c^f (N, t, l, u, \succeq).$$

Observe that $(bm)$ and $(ac)$ are independent. Both properties impose conditions on the behavior of the rule by comparing its choices across different pairs of problems $(N, t, l, u, \succeq)$ and $(N', t', l', u', \succeq')$ only if $c^f (N, t, l, u, \succeq) \in A (N', t', l', u', \succeq')$. However, $(bm)$ requires that the two problems differ only in a bound of one agent while $(ac)$ only requires that the two sets of agents coincide. Both properties say that the set of participants in the two problems coincide but $(bm)$ imposes natural (with respect to the change of the bound) additional conditions on the change of the vector of allotments at the two problems. Condition $(bm)$ restricts $f$ more, but less often, and $(ac)$ restricts $f$ less, but more often.

### 4 The Uniform Rule: Two Characterizations

In this section we present the two results of the paper.

We first define the uniform rule for the subclass of division problems under constraints, when the set of all agents is an admissible coalition. The uniform rule $U$ on problems without constraints (see Sprumont (1991)) tries to allocate the good as equally as possible, keeping the efficiency bounds biding (all agents have to be rationed in the same direction). The uniform rule $U^E$ on the subclass of division problems under constraints, when the set of all agents is an admissible coalition, does the same but it takes also into account the feasibility constraints. We show in Theorem 1 that the uniform rule $U^E$ is the unique rule satisfying efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity on the subclass of problems where the set of all agents is an admissible coalition.

Let $P$ be the set of division problems under constraints, where the set of all agents is an admissible coalition; namely,

$$P = \left\{ (N, t, l, u, \succeq) \in \mathcal{P} \mid \sum_{i \in N} l_i \leq t \leq \sum_{i \in N} u_i \right\}.$$

**Uniform Rule** The uniform rule $U^E$ on $P$ is defined as follows. For each $(N, t, l, u, \succeq) \in$
where \( \alpha \) is the unique number satisfying \( \sum_{j \in \mathcal{N}} U^E_j (N, t, l, u, \succeq) = t \).

**Remark 2** Consider the problem \((N, t, l, u, \succeq) \in \mathcal{P}\) and a division problem without constraints \((N, t, \succeq')\) (i.e., \(l_i' = 0\) and \(u_i \geq t\) for all \(i \in N\)) such that for each \(i \in N\), \(\succeq_i\) coincides with \(\succeq\) on \([l_i, u_i]\) and \(U(N, \succeq', t) \in FA(N, t, l, u, \succeq)\). Then, \(U(N, t, \succeq') = U^E(N, t, l, u, \succeq)\). Thus, the uniform rule \(U^E\) can be considered as an extension of the uniform rule \(U\) from division problems without constraints to \(\mathcal{P}\). Observe that the extension of the uniform rule to problems with voluntary participation when the bounds are idiosyncratic presented in Bergantiños, Massó, and Neme (SCW, 2011) does not have this property (elaborate more on that).

**Theorem 1** The uniform rule \(U^E\) on \(\mathcal{P}\) is the unique rule satisfying efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity on \(\mathcal{P}\). Moreover, the four properties are independent.

**Proof** See Subsection 6.1.

We now consider the general case. We first extend the uniform rule to the class of all problems in \(\mathcal{P}\). Given a problem \((N, t, l, u, \succeq)\) in \(\mathcal{P}\) the extended uniform rule selects the feasible set of participants by maximizing a given order \(\rho\) on \(\mathcal{N}\) (restricted to the family of admissible coalitions \(A(N, t, l, u, \succeq) \subset 2^N\)) and then it applies the uniform rule to this selected set of participants to choose their allotments. Formally, let \(\rho\) be a (linear) order on \(\mathcal{N}\); i.e., \(\rho\) is a complete, antisymmetric and transitive binary relation on \(\mathcal{N}\).

**Extended Uniform Rule** Let \(\rho\) be an order on \(\mathcal{N}\). The extended uniform rule on \(\mathcal{P}\) induced by the order \(\rho\), denoted by \(f^\rho\), is defined as follows. For each \((N, t, l, u, \succeq) \in \mathcal{P}\) and \(i \in N\),

\[
   f^\rho_i (N, t, l, u, \succeq) = \begin{cases} 
   U^E(c^{f^\rho}, t, (l_j)_{j \in c^{f^\rho}}, (u_j)_{j \in c^{f^\rho}}, (\succeq_j)_{j \in c^{f^\rho}}) & \text{if } i \in c^{f^\rho} \\
   0 & \text{if } i \notin c^{f^\rho},
   \end{cases}
\]

where \(c^{f^\rho} \rho S\) for all \(S \in A(N, t, l, u, \succeq) \setminus c^{f^\rho}\).

Obviously, the family of extended uniform rules on \(\mathcal{P}\) is extremely large. We are interested in the subfamily of rules that satisfy efficiency, strategy-proofness, equal treatment of
equals, bound monotonicity and admissible contraction. To identify it we restrict the order \( \rho \) to satisfy the properties of monotonicity and responsiveness.

**Definition 1** We say that an order \( \rho \) on \( N \) is

(i) monotonic (\( mo \)) if for all \( S \in \mathcal{N} \) and \( i \not\in S \), \((S \cup \{i\}) \rho S \); and

(ii) responsive (\( re \)) if for all \( S, T \in \mathcal{N} \) and \( i \not\in S \cup T \), \( S \rho T \) implies \((S \cup \{i\}) \rho (T \cup \{i\}) \).

