An Alternative Explicit Formula for the Hodrick–Prescott Filter in Finite Sample*

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September 2017

Abstract

By applying the Sherman–Morrison–Woodbury (SMW) formula and a discrete cosine transformation matrix, De Jong and Sakarya (2016, *Review of Economics and Statistics*, Vol. 98, No. 2, pp. 310–317) recently derived an explicit formula for the Hodrick–Prescott filter. More recently, by applying the SMW formula and the spectral decomposition of a symmetric tridiagonal Toeplitz matrix, Cornea-Madeira (2017, *Review of Economics and Statistics*, Vol. 99, pp. 314–318) provided a simpler formula. This paper provides an alternative simpler formula for it and explains the reason why our approach leads to a simpler formula. Wang et al. (2015, *Journal of Computational and Applied Mathematics*, Vol. 278, No. 15, pp. 12–18) recently developed a method for deriving the explicit inverse of a pentadiagonal (five-diagonal) Toeplitz matrix. Our approach may be regarded as an application of Wang et al. (2015).

2010 Mathematics Subject Classification: 62G05

JEL Classification: C22

Key Words: Hodrick–Prescott filter; Explicit inverse; Pentadiagonal Toeplitz matrix; Whittaker–Henderson method of graduation

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*The Japan Society for the Promotion of Science supported this work through KAKENHI Grant Number 16H03606.

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1 Introduction

The Hodrick–Prescott (HP) (1997) filter is a popular method to estimate the trend component of univariate time series.[1] It is described as a penalized least squares problem and a special case of the Whittaker–Henderson (WH) method of graduation:[2]

$$
\begin{aligned}
\widehat{\boldsymbol{x}} &= \underset{x_1,\ldots,x_T}{\arg\min} \left[\sum_{t=1}^{T}(y_t - x_t)^2 + \alpha \sum_{t=3}^{T}(\Delta^2 x_t)^2\right] \\
&= \underset{\boldsymbol{x}}{\arg\min} \left(\|\boldsymbol{y} - \boldsymbol{x}\|^2 + \alpha\|\boldsymbol{D}\boldsymbol{x}\|^2\right) \\
&= (\boldsymbol{I}_T + \alpha \boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{y}, \qquad (1)
\end{aligned}
$$

where $\boldsymbol{y} = [y_1, \ldots, y_T]'$ denotes univariate time series T observations, $\boldsymbol{x} = [x_1, \ldots, x_T]'$, $\alpha > 0$ is a smoothing/tuning parameter, $\boldsymbol{I}_m$ is the $m \times m$ identity matrix, and $\boldsymbol{D}$ denotes the $(T-2)\times T$ second-order difference matrix such that $\boldsymbol{D}\boldsymbol{x} = [\Delta^2 x_3, \ldots, \Delta^2 x_T]'$ with $\Delta^2 x_t = x_t - 2x_{t-1} + x_{t-2}$ for $t = 3, \ldots, T$. Explicitly, $\boldsymbol{D}$ is the $(T-2)\times T$ Toeplitz matrix of which the first and last rows are $[1, -2, 1, 0, \ldots, 0]$ and $[0, \ldots, 0, 1, -2, 1]$, respectively.

Letting $f(\boldsymbol{x}) = \|\boldsymbol{y}-\boldsymbol{x}\|^2 + \alpha\|\boldsymbol{D}\boldsymbol{x}\|^2$, it follows that $f(\boldsymbol{x}) = f(\widehat{\boldsymbol{x}}) + (\boldsymbol{x}-\widehat{\boldsymbol{x}})'(\boldsymbol{I}_T + \alpha\boldsymbol{D}'\boldsymbol{D})(\boldsymbol{x}-\widehat{\boldsymbol{x}})$. Since $\boldsymbol{I}_T + \alpha\boldsymbol{D}'\boldsymbol{D}$ is a positive-definite matrix, $\widehat{\boldsymbol{x}}$ is the unique global minimizer of $f(\boldsymbol{x})$. In addition, it is well-known that since $\boldsymbol{D}\boldsymbol{\iota} = \boldsymbol{0}$, it follows that $(\boldsymbol{I}_T + \alpha\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{\iota} = \boldsymbol{\iota}$ and then $\boldsymbol{\iota}'\widehat{\boldsymbol{x}}/T = \boldsymbol{\iota}'\boldsymbol{y}/T$, where $\boldsymbol{\iota} = [1, \ldots, 1]'$ is a T-dimensional column vector. [Note that $(\boldsymbol{I}_T + \alpha\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{\iota} = \boldsymbol{\iota}$ may be derived from that (a) $(\boldsymbol{I}_T + \alpha\boldsymbol{D}'\boldsymbol{D})\boldsymbol{\iota} = \boldsymbol{\iota}$ and (b) $(\boldsymbol{I}_T + \alpha\boldsymbol{D}'\boldsymbol{D})$ is a nonsingular matrix. Likewise, $(\boldsymbol{I}_T + \alpha\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{\tau} = \boldsymbol{\tau}$ follows, where $\boldsymbol{\tau} = [1, \ldots, T]'$.]

