Testing frequency domain causality for cointegrated time series

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Abstract

In this article, we first review the frequency domain causality test of Breitung and Candelon (2006) [Journal of Econometrics 132, 363–378] for cointegrated time series. Then, based on the causality measure of Granger and Lin (1995) [Econometric Theory 11 (3), 530–536] and Yao and Hosoya (2000) [Journal of Econometrics 98, 225–255] in the vector error correction model, we propose an alternative frequency domain causality test procedure for cointegrated time series. The test is based on a set of linear hypotheses on the loading and short-run coefficients of the vector error correction model. Thus, the null hypothesis corresponding to the proposed test can be tested easily by the usual Wald test statistic. We also show that, in essence, the proposed test is the same as the test of Breitung and Candelon (2006) for cointegrated time series. Finally, the power properties of the test are investigated by Monte Carlo experiments. Meanwhile, as an illustration, we analyze the predictive power of long-term interest rates for U.S. real GDP in the frequency domain. We find the predictive power of long-term interest rates for U.S. real GDP has declined significantly since the mid-1980s.

JEL Classification: C12; C32; E43

Key Words: Frequency domain causality test; Cointegrated time series; Vector error correction model; Interest rates.

1 Introduction

In the seminal paper of Granger (1969), he suggested investigating the variables' causal relationship by cross-spectral methods in the frequency domain. Inspired by the work of Granger (1969), Geweke (1982) and Hosoya (1991) respectively introduced the frequency domain causality measure in stationary vector autoregressive (VAR) systems. Furthermore, Granger and Lin (1995) extended the work of Geweke (1982) and Hosoya (1991) and gave the frequency domain causality measure for a non-stationary cointegrated process in a bivariate vector error correction (VEC) model. Similar to Granger and Lin (1995), Yao and Hosoya (2000) proposed the frequency domain causality measure for cointegrated vector time series in high dimensional VEC models. Yao and Hosoya (2000) also developed Wald test statistics that can be used to statistically test causality in the frequency domain for cointegrated time series. However, their Wald test statistics are constructed by using the numerical differentiation method. This is mainly because the null hypothesis corresponding to the Wald test statistic involves a set of nonlinear restrictions on the coefficients of the VEC model.

Based on the frequency domain causality measure of Geweke (1982) and Hosoya (1991), Breitung and Candelon (2006) proposed a simple statistical test procedure for the non-causality hypothesis at a given frequency. Specifically, they developed a Wald test statistic for the null hypothesis based on a set of linear restrictions on the coefficients of the stationary VAR model. Their test procedure can be easily generalized to allow for cointegration relationships. Because of the simplicity of the test, it has been broadly applied to study the causal relationship between macroeconomic variables in the frequency domain (e.g., Assenmacher-Wesche *et al.* 2008; Bodart and Candelon 2009; Gradojevic 2012; Wei 2015; Satish 2016; Wei and Guo 2016; Stolbov 2017). Moreover, recently, Breitung and Schreiber (2018) extended the frequency domain causality test of Breitung and Candelon (2006) to a more general null hypothesis that allows causality testing at unknown frequencies within a pre-specified range of frequencies.

In this paper, similar to Breitung and Candelon (2006), we propose a simple frequency domain causality test procedure for cointegrated time series. However, unlike Breitung and Candelon (2006), our test procedure originates from the frequency domain causality measure of Granger and Lin (1995) and Yao and Hosoya (2000) in a VEC model. The test procedure is based on a set of linear hypotheses on the loading and short-run coefficients of the VEC model. Thus, similar to the conventional causality test, the null hypothesis corresponding to the proposed test procedure can be easily tested by using a Wald test statistic. Meanwhile, we show that, essentially, the proposed test is the same as the test of Breitung and Candelon (2006) for cointegrated time series.

The power of the proposed test is analyzed by Monte Carlo experiments. We find the test has reasonable size power properties and, the power of the test increases as the sample size increases. We also notice that the power of the test is closely related to the frequency being considered. When the frequency being tested approaches 0, the power of the test increases significantly. In contrast, when the frequency being tested approaches π, the power of the test is relatively low. In addition, we reveal that the power of the proposed test is quite sensitive to the loading coefficient of the VEC model. Meanwhile, we uncover that the power of the test based on the VEC model is similar to that of the test of Breitung and Candelon (2006) for cointegrated time series when the loading coefficient and the sample size are relatively large. However, when the loading coefficient and the sample size become small, the power of the two tests shows certain different patterns at low frequencies.

The proposed test procedure is quite suitable for testing the causality of the cointegrated time series at low and medium frequencies. Using the test, we analyze if the predictive power of long-term interest rates for U.S. real GDP has declined since the mid-1980s. We find strong predictive power of long-term interest rates for real GDP at low and high frequencies before the mid-1980s. However, after the mid-1980s, the predictive content of long-term interest rates for real GDP decreases significantly in the frequency domain.

The remainder of this article is organized as follows. In Section 2, we give a brief illustration on the frequency domain causality test of Breitung and Candelon (2006) for cointegrated time series. In Section 3, we propose the frequency domain causality test procedure for cointegrated time series in a bivariate VEC model. Section 4 examines the power properties of the proposed test. An application of the test is illustrated in Section 5. Section 6 concludes the paper.

2 Breitung and Candelon's (2006) frequency domain causality test for cointegrated time series

In this section, we give a brief introduction of the frequency domain causality test proposed by Breitung and Candelon (2006) in a bivariate stationary VAR model, and then illustrate how their test procedure can be used to test the frequency domain causality for cointegrated time series.

Let us consider a two-dimensional vector of time series $\boldsymbol{z}_t = [x_t, y_t]'$ observed at $t = 1, \ldots, T$. It is supposed that $\boldsymbol{z}_t$ has a finite-order VAR representation of the form

$$\boldsymbol{\Theta}(L)\boldsymbol{z}_t = \boldsymbol{\varepsilon}_t, \tag{1}$$

where $\boldsymbol{\Theta}(L) = \boldsymbol{I} - \boldsymbol{\Theta}_1 L - \cdots - \boldsymbol{\Theta}_p L^p$ is a 2×2 lag polynomial with $L^k \boldsymbol{z}_t = \boldsymbol{z}_{t-k}$. We also assume that the error vector $\boldsymbol{\varepsilon}_t$ is white noise with zero mean and positive definite covariance matrix $\boldsymbol{\Sigma}$. Furthermore, we let $\boldsymbol{G}$ be the lower triangular matrix of the Cholesky decomposition $\boldsymbol{G}'\boldsymbol{G} = \boldsymbol{\Sigma}^{-1}$, such that $E(\boldsymbol{\eta}_t \boldsymbol{\eta}_t') = \boldsymbol{I}$ and $\boldsymbol{\eta}_t = \boldsymbol{G}\boldsymbol{\varepsilon}_t$. Then, if the VAR system is assumed to be stationary, the moving average (MA) representation of the system can be derived as

$$\begin{aligned} \boldsymbol{z}_t = \boldsymbol{\Phi}(L)\boldsymbol{\varepsilon}_t &= \begin{bmatrix} \Phi_{11}(L) & \Phi_{12}(L) \\ \Phi_{21}(L) & \Phi_{22}(L) \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \\ = \boldsymbol{\Psi}(L)\boldsymbol{\eta}_t &= \begin{bmatrix} \Psi_{11}(L) & \Psi_{12}(L) \\ \Psi_{21}(L) & \Psi_{22}(L) \end{bmatrix} \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix}, \end{aligned} \tag{2}$$

where $\boldsymbol{\Phi}(L) = \boldsymbol{\Theta}(L)^{-1}$ and $\boldsymbol{\Psi}(L) = \boldsymbol{\Phi}(L)\boldsymbol{G}^{-1}$. Based on this representation, the measure of causality defined by Geweke (1982) and Hosoya (1991) can be expressed as

$$M_{y \to x}(\omega) = \log\left[1 + \frac{|\Psi_{12}(\tau)|^2}{|\Psi_{11}(\tau)|^2}\right], \tag{3}$$

where $\tau = e^{-\mathrm{i}\omega}$. Within this framework, if $M_{y \to x}(\omega) = 0$, we say that y does not cause x at frequency ω.