Theorem 2 below characterizes the set of extended uniform rules with the property that they choose the admissible coalition according to a monotonic and responsive order as the class of rules satisfying efficiency, strategy-proofness, equal treatment of equals, bound monotonicity, and admissible contraction.

**Theorem 2** Let \( f \) be a rule on \( \mathcal{P} \). Then, \( f \) satisfies efficiency, strategy-proofness, equal treatment of equals, bound monotonicity, and admissible contraction if and only if \( f = f^\rho \) for some monotonic and responsive order \( \rho \) on \( \mathcal{N} \). Moreover, the five properties are independent.

**Proof.** See Subsection 6.2.

## 5 Final Remarks

Other properties, open questions, etc.

**OWN-PEAK MONOTONICITY (\( opm \))** A rule \( f \) on \( \mathcal{P} \) is own-peak monotonic if for all \((N, t, l, u, \succeq), (N, t, l, u, \succeq') \in \mathcal{P}, p'_j \leq p_j \) implies \( f_j(N, t, l, u, (\succeq_j', \succeq_{-j})) \leq f_j(N, t, l, u, \succeq) \).

**TOPS-ONLY (\( t-o \))** A rule \( f \) on \( \mathcal{P} \) is tops-only if for all \( \succeq_N \) and \( \succeq'_N \) such that \( p_i = p'_i \) for all \( i \in N \) then, \( f(N, t, l, u, \succeq) = f(N, t, l, u, \succeq') \).

## 6 Proofs

### 6.1 Proof of Theorem 1

To prove that \( U^E \) satisfies efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity is straightforward and therefore omitted.

We prove that \( U^E \) is the unique rule satisfying the four properties on \( P \). We do it by proving the following five lemmata.

**Lemma 1.1** Let \( f \) be an efficient and bound monotonic rule on \( P \) and let \((N, t, l, u, \succeq) \in \mathcal{P} \). If \( c^f(N, t, l, u, \succeq) \not\subseteq S \subseteq N \) then, \( S \not\in A(N, t, l, u, \succeq) \).
Proof To obtain a contradiction suppose that there exists \( j \in S \subseteq A(N, t, l, u, \succeq) \) and \( j \notin c^f(N, t, l, u, \succeq) \). Hence, \( f_j(N, t, l, u, \succeq) = 0 \) and \( f_j(N, t, l, u, \succeq) \notin [l_j, u_j] \). Thus, \( 0 < l_j \).

Let \((N, t, l^1, u^1, \succeq^1) \in \mathcal{P}\) be such that \((N, t, l^1, u^1, \succeq^1)\) coincides with \((N, t, l, u, \succeq)\) except that \( l^1_i = u^1_i = f_i(N, t, l, u, \succeq) \) for all \( i \in c^f(N, t, l, u, \succeq) \). By (bm),

\[
  f(N, t, l, u, \succeq) = f(N, \succeq^1, t, l^1, u^1) .
\]

Since \( \sum_{i \in c^f} f_i(N, t, l, u, \succeq) = t \) and \( f_i(N, t, l, u, \succeq) = 0 \) for all \( i \notin c^f(N, t, l, u, \succeq) \), \( \sum_{i \in S} f_i(N, t, l, u, \succeq) = t > 0 \). Let \( k \in c^f(N, t, l, u, \succeq) \) be such that \( f_k(N, t, l, u, \succeq) > 0 \). Let \((N, t, l^2, u^2, \succeq^2) \in \mathcal{P}\) be such that it coincides with \((N, t, l^1, u^1, \succeq^1)\) except that \( l_j > l^2_j = \varepsilon > 0 \) for \( \varepsilon \) sufficiently small, \( u^2_j = t \) and \( \succeq^2 \) coincides with \( \succeq^1 \) on \([l^1_j, u^1_j]\). By (bm),

\[
  f(N, t, l^2, u^2, \succeq^2) = f(N, t, l^1, u^1, \succeq^1) .
\]

Let \((N, t, l^3, u^3, \succeq^3) \in \mathcal{P}\) be such that it coincides with \((N, t, l^2, u^2, \succeq^2)\) except that \( 0 < l^3_k = p^3_k = f_k(N, t, l, u, \succeq) - \varepsilon \) and \( \succeq^3 \) coincides with \( \succeq^2 \) on \([l^2_k, u^2_k]\). By (bm)

\[
  f(N, t, l^3, u^3, \succeq^3) = f(N, t, l^2, u^2, \succeq^2) .
\]

Let \( x = (x_i)_{i \in N} \) be such that \( x_j = \varepsilon, x_k = f_k(N, t, l, u, \succeq) - \varepsilon, \) and \( x_i = f_i(N, t, l^3, u^3, \succeq^3) \) for all \( i \in N \setminus \{j, k\} \), if any. But \( x \in FA(N, t, l^3, u^3, \succeq^3) \) and \( x \) Pareto dominates \( f(N, t, l^3, u^3, \succeq^3) \), a contradiction with the efficiency of \( f \).