The first-order condition (FOC) for the HP filter consists of T equations, $(\boldsymbol{I}_T + \alpha\boldsymbol{D}'\boldsymbol{D})\widehat{\boldsymbol{x}} = \boldsymbol{y}$, of which $(T-4)$ equations are expressed by[3]

$$
\left[1 + \alpha(1-B)^2(1-F)^2\right]\widehat{x}_t = y_t, \quad t = 3, \ldots, T-2, \qquad (2)
$$

[1] Yamada (2017a) clarifies the reason why the trend extracted by the HP filter seems to be more plausible than the linear trend.

[2] In the actuarial sciences, the WH method of graduation has been used for mortality table construction. For a historical survey, see, e.g., Weinert (2007) and Phillips (2010).

[3] King and Rebelo (1993) derived the gain function corresponding to $1 - [1 + \alpha(1-B)^2(1-F)^2]^{-1}$, which is $W(\omega) = [4\alpha(1-\cos\omega)^2]/[1 + 4\alpha(1-\cos\omega)^2]$. See also Baxter and King (1999) and Gómez (2001).

where $\widehat{\boldsymbol{x}} = [\widehat{x}_1, \ldots, \widehat{x}_T]'$, $B\widehat{x}_t = \widehat{x}_{t-1}$, $F\widehat{x}_t = \widehat{x}_{t+1}$, and $BF = 1$. Based on (2), McElroy (2008) provided exact formulas of the HP filter. However, as pointed out in De Jong and Sakarya (2016), (2) is only a part of the FOC.

De Jong and Sakarya (2016, Theorem 1) provided an explicit formula for the HP filter, following which, Cornea-Madeira (2017, Theorem 1) provided a simpler explicit formula. Both of these works applied the Sherman–Morrison–Woodbury (SMW) formula to the following form of matrix:

$$(\boldsymbol{\Omega} + \alpha\boldsymbol{\zeta}_1\boldsymbol{\zeta}_1' + \alpha\boldsymbol{\zeta}_2\boldsymbol{\zeta}_2')^{-1},$$

where $\boldsymbol{\Omega}$ is a nonsingular matrix whose inverse is easily obtainable, and both $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ are column vectors. In this paper, we provide a simpler alternative formula for the HP filter. The reason such a simpler formula is obtainable is that in our approach both $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ are *unit vectors*. Recently, Wang et al. (2015) developed a method for deriving the explicit inverse of a pentadiagonal (five-diagonal) Toeplitz matrix. Our approach may be regarded as an application of Wang et al. (2015).

The paper is organized as follows. In Section 2, we provide a literature review. In Section 3, we show another explicit formula for the HP filter. In Section 4, we discuss a possible modification of the HP filter. Section 5 concludes.

2 A literature review

In this section, we briefly review two closely related papers: De Jong and Sakarya (2016) and Cornea-Madeira (2017).

2.1 De Jong and Sakarya (2016)

Let $\boldsymbol{x} = \boldsymbol{\Gamma}\boldsymbol{\theta}$, where $\boldsymbol{\Gamma}$ is a $T \times T$ nonsingular matrix and $\boldsymbol{\theta}$ is a T-dimensional column vector. Then, the HP filter defined by (1) may be represented as $\widehat{\boldsymbol{x}} = \boldsymbol{\Gamma}\widehat{\boldsymbol{\theta}}$, where

$$\begin{aligned} \widehat{\boldsymbol{\theta}} &= \arg\min_{\boldsymbol{\theta}} \left(\|\boldsymbol{y} - \boldsymbol{\Gamma}\boldsymbol{\theta}\|^2 + \alpha\|\boldsymbol{D}\boldsymbol{\Gamma}\boldsymbol{\theta}\|^2\right) \\ &= (\boldsymbol{\Gamma}'\boldsymbol{\Gamma} + \alpha\boldsymbol{\Gamma}'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}'\boldsymbol{y}. \end{aligned} \tag{3}$$