Breitung and Candelon (2006) try to test the hypothesis that y does not cause x at frequency ω by considering the null hypothesis

$$M_{y\to x}(\omega) = 0. \tag{4}$$

They propose a very simple test procedure for the above null hypothesis. Let $\theta_{12,k}$ denote the (1,2)-element of $\boldsymbol{\Theta}_k$ for $k = 1, \ldots, p$, Breitung and Candelon (2006) show that the null hypothesis $M_{y\to x}(\omega) = 0$ is equivalent to the linear restriction

$$H_0 : \boldsymbol{R\beta} = \boldsymbol{0}, \tag{5}$$

where $\boldsymbol{\beta} = [\theta_{12,1}, \theta_{12,2}, \ldots, \theta_{12,p}]'$ and

$$\boldsymbol{R} = \begin{bmatrix} \cos(\omega) & \cos(2\omega) & \cdots & \cos(p\omega) \\ \sin(\omega) & \sin(2\omega) & \cdots & \sin(p\omega) \end{bmatrix}.$$

The ordinary Wald test statistic for Eq. (5) is approximately distributed as $\chi^2(2)$ for $\omega \in (0, \pi)$.

The above test procedure can be easily extended to the cointegrated time series. Specifically, Breitung and Candelon (2006) argue that if the series x_t and y_t are $I(1)$ and cointegrated, then the autoregressive polynomial $\boldsymbol{\Theta}(L)$ has a unit root. The remaining roots are outside the unit circle. Subtracting $\boldsymbol{z}_{t-1}$ from both sides of Eq. (1) gives

$$\begin{aligned} \Delta \boldsymbol{z}_t &= (\boldsymbol{\Theta}_1 - \boldsymbol{I})\boldsymbol{z}_{t-1} + \boldsymbol{\Theta}_2 \boldsymbol{z}_{t-2} + \cdots + \boldsymbol{\Theta}_p \boldsymbol{z}_{t-p} + \boldsymbol{\varepsilon}_t \\ &= \widetilde{\boldsymbol{\Theta}}(L)\boldsymbol{z}_{t-1} + \boldsymbol{\varepsilon}_t, \end{aligned} \tag{6}$$

where $\widetilde{\boldsymbol{\Theta}}(L) = \boldsymbol{\Theta}_1 - \boldsymbol{I} + \boldsymbol{\Theta}_2 L + \cdots + \boldsymbol{\Theta}_p L^{p-1}$. Let $\widetilde{\boldsymbol{\Theta}}_1 = \boldsymbol{\Theta}_1 - \boldsymbol{I}$ and $\widetilde{\boldsymbol{\Theta}}_k = \boldsymbol{\Theta}_k$ for $k = 2, \cdots, p$. Meanwhile, let $\widetilde{\theta}_{12,k}$ denote the (1,2)-element of $\widetilde{\boldsymbol{\Theta}}_k$ for $k = 1, \cdots, p$. Then, based on Eq. (6), the null hypothesis that y does not cause x at frequency ω can be expressed as

$$H_0 : \boldsymbol{R}\widetilde{\boldsymbol{\beta}} = \boldsymbol{0}, \tag{7}$$

where $\widetilde{\boldsymbol{\beta}} = [\widetilde{\theta}_{12,1}, \widetilde{\theta}_{12,2}, \ldots, \widetilde{\theta}_{12,p}]'$ and

$$\boldsymbol{R} = \begin{bmatrix} \cos(\omega) & \cos(2\omega) & \cdots & \cos(p\omega) \\ \sin(\omega) & \sin(2\omega) & \cdots & \sin(p\omega) \end{bmatrix}.$$

The ordinary Wald test statistic for Eq. (7) is approximately distributed as $\chi^2(2)$ for $\omega \in (0, \pi)$. As argued in Breitung and Candelon (2006), this Wald test statistic can be used to statistically test the frequency domain causality between the two cointegrated time series.

3 An alternative frequency domain causality test procedure in cointegrated VAR systems

Breitung and Candelon's (2006) frequency domain causality test procedure for cointegrated time series is based on the causality measure of Geweke (1982) and Hosoya (1991) in a stationary VAR model. In this section, based on the causality measure of Granger and Lin (1995) and Yao and Hosoya (2000) in a VEC model, we propose an alternative frequency domain causality test procedure for cointegrated time series. In addition, we show that, in essence, these two test procedures are the same.

3.1 Frequency domain causality measure in cointegrated VAR systems

To simplify the terminology, following Lütkepohl (2006), we first define that a K-dimensional process $\boldsymbol{z}_t$ is integrated of order d, briefly, $\boldsymbol{z}_t \sim I(d)$, if $\Delta^d \boldsymbol{z}_t$ is stationary while $\Delta^{d-1}\boldsymbol{z}_t$ is non-stationary. Based on this definition, if there is just one $I(d)$ component in $\boldsymbol{z}_t$ and all other components are stationary, the vector $\boldsymbol{z}_t$ is still $I(d)$. Next, it is supposed that $\boldsymbol{z}_t = [x_t, y_t]'$ is $I(1)$ but $u_t = y_t - \beta_1 x_t$ is $I(0)$, then we can get the VEC representation of a bivariate cointegrated system

$$\begin{aligned} \Delta \boldsymbol{z}_t &= \boldsymbol{\Pi} \boldsymbol{z}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \boldsymbol{z}_{t-1} + \cdots + \boldsymbol{\Gamma}_{p-1} \Delta \boldsymbol{z}_{t-p+1} + \boldsymbol{\varepsilon}_t \\ &= \boldsymbol{\alpha}\boldsymbol{\beta}' \boldsymbol{z}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \boldsymbol{z}_{t-1} + \cdots + \boldsymbol{\Gamma}_{p-1} \Delta \boldsymbol{z}_{t-p+1} + \boldsymbol{\varepsilon}_t, \end{aligned} \tag{8}$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2]'$ is the loading matrix, $\boldsymbol{\beta} = [-\beta_1, 1]'$ is the cointegration matrix, $\boldsymbol{\Gamma}_1, \ldots, \boldsymbol{\Gamma}_{p-1}$ are short-run coefficient matrices. The error vector $\boldsymbol{\varepsilon}_t$ is white noise with $E(\boldsymbol{\varepsilon}_t) = \mathbf{0}$, $E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma}$ is positive definite. The VEC model can be further written as

$$\begin{bmatrix} \Delta x_t \\ \Delta y_t \end{bmatrix} = \begin{bmatrix} \alpha_1(y_{t-1} - \beta_1 x_{t-1}) \\ \alpha_2(y_{t-1} - \beta_1 x_{t-1}) \end{bmatrix} + \begin{bmatrix} \Gamma_{11}(L) & \Gamma_{12}(L) \\ \Gamma_{21}(L) & \Gamma_{21}(L) \end{bmatrix} \begin{bmatrix} \Delta x_t \\ \Delta y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}. \tag{9}$$

Let $\boldsymbol{G}$ be the lower triangular matrix associated with the Cholesky decomposition $\boldsymbol{G}'\boldsymbol{G} = \boldsymbol{\Sigma}^{-1}$ and $\boldsymbol{\eta}_t = \boldsymbol{G}\boldsymbol{\varepsilon}_t$, then $E(\boldsymbol{\eta}_t \boldsymbol{\eta}_t') = \boldsymbol{I}$. Based on this, according to Granger and Lin (1995), the above VEC model can be written as the Wold representation

$$\begin{aligned} \boldsymbol{z}_t = D(L)^{-1}\boldsymbol{A}(L)\boldsymbol{\varepsilon}_t = D(L)^{-1} \begin{bmatrix} A_{11}(L) & A_{12}(L) \\ A_{21}(L) & A_{22}(L) \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \\ = D(L)^{-1}\boldsymbol{B}(L)\boldsymbol{\eta}_t = D(L)^{-1} \begin{bmatrix} B_{11}(L) & B_{12}(L) \\ B_{21}(L) & B_{22}(L) \end{bmatrix} \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix}, \end{aligned} \tag{10}$$

where $D(L)$ is a complicated function arising from the inversion, $\boldsymbol{A}(L)$ and $\boldsymbol{B}(L)$ are 2×2 matrices with $\boldsymbol{B}(L) = \boldsymbol{A}(L)\boldsymbol{G}^{-1}$. Specifically, $\boldsymbol{A}(L)$ can be expressed as follows

$$\boldsymbol{A}(L) = \begin{bmatrix} (1-L)(1-\Gamma_{22}(L)) - \alpha_2 L & \alpha_1 L + \Gamma_{12}(L)(1-L) \\ (1-L)\Gamma_{21}(L) - \alpha_2\beta_1 L & \alpha_1\beta_1 L + (1-\Gamma_{11}(L))(1-L) \end{bmatrix}.$$

Based on Eq. (10), according to Granger and Lin (1995) and Yao and Hosoya (2000), the measure of causality running from y to x at frequency ω can be defined as

$$M_{y \to x}(\omega) = \log\left[1 + \frac{|B_{12}(\tau)|^2}{|B_{11}(\tau)|^2}\right], \tag{11}$$

where $\tau = e^{-\mathrm{i}\omega}$.

