

New Tests of Expectation Formation with Applications to Asset Pricing Models*

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Abstract

The paper develops new tests of expectation formation which are generally applicable in financial and macroeconomic models. The tests utilize cointegration restrictions among forecasts of model variables. Survey data suggests forecasts of stock prices are not cointegrated with forecasts of consumption and rejects this aspect of the formation of stock price expectations in a wide range of asset pricing models, including rational expectations and various learning or sentiment-based models. Potential resolutions are discussed for reconciling equity pricing models with the new survey evidence.

Keywords: Survey Expectation, Cointegration, Incomplete Information, Sentiment, Learning

JEL classifications: D84, G12, G17.

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1 Introduction

Asset prices are crucially determined by investors' expectations about future. Yet asset pricing models are usually silent about to what extent model-implied asset price forecasts resemble forecasts made by agents in reality. Recent research employs survey expectations data to guide the modeling of expectation formation in financial markets and/or to discipline the modeling of asset price dynamics, examples are Greenwood and Shleifer (2014), Barberis, Greenwood, Jin and Shleifer (2015), Adam, Marcet and Beutel (2017), Adam, Matveev and Nagel (2018), and Nagel and Xu (2018).

Along this line, the paper develops new tests of expectation formation which are generally applicable in financial and macroeconomic models. The tests utilize cointegration restrictions among forecasts of model variables. Applying to the context of asset pricing, a central new piece of evidence from expectations data uncovered by the paper is that forecasts of stock prices are not cointegrated with consumption forecasts. This evidence is robust to different sources of expectations data, forecasting horizons, statistical tests, using median or mean forecasts for testing, and using stock price forecasts data which is made at different dates from consumption forecasts.

We show that in a wide range of asset pricing models, stock price forecasts are cointegrated with consumption forecasts and forecasts of price-consumption ratios are stationary. Put differently, the long-run or trend component of stock price forecasts is anchored by consumption forecasts. Thus, the evidence rejects this aspect of the formation of stock price expectations in these models. Potential resolutions are discussed for reconciling asset pricing models with the new survey evidence, such as adding sentiment (or judgment) directly to subjective price forecasts.

Consider full-information rational expectations (RE) asset pricing models in an endowment economy setting, such as the habit model of Campbell and Cochrane (1999) and long-run risks model of Bansal, Kiku and Yaron (2012). The exogenous consumption process contains a stochastic trend. Agents have full information about the model economy, including the consumption process and all agents' beliefs and preferences. They are able to correctly deduce the equilibrium law of motion for stock prices and that stock prices are cointegrated with consumption. This knowledge is used to forecast future prices and hence the forecasts of stock prices and consumption are cointegrated. The cointegration relation

between the two forecasts is also present in production-based models (e.g., Jermann (1998), Boldrin, Christiano and Fisher (2001) and Croce (2014)). In these models, both stock prices and consumption are endogenous and their forecasts share a common trend with the exogenous productivity process which contains a unit root.

One may wonder if this new survey evidence is compatible with existing asset pricing models which relax the assumption of full information or RE. The paper further shows in various learning or sentiment-based models, the long-run component of stock price forecasts is still anchored by consumption forecasts and the two forecasts are cointegrated. This aspect of the formation of stock price expectations in these models is inconsistent with the survey evidence too.

In some asset pricing models, agents have RE but incomplete information about the exogenous driving process. They learn about the exogenous consumption process over time (e.g., Collin-Dufresne, Johannes and Lochstoer (2016)). Alternatively, in some sentiment-based models, agents may have misperception that certain (possibly extrinsic) variable influences the consumption process (e.g., Jin and Sui (2018)).¹ In both types of models, agents still possess the knowledge of the equilibrium pricing function. While agents may have misperception about the exogenous consumption process, the long-run component of stock price forecasts remains anchored by consumption forecasts.

In adaptive learning models, e.g., Adam, Marcet and Beutel (2017, henceforth AMB), agents' beliefs and preferences etc are not common knowledge. Agents cannot correctly deduce the equilibrium law of motion for asset prices. Instead, they form subjective price beliefs and learn from equilibrium prices. Despite having non-rational expectations, they make (internally) rational economic decisions. We show the specification of subjective stock price beliefs in existing adaptive learning models implies that the long-run component of stock price forecasts is *not* de-linked from consumption forecasts. For instance, in AMB, forecasts of price consumption ratio depend on current price consumption ratio and agents' forecasts about the difference in the growth rate of stock price and consumption. The cointegration between stock price forecasts and consumption forecasts in AMB arises from the model features that price consumption ratio is stationary and agents' beliefs about the trend growth rate of stock prices mean-revert to the trend growth rate of consumption, despite agents' lack of knowledge of both features.

¹Another example in this fashion is the exchange rate model of Yu (2013).

Our tests can – but are not limited to – test the RE hypothesis. On the one hand, realized price consumption ratio is stationary.² On the other hand, we show forecasts of price consumption ratio is non-stationary. The discrepancy between realized and forecasts of price consumption ratio can be interpreted as a rejection of the RE hypothesis, in line with Greenwood and Shleifer (2014) and AMB which reject the RE hypothesis using stock market survey expectations data.

Potential resolutions are discussed for reconciling asset pricing models with the new evidence. One new proposal is a modification of agents’ subjective stock price belief in AMB. We assume agents’ stock price forecasts consists of two components. The first component is forecasts generated by econometric learning from historical stock prices as in AMB. The new second component is an extrinsic variable – interpreted as sentiment (or judgment) – which contains a unit root and independent of stock prices and fundamentals. When the sentiment variable is positive (or negative), agents’ are more optimistic (or pessimistic) than the forecasts produced by the econometric model. The paper shows adding sentiment *directly* to stock price forecasts is crucial to break the tight link between the trend component of stock price forecasts and consumption forecasts and delivers consistency between models and the new evidence.³

The paper develops other tests of expectation formation by utilizing cointegration restrictions among forecasts of the same variable (e.g., stock prices) over different forecasting horizons. It finds that, for instance, forecasts of stock prices (or consumption) over different horizons in the data are cointegrated with each other, consistent with asset pricing models we considered. Moreover, surveys of expectation often ask participants about their expectation of the average value of economic variables over a number of periods. The paper develops cointegration tests of expectation formation utilizing the average expectations data as well.

The paper relates to recent work on utilizing survey expectations data to test expectation formation and discipline financial modeling. Malmendier and Nagel (2011) show that investors’ experience of macroeconomic shocks affects financial risk decisions. Greenwood

²It seems researchers who empirically test the stationarity of price consumption (or dividend) ratios agree on this property which is a feature of almost all asset pricing models. And whether realized price consumption (or dividend) ratio is stationary or not does not affect our new survey evidence and our derived testable implications for the formation of stock price expectations in various full- and incomplete-information asset pricing models.

³Note as discussed earlier and shown formally later, adding sentiment to the exogenous consumption process in asset pricing models cannot break this tight link.

and Shleifer (2014) and Adam, Marcet and Beutel (2017) both use survey stock return forecasts to reject RE hypothesis. Adam, Matveev and Nagel (2018) empirically reject that survey return expectations are formed by risk-neutral investors or ambiguity averse/robust investors. Based on survey data findings (Malmendier and Nagel 2011, 2016), Nagel and Xu (2018) build an asset pricing model with learning with fading memory about dividend process to replicate several stock market facts such as counter-cyclical risk premium. Coibion and Gorodnichenko (2015) study the relation between ex post forecast errors and ex ante forecast revisions. They document widespread rejection of full-information RE models in the direction predicted by models of informational rigidities. Bordalo, Gennaioli, Ma and Shleifer (2018) find that forecasters typically over-react to their individual news, while consensus forecasts under-react to average news on individual stock earnings. To reconcile the findings, they combine a diagnostic expectation model of belief formation with a noisy information model of belief dispersion.

Section 2 shows that stock prices are cointegrated with aggregate consumption in major asset pricing models with RE and full information. New tests of expectation formation in full-information RE models are derived in Section 3. New evidence on the formation of stock price expectations is provided in Section 4. Section 5 (Section 6) derives tests of expectation formation in asset pricing models with adaptive learning (with incomplete information or sentiment) and contrasts the models with the evidence. Section 7 discusses potential resolutions for reconciling asset pricing models with the new evidence. Section 8 concludes.

2 Asset pricing models with RE and full information

This section demonstrates that stock prices and aggregate consumption are cointegrated in several major (endowment or production economy) asset pricing models with RE and full information. The cointegration relation gives rise to a rich set of testable implications for the formation of stock price expectations, as is shown later.

Consider the long-run risks model of Bansal, Kiku and Yaron (2012) and the habit model of Campbell and Cochrane (1999). In both models, aggregate consumption, an exogenous driving process, contains a stochastic trend. Agents are endowed with full information about the model economy. Like the modeler, agents are able to correctly deduce the equilibrium

pricing function and form RE on stock prices. It is shown below that agents possess the knowledge that realized stock prices are cointegrated with aggregate consumption, irrespective of agents' preferences (represented by habit formation or Epstein-Zin utility). In addition, we argue this cointegration relation is a common feature of both endowment economy and production based asset pricing models with RE and full information.

2.1 The long-run risks model

Consider the long-run risks model studied in Bansal, Kiku and Yaron (2012). The representative agent with recursive preference maximizes his life-time utility given by

$$V_t = [(1 - \delta)C_t^{\frac{1-\gamma}{\theta}} + \delta(E_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}}. \quad (1)$$

The variable θ is defined as $\theta \equiv \frac{1-\gamma}{1-\psi}$ where the parameters γ and ψ represent relative risk aversion and the elasticity of intertemporal substitution. Log consumption c_t and dividend d_t have the following joint dynamics

$$\Delta c_{t+1} = \mu_c + x_t + \sigma_t \eta_{t+1}, \quad (2)$$

$$x_{t+1} = \rho x_t + \varphi_e \sigma_t \epsilon_{t+1}, \quad (3)$$

$$\sigma_{t+1}^2 = \bar{\sigma}^2 + \nu(\sigma_t^2 - \bar{\sigma}^2) + \sigma_w w_{t+1}, \quad (4)$$

$$\Delta d_{t+1} = \mu_d + \phi x_t + \pi \sigma_t \eta_{t+1} + \varphi \sigma_t u_{d,t+1}. \quad (5)$$

$\mu_c + x_t$ is the conditional expectation of the growth rate of aggregate consumption. x_t is a persistent component which captures long run risks in consumption and drives both the consumption and dividend process. ϕ captures a levered exposure of dividend to x_t . In addition, the i.i.d consumption shock η_{t+1} is allowed to influence the dividend process. It serves as an additional source of risk premia and π governs the magnitude of this influence.

Their paper provides the analytical solution for (log) price-consumption ratio

$$\log\left(\frac{P_t}{C_t}\right) = A_0 + A_1 x_t + A_2 \sigma_t^2, \quad (6)$$

where A_0, A_1, A_2 are all constants and functions of model parameters, see their p. 189. Stock prices and aggregate consumption are cointegrated as the right hand side of equation (6) is

stationary. The following proposition summarizes the result.

Proposition 1 *In the model of Bansal, Kiku and Yaron (2012), stock prices and aggregate consumption are cointegrated with cointegrating vector $(1, -1)$ and realized (log) stock price consumption ratio is a stationary process.*

We also simulate the long-run risks model for 948 periods (months) as in Bansal, Kiku and Yaron (2012) to confirm the stationarity. Table 1 shows the unit root testing results by applying the Phillips-Perron (PP) test (see Phillips and Perron (1988)) and the Augmented Dickey-Fuller Generalized Least Squares (DF-GLS) test to (log) price consumption ratio. Both test statistics are smaller than the corresponding 1% critical value, suggesting that realized stock price consumption ratio passes the unit root tests.