This representation was used in, e.g., Paige and Trindade (2010), which derived a ridge regression (Hoerl and Kennard, 1970) representation of the HP filter.[4] De Jong and Sakarya (2016) considered the case where

$$\boldsymbol{\Gamma} = \begin{bmatrix} \sqrt{\frac{1}{T}} & \sqrt{\frac{2}{T}}\cos\left(\frac{(2-1)(1-0.5)\pi}{T}\right) & \cdots & \sqrt{\frac{2}{T}}\cos\left(\frac{(T-1)(1-0.5)\pi}{T}\right) \\ \sqrt{\frac{1}{T}} & \sqrt{\frac{2}{T}}\cos\left(\frac{(2-1)(2-0.5)\pi}{T}\right) & \cdots & \sqrt{\frac{2}{T}}\cos\left(\frac{(T-1)(2-0.5)\pi}{T}\right) \\ \vdots & \vdots & & \vdots \\ \sqrt{\frac{1}{T}} & \sqrt{\frac{2}{T}}\cos\left(\frac{(2-1)(T-0.5)\pi}{T}\right) & \cdots & \sqrt{\frac{2}{T}}\cos\left(\frac{(T-1)(T-0.5)\pi}{T}\right) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \\ \vdots \\ \boldsymbol{\gamma}_T \end{bmatrix} \tag{4}$$

and showed that $\boldsymbol{\Gamma}'\boldsymbol{\Gamma} = \boldsymbol{I}_T$ and $\boldsymbol{\Gamma}'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{\Gamma} = \boldsymbol{\Sigma} - \boldsymbol{p}_1\boldsymbol{p}_1' - \boldsymbol{p}_T\boldsymbol{p}_T'$, where $\boldsymbol{\Sigma} = \mathsf{diag}(0, \sigma_2^2, \ldots, \sigma_T^2)$,

$$\sigma_j = 4\sin^2\left(\frac{(j-1)\pi}{2T}\right), \quad j = 2, \ldots, T,$$

$\boldsymbol{p}_1$ and $\boldsymbol{p}_T$ are T-dimensional column vectors such that $\boldsymbol{p}_1 = (\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)' = [0, p_{1,2}, \ldots, p_{1,T}]'$ and $\boldsymbol{p}_T = (\boldsymbol{\gamma}_T - \boldsymbol{\gamma}_{T-1})' = [0, p_{T,2}, \ldots, p_{T,T}]'$, where

$$p_{1,j} = \sqrt{\frac{32}{T}}\sin^2\left(\frac{(j-1)\pi}{2T}\right)\cos\left(\frac{(j-1)\pi}{2T}\right), \quad j = 2, \ldots, T, \tag{5}$$

$$p_{T,j} = \sqrt{\frac{32}{T}}\sin^2\left(\frac{(j-1)\pi}{2T}\right)\cos\left(\frac{(j-1)(T-0.5)\pi}{T}\right), \quad j = 2, \ldots, T. \tag{6}$$

For the proofs of (5) and (6), see the Appendix. It is noteworthy that $\boldsymbol{\Gamma}$ is an orthogonal matrix that represents a discrete cosine transformation (DCT-II) (Ahmed et al., 1974).[5] Accordingly,

[4] See also Kim et al. (2009) and Yamada (2015), the former of which gave a lasso (least absolute shrinkage and selection operator) regression (Tibshirani, 1996) representation of the ℓ_1 trend filter and the latter of which provided ridge regression representations of the WH method of graduation.

[5] See also Hamming (1973), Bierens (1997), and Strang (1999).

it follows that

$$\widehat{\boldsymbol{\theta}} = (\boldsymbol{A} - \alpha \boldsymbol{p}_1 \boldsymbol{p}_1' - \alpha \boldsymbol{p}_T \boldsymbol{p}_T')^{-1} \boldsymbol{\Gamma}' \boldsymbol{y}, \tag{7}$$

where $\boldsymbol{A} = \boldsymbol{I}_T + \alpha \boldsymbol{\Sigma}$. Since $\boldsymbol{A}$ is a diagonal matrix, $\boldsymbol{A}^{-1}$ is easily obtainable. By applying the SMW formula to $(\boldsymbol{A} - \alpha \boldsymbol{p}_1 \boldsymbol{p}_1' - \alpha \boldsymbol{p}_T \boldsymbol{p}_T')^{-1}$ in (7), De Jong and Sakarya (2016) derived an explicit formula of the HP filter.