Let

$$\boldsymbol{G}^{-1} = \begin{bmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{bmatrix},$$

we can get $B_{11}(L) = A_{11}(L)g_{11} + A_{12}(L)g_{21}$ and $B_{12}(L) = A_{12}(L)g_{22}$. Accordingly, the frequency domain causality measure defined in Eq. (11) can be further written as

$$\begin{aligned} M_{y\to x}(\omega) &= \log\left[1 + \frac{|A_{12}(\tau)g_{22}|^2}{|A_{11}(\tau)g_{11} + A_{12}(\tau)g_{21}|^2}\right] \\ &= \log\left[1 + \frac{|\alpha_1\tau + \Gamma_{12}(\tau)(1-\tau)|^2 g_{22}^2}{|\{(1-\tau)(1-\Gamma_{22}(\tau)) - \alpha_2\tau\}g_{11} + \{\alpha_1\tau + \Gamma_{12}(\tau)(1-\tau)\}g_{21}|^2}\right], \end{aligned} \tag{12}$$

where, again, $\tau = e^{-\mathrm{i}\omega}$.

3.2 Test procedures in cointegrated VAR systems

The above analysis indicates that the null hypothesis that y dose not cause x at frequency ω can be expressed as

$$M_{y\to x}(\omega) = 0. \tag{13}$$

Yao and Hosoya (2000) first propose a test procedure for the null hypothesis based on the causality measure defined in Eq. (11). Let $\boldsymbol{\varphi} = vec([\boldsymbol{\alpha}, \boldsymbol{\Gamma}_1, \ldots, \boldsymbol{\Gamma}_{p-1}, \boldsymbol{\Sigma}])$, since the causality measure $M_{y\to x}(\omega)$ in Eq. (11) can be viewed as a function of β_1 and $\boldsymbol{\varphi}$ for a fixed ω, it can be expressed alternatively as $M_{y\to x}(\beta_1, \boldsymbol{\varphi}|\omega)$. Meanwhile, let $\hat{\beta}_1$ and $\hat{\boldsymbol{\varphi}}$ denote the maximum likelihood (ML) estimate of β_1 and $\boldsymbol{\varphi}$, Johansen(1988, 1991) shows that $\sqrt{T}(\hat{\boldsymbol{\varphi}} - \boldsymbol{\varphi})$ has a multivariate normal distribution as $T \to \infty$, and therefore $M_{y\to x}(\hat{\beta}_1, \hat{\boldsymbol{\varphi}}|\omega)$ is a $\sqrt{T}$ consistent estimate of $M_{y\to x}(\beta_1, \boldsymbol{\varphi}|\omega)$. Using delta method, Yao and Hosoya (2000) illustrate that

$$\sqrt{T}[M_{y\to x}(\hat{\beta}_1, \hat{\boldsymbol{\varphi}}|\omega) - M_{y\to x}(\beta_1, \boldsymbol{\varphi}|\omega)] = \boldsymbol{D}'_{\beta_1,\boldsymbol{\varphi}}\sqrt{T}(\hat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) + o_p(1), \tag{14}$$

where $\boldsymbol{D}_{\beta_1,\boldsymbol{\varphi}}$ is the vector of derivatives of $M_{y\to x}(\beta_1, \boldsymbol{\varphi}|\omega)$ with respect to $[\beta_1, \boldsymbol{\varphi}']'$. Based on Eq. (14), Yao and Hosoya (2000) show that the Wald test statistic for Eq. (13) can be expressed as

$$W = T[M_{y\to x}(\hat{\beta}_1, \hat{\boldsymbol{\varphi}}|\omega)]^2 / \boldsymbol{H}(\hat{\beta}_1, \hat{\boldsymbol{\varphi}}), \tag{15}$$

where $\boldsymbol{H}(\hat{\beta}_1, \hat{\boldsymbol{\varphi}}) = \boldsymbol{D}'_{\hat{\beta}_1,\hat{\boldsymbol{\varphi}}}\boldsymbol{V}(\hat{\boldsymbol{\varphi}})\boldsymbol{D}_{\hat{\beta}_1,\hat{\boldsymbol{\varphi}}}$ and $\boldsymbol{V}(\hat{\boldsymbol{\varphi}})$ is the asymptotic variance-covariance matrix of $\sqrt{T}(\hat{\boldsymbol{\varphi}} - \boldsymbol{\varphi})$. This Wald test statistic is asymptotically distributed as $\chi^2(1)$ for $\omega \in (0, \pi)$.

However, in practice, the derivative $\boldsymbol{D}_{\hat{\beta}_1,\hat{\boldsymbol{\varphi}}}$ is difficult to evaluate owing to its non-linearity. Consequently, Yao and Hosoya (2000) propose a numerical differentiation instead of the exact analytic expression for $\boldsymbol{D}_{\hat{\beta}_1,\hat{\boldsymbol{\varphi}}}$. Accordingly, the Wald test statistic W defined in Eq. (15) is expressed in a numerical differentiation form.

In the present paper, similar to Breitung and Candelon (2006), we develop a much simpler approach to test the null hypothesis $M_{y\to x}(\omega) = 0$ in a cointegrated VAR system. According to Eq. (12), we can observe that $M_{y\to x}(\omega) = 0$ if $|\alpha_1 e^{-\mathrm{i}\omega} + \Gamma_{12}(e^{-\mathrm{i}\omega})(1 - e^{-\mathrm{i}\omega})|^2 g_{22}^2 = 0$. Meanwhile, note that g_{22} is positive because $\boldsymbol{\Sigma}$ is assumed to be positive definite. Thus, we can conclude that $M_{y\to x}(\omega) = 0$ if $|\alpha_1 e^{-\mathrm{i}\omega} + \Gamma_{12}(e^{-\mathrm{i}\omega})(1 - e^{-\mathrm{i}\omega})| = 0$. It follows that y does not cause x at frequency ω if

$$|\alpha_1 e^{-\mathrm{i}\omega} + \Gamma_{12}(e^{-\mathrm{i}\omega})(1 - e^{-\mathrm{i}\omega})| = 0. \tag{16}$$

Accordingly, let $\gamma_{12,k}$ denote the $(1,2)$-element of Γ_k for $k = 1, \ldots, p-1$, then a necessary and sufficient set of conditions for $|\alpha_1 e^{-\mathrm{i}\omega} + \Gamma_{12}(e^{-\mathrm{i}\omega})(1 - e^{-\mathrm{i}\omega})| = 0$ is

$$\alpha_1 \cos(\omega) + \sum_{k=1}^{p-1} \gamma_{12,k}(\cos(k\omega) - \cos((k+1)\omega)) = 0, \tag{17}$$

$$\alpha_1 \sin(\omega) + \sum_{k=1}^{p-1} \gamma_{12,k}(\sin(k\omega) - \sin((k+1)\omega)) = 0. \tag{18}$$

Therefore, the null hypothesis that y dose not cause x at frequency ω can be expressed as the linear restriction

$$H_0 : \boldsymbol{U}\boldsymbol{\gamma} = 0, \tag{19}$$

where $\boldsymbol{\gamma} = [\alpha_1, \gamma_{12,1}, \gamma_{12,2}, \cdots, \gamma_{12,p-1}]$ and

$$\boldsymbol{U} = \begin{bmatrix} \cos(\omega) & \cos(\omega) - \cos(2\omega) & \cos(2\omega) - \cos(3\omega) & \cdots & \cos((p-1)\omega) - \cos(p\omega) \\ \sin(\omega) & \sin(\omega) - \sin(2\omega) & \sin(2\omega) - \sin(3\omega) & \cdots & \sin((p-1)\omega) - \sin(p\omega) \end{bmatrix}.$$

Here, we need to note that the lag length p should be greater than two. This is because when $p = 1$ or 2, Eq. (19) has no nonzero solution for $\omega \in (0, \pi)$. From this linear restriction, we can observe that, if $\omega = 0$, the null hypothesis is equivalent to the condition $\alpha_1 = 0$, which exactly corresponds to the test of long-run causality (e.g., Toda and Phillips 1993; Bruneau and Jondeau 1999; Caporale and pittis 1999; Breitung and Candelon 2006; Breitung and Schreiber 2018).