Table 1: Stationarity of log price consumption ratio

<i>I(1) test (Long-run Risk)</i>		<i>I(1) test (Habit)</i>	
PP (Z_t statistics)	-5.962	PP (Z_t statistics)	-26.752
1% critical value	-3.455	1% critical value	-3.430
DF-GLS	-3.737	DF-GLS	-3.560
1% critical value	-3.480	1% critical value	-2.580

2.2 The habit model

Consider the habit model of Campbell and Cochrane (1999). The representative agent maximizes his life-time utility as

$$U = E \sum_{t=0}^{\infty} \delta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma},$$

where C_t is consumption at period t and X_t denotes external habit. The surplus consumption ratio is $S_t = (C_t - X_t)/C_t$. The intertemporal marginal rate of substitution is $M_{t+1} = \delta(\frac{S_{t+1}}{S_t} \frac{C_{t+1}}{C_t})^{-\gamma}$. The (log) surplus consumption ratio $s_t \equiv \log(S_t)$ evolves according to a heteroskedastic AR(1) process

$$s_{t+1} = (1 - \phi)\bar{s} + \phi s_t + \lambda(s_t)[\Delta c_{t+1} - E(\Delta c_{t+1})].$$

The sensitivity function $\lambda(s_t)$ is specified as

$$\lambda(s_t) = \begin{cases} (1/\bar{S})\sqrt{1 - 2(s_t - \bar{s})} - 1, & s_t \leq s_{\max} \\ 0, & s_t \geq s_{\max} \end{cases},$$

where \bar{S} is set to be $\bar{S} = \sigma\sqrt{\frac{\gamma}{1-\phi-B/\gamma}}$ and $s_{\max} = \bar{s} + \frac{1}{2}(1 - \bar{S}^2)$. The growth rate of aggregate consumption follows a log-normal process

$$\Delta c_{t+1} = g + v_{t+1},$$

where c is the logarithm of aggregate consumption, v_{t+1} is the *i.i.d.* normally distributed variables with mean zero and variances σ^2 . Then, the equilibrium price-consumption ratio as the function of state variable s_t satisfies

$$\frac{P_t}{C_t}(s_t) = E_t \left[M_{t+1} \frac{C_{t+1}}{C_t} \left(1 + \frac{P_t}{C_t}(s_{t+1}) \right) \right].$$

There is no analytical solution for the habit model. We simulate the habit formation model for 120, 000 months (as in Campbell and Cochrane (1999)) and then test the cointegration between realized quarterly (log) stock prices and (log) aggregate consumption. Using the PP test and the DF-GLS test, Table 1 shows that realized price consumption ratios pass the tests as the null hypothesis of the unit root tests are rejected.⁴

2.3 Production economy asset pricing models

In production-based asset pricing models with RE and full information, such as Jermann (1997), Boldrin, Christiano and Fisher (2001) and Croce (2014), the exogenous driving process is a productivity process which is assumed to contain a stochastic trend. Agents are endowed with full information about the economy and can deduce the equilibrium mapping from the exogenous productivity process to endogenous variables. Both stock prices and consumption are endogenous variables. Different numerical methods may be employed to solve these models. Yet a common feature is that (log) stock price consumption ratio can

⁴Figure 3 of Campbell and Cochrane (1999) shows that $\log(P_t) - \log(C_t)$ is approximately linear in the stationary state variable, i.e., consumption surplus ratio s_t . This also suggests the stationarity of $\log(P_t) - \log(C_t)$.

be well approximated by a polynomial function of stationary state variables and is again stationary. Thus, realized stock prices and aggregate consumption are cointegrated with cointegrating vector $(1, -1)$ and agents have this knowledge as a consequence of having RE.

3 Testing expectation formation in models with RE and full information

This section develops new tests of expectation formation in models with RE and full information, such as these discussed in the previous section. These models impose a large amount of cointegration restrictions among forecasts of model variables and the tests utilize the restrictions. In later sections, we show tests of expectation formation can be similarly derived in several classes of incomplete-information RE models and non-RE models.

Consider a variable $\{X_t\}$ from a full-information RE model, such as (log) stock price or aggregate consumption, which is generally represented by

$$X_t = X_t^P + X_t^C, \quad (7)$$

$$X_t^P = \mu + X_{t-1}^P + \sigma_{\epsilon,t}\epsilon_t, \quad (8)$$

$$(1 - \phi(L))X_t^C = (1 + \psi(L))\sigma_{\eta,t}\eta_t, \quad (9)$$

$$(1 - \tilde{\phi}(L))(\sigma_{\epsilon,t}^2 - \bar{\sigma}_\epsilon^2) = (1 + \tilde{\psi}(L))\tilde{\epsilon}_t, \quad (10)$$

$$(1 - \hat{\phi}(L))(\sigma_{\eta,t}^2 - \bar{\sigma}_\eta^2) = (1 + \hat{\psi}(L))\tilde{\eta}_t. \quad (11)$$

This variable contains a stochastic trend. The superscripts P and C stand for the permanent and cyclical component. ϵ_t , η_t , $\tilde{\epsilon}_t$ and $\tilde{\eta}_t$ are *i.i.d* innovations and independent to each other. The variance of ϵ_t and η_t are normalized to 1. $\sigma_{\epsilon,t}$ and $\sigma_{\eta,t}$ are allowed to be time-varying and their mean is constant and positive, i.e., $\bar{\sigma}_\epsilon^2$ and $\bar{\sigma}_\eta^2$. $\phi(L) = \phi_1L + \phi_2L^2 + \dots + \phi_pL^p$ and $\psi(L) = \psi_1L + \psi_2L^2 + \dots + \psi_qL^q$ where L is the lag operator. $\tilde{\phi}(L)$, $\tilde{\psi}(L)$, $\hat{\phi}(L)$ and $\hat{\psi}(L)$ are similarly defined.⁵ The roots of $1 - \phi(z) = 0$, $1 - \tilde{\phi}(z) = 0$, and $1 - \hat{\phi}(z) = 0$ are within the unit circle, so X_t^C is a stationary process.

⁵Specifically, $\tilde{\phi}(L) = \tilde{\phi}_1L + \tilde{\phi}_2L^2 + \dots + \tilde{\phi}_pL^p$, $\tilde{\psi}(L) = \tilde{\psi}_1L + \tilde{\psi}_2L^2 + \dots + \tilde{\psi}_qL^q$, $\hat{\phi}(L) = \hat{\phi}_1L + \hat{\phi}_2L^2 + \dots + \hat{\phi}_pL^p$ and $\hat{\psi}(L) = \hat{\psi}_1L + \hat{\psi}_2L^2 + \dots + \hat{\psi}_qL^q$.

3.1 Integration property of conditional forecasts

Given the assumption of RE and full information, agents know the law of motion for X_t (equation (7)-(11)) and make use of this knowledge to make forecasts. The following lemma shows that if the variable X_t is integrated of order 1 ($X_t \sim I(1)$), conditional forecasts of this variable over arbitrary forecasting horizons i (i.e., $E_t X_{t+i}$) contain a unit root. For instance, if stock prices is an I(1) process, 1-year ahead forecasts of stock prices also contain a unit root.

Lemma 2 *If X_t follows (7)-(11) (i.e., $X_t \sim I(1)$), $E_t X_{t+i} \sim I(1)$ for $i > 0$.*

Proof. Given (7)-(11), we have $E_t X_{t+i} = E_t X_{t+i}^P + E_t X_{t+i}^C = \mu i + X_t^P + E_t X_{t+i}^C$. $E_t X_{t+i}$ is the sum of a unit root process and a stationary process and hence a unit root process. ■

3.2 Cointegration among forecasts of different variables

This section establishes the cointegration relation among forecasts of different variables when their realizations are cointegrated. Suppose $y_t = (y_{1,t} \ y_{2,t} \ \dots \ y_{n,t})'$ is a $1 \times n$ vector which is cointegrated with cointegrating vector $a = (a_1 \ a_2 \ \dots \ a_n)'$ and $a'y_t$ is a stationary process (with possibly time-varying volatility). Mathematically,

$$\begin{aligned} (1 - \phi(L))a'y_t &= (1 + \psi(L))\sigma_{\eta,t}\eta_t, \\ (1 - \widehat{\phi}(L))(\sigma_{\eta,t}^2 - \bar{\sigma}_\eta^2) &= \left(1 + \widehat{\psi}(L)\right)\tilde{\eta}_t. \end{aligned}$$

where the roots of $1 - \phi(z) = 0$ and $1 - \widehat{\phi}(z) = 0$ are within the unit circle. We firstly establish a preliminary result which says the forecasts of an I(1) variable X made at date t over an arbitrary horizon i (i.e., $E_t X_{t+i}$) are cointegrated with X_k with cointegrating vector $(1, -1)$, where k can be identical to or different from t .

Lemma 3 *If X_t follows (7)-(11) (i.e., $X_t \sim I(1)$), $E_t X_{t+i} - X_k \sim I(0)$ for $i > 0$.*

Proof. Let

$$\begin{aligned} E_t X_{t+i} - X_k &= (E_t X_{t+i}^P + E_t X_{t+i}^C) - X_t + (X_t - X_k) \\ &= (E_t X_{t+i}^P - X_t^P) + (E_t X_{t+i}^C - X_t^C) + (X_t - X_k) \\ &= \mu i + (E_t X_{t+i}^C - X_t^C) + (X_t - X_k) \end{aligned}$$

$(E_t X_{t+i} - X_k)$ is stationary as $E_t X_{t+i}^C$, X_t^C and $(X_t - X_k)$ are stationary. Note a special case is $t = k$. ■

Denote by $E_{i_1} y_{1,i_1+j_1}$ j_1 -period ahead expectation of variable y_1 made at date i_1 .

Theorem 4 *If $a'y_t$ is a stationary process, $a_1 E_{i_1} y_{1,i_1+j_1} + a_2 E_{i_2} y_{2,i_2+j_2} + \dots + a_n E_{i_n} y_{n,i_n+j_n}$ is stationary for arbitrary $i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n > 0$.*

Proof. Let

$$\begin{aligned} & [a_1 E_{i_1} y_{1,i_1+j_1} + a_2 E_{i_2} y_{2,i_2+j_2} + \dots + a_n E_{i_n} y_{n,i_n+j_n}] \\ = & \left[\sum_{k=1}^n a_k (E_{i_k} y_{k,i_k+j_k} - y_{k,i_k}) + \sum_{k=1}^n a_k y_{k,i_k} \right] \\ = & \left[\sum_{k=1}^n a_k (E_{i_k} y_{k,i_k+j_k} - y_{k,i_k}) + \sum_{k=1}^n a_k y_{k,i_1} + \sum_{k=2}^n a_k (y_{k,i_k} - y_{k,i_1}) \right] \end{aligned}$$

Note Lemma 3 implies $(E_{i_k} y_{k,i_k+j_k} - y_{k,i_k})$ is stationary for $k = 1, 2, \dots, n$. In addition, the cointegration of the vector y_t yields $\sum_{k=1}^n a_k y_{k,i_1}$ is stationary and $y_{k,t} \sim I(1)$ gives $(y_{k,i_k} - y_{k,i_1})$ is stationary. Thus, we have $a_1 E_{i_1} y_{1,i_1+j_1} + a_2 E_{i_2} y_{2,i_2+j_2} + \dots + a_n E_{i_n} y_{n,i_n+j_n}$ is stationary. ■

The theorem contains a rich set of testable implications for expectation formation. For illustration, consider the asset pricing models discussed in the previous section in which realized stock prices and consumption are cointegrated with cointegrating vector $(1, -1)$. First, note a special case of the theorem is that forecasts of stock prices and consumption made at the same date (i.e., $i_1 = i_2 = \dots = i_n$) and over the same forecasting horizons (i.e., $j_1 = j_2 = \dots = j_n$) are cointegrated. And forecasts of stock price consumption ratio, i.e., $(E_t \log P_{t+j} - E_t \log C_{t+j})$ is stationary. This means, for example, 1-year ahead forecasts of stock prices and 1-year ahead forecasts of consumption (made at the same date) are cointegrated with cointegrating vector $(1, -1)$.

Second, the cointegration relation holds for forecasts of different variables *over different forecasting horizons* (i.e., j 's need not to be identical) as $(E_t \log P_{t+j_1} - E_t \log C_{t+j_2})$ is stationary for $j_1 \neq j_2$. This means, for instance, 10-year ahead forecast of stock prices and 1-year ahead forecast of consumption made at the same date are cointegrated. This result is particularly useful when the forecasting horizons of expectation data available to

researchers are different across different variables. For instance, researchers may have data on 10-year ahead forecasts of stock prices and 1-year ahead (but not 10-year ahead) forecasts of consumption.

Third, the cointegration relation also holds for forecasts of different variables *made at different dates* (i.e., i 's need not to be identical) as $(E_{i_1} \log P_{i_1+j_1} - E_{i_2} \log C_{i_2+j_2})$ is stationary for $i_1 \neq i_2$. This means, for instance, stock price forecasts made during 1960 – 1990 (over an arbitrary forecasting horizon) are cointegrated with consumption forecasts made during 1970 – 2000 (over an arbitrary forecasting horizon). This result is useful when the sample period of expectation data available to researchers is different (or do not exactly overlap) across different variables.

Perhaps surprisingly, all testable implications (i.e. cointegration restrictions) are also present in various learning and sentiment-based models, as is shown later.

3.3 Cointegration among forecasts of the same variable

The following theorem shows that the forecasts of the same $I(1)$ variable made at the same date i over two arbitrary and different horizons $j \neq l$ are cointegrated with cointegrating vector $(1 \ -1)$. This means, for instance, 1-year ahead and 10-year ahead forecasts of stock prices made at the same date are cointegrated. In addition, the forecasts of the same variable made at two different dates (i.e., $i \neq k$) over two arbitrary horizons (i.e., j and l) are cointegrated with cointegrating vector $(1 \ -1)$.