2.2 Cornea-Madeira (2017)

Let $\boldsymbol{Q}_m$ denote the $m \times m$ symmetric tridiagonal Toeplitz matrix where the first row is $[2, -1, 0, \ldots, 0]$, which is a well-known matrix (Strang and MacNamara, 2014), and $\boldsymbol{Q}_m = \boldsymbol{G}_m \boldsymbol{\Lambda}_m \boldsymbol{G}_m'$ denotes its spectral decomposition.[6] Letting $\boldsymbol{q}_1 = [-2, 1, 0, \ldots, 0]'$ be a T-dimensional column vector and $\boldsymbol{q}_T = \boldsymbol{J}_T \boldsymbol{q}_1$, where $\boldsymbol{J}_m$ is the $m \times m$ exchange matrix, it follows that

$$\boldsymbol{D}' \boldsymbol{D} = \boldsymbol{Q}_T^2 - \boldsymbol{q}_1 \boldsymbol{q}_1' - \boldsymbol{q}_T \boldsymbol{q}_T', \tag{8}$$

which indicates

$$\widehat{\boldsymbol{x}} = (\boldsymbol{B} - \alpha \boldsymbol{q}_1 \boldsymbol{q}_1' - \alpha \boldsymbol{q}_T \boldsymbol{q}_T')^{-1} \boldsymbol{y}, \tag{9}$$

where $\boldsymbol{B} = (\boldsymbol{I}_T + \alpha \boldsymbol{Q}_T^2) = \boldsymbol{G}_T (\boldsymbol{I}_T + \alpha \boldsymbol{\Lambda}_T^2) \boldsymbol{G}_T'$. Since $\boldsymbol{I}_T + \alpha \boldsymbol{\Lambda}_T^2$ is a diagonal matrix and $\boldsymbol{G}_T$ is an orthogonal matrix, $\boldsymbol{B}^{-1} = \boldsymbol{G}_T (\boldsymbol{I}_T + \alpha \boldsymbol{\Lambda}_T^2)^{-1} \boldsymbol{G}_T'$, which is easy to calculate. By applying the SMW formula to $(\boldsymbol{B} - \alpha \boldsymbol{q}_1 \boldsymbol{q}_1' - \alpha \boldsymbol{q}_T \boldsymbol{q}_T')^{-1}$ in (9), Cornea-Madeira (2017) derived an explicit formula of the HP filter.

3 Another explicit formula for the HP filter

The product of any two tridiagonal Toeplitz matrices is not a pentadiagonal Toeplitz matrix because the first and the last entries in the principal diagonal are different to the other ones (Marr and Vineyard, 1988; Montaner and Alfaro, 1995; Diele and Lopez, 1998; Wang et al.,

[6] For the explicit forms of $\boldsymbol{\Lambda}_m$ and $\boldsymbol{G}_m$, see (16) and (17).

2015). Accordingly, $\boldsymbol{Q}_{T-2}^2$ is not a pentadiagonal Toeplitz matrix. Explicitly, it is

$$\boldsymbol{Q}_{T-2}^2 = \begin{bmatrix} 5 & -4 & 1 & 0 & \cdots & 0 \\ -4 & 6 & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 6 & -4 \\ 0 & \cdots & 0 & 1 & -4 & 5 \end{bmatrix}. \tag{10}$$

Interestingly, the corresponding pentadiagonal Toeplitz matrix to $\boldsymbol{Q}_{T-2}^2$ is $\boldsymbol{DD}'$ and their relationship may be expressed as

$$\boldsymbol{DD}' = \boldsymbol{Q}_{T-2}^2 + \boldsymbol{UU}', \tag{11}$$

where

$$\boldsymbol{U} = [\boldsymbol{e}_1, \boldsymbol{e}_{T-2}]. \tag{12}$$

Here, $\boldsymbol{I}_{T-2} = [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{T-2}]$. Note that (11) corresponds to (3.16) of Wang et al. (2015).

By applying the SMW formula to $(\boldsymbol{I}_T + \alpha \boldsymbol{D}'\boldsymbol{D})^{-1}$ in (1), the HP filter may be alternatively expressed as[7]

$$\widehat{\boldsymbol{x}} = \boldsymbol{y} - \boldsymbol{D}'(\alpha^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{DD}')^{-1}\boldsymbol{Dy} = \boldsymbol{y} - \boldsymbol{D}'\boldsymbol{\Psi}^{-1}\boldsymbol{Dy}, \tag{13}$$

where $\boldsymbol{\Psi} = \alpha^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{DD}'$, which is a pentadiagonal Toeplitz matrix. From (11), $\boldsymbol{\Psi}$ may be represented as $\boldsymbol{\Psi} = \boldsymbol{C} + \boldsymbol{UU}'$, where $\boldsymbol{C} = \alpha^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{Q}_{T-2}^2$. As in Wang et al. (2015), applying