To test the null hypothesis expressed in Eq. (19), we note that the linear restrictions corresponding to the null hypothesis do not involve β_1. Accordingly, we can first estimate β_1 by the ML procedure, and thus get a super-consistent estimator $\hat{\beta}_1$. Then, we consider the following model

$$\Delta x_t = \alpha_1(y_{t-1} - \hat{\beta}_1 x_{t-1}) + \sum_{k=1}^{p-1} \gamma_{11,k} \Delta x_{t-k} + \sum_{k=1}^{p-1} \gamma_{12,k} \Delta y_{t-k} + \varepsilon_{1t}. \tag{20}$$

Here, we let $\hat{\boldsymbol{\gamma}} = (\boldsymbol{S}_2' \boldsymbol{Q}_{\boldsymbol{S}_1} \boldsymbol{S}_2)^{-1} \boldsymbol{S}_2' \boldsymbol{Q}_{\boldsymbol{S}_1} \boldsymbol{x}$, $\hat{\sigma}^2_{\boldsymbol{\varepsilon}_1} = \boldsymbol{\varepsilon}_1' \boldsymbol{\varepsilon}_1 / (T-p)$ with $\boldsymbol{Q}_{\boldsymbol{S}_1} = \boldsymbol{I}_{T-p} - \boldsymbol{S}_1(\boldsymbol{S}_1'\boldsymbol{S}_1)^{-1}\boldsymbol{S}_1'$, $\boldsymbol{\varepsilon}_1 = \boldsymbol{Q}_{\boldsymbol{S}_1}\boldsymbol{x} - \boldsymbol{Q}_{\boldsymbol{S}_1}\boldsymbol{S}_2\hat{\gamma}$, $\boldsymbol{x} = [\Delta x_{p+1}, \ldots, \Delta x_T]'$, $\boldsymbol{S}_1 = [\boldsymbol{s}_{1,p+1}, \ldots, \boldsymbol{s}_{1,T}]'$ with $\boldsymbol{s}_{1,t}' = [\Delta x_{t-1}, \ldots, \Delta x_{t-p+1}]$, and $\boldsymbol{S}_2 = [\boldsymbol{s}_{2,p+1}, \ldots, \boldsymbol{s}_{2,T}]'$ with $\boldsymbol{s}_{2,t}' = [y_{t-1} - \hat{\beta}_1 x_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-p+1}]$, then the hypothesis, $H_0 : \boldsymbol{U}\boldsymbol{\gamma} = 0$, can be tested by using a Wald test statistic such as

$$W = (\boldsymbol{U}\hat{\boldsymbol{\gamma}})'[\hat{\sigma}^2_{\boldsymbol{\varepsilon}_1} \boldsymbol{U}(\boldsymbol{S}_2' \boldsymbol{Q}_{\boldsymbol{S}_1} \boldsymbol{S}_2)^{-1} \boldsymbol{U}']^{-1} \boldsymbol{U}\hat{\boldsymbol{\gamma}}. \tag{21}$$

Under the null hypothesis, the Wald test statistic W is asymptotically distributed as $\chi^2(2)$ for $\omega \in (0, \pi)$.

Finally, we show that the proposed test is the same as the test of Breitung and Candelon (2006) for cointegrated time series. To see this, we rewrite Eq. (8) as follows

$$\Delta \boldsymbol{z}_t = (\boldsymbol{\alpha}\boldsymbol{\beta}' + \boldsymbol{\Gamma}_1)\boldsymbol{z}_{t-1} + (\boldsymbol{\Gamma}_2 - \boldsymbol{\Gamma}_1)\boldsymbol{z}_{t-2} + \cdots + (\boldsymbol{\Gamma}_{p-1} - \boldsymbol{\Gamma}_{p-2})\boldsymbol{z}_{t-p+1} - \boldsymbol{\Gamma}_{p-1}\boldsymbol{z}_{t-p} + \boldsymbol{\varepsilon}_t. \tag{22}$$

Based on this equation, according to Breitung and Candelon (2006), the null hypothesis that y does not cause x at frequency ω can be expressed as

$$(\alpha_1 + \gamma_{12,1})\cos(\omega) + \sum_{k=2}^{p-1} (\gamma_{12,k} - \gamma_{12,k-1})\cos(k\omega) - \gamma_{12,p-1}\cos(p\omega) = 0, \tag{23}$$

$$(\alpha_1 + \gamma_{12,1})\sin(\omega) + \sum_{k=2}^{p-1} (\gamma_{12,k} - \gamma_{12,k-1})\sin(k\omega) - \gamma_{12,p-1}\sin(p\omega) = 0. \tag{24}$$

The above two linear restrictions Eq. (23) and Eq. (24) can be further expressed as

$$\alpha_1 \cos(\omega) + \sum_{k=1}^{p-1} \gamma_{12,k}(\cos(k\omega) - \cos((k+1)\omega)) = 0, \tag{25}$$

$$\alpha_1 \sin(\omega) + \sum_{k=1}^{p-1} \gamma_{12,k}(\sin(k\omega) - \sin((k+1)\omega)) = 0. \tag{26}$$

We can observe that Eq. (25) and Eq. (26) are just the null hypothesis expressed in Eq. (19). Consequently, we can conclude that, in spirit, the proposed test is the same as the test of Breitung and Candelon (2006) for cointegrated time series.

4 Power of the test

To investigate the local power of the proposed frequency domain causality test for cointegrated time series, we consider the following model

$$\Delta x_t = \alpha_1(y_{t-1} - \beta_1 x_{t-1}) + \gamma_{12,1}\Delta y_{t-1} + \gamma_{12,2}\Delta y_{t-2} + \varepsilon_{1t}, \tag{27}$$

$$\Delta y_t = \alpha_2(y_{t-1} - \beta_1 x_{t-1}) + \varepsilon_{2t}, \tag{28}$$

where $\gamma_{12,1} = \frac{2\cos(\omega)-1}{2-2\cos(\omega)}\alpha_1$ and $\gamma_{12,2} = \frac{-1}{2-2\cos(\omega)}\alpha_1$. The error vector $\boldsymbol{\varepsilon}_t$ is white noise with $E(\boldsymbol{\varepsilon}_t) = \mathbf{0}$ and

$$E(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t') = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.5 \end{bmatrix}.$$

In this model, $\boldsymbol{\gamma} = [\alpha_1, \frac{2\cos(\omega)-1}{2-2\cos(\omega)}\alpha_1, \frac{-1}{2-2\cos(\omega)}\alpha_1]$ and

$$\boldsymbol{U} = \begin{bmatrix} \cos(\omega) & \cos(\omega) - \cos(2\omega) & \cos(2\omega) - \cos(3\omega) \\ \sin(\omega) & \sin(\omega) - \sin(2\omega) & \sin(2\omega) - \sin(3\omega) \end{bmatrix}$$

such that $\boldsymbol{U\gamma} = \mathbf{0}$ for $\omega \in (0, \pi)$.[1] Consequently, y does not cause x at frequency $\omega \in (0, \pi)$. Thus, for a specified frequency ω in Eq. (27), we can generate data according to Eq. (27) and Eq. (28). Then, for the generated data, y is not a cause of x only at the specified frequency ω.