An immediate consequence of Lemma 1.1 is that if \( N \in A(N, t, l, u, \succeq) \) then, \( c^f(N, t, l, u, \succeq) = N \). Hence, by (ete), for all \((N, t, l, u, \succeq) \in \mathcal{P}\) such that \( N \in A(N, t, l, u, \succeq) \), \( \succeq_i = \succeq_j \) implies \( f_i(N, t, l, u, \succeq) = f_j(N, t, l, u, \succeq) \).

Lemma 1.2 Let \( f \) be an efficient and strategy-proof rule on \( P \). Then, \( f \) is own-peak monotonic.

Proof Let \((N, t, l, u, \succeq), (N, t, l, u, \succeq') \in P \) and \( j \in N \) be such that \( p'_j \leq p_j \). To obtain a contradiction, assume

\[
  f_j(N, t, l, u, \succeq) < f_j(N, t, l, u, (\succeq', \succeq_{-j})) .
\]

We consider two cases.

Case 1: \( \sum_{i \in N} p_i \leq t \). By (ef), \( p_i \leq f_i(N, t, l, u, \succeq) \) for all \( i \in N \). Hence,

\[
  p'_j \leq p_j \leq f_j(N, t, l, u, \succeq) < f_j(N, t, l, u, (\succeq', \succeq_{-j})),
\]

which implies, by (P.2), that \( f_j(N, t, l, u, \succeq) \succ_j f_j(N, t, l, u, (\succeq', \succeq_{-j})) \), a contradiction with (sp).
Lemma 1.3

Let $f_i(N, t, l, u, \succeq) \leq p_i$ (4)

for all $i \in N$. We consider two subcases.

Subcase 2.1: $t \leq \sum_{i \neq j} p_i + p_j'$. By (ef), for all $i \neq j$; $f_i(N, t, l, u, (\succeq'_j, \succeq_j)) \leq p_i$ and $f_j(N, t, l, u, (\succeq'_j, \succeq_j)) \leq p_j'$. Hence, by (3),

$$f_j(N, t, l, u, \succeq) < f_j(N, t, l, u, (\succeq'_j, \succeq_j)) \leq p_j' \leq p_j,$$

which implies, by (P.2), that $f_j(N, t, l, u, (\succeq'_j, \succeq_j)) \succ_j f_j(N, t, l, u, \succeq)$, a contradiction with (sp).

Subcase 2.2: $\sum_{i \neq j} p_i + p_j' < t$. By (ef), for all $i \neq j$; $p_i \leq f_i(N, t, l, u, (\succeq'_j, \succeq_j))$ and $p_j' \leq f_j(N, t, l, u, (\succeq'_j, \succeq_j))$. Thus, $p_j' \leq f_j(N, t, l, u, \succeq)$; otherwise, by (4),

$$t = \sum_{i \in N} f_i(N, t, l, u, \succeq) < \sum_{i \neq j} p_i + p_j'$$

a contradiction. Hence,

$$p_j' \leq f_j(N, t, l, u, \succeq) < f_j(N, t, l, u, (\succeq'_j, \succeq_j)),$$

which implies, by (P.2), that $f_j(N, t, l, u, \succeq) \succ_j f_j(N, t, l, u, (\succeq'_j, \succeq_j))$, a contradiction with (sp).

Lemma 1.3 Let $f$ be an efficient and strategy-proof rule. Then, for all $j \in N$:

(a) If $p_j < f_j(N, t, l, u, \succeq)$ and $\succeq_j$ satisfies $0 \leq p_j' \leq f_j(N, t, l, u, \succeq)$ then, $f_j(N, t, l, u, (\succeq'_j, \succeq_j)) = f_j(N, t, l, u, \succeq)$.

(b) If $f_j(N, t, l, u, \succeq) < p_j$ and $\succeq'_j$ satisfies $f_j(N, t, l, u, \succeq) \leq p_j' \leq t$, then $f_j(N, t, l, u, (\succeq'_j, \succeq_j)) = f_j(N, t, l, u, \succeq)$.

Proof Let $f$ be an efficient and strategy-proof rule and $j$ be any agent.

(a) Assume $p_j < f_j(N, t, l, u, \succeq)$ and let $\succeq'_j$ be such that $0 \leq p_j' \leq f_j(N, t, l, u, \succeq)$. By (ef), Remark 1 implies that for all $i \in c'(N, t, l, u, \succeq)$,

$$p_i \leq f_i(N, t, l, u, \succeq).$$

Since $p_j' \leq f_j(N, t, l, u, \succeq)$, (5) implies

$$\sum_{i \in c'(N, t, l, u, \succeq)} p_i + p_j' \leq \sum_{i \in c'} f_i(N, t, l, u, \succeq) = t.$$
We now show that \( f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) = f_j(N, t, l, u, \succeq) \). To obtain a contradiction, assume otherwise and consider two cases.

**Case 1**: \( f_j(N, t, l, u, \succeq) < f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) \). Then
\[
p_j' \leq f_j(N, t, l, u, \succeq) < f_j(N, t, l, u, (\preceq_{,} \preceq_{,})),
\]
which implies, by (P.2), that
\[
f_j(N, t, l, u, \succeq) \succ_j f_j(N, t, l, u, (\preceq_{,} \preceq_{,})),
\]
contradicting \((sp)\).