[7] It is of interest that a ridge regression exists in (13): $\widehat{\boldsymbol{x}} = \boldsymbol{y} - \boldsymbol{D}'\widehat{\boldsymbol{\phi}}$, where

$$\widehat{\boldsymbol{\phi}} = \arg\min_{\boldsymbol{\phi}} \left(\|\boldsymbol{y} - \boldsymbol{D}'\boldsymbol{\phi}\|^2 + \alpha^{-1}\|\boldsymbol{\phi}\|^2\right) = (\boldsymbol{DD}' + \alpha^{-1}\boldsymbol{I}_{T-2})^{-1}\boldsymbol{Dy}.$$

Yamada (2017b) listed several penalized/unpenalized least squares problems related to the HP filter.

the SMW formula to $(\boldsymbol{C}+\boldsymbol{U}\boldsymbol{U}')^{-1}$, it follows that

$$\boldsymbol{\Psi}^{-1} = (\boldsymbol{C}+\boldsymbol{U}\boldsymbol{U}')^{-1} = \boldsymbol{C}^{-1} - \boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}. \tag{14}$$

It is noteworthy that $\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U}$ in (14) is a 2×2 matrix and hence its inverse is easily obtainable.[8] In addition, for obtaining the explicit formula for $\boldsymbol{C}^{-1}$, it is possible to apply the spectral decomposition of $\boldsymbol{Q}_{T-2}$ as in Cornea-Madeira (2017):

$$\boldsymbol{C}^{-1} = \boldsymbol{G}_{T-2}(\alpha^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{\Lambda}_{T-2}^2)^{-1}\boldsymbol{G}_{T-2}', \tag{15}$$

where $(\alpha^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{\Lambda}_{T-2}^2)^{-1}$ in $\boldsymbol{C}^{-1} = \boldsymbol{G}_{T-2}(\alpha^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{\Lambda}_{T-2}^2)^{-1}\boldsymbol{G}_{T-2}'$ is a diagonal matrix, where $\boldsymbol{\Lambda}_{T-2} = \mathsf{diag}(\lambda_1, \ldots, \lambda_{T-2})$ is

$$\lambda_j = 4\sin^2\left(\frac{j\pi}{2(T-1)}\right), \quad j = 1, \ldots, T-2, \tag{16}$$

and (i,j)-entry of $\boldsymbol{G}_{T-2}$, denoted by $g_{i,j}$, is

$$g_{i,j} = \sqrt{\frac{2}{T-1}}\sin\left(\frac{ij\pi}{T-1}\right), \quad i,j = 1, \ldots, T-2. \tag{17}$$

See, e.g., Strang and MacNamara (2014).

We may summarize the above results as follows:

Theorem 3.1. $\widehat{\boldsymbol{x}}$ *in* (1) *may be expressed as*

$$\begin{aligned}
\widehat{\boldsymbol{x}} &= (\boldsymbol{I}_T + \alpha\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{y} \\
&= \left[\boldsymbol{I}_T - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D} + \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{D}\right]\boldsymbol{y} \qquad (18) \\
&= \boldsymbol{y} - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{y} + \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{y}, \qquad (19)
\end{aligned}$$

where $\boldsymbol{U}$ *and* $\boldsymbol{C}^{-1}$ *are defined by* (12) *and* (15)*, respectively.*

Proof. (18) immediately follows from (13) and (14). □

[8] Likewise, letting $\boldsymbol{V} = [\boldsymbol{q}_1, \boldsymbol{q}_T]$, it follows that $\boldsymbol{V}\boldsymbol{V}' = \boldsymbol{q}_1\boldsymbol{q}_1' + \boldsymbol{q}_T\boldsymbol{q}_T'$, and accordingly, (9) becomes $\widehat{\boldsymbol{x}} = (\boldsymbol{B} - \alpha\boldsymbol{V}\boldsymbol{V}')^{-1}\boldsymbol{y}$. The proof of Cornea-Madeira (2017) may become simpler by applying the SMW formula to $(\boldsymbol{B} - \alpha\boldsymbol{V}\boldsymbol{V}')^{-1}$. See the Appendix for details.

Remarks. Since $\boldsymbol{U}\boldsymbol{U}' = \boldsymbol{e}_1\boldsymbol{e}_1' + \boldsymbol{e}_{T-2}\boldsymbol{e}_{T-2}'$, the trend extracted by the HP filter may be rewritten as $\widehat{\boldsymbol{x}} = \boldsymbol{y} - \boldsymbol{D}'(\boldsymbol{C} + \boldsymbol{e}_1\boldsymbol{e}_1' + \boldsymbol{e}_{T-2}\boldsymbol{e}_{T-2}')^{-1}\boldsymbol{D}\boldsymbol{y}$. Then, it is possible to obtain the result in Theorem 3.1 by applying the SMW formula to $(\boldsymbol{C} + \boldsymbol{e}_1\boldsymbol{e}_1' + \boldsymbol{e}_{T-2}\boldsymbol{e}_{T-2}')^{-1}$. Nevertheless, in this case, its derivation becomes longer.