We investigate the power of the test by means of Monte Carlo simulations. Thus,

[1] See Appendix for details.

we simulate the rejection frequencies for the proposed frequency domain causality test based on 10,000 replications of the process in Eq. (27) and Eq. (28), with the sample size $T = 100$ and $T = 300$. We choose $\beta_1 = 4$, $\alpha_1 = 0.5$, $\alpha_2 = 0.02$. Meanwhile, the 0.05 significance level is employed. In Figure 1, we report the simulated rejection frequencies for $\omega = \{j\pi/8; j = 1, \ldots, 8\}$. Since we find the power of the test is symmetric around $\omega = \pi$ for $\omega \in (0, 2\pi)$, we do not present the simulated rejection frequencies for $\omega \in (\pi, 2\pi)$.

Four interesting insights emerge from Figure 1. First, the proposed test fails to distinguish causal links at frequencies that are close to each other. Second, the minimum value of all graphs is very close to 0.05, which suggests that the test has reasonable size properties. Third, the power of the test increases as the sample size increases. Fourth, the power of the test is closely related to the frequency being considered. When the frequency being tested is close to 0, the power of the test increases significantly. On the contrary, when the frequency being tested is close to π, the power of the test is relatively low. This may be partly because the spectrum of the integrated series is mainly concentrated in low frequencies, and partly because there exists a long-run equilibrium relationship between the cointegrated series. Fortunately, in practice, for the cointegrated time series, we are mainly interested in the casual relationship at low frequencies. Therefore, the low power of test at high frequencies does not have largely negative effects on the application of the test.

Since the proposed test is closely related to the loading coefficient α_1, we further investigate the power properties of the test when the loading coefficient α_1 is small. Consequently, we set $\alpha_1 = 0.05$, with $\beta_1 = 4$, $\alpha_2 = 0.02$, and then simulate the rejection frequencies for the proposed frequency domain causality test based on 10,000 replications of the process in Eq. (27) and Eq. (28). Similarly, the 0.05 significance level is employed. From Figure 2, we notice that when α_1 decreases from 0.5 to 0.05, the power of the test becomes relatively low at high frequencies, whatever the sample size $T = 100$ or $T = 300$. This suggests that the power of the proposed test is quite sensitive to the loading coefficient of the VEC model.

As pointed out previously, the proposed test is consist with the test of Breitung and Candelon (2006) for cointegrated time series. Consequently, the power of the two tests may be similar. Therefore, to go deeper, we further verify if the power of the proposed test is similar to that of the test of Breitung and Candelon (2006) for cointegrated time series. Thus, we simulate the rejection frequencies for the two kinds of frequency domain causality tests based on 10,000 replications of the process in Eq. (27) and Eq. (28), with sample size $T = 100$ and $T = 300$. Again, we set $\beta_1 = 4$, $\alpha_1 = 0.5$ (or 0.05), $\alpha_2 = 0.02$, with the significance level being equal to 0.05. Here, for the data generated in Eq. (27) and Eq. (28), the test of Breitung and Candelon (2006) can be conducted in the following VAR model

$$\Delta \boldsymbol{z}_t = \widetilde{\boldsymbol{\Theta}}_1 \boldsymbol{z}_{t-1} + \widetilde{\boldsymbol{\Theta}}_2 \boldsymbol{z}_{t-2} + \widetilde{\boldsymbol{\Theta}}_3 \boldsymbol{z}_{t-3} + \boldsymbol{\varepsilon}_t, \tag{29}$$

where $\Delta \boldsymbol{z}_t = [\Delta x_t, \Delta y_t]'$, $\widetilde{\boldsymbol{\Theta}}_1, \widetilde{\boldsymbol{\Theta}}_2$ and $\widetilde{\boldsymbol{\Theta}}_3$ are the coefficient matrices and $\boldsymbol{\varepsilon}_t$ is the error vector. The simulation results are presented in Figure 3 and Figure4. From Figure 3, we can observe that when the loading coefficient $\alpha_1 = 0.5$, the power properties of the two tests are almost the same, whatever the sample $T = 100$ or $T = 300$. Figure 4 shows that when α_1 decreases from 0.5 to 0.05 while the sample size $T = 300$, the power properties of the two tests are still quite similar. However, when $\alpha_1 = 0.05$ while the sample size $T = 100$, the power of the two tests shows quite different patterns. Compared with the test of Breitung and Candelon (2006), the test based on the VEC model seems to be more likely to reject the null hypothesis at low frequencies. As a consequence, on the one hand, the test based on the VEC model tends to perform better than the test of Breitung and Candelon (2006) at low frequencies. On the other hand, when the frequency ω corresponding to the non-causality is small, the test based on the VEC model may lead to over-rejection of the null hypothesis of non-causality at ω. This phenomenon is further confirmed by the following empirical applications of the two types of tests.

5 Empirical applications

A great number of studies have shown that the relationship between interest rate spreads and U.S. real economic activity has significantly diminished from the mid-1980s (e.g., Bordo and Haubrich 2008; Giacomini and Rossi 2006; Morell 2018). In light of this finding, using frequency domain causality tests, this section attempts to analyze the predictive power of long-term interest rates for U.S. real GDP during the period from

1960Q1 to 2003Q4. Meanwhile, we split the full sample into two sub-samples 1960Q1-1984Q4 and 1985Q1-2003Q4 and compare if the predictive content of long-term interest rates for U.S. real GDP is the same in the two sub-periods. The data set used in this empirical analysis includes U.S. 10-year long-term government bond yields (R) and the real GDP. Plots of these two series are shown in Figure 5.

Table 1 indicates that both R and real GDP are $I(1)$ for the three different samples. Consequently, we apply Johansen's (1998) cointegration tests to investigate if R and real GDP are cointegrated. The cointegration tests are performed in VAR systems with no deterministic part. The results listed in Table 2 show that R and real GDP are cointegrated at a 5% significance level, which is consistent with the finding of Jardet (2004). We thus conduct frequency domain causality tests in bivariate VEC models. For comparison, we also carry out Breitung and Candelon's (2006) frequency domain causality tests for cointegrated time series.

Figure 6 graphs the results of the frequency domain causality tests for the full sample. Both of the two types of tests reveal that long-term interest rates are able to predict real GDP at frequencies below 1.28, corresponding to periods longer than 4.91 quarters. From these results, we can see that long-term interest rates are a powerful indicator of U.S. real GDP in the long run.

Next, we investigate if the predictive power of long-term interest rates for U.S. real GDP is similar during the two subsamples 1960Q1-1984Q4 and 1985Q1-2003Q4. The corresponding empirical results are presented in Figure 7 and Figure 8. For the first subsample 1960Q1-1984Q4, both of the two kinds of tests uncover that long-term interest rates can predict real GDP at frequencies less than about 2.59, corresponding to periods longer than 2.43 quarters. However, for the second subsample 1960Q1-1984Q4, the frequency domain causality test based on the VEC model reveals that long-term interest rates can predict real GDP at frequencies below 0.25, corresponding to periods longer than 25.13 quarters. On the contrary, the frequency domain causality test of Breitung and Candelon (2006) shows that long-term interest rates have no predictive power for real GDP at any frequencies.

From the above results, we can observe that the predictive power of long-term interest rates for U.S. real GDP has decreased substantially since the mid-1980s. In addition, we note that the frequency domain causality test based on the VEC model is more likely to reject the null hypothesis of no causality at low frequencies than the test of Breitung and Candelon (2006), which is consistent with the above power analysis results of the two types of tests.

6 Conclusions

Based on the work of Granger and Lin (1995) and Yao and Hosoya (2000), this paper proposes a simple test procedure that can be used to statistically test the frequency domain causality for cointegrated time series. Unlike the frequency domain causality test of Breitung and Candelon (2006) for cointegrated time series, the null hypothesis of the proposed test is based on a set of linear restrictions on the loading and short-run coefficients of the VEC model. Since the null hypothesis does not involve the cointegration vector, it can be easily tested by the usual Wald test statistic. Furthermore, we show that, in essential, the proposed test is the same as the test of Breitung and Candelon (2006) for cointegrated time series.