Theorem 5 *If X_t follows (7)-(11) (i.e., $X_t \sim I(1)$), $E_i X_{i+j} - E_k X_{k+l} \sim I(0)$ for (a) $i = k$, $j \neq l$ or (b) $i \neq k$, $j > 0$, $l > 0$.⁶*

Proof. First, consider case (a) when $i = k$ and $j \neq l$. Let $E_i X_{i+j} - E_i X_{i+l} = (\mu j + X_i^P + E_i X_{i+j}^C) - (\mu l + X_i^P + E_i X_{i+l}^C) = \mu(j - l) + (E_i X_{i+j}^C - E_i X_{i+l}^C)$. $(E_i X_{i+j} - E_i X_{i+l})$ is stationary because $(E_i X_{i+j}^C - E_i X_{i+l}^C)$ is stationary. Turning to case (b) when $i \neq k$. Let $E_i X_{i+j} - E_k X_{k+l} = (E_i X_{i+j} - X_i) - (E_k X_{k+l} - X_k) + (X_i - X_k)$. Lemma 3 yields that $(E_i X_{i+j} - X_i)$ and $(E_k X_{k+l} - X_k)$ are stationary. Moreover, given $X_t \sim I(1)$, $(X_i - X_k)$ is stationary. Thus, $(E_i X_{i+j} - E_k X_{k+l})$ is stationary. ■

⁶Note if $i = k$ and $j = l$, $E_i X_{i+j}$ and $E_k X_{k+l}$ are identical to each other.

3.4 Tests using average forecasts over many periods

Economic surveys often ask participants their forecast of the average value of economic variables X_t over the next m periods, for instance, average unemployment rate over next five years. This section provides testable implications for average forecasts when $X_t \sim I(1)$. The tests are useful when researchers have data on average expectations over a number of periods.⁷

Define the average forecast $\bar{X}_t^m = \frac{1}{m} \sum_{i=1}^m E_t X_{t+i}$. Note the average is calculated over a number of time periods (rather than across different survey participants). Part (1) of the following Lemma shows that the average forecasts \bar{X}_t^m contain a unit root. Part (2) shows that the average forecasts of X over the next m periods made at an arbitrary date h are cointegrated with conditional forecasts of X over horizon l made at an arbitrary date j with cointegrating vector $(1, -1)$. Part (3) shows that $\bar{X}_t^m - X_j \sim I(0)$.

Lemma 6 *If X_t follows (7)-(11) (i.e., $X_t \sim I(1)$), then (1) $\bar{X}_t^m \sim I(1)$ for $m > 0$; (2) $\bar{X}_h^m - E_j X_{j+l} \sim I(0)$ for arbitrary h, j, m and $l > 0$; (3) $\bar{X}_t^m - X_j \sim I(0)$ for arbitrary $t, j, m > 0$.*

Proof. (1) Let $\bar{X}_t^m - \bar{X}_{t-1}^m = \frac{1}{m} \sum_{i=1}^m E_t X_{t+i} - \frac{1}{m} \sum_{i=1}^m E_{t-1} X_{t-1+i} = \frac{1}{m} \sum_{i=1}^m (E_t X_{t+i} - E_{t-1} X_{t-1+i})$.

Lemma 2 implies that $E_t X_{t+i} \sim I(1)$ and hence $(E_t X_{t+i} - E_{t-1} X_{t-1+i})$ is stationary. Thus, $(\bar{X}_t^m - \bar{X}_{t-1}^m)$ is stationary.

(2) For arbitrary h, j, m and l , let $\bar{X}_h^m - E_j X_{j+l} = \frac{1}{m} \sum_{i=1}^m E_h X_{h+i} - E_j X_{j+l} = \frac{1}{m} \sum_{i=1}^m (E_h X_{h+i} - E_j X_{j+l})$. Theorem 5 shows that $(E_h X_{h+i} - E_j X_{j+l})$ is stationary. Thus, $\bar{X}_h^m - E_j X_{j+l}$ is stationary.

(3) Let $\bar{X}_t^m - X_j = \frac{1}{m} \sum_{i=1}^m E_t X_{t+i} - X_j = \frac{1}{m} \sum_{i=1}^m (E_t X_{t+i} - X_j)$. Lemma 3 implies that $(E_t X_{t+i} - X_j)$ is stationary. Thus, we have $(\bar{X}_t^m - X_j)$ is stationary. ■

Let $y_t = (y_{1,t} \ y_{2,t} \ \dots \ y_{n,t})'$ be a $1 \times n$ vector which is cointegrated with cointegrating vector $a = (a_1 \ a_2 \ \dots \ a_n)'$ and denote $Z_{k,i_k}^{j_k} = E_{i_k} y_{k,i_k+j_k}$ or $\bar{y}_{k,i_k}^{j_k}$ where $\bar{y}_{k,i_k}^{j_k} = \frac{1}{j_k} \sum_{l=1}^{j_k} E_{i_k} y_{k,i_k+l}$.

⁷Note they are not used in empirical testing of the paper as we do not have average expectations data in the current context.

Theorem 7 $a_1 Z_{1,i_1}^{j_1} + a_2 Z_{2,i_2}^{j_2} + \dots + a_n Z_{n,i_n}^{j_n}$ is stationary for arbitrary $i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n$.

Proof. Let $a_1 Z_{1,i_1}^{j_1} + a_2 Z_{2,i_2}^{j_2} + \dots + a_n Z_{n,i_n}^{j_n} = \left[\sum_{k=1}^n a_k (Z_{k,i_k}^{j_k} - y_{k,i_k}) + \sum_{k=1}^n a_k y_{k,i_k} \right]$. It is stationary for two reasons. First, Lemma 3 and part (3) of Lemma 6 imply that $(Z_{k,i_k}^{j_k} - y_{k,i_k})$ is stationary no matter $Z_{k,i_k}^{j_k} = E_{i_k} y_{k,i_k+j_k}$ or $\bar{y}_{k,i_k}^{j_k}$. Second, $\sum_{k=1}^n a_k y_{k,i_k}$ is stationary as is shown in the proof of Theorem 4. ■

Theorem 7 shows that if a vector of variables are cointegrated, a linear combination of the average and conditional forecasts of these variables (with the same cointegrating vector) is stationary. Note a special case is when the forecasts of all variables are made at the same date ($i_1 = i_2 = \dots = i_n$).

4 New evidence on the formation of stock price expectations

Using the tests developed from previous section, this section presents new evidence on the formation of stock price expectations. A central piece of evidence from expectation data is that stock price forecasts are not cointegrated with consumption forecasts (with cointegrating vector $(1, -1)$), as opposed to asset pricing models with RE and full information.⁸ Put differently, the long-run/trend component of stock price forecasts is *not* anchored by consumption forecasts. This evidence is robust to different sources of expectations data, forecasting horizons, statistical tests, using median or mean forecasts for testing, and using stock price forecasts data which is made at different dates from consumption forecasts. In subsequent sections, we show several classes of asset pricing models with incomplete information or non-RE appear inconsistent with this evidence too.

⁸Other evidence includes that forecasts of stock prices (or consumption) over different forecasting horizons are cointegrated with each other, consistent with all asset pricing models considered in the paper.

4.1 Data

Two sources of survey forecasts of US stock prices are used. One is the Livingston Survey managed by the Federal Reserve Bank of Philadelphia. The survey contains forecasts of S&P 500 index made by professional economists from industry, government banking and academia. The stock price forecast data is semi-annual and covers from 1952 to the second half of 2017.⁹ Two forecasting horizons are available: 2- and 4-quarter ahead. The other source is Robert Shiller’s survey of individual investors. This forecast of stock prices is measured by forecasts of the Dow Jones index and available at quarterly frequency. The data covers from the first quarter of 1999 to the second quarter of 2015. Four forecasting horizons are available: 1-quarter, 2-quarter, 4-quarter and 10-year ahead. Both survey forecasts of stock prices are deflated by forecasts of inflation rate obtained from the Survey of Professional Forecasters (SPF) conducted by the Philadelphia Fed. The forecasting horizons of inflation forecast data are 1- to 4- quarter ahead as well as 10-year ahead.¹⁰

Two sources of US aggregate consumption forecasts are used. One is SPF forecasts of the chain-weighted real personal consumption expenditures. It is available at quarterly frequency and from 1981 Q3 onwards. SPF consumption forecasts data is provided with varying base years. Appendix A explains the rebasing of consumption forecast data. As an alternative, consumption forecasts from the US Federal Reserve Board’s Greenbook datasets is employed. In the text, we report testing results using SPF consumption forecasts. Most results reported in the text use median survey forecasts. Appendix B (Appendix C) shows our results are robust to using mean forecasts (Greenbook consumption forecasts). Figure 1 plots the (normalized) median forecasts of (log) stock prices and rebased aggregate consumption for all available forecasting horizons.

⁹In all cases (with one exception, i.e., Table 5), we use the data from 1981 onwards which corresponds to the longest sample of consumption forecasts in the Survey of Professional Forecasters.

¹⁰For robustness analysis, we also deflate 1-year ahead stock price forecasts using 1-year ahead inflation forecasts using the Michigan Survey of Consumers. Our results are robust to this alternative measure of inflation expectation.

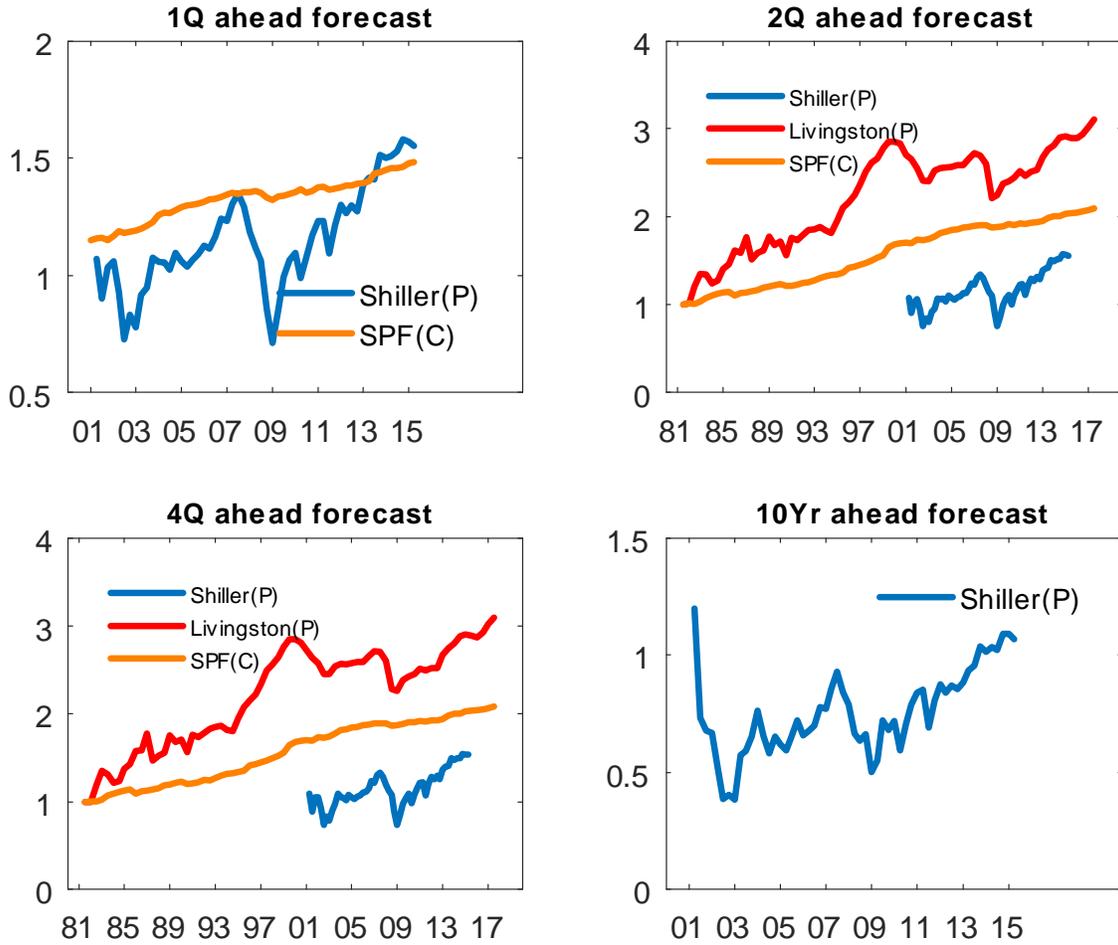


Figure 1: Median forecasts of (log) stock price and consumption

Before proceeding to the test results, we discuss three issues. First, it is pervasive in the literature to use mean or median forecasts from surveys of expectations (e.g., SPF, Livingston, Shiller) as a proxy for the expectations of the representative agent, such as Piazzesi, Salomao and Schneider (2015), and Coibion and Gorodnichenko (2015). With this view, it may be appropriate for us to use stock price forecasts from one survey and consumption forecasts from another survey to test the cointegration between the forecasts of the two variables made by the representative agent in asset pricing models, despite that the set of forecasters are not the same in different surveys.¹¹ Many papers similarly follow this

¹¹Carroll (2003) develops a model of expectation formation in which households derive their expectations from professional forecasters and provides evidence supporting this model using inflation expectations data. If this model is generally true (for other variables), forecasts of a variable (e.g., stock prices or consumption)

practice. For instance, Adam, Marcet and Beutel (2017) uses median (or mean) nominal stock price forecasts from Shiller Survey and inflation forecasts from SPF and Michigan Survey of Consumers to compute median (or mean) forecasts of real stock capital gains which is then used as a proxy for forecasts of real stock capital gains made by the representative agent in their model. Another example is Malmendier and Nagel (2011) which documents heterogeneity in inflation expectations using Michigan Survey of Consumers and then studies empirically the implications of this heterogeneity for borrowing and lending decisions using Survey of Consumer Finances.