Denote (i,j)-entry of $\boldsymbol{C}^{-1}$ in (15) by $c_{i,j}$. In addition, let $\boldsymbol{c}_1$ and $\boldsymbol{c}_{T-2}$ denote the first and last column of $\boldsymbol{C}^{-1}$, respectively. Then, since $\boldsymbol{e}_i'\boldsymbol{C}^{-1}\boldsymbol{e}_j = c_{i,j}$ for $i,j = 1, T-2$ and $\boldsymbol{C}^{-1}\boldsymbol{U} = [\boldsymbol{C}^{-1}\boldsymbol{e}_1, \boldsymbol{C}^{-1}\boldsymbol{e}_{T-2}] = [\boldsymbol{c}_1, \boldsymbol{c}_{T-2}]$, it follows that

$$(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1} = \frac{1}{(1+c_{1,1})(1+c_{T-2,T-2}) - c_{1,T-2}c_{T-2,1}}\begin{bmatrix} 1+c_{T-2,T-2} & -c_{1,T-2} \\ -c_{T-2,1} & 1+c_{1,1}\end{bmatrix}$$

and we accordingly obtain

$$\begin{aligned}&\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1} \\ &\quad= \frac{(1+c_{T-2,T-2})\boldsymbol{c}_1\boldsymbol{c}_1' - c_{1,T-2}\boldsymbol{c}_1\boldsymbol{c}_{T-2}' - c_{T-2,1}\boldsymbol{c}_{T-2}\boldsymbol{c}_1' + (1+c_{1,1})\boldsymbol{c}_{T-2}\boldsymbol{c}_{T-2}'}{(1+c_{1,1})(1+c_{T-2,T-2}) - c_{1,T-2}c_{T-2,1}},\end{aligned} \tag{20}$$

where $\boldsymbol{C}^{-1} = [\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{T-2}] = [c_{i,j}]_{i,j=1,\ldots,T-2}$ is calculated by

$$c_{i,j} = \sum_{k=1}^{T-2} \frac{g_{i,k}g_{j,k}}{\alpha^{-1} + \lambda_k^2}, \quad i,j = 1,\ldots,T-2. \tag{21}$$

From (10), $\boldsymbol{Q}_{T-2}^2$ is a centrosymmetric matrix and accordingly $\boldsymbol{C} = \alpha^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{Q}_{T-2}^2$ is also a centrosymmetric matrix. Since the inverse of a nonsingular centrosymmetric matrix is also a centrosymmetric matrix (Graybill, 2001, Theorem 8.15.7), $\boldsymbol{C}^{-1}$ is a centrosymmetric matrix. Then, it follows that $c_{1,1} = c_{T-2,T-2}$, $c_{1,T-2} = c_{T-2,1}$, and $\boldsymbol{c}_{T-2} = \boldsymbol{J}_{T-2}\boldsymbol{c}_1$.

Combining (18) and (20), it follows that

$$\begin{aligned}\widehat{\boldsymbol{x}} &= \boldsymbol{y} - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{y} \\ &\quad + \boldsymbol{D}'\left(\frac{(1+c_{T-2,T-2})\boldsymbol{c}_1\boldsymbol{c}_1' - c_{1,T-2}\boldsymbol{c}_1\boldsymbol{c}_{T-2}' - c_{T-2,1}\boldsymbol{c}_{T-2}\boldsymbol{c}_1' + (1+c_{1,1})\boldsymbol{c}_{T-2}\boldsymbol{c}_{T-2}'}{(1+c_{1,1})(1+c_{T-2,T-2}) - c_{1,T-2}c_{T-2,1}}\right)\boldsymbol{D}\boldsymbol{y},\end{aligned}$$

Denote (i,j)-entry of $\boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}$ by $\xi_{i,j}$ for $i,j = 1,\ldots,T$ and i-th entry of $\boldsymbol{D}'\boldsymbol{c}_j$ for $j = 1, T-2$

by $\upsilon_i^{(j)}$ for $i = 1, \ldots, T$. Then, it follows that

$$\xi_{i,j} = \sum_{k=1}^{T-2} \frac{(\Delta^2 g_{i,k})(\Delta^2 g_{j,k})}{\alpha^{-1} + \lambda_k^2}, \quad i, j = 1, \ldots, T, \tag{22}$$
$$\upsilon_i^{(j)} = \sum_{k=1}^{T-2} \frac{(\Delta^2 g_{i,k}) g_{j,k}}{\alpha^{-1} + \lambda_k^2}, \quad i = 1, \ldots, T, \quad j = 1, T-2,$$

where $g_{-1,j} = g_{0,j} = g_{T-1,j} = g_{T,j} = 0$ for $j = 1, \ldots, T-2$ and these are introduced for notational convenience.