The power of the proposed test is studied by means of Monte Carlo simulations. It is shown that the test has reasonable size properties.. As the sample size increases, the power of the test increases significantly. Meanwhile, the test has very high power at low frequencies. However, the power of the proposed test is quite sensitive to the loading coefficient of the VEC model. When the loading coefficient becomes small, the power of the test decreases significantly at high frequencies. We further compare the power of the proposed test with the test of Breitung and Candelon (2006) for cointegrated times series. We observe that the power properties of the two tests are quite similar. Nevertheless, when the loading coefficient and the sample size are small, the power of the two tests shows quite different patterns. Compared with the test of Breitung and Candelon (2006), the test based on the VEC model seems to be more likely to reject the null hypothesis at low frequencies.

We apply the frequency domain causality tests to analyze the predictive power of long-term interest rates for U.S. real GDP. We find long-term interest rates are a good predictor of U.S. real GDP at low and medium frequencies before the mid-1980s. Nevertheless, after the mid-1980s, the predictive content of long-term interest rates for U.S. real GDP has declined significantly.

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Appendix

Proof of $\boldsymbol{U}(\omega)\boldsymbol{\gamma} = \mathbf{0}$ in Eq. (27) and Eq. (28)

When $p = 3$, $\boldsymbol{U}(\omega)\boldsymbol{\gamma} = \mathbf{0}$ can be expressed as

$$\begin{bmatrix} \cos(\omega_0) & \cos(\omega_0) - \cos(2\omega_0) & \cos(2\omega_0) - \cos(3\omega_0) \\ \sin(\omega_0) & \sin(\omega_0) - \sin(2\omega_0) & \sin(2\omega_0) - \sin(3\omega_0) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \gamma_{12,1} \\ \gamma_{12,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A.1}$$

Eq. (A.1) can be further written as

$$\begin{bmatrix} (\alpha_1 + \gamma_{12,1})\cos(\omega_0) + (\gamma_{12,2} - \gamma_{12,1})\cos(2\omega_0) - \gamma_{12,2}\cos(3\omega_0)) \\ (\alpha_1 + \gamma_{12,1})\sin(\omega_0) + (\gamma_{12,2} - \gamma_{12,1})\sin(2\omega_0) - \gamma_{12,2}\sin(3\omega_0)) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A.2}$$

Based on Eq. (A.2), $\boldsymbol{U}(\omega)\boldsymbol{\gamma} = \mathbf{0}$ can be expressed alternatively as

$$\begin{bmatrix} \cos(\omega_0) & \cos(2\omega_0) & \cos(3\omega_0) \\ \sin(\omega_0) & \sin(2\omega_0) & \sin(3\omega_0) \end{bmatrix} \begin{bmatrix} \alpha_1 + \gamma_{12,1} \\ \gamma_{12,2} - \gamma_{12,1} \\ -\gamma_{12,2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A.3}$$

Let $-\gamma_{12,2} = \nu$, $\alpha_1 + \gamma_{12,1} = m_1$ and $\gamma_{12,2} - \gamma_{12,1} = m_2$, the above system of linear equations above can be represented as

$$\begin{aligned} \cos(\omega_0)m_1 + \cos(2\omega_0)m_2 &= -\nu\cos(3\omega_0), \\ \sin(\omega_0)m_1 + \sin(2\omega_0)m_2 &= -\nu\sin(3\omega_0). \end{aligned}$$

By using formulas such as

$$\begin{aligned} &\cos(2\omega_0) = 2\cos(\omega_0)^2 - 1, \quad \sin(2\omega_0) = 2\sin(\omega_0)\cos(\omega_0), \\ &\cos(3\omega_0) = 4\cos(\omega_0)^3 - 3\cos(\omega_0), \quad \sin(3\omega_0) = 3\sin(\omega_0) - 4\sin(\omega_0)^3, \end{aligned}$$

we can see $m_1 = \nu$ and $m_2 = -2\nu\cos(\omega_0)$. By solving the equations

$$
\begin{aligned}
-\gamma_{12,2} &= \nu \\
\alpha_1 + \gamma_{12,1} &= \nu \\
\gamma_{12,2} - \gamma_{12,1} &= -2\nu\cos(\omega_0).
\end{aligned}
$$

we can get $\nu = \frac{1}{2-2\cos(\omega_0)}\alpha_1$, $\gamma_{12,1} = \frac{2\cos(\omega_0)-1}{2-2\cos(\omega_0)}\alpha_1$ and $\gamma_{12,2} = \frac{-1}{2-2\cos(\omega_0)}\alpha_1$.

Table 1: Unit root tests

Sample	Variable	ADF	95% level	PP	95% level
1960Q1-2003Q4					
	Real GDP	-1.215074	-2.878015	-1.003815	-2.877919
	R	-1.550792	-2.878311	-1.571474	-2.877919
1960Q1-1984Q4					
	Real GDP	-2.107422	-3.456319	-1.860700	-3.455842
	R	-0.853264	-2.891234	-0.610584	-2.890926
1985Q1-2003Q4					
	Real GDP	-2.408633	-3.472558	-1.781965	-3.470851
	R	-2.147924	-2.901217	-2.586919	-2.900670

Note: This table presents unit root test statistics. ADF: Augmented Dickey–Fuller; PP: Phillips–Perron.

Table 2: Johansen cointegration tests

	H_0	Trace	95% level	$\lambda_{\max}$	95% level
1960Q1-2003Q4					
	$r \leq 0$	23.387	12.321	21.545	11.225
	$r \leq 1$	1.842	4.130	1.842	4.130
1960Q1-1984Q4					
	$r \leq 0$	17.559	12.321	17.366	11.225
	$r \leq 1$	0.193	4.130	0.193	4.130
1985Q1-2003Q4					
	$r \leq 0$	14.738	12.321	11.348	11.225
	$r \leq 1$	3.391	4.130	3.391	4.130

Note: This table presents results of the Johansen cointegration tests. r is the cointegration rank.

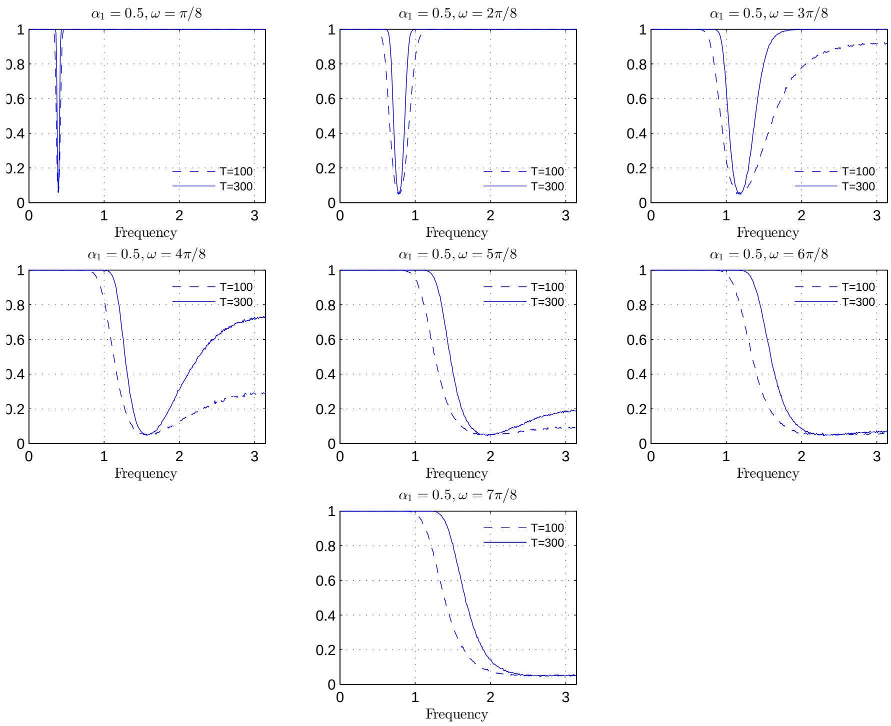


Figure 1: Empirical power of the frequency domain causality test based on the VEC model. The significance level corresponding to the rejection frequencies is 0.05.