Second, survey expectations data may contain measurement errors. If the measurement errors are *i.i.d* over time or follow a stationary process, all our testing results below are not affected because such measurement errors do not change the integration and cointegration properties of the forecasts data. Third, the power of standard Dickey-Fuller class of unit root tests were frequently criticized in 1980s and 1990s. Subsequent work has made great advances in improving the power of the tests. Ng and Perron (2001) and Haldrup and Jansson (2006) argue some subsequently developed tests have much improved or excellent power. The paper uses the more powerful tests like the DF-GLS test.¹² Another way to alleviate the problem is applying the KPSS test which tests the null hypothesis of a stationary process against the alternative of a unit root. We find this test leads to the same conclusion.

4.2 Integration properties of the forecasts

This section studies the integration properties of forecasts of aggregate stock price index and aggregate consumption. Table 2 reports the test statistics and critical value of the PP and DF-GLS test for median forecasts of stock prices. Panel A shows that for both surveys and all forecast horizons, both tests cannot reject that stock price forecasts is integrated of order 1, i.e., $I(1)$, at 10% significance level.¹³ Panel B shows that for both surveys and all

made by professional forecasters will be cointegrated with forecasts of the same variable made by households. This may help to justify our use of different surveys where forecasts of stock prices and consumption are not made by the same set of forecasters.

¹²One might still think failing to reject the cointegration between stock price forecasts and consumption forecasts does not necessarily mean forecasts of stock price consumption ratio is non-stationary. In this case, our tests can be interpreted as providing test statistics which can be used to discipline the modeling of asset prices.

¹³DF-GLS test gives all test statistics for a series of models that include 1 to k lags of the first differenced, detrended variable, where k is set by default. We report the statistics produced with the number of lags

forecasting horizons (with one exception), stock price forecasts is not integrated of order 2, i.e., $I(2)$.¹⁴

Table 2: Integration properties: forecasts of $\log P$

	1Q ahead	2Q ahead	4Q ahead	10-yr ahead
<i>Panel A: I(1) test</i>				
Shiller (PP Z_t stat.)	-2.231	-2.183	-2.242	-1.997
10% critical value	-3.172	-3.172	-3.172	-3.172
Shiller (DF-GLS)	-2.236	-2.150	-2.183	-1.317
10% critical value	-2.851	-2.851	-2.851	-2.851
Livingston (PP Z_t stat.)	<i>n.a.</i>	-2.195	-2.118	<i>n.a.</i>
10% critical value	<i>n.a.</i>	-3.167	-3.167	<i>n.a.</i>
Livingston (DF-GLS)	<i>n.a.</i>	-1.714	-1.756	<i>n.a.</i>
10% critical value	<i>n.a.</i>	-2.818	-2.818	<i>n.a.</i>
<i>Panel B: I(2) test</i>				
Shiller (PP Z_t stat.)	-7.618	-7.596	-8.012	-9.351
1% critical value	-2.615	-2.615	-2.615	-2.615
Shiller (DF-GLS)	-3.607	-3.505	-3.311	-1.518
1% critical value	-2.615	-2.615	-2.615	-2.615
Livingston (PP Z_t stat.)	<i>n.a.</i>	-7.206	-6.950	<i>n.a.</i>
1% critical value	<i>n.a.</i>	-2.612	-2.612	<i>n.a.</i>
Livingston (DF-GLS)	<i>n.a.</i>	-4.528	-3.084	<i>n.a.</i>
1% critical value	<i>n.a.</i>	-2.611	-2.611	<i>n.a.</i>

Table 3 reports the test statistic value and critical value of the unit root tests for aggregate consumption forecasts. Similarly, for all forecasting horizons, both tests suggest that consumption forecasts is an $I(1)$ but not $I(2)$ process.

Lemma 2 suggests that in the habit model and long-run risks model, forecasts of stock prices and consumption are $I(1)$ process but not $I(2)$ process. This is consistent with the evidence from survey data presented here.

leading to the lowest mean squared errors. And the results are robust to alternative lags.

¹⁴The only exception is the DF-GLS test cannot reject that 10-year ahead median forecast of stock prices from the Shiller Survey follows an $I(2)$ process. Yet we show that it is rejected using the mean forecast at 1% significance level, see column 4 of Table A1 in the Appendix B.

Table 3: Integration properties: forecasts of $\log C$ (SPF)

	1Q ahead	2Q ahead	4Q ahead
<i>Panel A: I(1) test</i>			
PP (Z_t stat.)	-1.324	-1.327	-1.340
10% critical value	-3.167	-3.167	-3.167
DF-GLS	-1.223	-1.227	-1.238
10% critical value	-2.818	-2.818	-2.818
<i>Panel B: I(2) test</i>			
PP (Z_t stat.)	-4.725	-4.805	-4.837
1% critical value	-2.612	-2.612	-2.612
DF-GLS	-3.809	-4.082	-3.862
1% critical value	-2.611	-2.611	-2.611

4.3 No cointegration between forecasts of stock prices and aggregate consumption

Figure 2 displays the difference between median forecasts of $\log P$ and $\log C$ made at the same date using both stock price surveys. “1Q ahead forecast” corresponds to (normalized) 1-year ahead stock price forecasts minus 1-year ahead SPF consumption forecasts; similarly for 2Q and 4Q ahead forecast. The exception is “10 Yr ahead forecast” which corresponds to 10-year ahead stock price forecasts minus 1-year ahead consumption forecasts, given the unavailability of 10-year ahead consumption forecasts in the SPF.

Recall Theorem 4 implies that in the asset pricing models with RE and full information considered in Section 2, stock prices forecasts and consumption forecasts made at the same dates (and over possibly different horizons) are cointegrated with cointegrating vector $(1, -1)$. These models imply, for instance, 1-quarter ahead forecasts of stock prices are cointegrated with 1-quarter ahead forecast of aggregate consumption and 10-year ahead forecast of stock prices are cointegrated with 1-year ahead forecast of consumption.

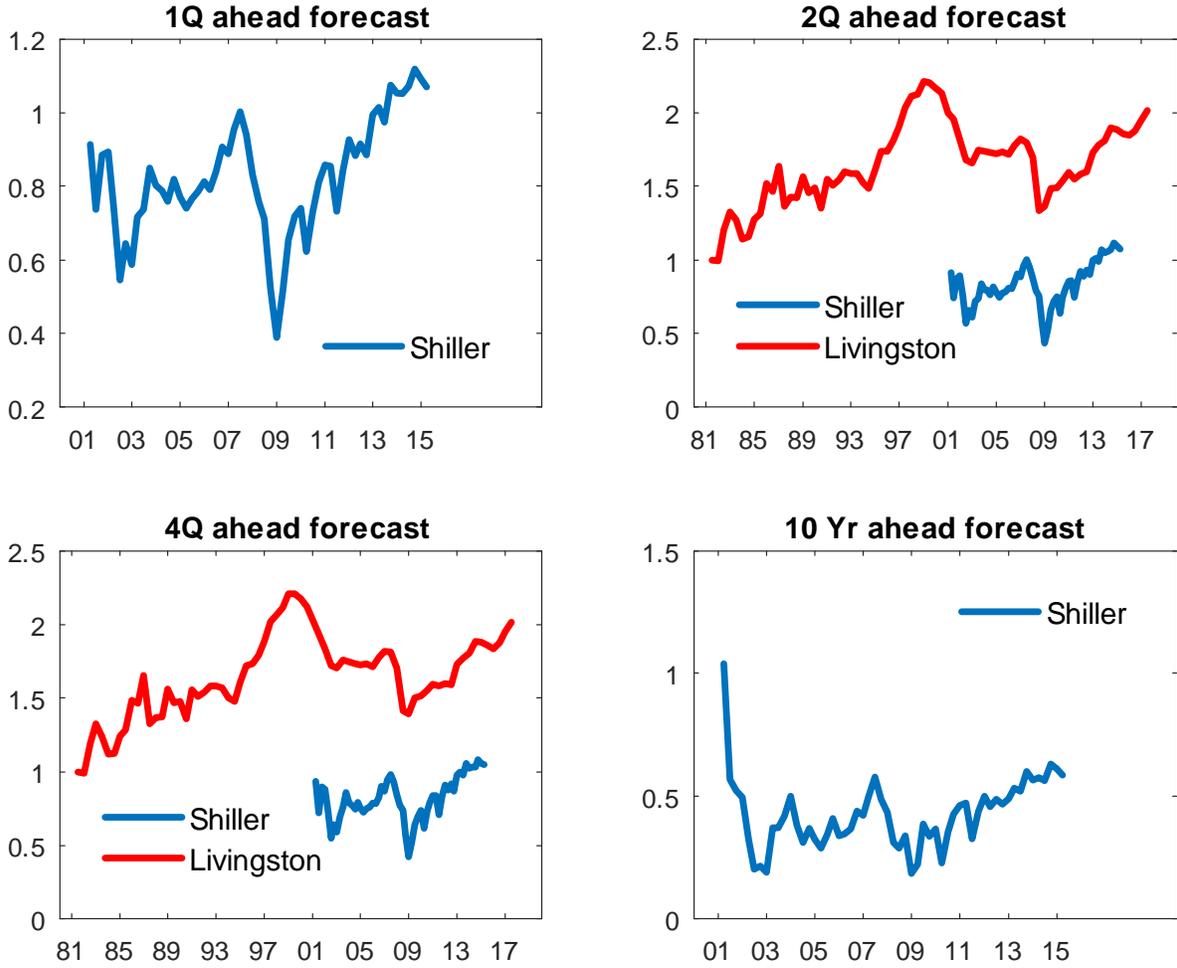


Figure 2: Median forecast of (log) price consumption ratio

Note: the bottom right panel plots the (normalized) difference between 10-year ahead stock price forecasts and 1-year ahead consumption forecasts.

Table 4 reports the test results of whether median forecasts of aggregate consumption are cointegrated with median forecasts of stock prices made at the same date and over the same forecasting horizon (with cointegrating vector $(1, -1)$);¹⁵ the only exception is that the column “10-yr ahead” is the test results on the cointegration between 10-year ahead forecasts of stock prices and 1-year ahead forecasts of consumption, given the unavailability of 10-year

¹⁵For DF-GLS test, we report the test statistics produced with the number of lags leading to the lowest mean squared errors. The results are robust to different choices of lags.

ahead consumption forecasts data. Both PP and DF-GLS tests show that we cannot reject the null hypothesis that stock price forecasts are not cointegrated with consumption forecasts with cointegrating vector $(1, -1)$, robust to different data sources and forecasting horizons. The same conclusion is reached with mean forecasts, see Table A3 of the Appendix. We also applied the KPSS test which tests the null hypothesis of a stationary process against the alternative of a unit root. The tests yield the same conclusion that forecasts of stock price to consumption ratio is non-stationary, robust to different sources of forecasts data, different forecasting horizons, different choices of lags in the testing, etc.¹⁶

Table 4: No cointegration between forecasts of $\log P$ and $\log C$

	1Q ahead	2Q ahead	4Q ahead	10-yr ahead*
<i>I(1) test</i>				
Shiller (PP Z_t stat.)	-2.039	-1.962	-2.097	-2.304
10% critical value	-2.595	-2.595	-2.595	-2.595
Shiller (DF-GLS)	-1.665	-1.569	-1.602	-0.814
10% critical value	-1.929	-1.929	-1.929	-1.929
Livingston (PP Z_t stat.)	<i>n.a.</i>	-2.234	-2.161	<i>n.a.</i>
10% critical value	<i>n.a.</i>	-2.591	-2.591	<i>n.a.</i>
Livingston (DF-GLS)	<i>n.a.</i>	-0.246	-0.227	<i>n.a.</i>
10% critical value	<i>n.a.</i>	-1.895	-1.895	<i>n.a.</i>

*Note: for the first three columns, forecasts of stock prices and consumption are made at the same date and over the same forecasting horizons. For the fourth column, the forecasts of stock prices and consumption are made at the same dates but different forecasting horizons (i.e., 10-year ahead stock price forecasts vs 1-year ahead consumption forecasts).