Accordingly, we obtain the following result:

Corollary 3.2. *Let $z_{i,j}$ denote (i,j)-entry of $(\boldsymbol{I}_T + \alpha \boldsymbol{D}'\boldsymbol{D})^{-1}$ in* (1)*. Then, $z_{i,j}$ is expressed as*

$$z_{i,j} = \delta_{i,j} - \xi_{i,j} + \mu_{i,j}, \quad i, j = 1, \ldots, T, \tag{23}$$

where $\delta_{i,j}$ denotes the Kronecker delta, $\xi_{i,j}$ is defined in (22)*, and*

$$\mu_{i,j} = \frac{(1 + c_{T-2,T-2})\upsilon_i^{(1)}\upsilon_j^{(1)} - c_{1,T-2}\upsilon_i^{(1)}\upsilon_j^{(T-2)} - c_{T-2,1}\upsilon_i^{(T-2)}\upsilon_j^{(1)} + (1 + c_{1,1})\upsilon_i^{(T-2)}\upsilon_j^{(T-2)}}{(1 + c_{1,1})(1 + c_{T-2,T-2}) - c_{1,T-2}c_{T-2,1}}.$$

Remarks. (a) $\sum_{j=1}^{T} z_{i,j} = 1$ for $i = 1, \ldots, T$ because $\boldsymbol{D}\boldsymbol{\iota} = \boldsymbol{0}$. (b) Since $\boldsymbol{D}\boldsymbol{J}_T = \boldsymbol{J}_{T-2}\boldsymbol{D}$ and $\boldsymbol{C}^{-1}$ is a centrosymmetric matrix, it follows that

$$\boldsymbol{J}_T \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{J}_T = \boldsymbol{D}'\boldsymbol{J}_{T-2}\boldsymbol{C}^{-1}\boldsymbol{J}_{T-2}\boldsymbol{D} = \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}, \tag{24}$$

which indicates that $\boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}$ is a centrosymmetric matrix. Likewise, since $\boldsymbol{J}_T\boldsymbol{U} = \boldsymbol{U}\boldsymbol{J}_2$, it follows that $\boldsymbol{J}_2\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U}\boldsymbol{J}_2 = \boldsymbol{U}'\boldsymbol{J}_T\boldsymbol{C}^{-1}\boldsymbol{J}_T\boldsymbol{U} = \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U}$, which indicates $(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}$ is a centrosymmetric matrix. Accordingly, it follows that

$$\begin{aligned}
\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1} &= \boldsymbol{J}_T\boldsymbol{C}^{-1}\boldsymbol{J}_T\boldsymbol{U}(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{J}_T\boldsymbol{C}^{-1}\boldsymbol{J}_T \\
&= \boldsymbol{J}_T\boldsymbol{C}^{-1}\boldsymbol{U}\boldsymbol{J}_2(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{J}_2\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{J}_T \\
&= \boldsymbol{J}_T\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{J}_T,
\end{aligned}$$

which indicates that $\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}$ is a centrosymmetric matrix. From these results, we obtain, e.g., $\xi_{1,1} = \xi_{T,T}$ and $\mu_{1,1} = \mu_{T,T}$ in (23). (c) `calc_HP_hat_matrix` in the Appendix is a MATLAB/GNU Octave function to calculate $(\boldsymbol{I}_T + \alpha \boldsymbol{D}'\boldsymbol{D})^{-1}$ in (1) based on (23).

Finally, we emphasize that our approach leads to a simpler formula because we apply the SMW formula to $(\boldsymbol{C} + \boldsymbol{U}\boldsymbol{U}')^{-1} = (\boldsymbol{C} + \boldsymbol{e}_1\boldsymbol{e}_1' + \boldsymbol{e}_{T-2}\boldsymbol{e}_{T-2}')^{-1}$, where both $\boldsymbol{e}_1$ and $\boldsymbol{e}_{T-2}$ are *unit vectors*. Observe that $\boldsymbol{p}_i$ in (7) and $\boldsymbol{q}_i$ in (9), where $i = 1, T$, are not unit vectors.