**Case 2**: \( f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) < f_j(N, t, l, u, \succeq) \). We consider two subcases.

**Subcase 2.1**: \( p_j \leq f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) \). Then
\[
p_j \leq f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) < f_j(N, t, l, u, \succeq) .
\]
Hence, by (P.2),
\[
f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) \succ_j f_j(N, t, l, u, \succeq) ,
\]
contradicting \((sp)\).

**Subcase 2.2**: \( f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) < p_j \). Then, \( p_j' > 0 \) and
\[
f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) < p_j = p_j' < f_j(N, t, l, u, \succeq) . \tag{6}
\]
If \( f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) \notin [l_j, u_j] \) then, \( f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) = 0 \). By (P.3), \( j \) manipulates \( f \) at \((N, t, l, u, (\preceq_{,} \preceq_{,}))\) via \( \preceq_j \), contradicting \((sp)\). Assume \( f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) \in [l_j, u_j] \) and let \( \preceq_{,}'' \) be such that \( p_j'' = p_j \) and
\[
f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) \succ_j f_j(N, t, l, u, \succeq) . \tag{7}
\]
By Lemma 1.2, \( f \) is own-peak monotonic. Hence, for all \( i \in N \),
\[
f_i(N, t, l, u, (\preceq_{,} \preceq_{,})) = f_i(N, t, l, u, \succeq) .
\]
By (7),
\[
f_j(N, t, l, u, (\preceq_{,} \preceq_{,})) \succ_j f_i(N, t, l, u, (\preceq_{,} \preceq_{,})),
\]
contradicting \((sp)\).

(b) We omit the proof since it follows a symmetric argument to the one used to prove (a).
A consequence of Lemma 1.3 is that if $f$ is (ef) and (sp) on $P$ then, $f$ is tops-only on $P$.

**Lemma 1.4** Let $f$ be a rule satisfying (ete), and (bm) on $P$ and assume $(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) \in P$ is such that $u_k = t$ for all $k \in N$ and $\succeq'_i$ coincides with $\succeq'_j$ on $[\max \{i, l_j\}, t]$. Then, it is not possible that simultaneously

\[
f_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) \leq U^E_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) \text{ and } \]

\[
f_j(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) > U^E_j(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) \]

hold.

**Proof** We consider separately the three possible cases.

**Case 1:** $l_i = l_j$. The statement follows since $\succeq'_i = \succeq'_j$ and $f$ and $U^E$ satisfy (ete).

**Case 2:** $l_i < l_j$. Assume

\[
f_j(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) > U^E_j(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})). \tag{8}
\]

We want to show that $f_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) < U^E_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}}))$ does not hold. By (8), $l_j \leq U^E_j(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) < f_j(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}}))$. Let $l'^*_j = l_i$ and consider the preference $\succeq'_{j}$ of agent $j$ on $[l'^*_i, u_j]$ that coincides with $\succeq'_{i}$ on $[l_i, u_i] = [l'^*_i, u_j]$. By (bm),

\[
f(N, t, (l'^*_j, l_{-j}), u, (\succeq'_{i}, \succeq'_{j}, \succeq_{N\setminus\{i,j\}})) = f(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})). \tag{9}
\]

Since $\succeq'^*_j = \succeq'_{i,j}$ by (ete), $f_j(N, t, (l'^*_j, l_{-j}), u, (\succeq'_{i}, \succeq'_{j}, \succeq_{N\setminus\{i,j\}})) = f_i(N, t, (l'^*_j, l_{-j}), u, (\succeq'_{i}, \succeq'_{j}, \succeq_{N\setminus\{i,j\}}))$. By (9),

\[
f_j(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) = f_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})). \tag{10}
\]

By (ete),

\[
U^E_j(N, t, (l'^*_j, l_{-j}), u, (\succeq'_{i}, \succeq'_{j}, \succeq_{N\setminus\{i,j\}})) = U^E_i(N, t, (l'^*_j, l_{-j}), u, (\succeq'_{i}, \succeq'_{j}, \succeq_{N\setminus\{i,j\}})). \tag{11}
\]

By (rm), $U^E(N, t, (l'^*_j, l_{-j}), u, (\succeq'_{i}, \succeq'_{j}, \succeq_{N\setminus\{i,j\}})) = U^E(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}}))$. By (11),

\[
U^E_j(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) = U^E_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})). \tag{11}
\]

By (8) and (10), $f_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) \leq U^E_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}}))$. Thus, $f_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) \leq U^E_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}}))$. By (8), $f_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}})) = U^E_i(N, t, l, u, (\succeq'_{i,j}, \succeq_{N\setminus\{i,j\}}))$ does not hold.

**Case 3:** $l_i > l_j$. The argument is similar to the one used to prove Case 2 and omitted.

**Lemma 1.5.** Let $f$ be a rule satisfying (ef), (sp), (ete), and (bm) on $P$. Then, $f = U^E$. 

17
Proof Let \((N, t, l, u, \succeq) \in P\) be arbitrary. We want to show that \(f(N, t, l, u, \succeq) = U^E(N, t, l, u, \succeq)\). Assume that \(\sum_{i \in N} p_i \geq t\). The case \(\sum_{i \in N} p_i < t\) is similar and omitted.

By (ef), \(f_i(N, t, l, u, \succeq) \leq p_i\) for all \(i \in N\).