Theorem 4 also suggests stock price forecasts and consumption forecasts *made at different dates* are cointegrated with cointegrating vector $(1, -1)$. For illustration, we test the existence of cointegration between 1-year ahead Livingston median stock price forecasts made during 1978 to 2014 with 1-year ahead SPF median consumption forecasts made during 1981 to 2017. Table 5 reports the test results using PP and DF-GLS test, suggesting no cointegration between forecasts of stock prices and consumption made at different dates.

¹⁶The test results are not reported here due to space limitation and available upon request.

The long-run/trend component of stock price forecasts made by agents in reality is not anchored by consumption forecasts. The survey evidence rejects this particular aspect of the formation of stock price expectations in asset pricing models with RE and full information and in various learning or sentiment-based models (shown later).

Table 5: Testing the cointegration between 1-year ahead stock price forecasts made during 1978 - 2014 and 1-year ahead consumption forecasts made during 1981 - 2017

	Median forecasts	Mean forecasts
<i>I(1) test</i>		
PP (Z_t statistics)	-1.628	-1.603
10% critical value	-2.591	-2.591
DF-GLS	-1.080	-0.964
10% critical value	-1.895	-1.895

We also test if stock price forecasts are cointegrated with consumption forecasts without imposing the restriction of cointegrating vector $(1, -1)$. Using the Livingston Survey data, we find that the null hypothesis that stock price forecasts are not cointegrated with consumption forecasts for any forecasting horizons cannot be rejected at conventional significance level (e.g., 5%). Using the shorter Shiller Survey data, we reject the null hypothesis that stock price forecasts are not cointegrated with consumption forecasts at 5% significance level. The estimated cointegrating vector is $(1, -2.55)$ for 1-year ahead forecasts of stock prices and consumption; similar cointegrating vectors are found for other forecasting horizons. This also rejects the formation of stock price expectations in the asset pricing models studied in the paper which has $(1, -1)$ as the cointegration vector.

4.4 Cointegration among forecasts of stock prices (or consumption)

Recall in the models discussed in Section 2, realized stock prices (or consumption) contains a unit root. Theorem 5 implies that forecasts of stock prices (or consumption) over two different horizons should be cointegrated with cointegrating vector $(1, -1)$. This section shows that this aspect of expectation formation in these models is broadly consistent with survey forecasts of stock prices and consumption.

Table 6 reports the p-value of PP test on whether forecasts of stock prices (or aggregate consumption) over two different horizons are cointegrated with each other with cointegrating vector $(1, -1)$. Using median and mean forecasts data, PP test shows that at 1% significance level, we can reject the null hypothesis that the difference between the forecasts of stock prices, i.e., $E_t \log(p_{t+i}) - E_t \log(p_{t+j})$, contains a unit root, for various pairs of forecasting horizons $(i, j) = (1, 2)$, $(i, j) = (1, 4)$ and $(i, j) = (2, 4)$.¹⁷ Similarly, forecasts of consumption over two different horizons are cointegrated with cointegrating vector $(1, -1)$.

Table 6: P-value of testing the stationarity of $E_t \log(X_{t+i}) - E_t \log(X_{t+j})$
(X stands for stock prices or consumption)

PP test	1 & 2Q	1 & 4Q	2 & 4Q
Stock prices (Shiller median)	0.0019	0.0005	0.0047
Stock prices (Shiller mean)	0.0019	0.0005	0.0047
Consumption (SPF median)	0.0004	0.0228	0.0011
Consumption (SPF mean)	0.0006	0.0035	0.0000

5 Testing the formation of stock price expectations in adaptive learning models

In a similar way as in Section 3, this section derives tests of expectation formation in a type of asset pricing models with non-rational expectations (i.e., adaptive learning models). Agents' perceived law of motion (PLM) for asset prices is generally not the same as the actual law of motion (ALM), in contrast to RE models. As is shown later, the tests need to use both the PLM and ALM for model variables.¹⁸ This section shows the typical specifications for subjective stock price beliefs in existing adaptive learning models imply that agents' forecasts of stock prices are cointegrated with consumption forecasts, as opposed to the survey evidence in Section 4.3.

¹⁷DF-GLS test rejects the null hypothesis that aggregate price index forecasts across different horizons are not cointegrated at 1 percent level. It also accepts the cointegration between 1 quarter ahead and 2 quarter ahead consumption forecast. However it fails to reject that 1- (or 2-) quarter ahead consumption forecast is not cointegrated with 4 quarter ahead consumption forecast.

¹⁸In models with full information and RE, agents' PLM for economic variables is identical to the corresponding ALM.

5.1 Setup of the AMB Model

We firstly consider the endowment economy asset pricing model of Adam, Marcet and Beutel (2017, henceforth AMB). The model quantitatively replicates many asset-pricing moments and endogenously generates boom-bust asset price dynamics. Importantly, the model is consistent with survey expectation data that price dividend ratio comoves positively with survey return expectations which cannot be matched by RE models (shown in AMB).

The model has a unit mass of infinitely lived investors $i \in [0, 1]$ who have time-separable preferences. They trade one unit of a stock in a competitive stock market. Investor i maximizes utility subject to a budget constraint,

$$\begin{aligned} & \max_{\{C_t^i \geq 0, S_t^i \in \mathcal{S}\}_{t=0}^\infty} E_0^{\mathcal{P}^i} \sum_{t=0}^{\infty} \delta^t u(C_t^i) & (12) \\ \text{s.t.} \quad & S_t^i P_t + C_t^i = S_{t-1}^i (P_t + D_t) + W_t \quad \text{for all } t \geq 0, \\ & S_{-1}^i = 0, & (13) \end{aligned}$$

where u is the instantaneous utility, C^i denotes consumption, S^i the agent's stock holdings and $P \geq 0$ the (ex-dividend) price of the stock. Each period they earn an exogenous non-dividend income $W_t \geq 0$ as 'wages'. Stocks deliver the exogenous dividend $D_t \geq 0$. Wage and dividend are in the form of perishable consumption goods. \mathcal{P} denotes the investor's subjective probability measure specified below.

Dividends grow at a constant rate

$$\log D_t = \log \beta^D + \log D_{t-1} + \log \varepsilon_t^D, \quad (14)$$

where $\beta^D \geq 1$ stands for the mean growth rate and $\ln \varepsilon_t^D$ an *i.i.d.* growth innovation described further below. The exogenous wage income process W_t is

$$\log \left(1 + \frac{W_t}{D_t} \right) = (1 - p) \log(1 + \rho) + p \log \left(1 + \frac{W_{t-1}}{D_{t-1}} \right) + \ln \varepsilon_t^W, \quad (15)$$

where $1 + \rho$ is the average consumption-dividend ratio and $p \in [0, 1)$ its quarterly persistence.

The innovations are given by

$$\begin{pmatrix} \log \varepsilon_t^D \\ \log \varepsilon_t^W \end{pmatrix} \sim iin \left(-\frac{1}{2} \begin{pmatrix} \sigma_D^2 \\ \sigma_W^2 \end{pmatrix}, \begin{pmatrix} \sigma_D^2 & \sigma_{DW} \\ \sigma_{DW} & \sigma_W^2 \end{pmatrix} \right), \quad (16)$$

with $E\varepsilon_t^D = E\varepsilon_t^W = 1$. Aggregate consumption is

$$C_t = W_t + D_t \quad (17)$$

The investor's expectation is taken using subjective probability measure \mathcal{P} that assigns probabilities to all external variables including stock price P_t , dividend D_t and wage W_t . The underlying sample space Ω consists of the space of realizations for prices, dividends and endowment. Specifically, a typical element $\omega \in \Omega$ is an infinite sequence $\omega = \{P_t, W_t, D_t\}_{t=0}^\infty$. The probability space $(\Omega, \mathcal{B}, \mathcal{P})$ is defined with \mathcal{B} denoting the corresponding σ -Algebra of Borel subsets of Ω , and \mathcal{P} is the agent's subjective probability measure over (Ω, \mathcal{B}) . In RE models, stock price P_t equals the discounted sum of future dividends, so P_t carries only redundant information. Agents' beliefs and preferences etc are not common knowledge. The agent with "Internal Rationality" does not know the mapping from D_t and W_t to P_t . The agent forms subjective price belief and learns from market prices. She cannot correctly deduce the equilibrium law of motion for asset prices. In this case, P_t should be included in the state space.

The first order condition for the representative investor is

$$u'(C_t) = \delta E_t^{\mathcal{P}} \left[u'(C_{t+1}) \frac{P_{t+1} + D_{t+1}}{P_t} \right]$$

5.2 Testable implications

Consider the baseline I(2) specification for subjective stock price beliefs in AMB

$$\Delta \log P_{t+1} = \log \beta_{t+1} + \log \epsilon_{t+1} \quad (18)$$

$$\log \beta_{t+1} = \log \beta_t + \log \nu_{t+1} \quad (19)$$

Agents are uncertain and learn about $\log \beta_t$ over time.

They are assumed to know the exogenous driving processes, (14), (15) and (17). These

equations give that

$$\log C_t = \log(D_t + W_t) = \log D_t + \log\left(1 + \frac{W_t}{D_t}\right). \quad (20)$$

On the one hand, agents' PLM for stock prices (18) - (19) is I(2). On the other hand, the consumption process (20) is I(1). One may think that the forecast of stock prices will not be cointegrated with forecasts of consumption. Yet the following proposition shows this thinking turns out to be incorrect.

Proposition 8 *Suppose agents' perceived law of motion for stock prices is (18) - (19), agents' stock price forecasts $E_i \log P_{i+j}$ are cointegrated with consumption forecasts $E_k \log C_{k+l}$ with cointegrating vector $(1, -1)$ for arbitrary $i, j, k, l > 0$.*

Proof. Denote by $\log m_i$ agents' beliefs about the growth rate of stock prices at period i . Given agents' PLM (18) - (19), we have

$$E_i \log P_{i+j} = \log P_i + j \log m_i.^{19}$$

Let forecasts of consumption be

$$\begin{aligned} E_k \log C_{k+l} &= E_k \left(\log D_{k+l} + \log\left(1 + \frac{W_{k+l}}{D_{k+l}}\right) \right) \\ &= \log D_k + l \log \beta^D + E_k \left(\log\left(1 + \frac{W_{k+l}}{D_{k+l}}\right) \right) \\ &= \log D_k + l \log \beta^D + E_k \left(\log\left(1 + \frac{W_{k+l}}{D_{k+l}}\right) \right) \end{aligned} \quad (21)$$

¹⁹Note stock price forecasts in the model contain a unit root because realized stock prices contain a unit root (sharing a common trend with dividend) and $\log m_t$ is stationary.

So

$$\begin{aligned}
E_i \log P_{i+j} - E_k \log C_{k+l} &= (\log P_i + j \log m_i) - \\
&\quad \left(\log D_k + l \log \beta^D + E_k \left(\log \left(1 + \frac{W_{k+l}}{D_{k+l}} \right) \right) \right) \\
&= (\log P_i - \log D_k) + (j \log m_i - l \log \beta^D) \\
&\quad - E_k \left(\log \left(1 + \frac{W_{k+l}}{D_{k+l}} \right) \right) \tag{22}
\end{aligned}$$

$(E_i \log P_{i+j} - E_k \log C_{k+l})$ is stationary for three reasons. First, Lemma 1 of AMB shows that without uncertainty $\log m_t$ is mean-reverting and converges to the corresponding RE value. In particular, Appendix A10 of AMB shows that under certain condition, the ALM for $\log m_t$ is a second-order difference equation and the eigenvalues determining the stability of the solution are inside the unit circle. Thus, $\log m_t$ is a stationary process. Second, price dividend ratio is stationary; see Proposition 3 of AMB.²⁰ Note $\log P_i - \log D_k = (\log P_i - \log P_k) + (\log P_k - \log D_k)$. $(\log P_i - \log D_k)$ is stationary because both $(\log P_i - \log P_k)$ and $(\log P_k - \log D_k)$ are stationary. Third, $E_k \left(\log \left(1 + \frac{W_{k+l}}{D_{k+l}} \right) \right)$ is stationary because $\log \left(1 + \frac{W_{k+l}}{D_{k+l}} \right)$ is stationary. ■

Proposition 8 contains a set of testable implications. A special case is when forecasts of stock prices and consumption are made at the same dates and over the same forecasting horizon, i.e., $i = k$ and $j = l$. The tests of the formation of agents' stock price expectation make use of both agents' PLM and the ALM for stock prices. First, agents' PLM for stock prices gives us the forecasts of price consumption ratio as a function of agents' belief about the difference in the growth rate of stock prices and consumption and of current price consumption ratio. Second, one feature of the model is that agents' belief about the long-run growth rate of stock prices will mean-revert to the long-run growth rate of consumption. Moreover, current price-consumption ratio is stationary, as implied by the ALM for stock prices. Taking the two features together, the forecast of price consumption ratio is stationary, despite agents' lack of knowledge of both features.