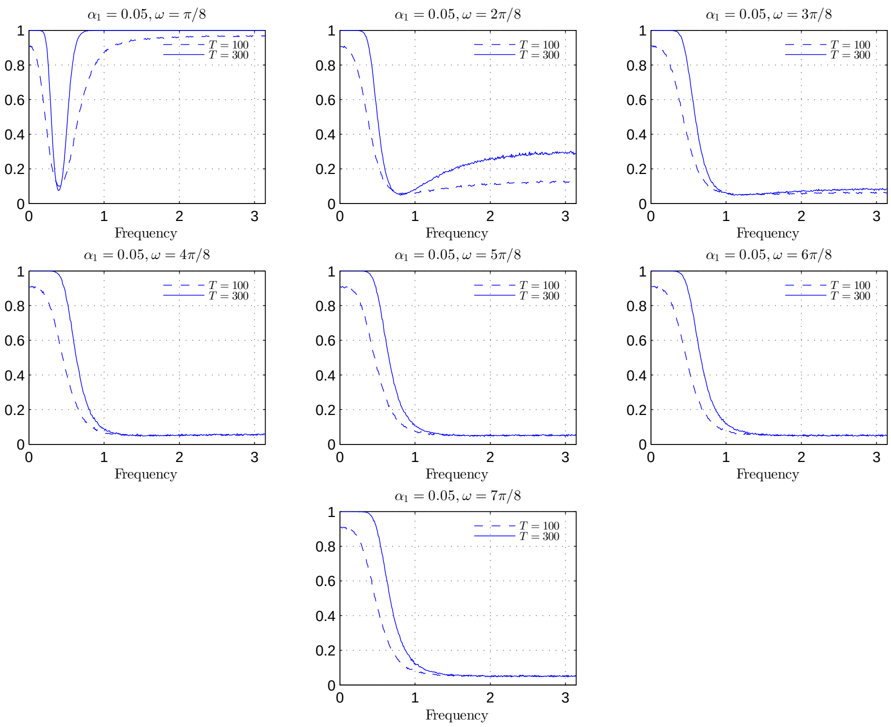


Figure 2: Empirical power of the frequency domain causality test based on the VEC model. The significance level corresponding to the rejection frequencies is 0.05.

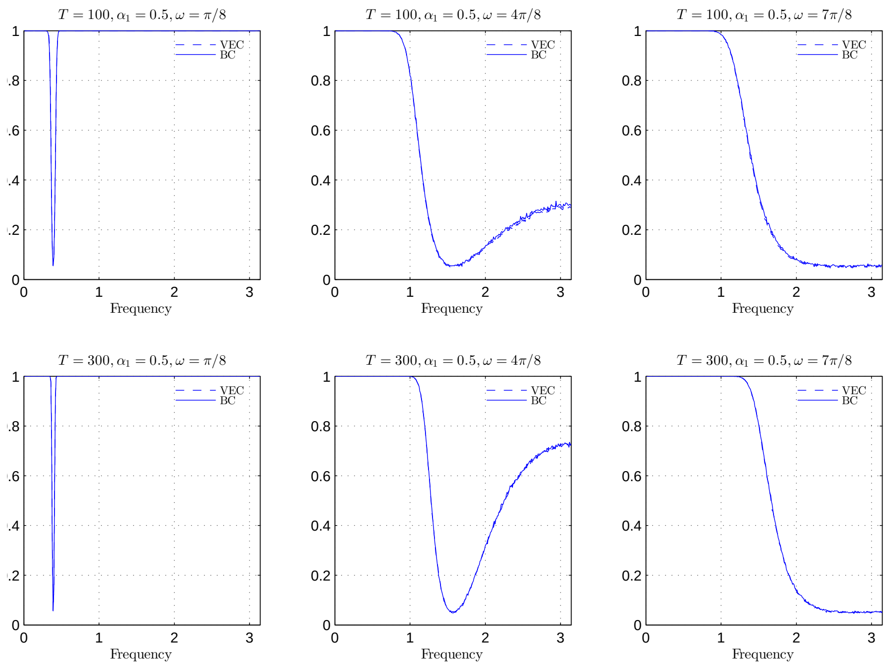


Figure 3: Empirical power of the frequency domain causality test for cointegrated time series. The significance level corresponding to the rejection frequencies is 0.05. VEC: the frequency domain causality test based on the VEC model. BC: the frequency domain causality test of Breitung and Candelon (2006) for cointegrated time series.

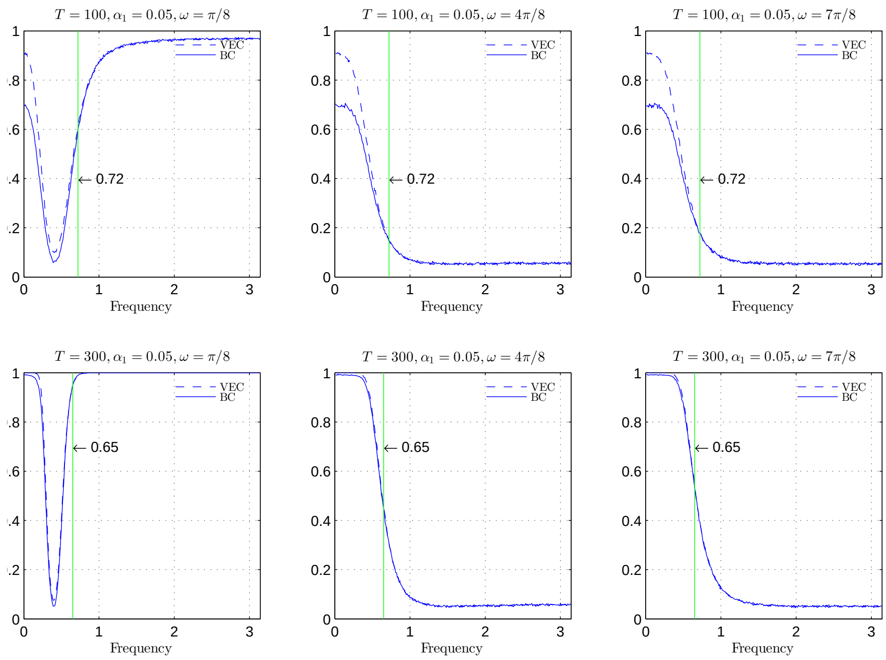


Figure 4: Empirical power of the frequency domain causality test for cointegrated time series. The significance level corresponding to the rejection frequencies is 0.05. The green vertical line indicates a specified frequency. When the frequency being tested is larger than the specified frequency, the rejection frequency of the test based on the VEC model is greater than that of the test of Breitung and Candelon (2006) for cointegrated time series. VEC: the frequency domain causality test based on the VEC model. BC: the frequency domain causality test of Breitung and Candelon (2006) for cointegrated time series.

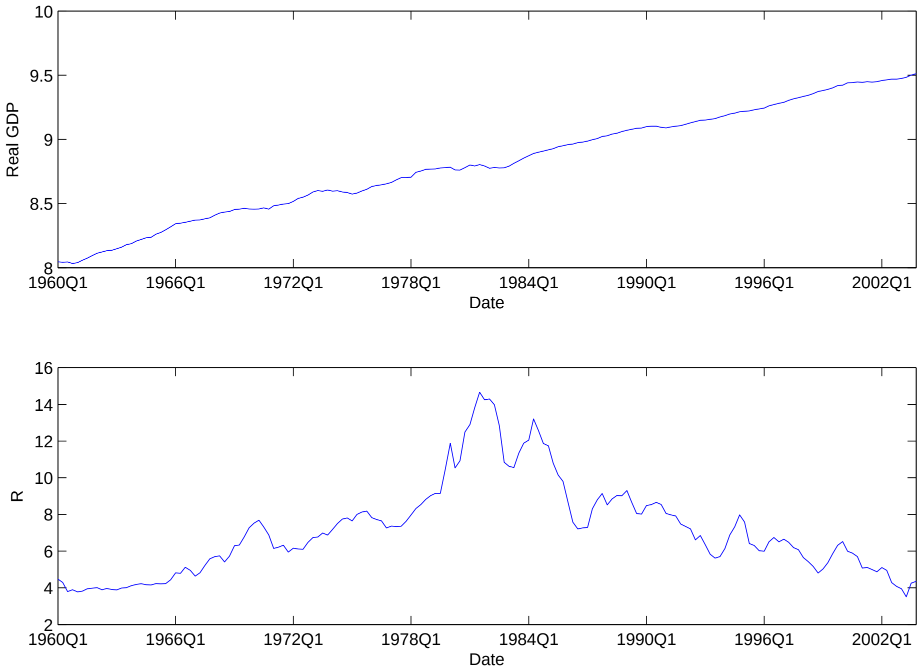


Figure 5: Plots of real GDP and R. The real GDP has been taken in logarithms.