Let \(u_1 = +\infty\) and consider any \(\succeq_1^o\) defined on \([l_1, +\infty]\) that coincides with \(\succeq_1\) on \([l_1, u_1]\). For each \(i \in N \setminus \{1\}\) define \(\succeq_1^i = \succeq_1\), and \(u_1^i = u_i\). Applying (bm) to \((N, t, l, u, \succeq)\) and \((N, t, l, u^1, \succeq^1)\) we obtain that for each \(i \in N\),

\[
 f_i(N, t, l, u, \succeq) \leq \max \{ f_i(N, t, l, u^1, \succeq^1), l_i \} = f_i(N, t, l, u^1, \succeq^1). 
\]

By (ef) and feasibility, \(f_i(N, t, l, u^1, \succeq^1) = f_i(N, t, l, u, \succeq)\).

Let \(u_2 = +\infty\) and consider any \(\succeq_2^o\) defined on \([l_2, +\infty]\) that coincides with \(\succeq_2\) on \([l_2, u_2]\). For each \(i \in N \setminus \{2\}\) define \(\succeq_2^i = \succeq_1\) and \(u_2^i = u_1^i\). Proceeding as in the previous case we obtain that for each \(i \in N\), \(f_i(N, t, l, u^2, \succeq^2) = f_i(N, t, l, u^1, \succeq^1)\).

Repeating this argument we obtain that \(f(N, t, l, w^a, \succeq^a) = f(N, t, l, u, \succeq)\). Thus, we can assume that \(u_i = +\infty\) for all \(i \in N\).

Without loss of generality assume that \(p_1 \geq p_2 \geq \ldots \geq p_n\). To obtain a contradiction, assume that \(U^E(N, t, l, u, \succeq) \neq f(N, t, l, u, \succeq)\). Then, there exists \(i^1 \in N\) such that

\[
U^E_{i^1}(N, t, l, u, \succeq) < f_{i^1}(N, t, l, u, \succeq) \leq p_{i^1} \leq p_1. 
\]

**Step 1:** Take \(\succeq'_{i^1}\) defined on \([l_{i^1}, u_{i^1}]\) and that it coincides with \(\succeq_1\) on \([\max \{l_1, l_{i^1}\}, t]\).

By Lemma 1.3,

\[
U^E_{i^1}(N, t, l, u, (\succeq'_{i^1}, \succeq_{N \setminus \{i^1\}})) = U^E_{i^1}(N, t, l, u, \succeq). 
\]

By Lemma 1.2, \(f\) is own-peak monotonic. Since \(p'_{i^1} \leq p_{i^1}\),

\[
f_{i^1}(N, t, l, u, \succeq) \leq f_{i^1}(N, t, l, u, (\succeq'_{i^1}, \succeq_{N \setminus \{i^1\}})).
\]

Thus,

\[
U^E_{i^1}(N, t, l, u, (\succeq'_{i^1}, \succeq_{N \setminus \{i^1\}})) < f_{i^1}(N, t, l, u, (\succeq'_{i^1}, \succeq_{N \setminus \{i^1\}})). 
\]

**Step 2:** Then, there exists \(i^2 \in N \setminus \{i^1\}\) such that

\[
f_{i^2}(N, t, l, u, (\succeq'_{i^1}, \succeq_{N \setminus \{i^1\}})) < U^E_{i^2}(N, t, l, u, (\succeq'_{i^1}, \succeq_{N \setminus \{i^1\}})) \leq p_{i^2}.
\]

Take \(\succeq'_{i^2}\) defined on \([l_{i^2}, u_{i^2}]\) and that it coincides with \(\succeq_1\) on \([\max \{l_1, l_{i^2}\}, t]\). By Lemma 1.3,

\[
f_{i^2}(N, t, l, u, (\succeq'_{i^1}, \succeq_{N \setminus \{i^1, i^2\}})) = f_{i^2}(N, t, l, u, (\succeq'_{i^1}, \succeq_{N \setminus \{i^1\}})). 
\]
By Lemma 1.2, \( f \) is own-peak monotonic. Since \( p_{i^2} \leq p_i \),
\[
U_{i^2}^E(N, t, l, u, (\succeq_{i^1}, \succeq_{N \setminus \{i^1\}})) \leq U_{i^2}^E(N, t, l, u, (\succeq_{\{i^1, i^2\}}, \succeq_{N \setminus \{i^1, i^2\}})).
\]

Thus,
\[
f_{i^2}(N, t, l, u, (\succeq_{\{i^1, i^2\}}, \succeq_{N \setminus \{i^1, i^2\}})) < U_{i^2}^E(N, t, l, u, (\succeq_{\{i^1, i^2\}}, \succeq_{N \setminus \{i^1, i^2\}})).
\]

By Lemma 1.4,
\[
f_{i^3}(N, t, l, u, (\succeq_{\{i^1, i^2\}}, \succeq_{N \setminus \{i^1, i^2\}})) \leq U_{i^3}^E(N, t, l, u, (\succeq_{\{i^1, i^2\}}, \succeq_{N \setminus \{i^1, i^2\}})).
\]