Note the forecasts of stock prices and consumption are cointegrated even if they are made at different dates ($i \neq k$) and/or over different forecasting horizons ($j \neq l$), as in the case of

²⁰In AMB, the stationarity of the price dividend ratio is essential for rejecting RE using survey expectation data and for calculating the statistical moments of price dividend ratio in the learning model.

full-information RE models (see Theorem 4). This is again because of the mean-reversion of price growth beliefs to $\log \beta^D$ and the stationarity of price to consumption (or price to dividend) ratio.

AMB considers another general belief specification which features mean-reversion in stock price growth rates. Appendix D shows in this case, stock price forecasts remain cointegrated with consumption forecasts. Thus, there appears a discrepancy between both belief specifications for subjective stock price beliefs and the evidence presented in Section 4.3.

Many other papers also relax the assumption of RE and incorporate adaptive learning into asset pricing models with endowment or production economies, e.g., Carceles-Poveda and Giannitsarou (2008). Agents learn about detrended stock prices in these models. This implies agents know exactly the evolution of the trend growth rate of stock prices and consumption as in RE models with full information; this can be shown following Section 7 of Kuang and Mitra (2016). Thus, in these adaptive learning models, forecasts of stock prices are cointegrated with consumption forecasts as in full-information RE models. Again, this aspect of expectation formation in these models appears inconsistent with the survey evidence in Section 4.3.

6 Testing expectation formation in incomplete information models

This section provides testable implications for the formation of stock price expectations in some sentiment-based asset pricing models as well as models with RE and learning about exogenous consumption process. It shows that although agents in these models may have (possibly time-varying) misperception about the exogenous consumption process, forecasts of stock prices are cointegrated with forecasts of aggregate consumption and forecasts of stock price consumption ratio are stationary, appearing inconsistent with the survey evidence in Section 4.3.

6.1 Sentiment-based models

Some papers introduce sentiment into asset pricing models, such as Jin and Sui (2018) and the exchange rate model of Yu (2013). As an example, suppose the representative agent's

preferences are represented by the Epstein-Zin utility (1) and the actual exogenous driving processes are

$$\Delta c_{t+1} = \mu_c + \sigma_t \eta_{t+1}, \quad (23)$$

$$\sigma_{t+1}^2 = \bar{\sigma}^2 + \nu(\sigma_t^2 - \bar{\sigma}^2) + \sigma_w w_{t+1}, \quad (24)$$

$$\Delta d_{t+1} = \mu_d + \pi \sigma_t \eta_{t+1} + \varphi \sigma_t u_{d,t+1}. \quad (25)$$

Comparing (23) - (25) with (2) - (5) in the long-run risks model of Bansal, Kiku and Yaron (2012), we drop the persistent component x_t in the actual exogenous driving processes (because sentiment plays the role of the persistent component x_t).²¹

Now assume agents have misperception about the exogenous consumption and dividend process. They perceive consumption and dividend processes as

$$\Delta c_{t+1} = \mu_c + a_t + \sigma_t \hat{\eta}_{t+1}, \quad (26)$$

$$a_{t+1} = \rho a_t + \varphi_e \sigma_t e_{t+1}, \quad (27)$$

$$\sigma_{t+1}^2 = \bar{\sigma}^2 + \nu(\sigma_t^2 - \bar{\sigma}^2) + \sigma_w w_{t+1}, \quad (28)$$

$$\Delta d_{t+1} = \mu_d + \phi a_t + \pi \sigma_t \eta_{t+1} + \varphi \sigma_t u_{d,t+1}. \quad (29)$$

where $(\hat{\eta}_{t+1}, e_{t+1})$ are i.i.d joint standard normal under agents' belief. a_t is an AR(1) process and does not appear in the true driving processes (called "sentiment"). When a_t is positive (negative), agents are optimistic (pessimistic). Assuming $0 < \rho < 1$.²² Note in this type of models, despite agents' misperception of the exogenous driving process, they know the equilibrium pricing function. The following result provides testable implications for the formation of stock price expectations.

Proposition 9 *Given agents' beliefs (26) - (29), agents' stock price forecasts $E_i \log P_{i+j}$ are cointegrated with their forecasts of aggregate consumption $E_k \log C_{k+l}$ with cointegrating vector $(1, -1)$ for arbitrary $i, j, k, l > 0$.*

Proof. Following Bansal, Kiku and Yaron (2012), the (approximate) analytical solution for

²¹Our proposition below is not affected by adding x_t in the exogenous driving processes, as is argued later.

²²If $\rho = 1$, consumption is an I(2) process and $\log(P_t/C_t)$ will have unbounded volatility. Both seem rejected by the data. Thus, we require $\rho < 1$.

(log) price consumption ratio can be derived as

$$\log\left(\frac{P_t}{C_t}\right) = A_0 + A_1 a_t + A_2 \sigma_t^2, \quad (30)$$

where A_0, A_1, A_2 remain the same constant as in Bansal, Kiku and Yaron (2012). The trend growth rate of both stock prices forecasts and consumption forecasts are identical to μ_c (noting sentiment is a stationary process). Thus, stock price forecasts and consumption forecasts can be expressed as $E_i \log P_{i+j} = \log P_i + j\mu_c + s(i, j)$ and $E_k \log C_{k+l} = \log C_k + l\mu_c + \tilde{s}(k, l)$ where $s(i, j)$ and $\tilde{s}(k, l)$ are stationary terms and omitted. Let

$$\begin{aligned} E_i \log P_{i+j} - E_k \log C_{k+l} &= (\log P_t + j\mu_c + s(i, j)) \\ &\quad - (\log C_k + l\mu_c + \tilde{s}(k, l)) \\ &= (\log P_t - \log C_t) + (\log C_t - \log C_k) \\ &\quad + (j - l)\mu_c + (s(i, j) - \tilde{s}(k, l)) \end{aligned}$$

$(E_i \log P_{i+j} - E_k \log C_{k+l})$ is stationary because $(\log P_t - \log C_t)$ is stationary (see equation (30)) and $(\log C_t - \log C_k)$ is stationary. ■

If the consumption driving process (23) - (25) contains a persistent and stationary component x_t , (log) price consumption ratio will be a linear function of x_t, a_t and σ_t^2 with constant coefficients. The above proposition will still hold as the trend growth rate of both stock prices forecasts and consumption forecasts are identical to μ_c .

6.2 Learning consumption dynamics

Many asset pricing models maintain the assumption of RE but assume agents have incomplete information and learn about the exogenous consumption process.²³ In this type of learning models, agents know the equilibrium pricing mapping and form RE about stock prices. This is in contrast to Adam, Marcet and Beutel (2017) where agents do not have this knowledge.

Suppose the representative agent's preferences is represented by the Epstein-Zin utility

²³Collin-Dufresne, Johannes and Lochstoer (2016) and Johannes, Lochstoer and Mou (2016) are examples of asset pricing models along this line.

(1). For illustration, the consumption process is

$$\Delta c_{t+1} = \mu_c + \bar{\sigma}\eta_{t+1}, \quad (31)$$

where η_{t+1} is an i.i.d process. Agents do not know the consumption growth rate but know the constant variance $\bar{\sigma}^2$. Agents learn μ_c over time and beliefs about μ_c is updated by

$$\mu_{c,t} = \mu_{c,t-1} + g_t (\Delta c_t - \mu_{c,t-1}) \quad (32)$$

Assuming constant gain or Kalman filter learning (under steady state variance ratio) is used, i.e., $g_t = g \in (0, 1)$. Substituting (31) into (32) yields

$$\mu_{c,t} = (1 - g)\mu_{c,t-1} + g(\mu_c + \bar{\sigma}\eta_t)$$

is a stationary process.

Proposition 10 *Given the beliefs (31), agents' forecasts of stock prices $E_i \log P_{i+j}$ are cointegrated with their forecasts of aggregate consumption $E_k \log C_{k+l}$ with cointegrating vector $(1, -1)$ for arbitrary $i, j, k, l > 0$.*

Proof. The RE version of the model here is a special case of Bansal, Kiku and Yaron (2012) with setting $\rho = 0$, $\nu = 0$, $x_t = 0$, $\sigma_t = \bar{\sigma}$, $\sigma_w = 0$, $\varphi_e = 0$. From the RE version to the learning model, we replace the actual growth rate of consumption μ_c by agents' beliefs about consumption growth rate $\mu_{c,t}$ in the analytical solution. The (analytical) solution for log price consumption ratio in the learning model is

$$\log\left(\frac{P_t}{C_t}\right) = \tilde{A}_{0,t} + \tilde{A}_2 \bar{\sigma}^2, \quad (33)$$

where $\tilde{A}_{0,t} = \frac{1}{1-\kappa_{1,t}} \left(\log \delta + \kappa_{0,t} + \left(1 - \frac{1}{\psi}\right) \mu_{c,t} + \kappa_{1,t} \tilde{A}_2 \bar{\sigma}^2 \right)$, $\tilde{A}_2 = -\frac{(\gamma-1)(1-\frac{1}{\psi})}{2}$, $\kappa_{0,t} = \log(1 + \exp(\bar{z}_t)) - \kappa_{1,t} \bar{z}_t$, $\kappa_{1,t} = \frac{\exp(\bar{z}_t)}{1 + \exp(\bar{z}_t)}$, $\bar{z}_t = \tilde{A}_{0,t}(\bar{z}_t) + \tilde{A}_2 \bar{\sigma}^2$. Note $\tilde{A}_{0,t}$ is a nonlinear function of $\mu_{c,t}$. Using Taylor expansion, the right hand side of the process (33) can be well approximated by a polynomial function of the AR(1) process $\mu_{c,t}$ and is again a stationary process.²⁴ In

²⁴If agents learn use least squares (i.e., $g_t = 1/t$), $\mu_{c,t}$ will converge to μ_c and $\tilde{A}_{0,t}$ will converge to a constant.

this model, agents' beliefs about the trend growth rate of both stock prices and consumption mean-revert to μ_c . Thus, stock price forecasts and consumption forecasts can be expressed as $E_i \log P_{i+j} = \log P_i + j\mu_c + s(i, j)$ and $E_k \log C_{k+l} = \log C_k + l\mu_c + \tilde{s}(k, l)$ where $s(i, j)$ and $\tilde{s}(k, l)$ are stationary terms and omitted. Again, $(E_i \log P_{i+j} - E_k \log C_{k+l})$ is stationary because realized stock price and consumption are cointegrated and the trend growth rate of stock price forecasts and consumption forecasts are identical to each other. ■

To sum up, in both sentiment-based models and models with RE and learning consumption process we considered here, stock price forecasts and consumption forecasts are cointegrated. Notice this cointegration relation also holds if the two forecasts are made at different dates and/or over different forecasting horizons. This aspect of the formation of stock price expectations in these models appears inconsistent with the evidence in Section 4.3.

7 Reconciling models with the new survey evidence

The paper has shown that a wide range of asset pricing models appear inconsistent with the survey evidence in Section 4.3. To emphasize, the survey evidence is incompatible with existing asset pricing models with maintaining the assumption of RE and simultaneously relaxing the assumption of full information (i.e., allowing for agents' learning about the exogenous consumption process), or relaxing the RE assumption (i.e., adaptive learning), or adding sentiment to the exogenous consumption process.

How can we break the tight link between the trend component of stock price forecasts and consumption forecasts in asset pricing models? This section discusses potential resolutions for reconciling models with the new survey evidence. We firstly propose a new way to specify subjective price beliefs in adaptive learning models and discuss the mechanism how the modified learning model is consistent with the new evidence.

7.1 Adding sentiment directly to stock price forecasts

Consider the AMB model discussed in Section 5, investors' subjective price belief is modified in the following way. Agents' subjective stock forecasts ($\log P_{t+1}^e$) consist of two components. The first component of stock price forecasts (denoted by $E_t \log P_{t+1}$) is produced as in AMB.