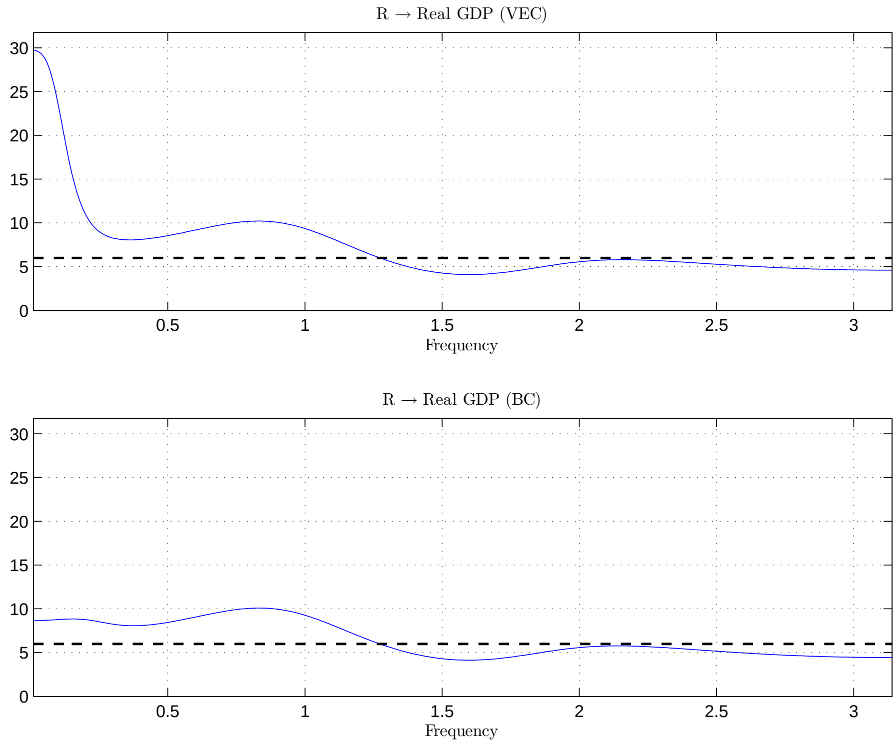
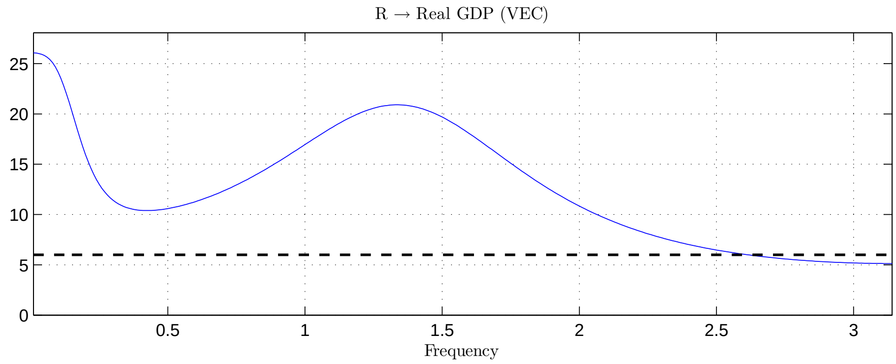


Figure 6: Frequency domain causality running from R to real GDP (1960Q1-2003Q1). The significance level corresponding to the rejection frequencies is 0.05. VEC: the frequency domain causality test based on the VEC model. BC: the frequency domain causality test of Breitung and Candelon (2006) for cointegrated time series.



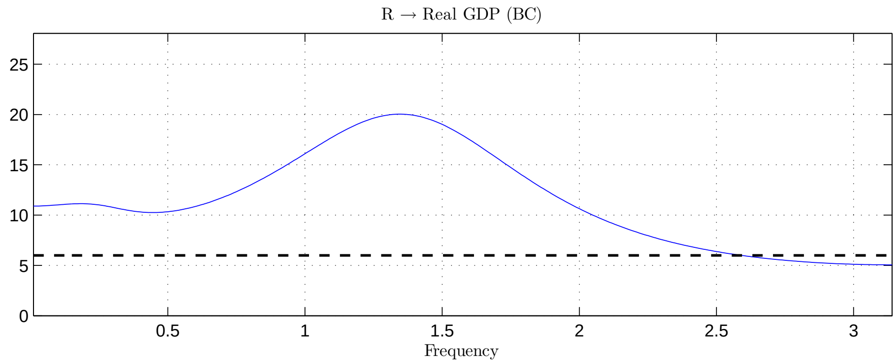


Figure 7: Frequency domain causality running from real GDP to R (1960Q1-1984Q4). The significance level corresponding to the rejection frequencies is 0.05. VEC: the frequency domain causality test based on the VEC model. BC: the frequency domain causality test of Breitung and Candelon (2006) for cointegrated time series.

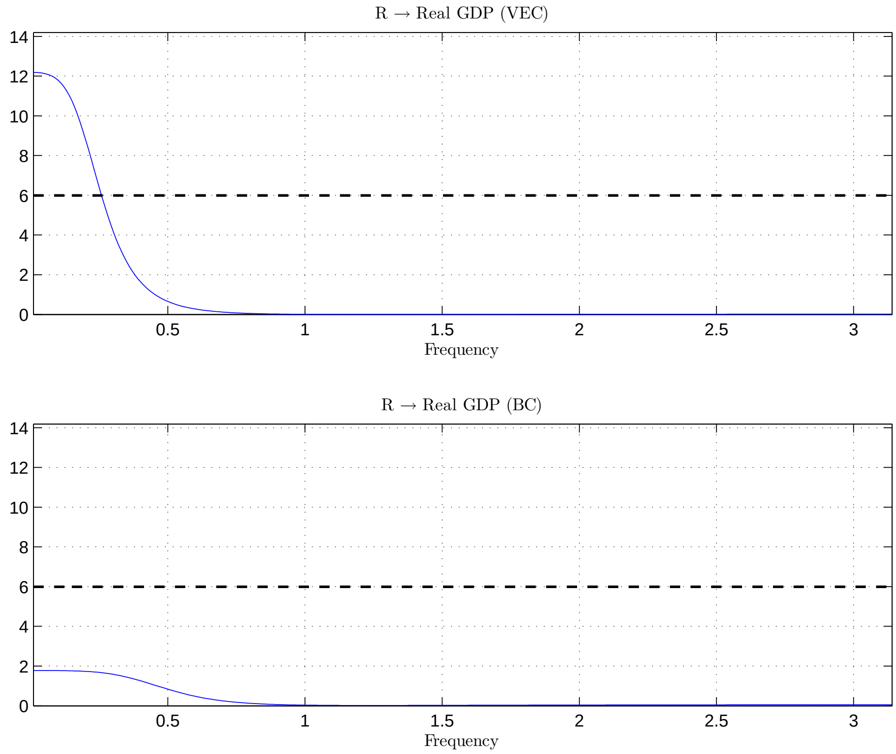


Figure 8: Frequency domain causality running from real GDP to R (1985Q1-2003Q4). The significance level corresponding to the rejection frequencies is 0.05. VEC: the frequency domain causality test based on the VEC model. BC: the frequency domain causality test of Breitung and Candelon (2006) for cointegrated time series.