**Step 3:** Then, by feasibility, there must exist \( i^3 \in N \setminus \{i^1, i^2\} \) such that
\[
U_{i^3}^E(N, t, l, u, (\succeq_{\{i^1, i^2\}}, \succeq_{N \setminus \{i^1, i^2\}})) < f_{i^3}(N, t, l, u, (\succeq_{\{i^1, i^2\}}, \succeq_{N \setminus \{i^1, i^2\}})).
\]

Take \( \succeq_{i^3} \) defined on \([l_{i^3}, u_{i^3}]\) and that it coincides with \( \succeq_1 \) on \([\max \{l_1, l_{i^3}\}, t] \). By Lemma 1.3,
\[
U_{i^3}^E(N, t, l, u, (\succeq_{\{i^1, i^2, i^3\}}, \succeq_{N \setminus \{i^1, i^2, i^3\}})) = U_{i^3}^E(N, t, l, u, (\succeq_{\{i^1, i^2\}}, \succeq_{N \setminus \{i^1, i^2\}})).
\]

By Lemma 1.2,
\[
f_{i^3}(N, t, l, u, (\succeq_{\{i^1, i^2, i^3\}}, \succeq_{N \setminus \{i^1, i^2, i^3\}})) \leq f_{i^3}(N, t, l, u, (\succeq_{\{i^1, i^2, i^3\}}, \succeq_{N \setminus \{i^1, i^2, i^3\}})).
\]

Thus,
\[
U_{i^3}^E(N, t, l, u, (\succeq_{\{i^1, i^2, i^3\}}, \succeq_{N \setminus \{i^1, i^2, i^3\}})) < f_{i^3}(N, t, l, u, (\succeq_{\{i^1, i^2, i^3\}}, \succeq_{N \setminus \{i^1, i^2, i^3\}})).
\]

By applying Lemma 1.4 twice, we obtain that
\[
U_{i^3}^E(N, t, l, u, (\succeq_{\{i^1, i^2, i^3\}}, \succeq_{N \setminus \{i^1, i^2, i^3\}})) \leq f_{i^3}(N, t, l, u, (\succeq_{\{i^1, i^2, i^3\}}, \succeq_{N \setminus \{i^1, i^2, i^3\}})).
\]

and
\[
U_{i^3}^E(N, t, l, u, (\succeq_{\{i^1, i^2, i^3\}}, \succeq_{N \setminus \{i^1, i^2, i^3\}})) \leq f_{i^3}(N, t, l, u, (\succeq_{\{i^1, i^2, i^3\}}, \succeq_{N \setminus \{i^1, i^2, i^3\}})).
\]

Continuing with this procedure, at Stage \( n \), we obtain that either
\[
U_{i^n}^E(N, t, l, u, (\succeq_N)) < f_{i^n}(N, t, l, u, (\succeq_N)) \quad \text{and for all } j \in N \setminus \{i^n\},
\]
\[
U_{j^n}^E(N, t, l, u) \leq f_{j^n}(N, t, l, u) \quad \text{or else}
\]
\[
f_{i^n}(N, t, l, u, (\succeq_N)) < U_{i^n}^E(N, t, l, u, (\succeq_N)) \quad \text{and for all } j \in N \setminus \{i^n\}
\]
\[
f_{j^n}(N, t, l, u, (\succeq_N)) \leq U_{j^n}^E(N, t, l, u, (\succeq_N)).
\]
In both cases we have a contradiction because

$$\sum_{i \in N} f_i(N, \succeq'_N, t, l, u) = \sum_{i \in N} U_i^E(N, \succeq'_N, t, l, u) = t.$$  

We now prove that the four properties are independent.

- (ef) is independent of the other three properties.

  We define the rule $f^1$ as follows. Let $(N, t, l, u, \succeq) \in P$. For each $i \in N$,

  $$f^1_i(N, t, l, u, \succeq) = \text{median } \{l_i, \alpha, u_i\}$$

  where $\alpha$ is such that $\sum_{i \in N} f_i(N, u, \succeq, t) = t$. Then, $f^1$ satisfies (sp), (ete), and (bm) by fails (ef).

- (sp) is independent of the other three properties.

  We define the rule $f^2$ as follows. Let $(N, t, l, u, \succeq) \in P$. For each $i \in N$,

  $$f^2_i(N, t, l, u, \succeq) = \begin{cases} 
  p_i + \min(\alpha, u_i - p_i) & \text{if } \sum_{i \in N} p_i < t \\
  U_i^E(N, t, l, u, \succeq) & \text{if } \sum_{i \in N} p_i \geq t
  \end{cases}$$

  where $\alpha$ is such that $\sum_{i \in N} f_i(N, u, \succeq, t) = t$. Then, $f^2$ satisfies (ef), (ete), and (bm) by fails (sp).

- (ete) is independent of the other three properties.

  We define $f^3$ as the priority rule given by the order $(1, 2, ..., n)$ applied to the set of efficient allocations. Namely, let $(N, t, l, u, \succeq) \in P$. We define $f^3$ formally, by considering separately the two following cases.