Specifically, agents have an econometric model of stock prices but do not know the parameters of the model. They need to learn about the parameters using historical stock price data and then make forecasts using the estimated model. The second component is new and denoted by $\log \gamma_t$. Denote by $\log P_{t+1}^e$ agents' forecast of stock prices in period $t + 1$ made at period t . Mathematically,

$$\log P_{t+1}^e = E_t \log P_{t+1} + \log \gamma_t \quad (34)$$

$$\log \gamma_t = \log \gamma_{t-1} + \log \xi_t \quad (35)$$

$\log \xi_t$ is assumed as an i.i.d process. When $\log \gamma_t$ is positive (or negative), agents are more optimistic (or pessimistic) than the stock price forecasts produced by the econometric model. We assume $\log \gamma_t$ is, for simplicity, a random walk process and it is private information and observable by individual agent each period.²⁵ The non-stationarity of $\log \gamma_t$ is the key to reconcile asset pricing models with the evidence. $\log \gamma_t$ is assumed to be independent of price, dividend and consumption.²⁶

Relating to the literature, three interpretations of $\log \gamma_t$ are provided. First, $\log \gamma_t$ can be interpreted as “sentiment,” see Yu (2013) for a sentiment-based exchange rate model. While adding sentiment to the exogenous consumption process – like in Yu (2013) – cannot reconcile models with the new survey evidence (as is shown in Section 6.1), adding sentiment directly to stock price forecasts (like here) is crucial. Second, $\log \gamma_t$ can be viewed as (the “guesswork” component of) judgment made by forecasters. This is known as “add-factoring” the forecast in the forecasting community. For instance, Bullard, Evans and Honkapohja (2008) examines the role of agents' judgmental adjustment to forecasts in self-referential learning models. They show this may lead to self-fulfilling fluctuations in New Keynesian models. Third, $\log \gamma_t$ can be regarded as “expectation shocks”, see Milani (2011) for an estimated New Keynesian model with learning and expectation shocks.

The first component of stock price forecasts ($E_t \log P_{t+i}$) is generated from the following

²⁵Two remarks are as follows. First, $\log \gamma_t$ can be specified as a more general $I(1)$ process and the proposition below – stock price forecasts and consumption forecasts are not cointegrated – will not be affected. Second, we can alternatively assume $\log \gamma_t$ is common knowledge. For the purpose of replicating the evidence in the representative agent setting, whether $\log \gamma_t$ is private information or common knowledge does not matter.

²⁶A model-implied sequence of sentiment $\log \gamma_t$ may be calculated as the difference between survey stock price forecast and stock price forecasts produced from an econometric forecasting model.

forecasting model

$$\Delta \log P_t = \log \beta_t + \log \epsilon_t, \quad (36)$$

$$\log \beta_t = (1 - \eta_\beta) \log \beta^D + \eta_\beta \log \beta_{t-1} + \log \nu_t, \quad (37)$$

where $\log \epsilon_t$ and $\log \nu_t$ are i.i.d. innovations. $\log \xi_t$, $\log \epsilon_t$, and $\log \nu_t$ are independent to each other. Agents' beliefs about $\log \beta_t$ is updated by

$$\begin{aligned} \log m_t &= (1 - \eta_\beta) \log \beta^D + \eta_\beta \log m_{t-1} \\ &\quad + g (\log P_{t-1} - \log P_{t-2} - \log m_{t-1}) \end{aligned}$$

where g is the Kalman gain parameter.

Proposition 11 *Given agents' price belief (34) - (37), agents' stock price forecasts $\log P_{i+j}^e$ are $I(1)$ and stock prices forecasts $\log P_{i+j}^e$ and consumption forecasts $\log C_{k+l}^e$ are not cointegrated with cointegrating vector $(1, -1)$ for arbitrary $i, j, k, l > 0$, consistent with the survey evidence in Section 4.3.*

Proof. Given (34) - (37), the forecasts of stock prices are

$$\begin{aligned} \log P_{i+j}^e &= E_i \log P_{i+j} + \log \gamma_i \\ &= \log P_i + j \log \beta^D + \tilde{s}(i, j) + \log \gamma_i \end{aligned}$$

where $\tilde{s}(i, j)$ is a stationary term and omitted because it is irrelevant for the proof. Denote by L the lag operator. Taking the difference of $\log P_{i+j}^e$ yields

$$\begin{aligned} (1 - L) \log P_{i+j}^e &= (E_i \log P_{i+j} + \log \gamma_i) - (E_{i-1} \log P_{i-1+j} + \log \gamma_{i-1}) \\ &= \log P_i + j \log \beta^D + \tilde{s}(i, j) + \log \gamma_i \\ &\quad - (\log P_{i-1} + j \log \beta^D + \tilde{s}(i-1, j) + \log \gamma_{i-1}) \\ &= \Delta \log P_i + \log \xi_i + \Delta \tilde{s}(i, j) \end{aligned}$$

Given $\Delta \log P_i$ is stationary (as in the data) and $\Delta \tilde{s}(i, j)$ is stationary, we have shown stock price forecasts are $I(1)$, consistent with the evidence in Section 3.1. And agents' forecasts of

consumption is again (21). Let

$$\begin{aligned}
\log P_{i+j}^e - \log C_{k+l}^e &= (\log P_i + j \log \beta^D + \tilde{s}(i, j) + \log \gamma_i) \\
&\quad - \left(\log D_k + l \log \beta^D + E_k \left(\log \left(1 + \frac{W_{k+l}}{D_{k+l}} \right) \right) \right) \\
&= (\log P_i - \log D_k) + \log \gamma_i + (j - l) \log \beta^D + \tilde{s}(i, j) \\
&\quad - E_k \left(\log \left(1 + \frac{W_{k+l}}{D_{k+l}} \right) \right)
\end{aligned}$$

Because $\log \gamma_i$ is I(1), we have shown that $(\log P_{i+j}^e - \log C_{k+l}^e)$ is I(1), consistent with the evidence in Section 4.3. ■

In this proposed resolution, introducing a non-stationary sentiment ($\log \gamma_t$) directly to stock price forecasts is the key to break the tight link between the trend component of stock price forecasts and consumption forecasts and reconcile models with the new survey evidence.²⁷ Note the proposition holds even if the forecasts of stock prices and consumption are made at different dates and/or over different forecasting horizons.

7.2 Further discussions

Some earlier models of stock prices, e.g., De Long, Shleifer, Summers and Waldmann (1990), avoid modelling consumption so that consumption is not a component for understanding stock price movement. Similarly, consumption is not an important driving force of stock prices in the X-CAPM model of Barberis, Greenwood, Jin and Shleifer (2015). We note these models are not inconsistent with the new survey evidence here.

In many asset pricing models, aggregate consumption is not cointegrated with aggregate dividends. Thus, in these models, stock prices are not necessarily cointegrated with aggregate consumption. However, the assumption of no cointegration between consumption and dividends is not made based on empirical evidence but for the purpose of simplifying analysis. In the basic habit formation model of Campbell and Cocharane (1999), aggregate consump-

²⁷Although we add a non-stationary sentiment variable into stock price forecasts, the learning model can still produce a stationary price to consumption ratio. This is because when expectations about stock price appreciation are sufficiently high, the wealth effect becomes as strong as (or even stronger than) the substitution effect as in AMB. We leave a quantitative evaluation of an asset pricing model with this sentiment shock for future research.

tion and aggregate dividends are not cointegrated. They, nevertheless, say “*It would be better to make dividends and consumption cointegrated. We have explored a model in which the log dividend/consumption ratio is i.i.d. and the correlation of one-period dividend and consumption growth rates is low as in the data... A cointegrated model with a persistent log dividend/consumption ratio would be more realistic, but this modification would require an additional state variable*” (p. 217). With this view, we tend to think that allowing no cointegration between consumption and dividends may not be a promising avenue to reconcile models with the new survey evidence.

One may think structural breaks is a potential reason why stock price forecasts and consumption forecasts are not cointegrated. The focus of our paper is testing expectation formation in major asset pricing models which do not usually include structural breaks. Yet we think our testing results here are robust to some types of structural breaks. For instance, the 2007 - 2008 Global Financial Crisis may be viewed as a structural break that the trend growth rate of consumption is reduced. Rational agents will observe this and the trend growth rate of equilibrium stock prices will have a one-to-one decline. Nevertheless, the cointegration relation between stock prices and consumption (including the cointegrating vector) is unaltered. Thus, our results of no cointegration between stock price forecasts and consumption forecasts will continue to hold if this type of structural break is allowed.²⁸

The models considered in the paper imply forecasts of stock prices and forecasts of dividends are cointegrated.²⁹ It would be interesting to test this cointegration relation. However, we do not pursue this analysis in the paper and leave it for future work due to lack of high quality data on aggregate dividend forecasts. To our knowledge, De La O and Myers (2018) is an example which makes effort on constructing analysts’ forecasts of aggregate dividend index using Thomson Reuters I/B/E/S Estimates Database. However, this data may not be appropriate for our purpose as their constructed data is relatively short (i.e., from 2003 to 2015).

²⁸Similarly, our empirical testing results will still hold if the Financial Crisis is viewed as, for example, a shift in the level of the trend of consumption.

²⁹Note in the X-CAPM model of Barberis, Greenwood, Jin and Shleifer (2015), stock price forecasts and dividend forecasts made by extrapolators (or rational agents) are cointegrated. This implication can be tested if dividend forecasts data is available.

8 Conclusion

Given the vital role for investors' expectations, many recent work employs expectations data to guide the modeling of expectation formation in financial markets and discipline the modeling of asset pricing dynamics. The paper provides new tests on expectation formation which are generally applicable in financial and macroeconomic models. The tests utilize cointegration restrictions on forecasts of model variables. The tests developed in the paper are informative and can provide guidance on modeling expectation formation.

We show stock price forecasts are cointegrated with consumption forecasts in a wide range of asset pricing models, including rational expectations models and several classes of incomplete information or non-rational expectations models. Yet survey data suggests stock price forecasts made by agents in reality are not cointegrated with consumption forecasts and rejects this aspect of modeling expectation formation in these models. This evidence is robust to different sources of expectations data, forecasting horizons, statistical tests, using median or mean forecasts for testing, and using stock price forecasts data which is made at different dates from consumption forecasts.

Potential resolutions are discussed for reconciling asset pricing models with the new survey evidence. The new evidence suggests major asset pricing models may miss an important persistent component which independently drives stock price forecasts but not consumption forecasts. We show adding a non-stationary sentiment (or judgment) component to subjective stock price forecasts in asset pricing models with adaptive learning reconciles the models with the new evidence. It would be interesting to evaluate quantitatively the role of this new component in driving the dynamics of stock prices.

The tests developed in the paper can be applied in or adapted to other settings. First, in the presence of structural breaks, financial and macroeconomic models impose cointegration restrictions among forecasts of model variables. Our tests can be adapted to test the modeling of agents' expectation formation in this setting.

Second, they can be implemented in other models, such as macroeconomic models and exchange rate models. An ongoing project finds that using median or mean SPF forecast data, output forecasts are not cointegrated with consumption forecasts (or investment forecasts) even if we do not impose any restriction on the cointegrating vector. This evidence appears incompatible with a wide range of stochastic growth models, such as full-information RE

models (including news shocks models) and incomplete-information models. The evidence may suggest there are independent driving forces for forecasts of different macroeconomic variables and mainstream stochastic growth models may have missed these important drivers in the modeling of macroeconomic forecasts.

Last but not least, while mean or median forecasts is used for testing in the paper, individual level data (where available) or data from Lab experiments can be used for testing. Moreover, the tests can be employed to study the relationship between forecasts made by economic agents with different characteristics (e.g., age, education, income, occupation). We leave these for future work.

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Appendix

A Rebasing consumption forecasts data

Since the Survey of Professional Forecasters (SPF) began, there have been a number of changes of base year in the national income and product accounts (NIPA). The forecasts for levels of consumption (SPF variable name: RCONSUM) use the base year that was in effect when the forecasters received the survey questionnaire. This Appendix explains how consumption forecasts data are rebased.

Table A0 provides the base year in effect for NIPA variables (including consumption expenditures), reproduced from Table 4 of the documentation of Survey of Professional Forecasters (p. 23). For rebasing, we use real consumption expenditures data of different vintages from the Real-Time Data Set for Macroeconomists managed by the Federal Reserve Bank of Philadelphia. Year 1996 is used as the common base year for all consumption forecast data. The data in each window needs to be rebased by multiplying a base ratio. For instance the 1959:Q4 real consumption in window from 1996:Q1 to 1999:Q3 is 1409.5 while it is 1469.5 in 1999:Q4 to 2003:Q4 window and hence the ratio is 1469.5/1409.5.