  1. $\sum_{i \in N} p_i \geq t$. Take $k$ as the unique agent satisfying that $\sum_{i=1}^k p_i \leq t < \sum_{i=1}^{k+1} p_i$. For each $i \in N$,

     $$f^3_i(N, t, l, u, \succeq) = \begin{cases} 
  p_i & \text{if } i \leq k \\
  t - \sum_{i=1}^k p_i & \text{if } i = k + 1 \\
  l_i & \text{if } i > k.
  \end{cases}$$

    

20
2. \( \sum_{i \in N} p_i < t \). Take \( k \) as the unique agent satisfying that \( \sum_{i=1}^{k} p_i + \sum_{i=k+1}^{n} u_i < t < \sum_{i=1}^{k} p_i + \sum_{i=k+1}^{n} u_i \). For each \( i \in N \),

\[
 f^3_i (N, t, l, u, \succeq) = \begin{cases} 
    p_i & \text{if } i \leq k \\
    t - \sum_{i=1}^{k} p_i - \sum_{i=k+1}^{n} u_i & \text{if } i = k + 1 \\
    u_i & \text{if } i > k
\end{cases}
\]

Then, \( f^3 \) satisfies \((ef)\), \((sp)\), and \((bm)\) by fails \((ete)\).

- \((ete)\) is independent of the other properties.

We define the rule \( f^4 \) inspired by the Constant Equal Losses rule used in bankruptcy problems. Let \( (N, t, l, u, \succeq) \in \mathcal{P} \). For each \( i \in N \),

\[
 f^4_i (N, t, l, u, \succeq) = \begin{cases} 
    \max\{u_i - \alpha, p_i\} & \text{if } \sum_{i \in N} p_i < t \\
    \min\{\max\{l_i, u_i - \alpha\}, p_i\} & \text{if } \sum_{i \in N} p_i \geq t
\end{cases}
\]

where \( \alpha \) is such that \( \sum_{i \in N} f_i (N, t, l, u, \succeq) = t \). Then, \( f^4 \) satisfies \((ef)\), \((sp)\), and \((ete)\) by fails \((bm)\).

### 6.2 Proof of Theorem 2

Let \( \rho \) be any monotonic and responsive order on \( \mathcal{N} \). To prove that \( f^\rho \) satisfies \((ef)\), \((sp)\), \((ete)\), \((bm)\), and \((ac)\) on \( \mathcal{P} \) is straightforward and therefore omitted.

Let \( f \) be a rule satisfying \((ef)\), \((sp)\), \((ete)\), \((bm)\), and \((ac)\). We prove that there exists a monotonic and responsive order \( \rho \) on \( \mathcal{N} \) for which \( f^\rho = f \).

We first define (using \( f \)) a binary relation \( \rho \) on \( \mathcal{N} \). Let \( S, S' \in \mathcal{N} \). Three cases are possible.

**Case 1:** \( S \supset S' \). Then, set \( S \rho S' \).

**Case 2:** \( S' \supset S \). Then, set \( S' \rho S \).

**Case 3:** There exist agents \( j \in S \setminus S' \) and \( j' \in S' \setminus S \). Consider any problem \( (N, t, l, u, \succeq) \in \mathcal{P} \) where \( S, S' \subseteq N \) and for each \( i \in N \), \( l_i = p_i = u_i \), and

\[
p_i = \begin{cases} 
    \varepsilon & \text{if } i \in S \cap S' \\
    \varepsilon^2 & \text{if } i \in S \setminus (S' \cup \{j\}) \\
    t - \varepsilon |S \cap S'| - \varepsilon^2 |S \setminus (S' \cup \{j\})| & \text{if } i = j \\
    \varepsilon^3 & \text{if } i \in S' \setminus (S \cup \{j'\}) \\
    t - \varepsilon |S \cap S'| - \varepsilon^3 |S' \setminus (S \cup \{j'\})| & \text{if } i = j' \\
    \varepsilon^4 & \text{if } i \in N \setminus (S \cup S').
\end{cases}
\]
Moreover, choose \( \varepsilon > 0 \) small enough to make sure that \( 0 < p_i < t \) for all \( i \in N \) and \( A(N,t,l,u,\succeq) = \{S,S'\} \). Observe that such \( \varepsilon > 0 \) exists. Since \( f \) is efficient, \( c^f(N,t,l,u,\succeq) \in \{S,S'\} \). Then, if \( c^f(N,t,l,u,\succeq) = S \) set \( S \rho S' \) and if \( c^f(N,t,l,u,\succeq) = S' \) set \( S' \rho S \).

Since \( f \) satisfies \((ac)\), the order \( \rho \) does not depend on the particular chosen problem \( (N,t,l,u,\succeq) \in \mathcal{P} \). Namely, let \( (N,\succeq',t',l',u') \in \mathcal{P} \) be such that \( AC(N,\succeq',t',l',u') = \{S,S'\} \). Then, \( c^f(N,\succeq',t',l',u') = c^f(N,t,l,u,\succeq) \). Thus, \( \rho \) is well defined.

It is immediate to see that the binary relation \( \rho \) on \( \mathcal{N} \) is complete, antisymmetric, monotonic and responsive. Using similar arguments to those used in the proof of Lemma 13 in Bergantiños, Massó, and Neme (2012b) it is possible to show that \( \rho \) is transitive.

**Lemma 2.1** Let \( f \) be a rule satisfying \((ef),(sp)\), \((ete)\), \((bm)\), and \((ac)\) and let \( \rho \) be its corresponding complete, antisymmetric, transitive, monotonic and responsive order on \( \mathcal{N} \) defined as in Cases 1, 2, and 3 above. Then, \( f(N,t,l,u,\succeq) = f^\rho(N,t,l,u,\succeq) \) for all \((N,t,l,u,\succeq) \in \mathcal{P} \).

**Proof** Let \((N,t,l,u,\succeq) \in \mathcal{P} \) be arbitrary and suppose that \( f \) and \( \rho \) satisfy the hypothesis of Lemma 2.1. If \( A(N,t,l,u,\succeq) = \emptyset \) then, \( c^f(N,t,l,u,\succeq) = c^\rho(N,t,l,u,\succeq) \) and \( f(N,t,l,u,\succeq) = f^\rho(N,t,l,u,\succeq) = (0,\ldots,0) \). Assume \( A(N,t,l,u,\succeq) \neq \emptyset \). By \((ef)\), \( c^f(N,t,l,u,\succeq) \) and \( c^\rho(N,t,l,u,\succeq) \) are non-empty. Since \( S \in A(S,t,(l_i)_{i \in S},(u_i)_{i \in S},(\succeq_i)_{i \in S}) \), implies \( S \in A(N,t,l,u,\succeq) \), we have that \( A(S,t,(l_i)_{i \in S},(u_i)_{i \in S},(\succeq_i)_{i \in S}) \subseteq A(N,t,l,u,\succeq) \). In particular, \( c^f(N,t,l,u,\succeq) \equiv c^f(S,t,(l_i)_{i \in \mathcal{E}},(u_i)_{i \in \mathcal{E}},(\succeq_i)_{i \in \mathcal{E}}) \subseteq A(N,t,l,u,\succeq) \). Hence, by \((ac)\), \( c^f(S,t,(l_i)_{i \in \mathcal{E}},(u_i)_{i \in \mathcal{E}},(\succeq_i)_{i \in \mathcal{E}}) = c^f(N,t,l,u,\succeq) \). Since \((c^f,t,(l_i)_{i \in \mathcal{E}},(u_i)_{i \in \mathcal{E}},(\succeq_i)_{i \in \mathcal{E}}) \in \mathcal{P} \) and \( f \) satisfies \((ef),(sp)\), \((ete)\), and \((bm)\), by Theorem 1,

\[
f_i(N,t,l,u,\succeq) = \begin{cases} U_i^E(c^f,t,(l_j)_{j \in \mathcal{E}},(u_j)_{j \in \mathcal{E}},(\succeq_j)_{j \in \mathcal{E}}) & \text{if } i \in c^f \\ 0 & \text{if } i \notin c^f. \end{cases}
\]

We show that \( c^f \rho S \) for all \( S \in A(N,t,l,u,\succeq) \setminus c^f \) by considering separately the following three cases.

**Case 1:** \( S \nsubseteq c^f \). Then, by Case 1 in the definition of \( \rho \), \( c^f \rho S \).

**Case 2:** \( c^f \nsubseteq S \). We obtain a contradiction. Since \( S \in A(N,t,l,u,\succeq) \), \( S \in A(S,t,(l_i)_{i \in S},(u_i)_{i \in S},(\succeq_i)_{i \in S}) \). Thus, \( (S,t,(l_i)_{i \in S},(u_i)_{i \in S},(\succeq_i)_{i \in S}) \in \mathcal{P} \). By Theorem 1, \( f(S,t,(l_i)_{i \in S},(u_i)_{i \in S},(\succeq_i)_{i \in S}) = U_i^E(S,t,(l_i)_{i \in S},(u_i)_{i \in S},(\succeq_i)_{i \in S}) \). Hence, \( c^f(S,t,(l_i)_{i \in S},(u_i)_{i \in S},(\succeq_i)_{i \in S}) = S \). Since \( c^f \nsubseteq S \), \( c^f \in A(S,t,(l_i)_{i \in S},(u_i)_{i \in S},(\succeq_i)_{i \in S}) \). Moreover, \( A(S,t,(l_i)_{i \in S},(u_i)_{i \in S},(\succeq_i)_{i \in S}) \subseteq A(N,t,l,u,\succeq) \). By \((ac)\), \( c^f = c^f(S,t,(l_i)_{i \in S},(u_i)_{i \in S},(\succeq_i)_{i \in S}) = S \), a contradiction.
Case 3: \( c^f \setminus S \neq \emptyset \) and \( S \setminus c^f \neq \emptyset \). Let \((N', t', l', u', \succeq') \in P\) be as in the definition of \( \rho \) where \( S' = c^f \). Thus, \( A(N', t', l', u', \succeq') = \{ c^f, S \} \subseteq A(N, t, l, u, \succeq) \). By \((ac)\), \( c^f(N', t', l', u', \succeq') = c^f(N, t, l, u, \succeq) \). Hence, by the definition of \( \rho \), \( c^f \rho S \).

Thus, \( c^f(N, t, l, u, \succeq) = c^f(N, t, l, u, \succeq) \). By \( (12)\), \( f(N, t, l, u, \succeq) = f^\rho(N, t, l, u, \succeq) \).

We now prove that the five properties are independent.

- \((ef)\) is independent of the other four properties.
- \((sp)\) is independent of the other four properties.
- \((ete)\) is independent of the other four properties.
- \((bm)\) is independent of the other four properties.
- \((ac)\) is independent of the other four properties.

References