Table A0: Base years and Ratios for rebasing

Range of Survey Dates	Base Year	Ratio
1976:Q1 to 1985:Q4	1972	3.31
1986:Q1 to 1991:Q4	1982	1.48
1992:Q1 to 1995:Q4	1987	1.23
1996:Q1 to 1999:Q3	1992	1.04
1999:Q4 to 2003:Q4	1996	1
2004:Q1 to 2009:Q2	2000	0.94
2009:Q3 to 2013:Q2	2005	0.84
2013:Q3 to present	2009	0.79

B Results using mean forecasts

Table A1 and A2 test the integration properties of mean forecasts of (log) stock prices and aggregate consumption, respectively. We consider all sources of forecasts, different forecast-

ing horizons and tests. The results suggest mean forecasts of stock prices and aggregate consumption are I(1) and not I(2) at 10% significance level.

Table A1: Integration properties: forecasts of $\log P$

	1Q ahead	2Q ahead	4Q ahead	10-yr ahead
<i>Panel A: I(1) test statistics</i>				
Shiller (PP Z_t stat.)	-2.196	-2.225	-2.215	-2.400
10% critical value	-3.172	-3.172	-3.172	-3.172
Shiller (DF-GLS)	-2.218	-2.247	-2.163	-1.683
10% critical value	-2.851	-2.851	-2.851	-2.851
Livingston (PP Z_t stat.)	<i>n.a.</i>	-2.297	-2.086	<i>n.a.</i>
10% critical value	<i>n.a.</i>	-3.167	-3.167	<i>n.a.</i>
Livingston (DF-GLS)	<i>n.a.</i>	-1.698	-1.717	<i>n.a.</i>
10% critical value	<i>n.a.</i>	-2.818	-2.818	<i>n.a.</i>
<i>Panel B: I(2) test (p-value)</i>				
Shiller (PP Z_t stat.)	-7.576	-7.714	-7.681	-9.262
1% critical value	-2.615	-2.615	-2.615	-2.615
Shiller (DF-GLS)	-3.737	-3.724	-3.785	-3.212
1% critical value	-2.615	-2.615	-2.615	-2.615
Livingston (PP Z_t stat.)	<i>n.a.</i>	-8.506	-6.590	<i>n.a.</i>
1% critical value	<i>n.a.</i>	-2.612	-2.612	<i>n.a.</i>
Livingston (DF-GLS)	<i>n.a.</i>	-5.933	-3.154	<i>n.a.</i>
1% critical value	<i>n.a.</i>	-2.611	-2.611	<i>n.a.</i>

Table A3 shows that the mean forecasts of stock prices are not cointegrated with mean forecasts of aggregate consumption with cointegrating vector $(1, -1)$ when testing at 10% significance level with one exception. That is, for 10-year ahead stock price forecasts from the Shiller survey, the null hypothesis that forecasts of 10-year ahead stock price forecast are not cointegrated with forecasts of 1-year ahead consumption with cointegrated vector $(1, -1)$ is rejected by the PP test at 10% significance level but not at 5% significance level (because the 5% critical value is -2.920).

Table A2: Integration properties: forecasts of $\log C$

	1Q ahead	2Q ahead	4Q ahead
<i>Panel A: I(1) test</i>			
PP (Z_t stat.)	-1.316	-1.323	-1.323
10% critical value	-3.167	-3.167	-3.167
DF-GLS	-1.225	-1.226	-1.188
10% critical value	-2.818	-2.818	-2.818
<i>Panel B: I(2) test</i>			
PP (Z_t stat.)	-4.696	-4.769	-4.747
1% critical value	-2.612	-2.612	-2.612
DF-GLS	-3.844	-4.006	-4.215
1% critical value	-2.611	-2.611	-2.611

Table A3: No cointegration between forecasts of $\log P$ and $\log C$

Mean	1Q ahead	2Q ahead	4Q ahead	10-yr ahead
<i>I(1) test</i>				
Shiller (PP Z_t stat.)	-2.430	-2.444	-2.389	-2.673
10% critical value	-2.595	-2.595	-2.595	-2.595
Shiller (DF-GLS)	-1.653	-1.664	-1.596	-0.917
10% critical value	-1.929	-1.929	-1.929	-1.929
Livingston (PP Z_t stat.)	<i>n.a.</i>	-2.234	-2.213	<i>n.a.</i>
10% critical value	<i>n.a.</i>	-2.591	-2.591	<i>n.a.</i>
Livingston (DF-GLS)	<i>n.a.</i>	-0.233	-0.185	<i>n.a.</i>
10% critical value	<i>n.a.</i>	-1.895	-1.895	<i>n.a.</i>

C Fed Greenbook Consumption Forecast and Cointegration Tests

This Appendix shows the result of no cointegration between forecasts of stock prices and consumption still holds when we use consumption forecasts from the Greenbook data sets instead of SPF data. Our test results in the main text are robust to this alternative consumption forecast.

The Greenbook contains projections on the US economy in future quarters and is produced before each meeting of the Federal Open Market Committee. It includes projections for a large number of macroeconomic variables including real consumption growth. Four forecasting horizons are reported in each projection: 1- to 4-quarter ahead (while more horizons are issued from time to time). The dataset is published with a five-year lag. The sample of Greenbook consumption growth forecast is from 1967 to 2012. We obtain real consumption level forecast by multiplying the consumption growth forecast by (rebased) consumption level; the latter is obtained from real-time datasets for the US economy maintained by the Philadelphia Fed. To conduct the tests, we use the vintage of Greenbook forecasts in the way that the corresponding FOMC meeting date is closest to the date of the Livingston survey.

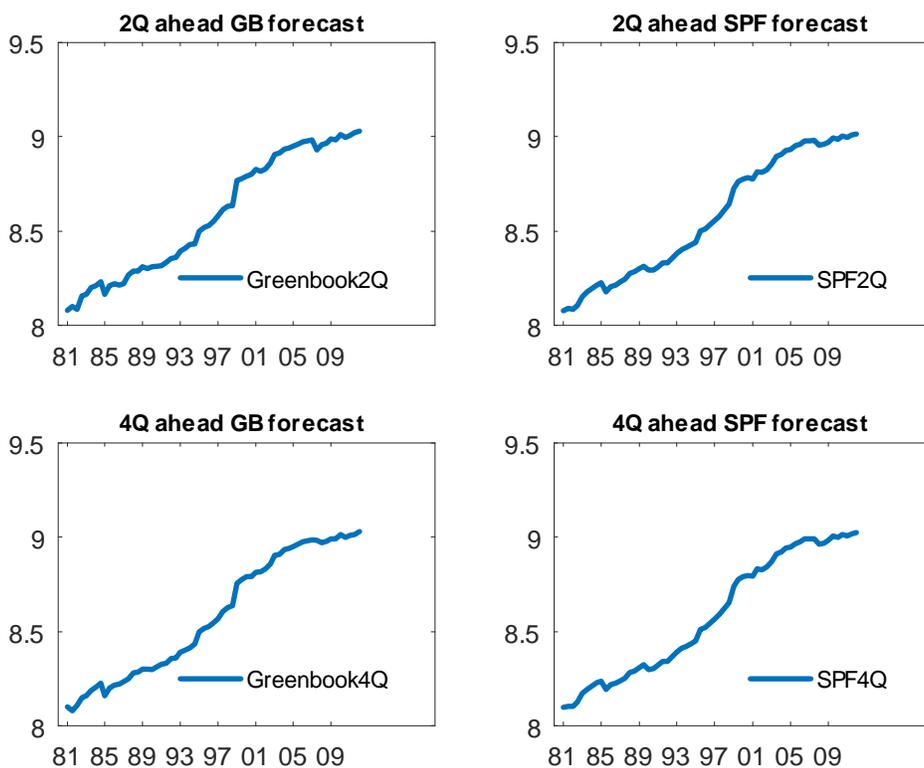


Figure A1: Greenbook and median SPF consumption forecasts

The tests are conducted using Livingston survey stock price forecasts and Greenbook consumption forecasts.³⁰ Figure A1 displays 2Q- and 4Q- ahead forecast of (log) consumption

³⁰The sample period which the Shiller Survey and the Greenbook datasets overlap is relatively short. Thus, we do not conduct the test using the Shiller survey data.

from the Greenbook (GB) datasets and the SPF. “Greenbook2Q” and SPF2Q” correspond to the 2-Quarter ahead Greenbook and median SPF consumption forecast respectively; similarly for 4Q ahead forecast. The forecasts from the two sources look quite similar.

Table A4 reports the the test statistics value and critical value for the unit root tests of forecasts of (log) aggregate consumption. For all forecasting horizons, both tests suggest that consumption forecasts is an I(1) but not I(2) process.

Table A4: Integration properties: forecasts of $\log C$

	2Q ahead	4Q ahead
<i>Panel A: I(1) test statistics</i>		
PP (Z_t stat.)	-1.320	-1.275
10% critical value	-3.173	-3.173
DF-GLS	-1.195	-1.091
10% critical value	-2.825	-2.825
<i>Panel B: I(2) test statistics</i>		
PP (Z_t stat.)	-9.340	-8.416
1% critical value	-3.565	-3.565
DF-GLS	-5.469	-2.995
1% critical value	-2.615	-2.615

Table A5 shows that both PP and DF-GLS tests show that we cannot reject the null hypothesis that forecasts of $\log(p)$ are not cointegrated with forecasts of $\log(c)$ with cointegrating vector $(1, -1)$. This is robust to both forecasting horizons (2Q-ahead and 4Q-ahead) and using median or mean stock price forecasts.

Table A5: No cointegration between forecasts of $\log P$ and $\log C$

	Median 2Q ahead	Median 4Q ahead	Mean 2Q ahead	Mean 4Q ahead
<i>I(1) test statistics</i>				
PP (Z_t stat.)	-2.370	-2.328	-2.442	-2.321
10% critical value	-2.595	-2.595	-2.595	-2.595
DF-GLS	-0.794	-0.788	-0.808	-0.754
10% critical value	-1.903	-1.903	-1.903	-1.903

D Testing expectation formation in the AMB model with mean-reversion stock price belief specification

Turning to the generalized price belief specification, see p. 2401 in AMB. Agents are assumed to perceive prices to evolve according to the process

$$\Delta \log P_{t+1} = \log \beta_{t+1} + (1 - \eta_{PD}) (\log PD - \log P_t/D_t) + \log \epsilon_{t+1}, \quad (38)$$

$$\log \beta_{t+1} = (1 - \eta_\beta) \log \beta^D + \eta_\beta \log \beta_t + \log \nu_{t+1}, \quad (39)$$

where ϵ_{t+1} denotes a transitory shock to price growth and β_{t+1} a persistent price growth component. $\log PD$ denotes the perceived long-run mean of the log PD ratio and $\eta_{PD} \in [0, 1]$, $\eta_\beta \in [0, 1]$ are given parameters. Agents perceive the innovations $\ln \epsilon_{t+1}$ and $\ln \nu_{t+1}$ to be jointly normally distributed according to

$$\begin{pmatrix} \ln \epsilon_{t+1} \\ \ln \nu_{t+1} \end{pmatrix} \sim iiN \left(\begin{pmatrix} -\frac{\sigma_\epsilon^2}{2} \\ -\frac{\sigma_\nu^2}{2} \end{pmatrix}, \begin{pmatrix} \sigma_\epsilon^2 & 0 \\ 0 & \sigma_\nu^2 \end{pmatrix} \right). \quad (40)$$

Proposition 12 *Suppose agents' perceived law of motion for stock prices is (38) - (39), agents' forecasts of stock prices $E_i \log P_{i+j}$ are cointegrated with their forecasts of aggregate consumption $E_k \log C_{k+l}$ with cointegrating vector $(1, -1)$ for arbitrary $i, j, k, l > 0$.*

Proof. Given agents' perceived law of motion (18) - (19) and that $\log P_t/D_t$ is stationary, we have

$$E_i \log P_{i+j} = \log P_i + j \log \beta^D + s(i, j) \quad (41)$$

where $s(i, j)$ is a stationary term depending on the forecasting horizon j and time i variables or beliefs. The forecasts of consumption remain equation (21). With (41) and (21), we get

$$\begin{aligned} E_i \log P_{i+j} - E_k \log C_{k+l} &= (\log P_i + j \log \beta^D + s(i, j)) \\ &\quad - \left(\log D_k + l \log \beta^D + E_k \left(\log \left(1 + \frac{W_{k+l}}{D_{k+l}} \right) \right) \right) \\ &= (\log P_i - \log D_k) + s(i, j) \\ &\quad + (j - l) \log \beta^D - E_k \left(\log \left(1 + \frac{W_{k+l}}{D_{k+l}} \right) \right) \end{aligned}$$

Now given that $(\log P_i - \log D_k)$, $s(i, j)$ and $E_k \left(\log \left(1 + \frac{W_{k+l}}{D_{k+l}} \right) \right)$ are stationary, we get $(E_i \log P_{i+j} - E_k \log C_{k+l})$ is stationary. ■

In the AMB model with generalized belief specification, the way of specifying subjective stock price beliefs implies that stock price forecasts are cointegrated with consumption forecasts.