4 Discussion on a possible modification of the HP filter

From Theorem 3.1, letting $\widehat{\boldsymbol{x}}_1 = \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{y}$ and $\widehat{\boldsymbol{x}}_2 = \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{y}$, $\widehat{\boldsymbol{x}}$ in (1) may be represented as $\widehat{\boldsymbol{x}} = \boldsymbol{y} - \widehat{\boldsymbol{x}}_1 + \widehat{\boldsymbol{x}}_2$. Interestingly, since $\boldsymbol{D}\boldsymbol{\iota} = \boldsymbol{0}$, it follows that (a) $\boldsymbol{\iota}'\widehat{\boldsymbol{x}}/T = \boldsymbol{\iota}'(\boldsymbol{y} - \widehat{\boldsymbol{x}}_1)/T \, (= \boldsymbol{\iota}'\boldsymbol{y}/T)$, which shows that the average of $\boldsymbol{y} - \widehat{\boldsymbol{x}}_1 = (\boldsymbol{I}_T - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D})\boldsymbol{y}$ equals that of $\widehat{\boldsymbol{x}}$, and (b) $(\boldsymbol{I}_T - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D})\boldsymbol{\iota} = \boldsymbol{\iota}$, which indicates that, as in the case of the HP filter, every sum of weights equals unity.[9] These properties urge us to examine whether $\boldsymbol{y} - \widehat{\boldsymbol{x}}_1$ may be a favorable alternative to $\widehat{\boldsymbol{x}}$ or not. This is because $\boldsymbol{y} - \widehat{\boldsymbol{x}}_1$ could reduce the endpoint problem of the HP filter.

As a result, unfortunately, that is not the case. Blue and red lines in Figure 1 respectively plot the T-th row of $(\boldsymbol{I}_T + \alpha\boldsymbol{D}'\boldsymbol{D})^{-1}$ and $\boldsymbol{I}_T - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}$ when $T = 100$ and $\alpha = 1600$. From the figure, it is observable that the endpoint problem of the modified filter is more terrible than that of the HP filter. The weights for y_{T-1} and y_T are 0.8885 and -2.9861, respectively. Figure 2 shows the results for the case where $T = 100$ and $\alpha = 6.25$.[10] From the figure, we observe that the endpoint problem is less severe but another endpoint problem occurs: $\widehat{x}_T$ heavily depends on y_{T-1}, more precisely, the corresponding weight is 0.5746.

From Figures 1 and 2, we also observe that the weights of the modified filter considerably differ depending on α. Let us consider the reason why the phenomenon occurs. From Corollary 3.2, (T, T)-entry of $\boldsymbol{I}_T - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}$ may be expressed as $1 - \xi_{T,T}$, where

$$\xi_{T,T} = \sum_{k=1}^{T-2} \frac{(\Delta^2 g_{T,k})^2}{\alpha^{-1} + \lambda_k^2} = \sum_{k=1}^{T-2} \frac{g_{T-2,k}^2}{\alpha^{-1} + \lambda_k^2},$$

[9] Likewise, since $\boldsymbol{D}\boldsymbol{\tau} = \boldsymbol{0}$, as in the case of the HP filter, it follows that $(\boldsymbol{I}_T - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D})\boldsymbol{\tau} = \boldsymbol{\tau}$.

[10] Ravn and Uhlig (2002) suggests 6.25 as a value of α for annual data.

which is $(T-2, T-2)$-entry of $\boldsymbol{C}^{-1}$. Since $\lambda_1 = 4\sin^2[\pi/(2(T-1))] \to 0$ as $T \to \infty$, it follows that $\alpha^{-1} + \lambda_1^2 \to \alpha^{-1}$ as $T \to \infty$, which implies that $\xi_{T,T}$ may become larger as α increases when T is large. Actually, it follows that

$$\frac{1}{1600} + 4\sin^2\left(\frac{\pi}{2(100-1)}\right) = 0.0016319 \ll \frac{1}{6.25} + 4\sin^2\left(\frac{\pi}{2(100-1)}\right) = 0.16101.$$

Note that as shown in (24), $\boldsymbol{I}_T - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}$ is a centrosymmetric matrix, it follows that $\xi_{1,1}$ equals $\xi_{T,T}$.

5 Concluding remarks

By applying the SMW formula and a discrete cosine transformation matrix, De Jong and Sakarya (2016) derived an explicit formula for the HP filter. Then, by applying the SMW formula and the spectral decomposition of a symmetric tridiagonal Toeplitz matrix, Cornea-Madeira (2017) provided a simper formula. In this paper, we provided an alternative simpler formula and explained why our approach leads to a simpler formula. Wang et al. (2015) developed a method for deriving the explicit inverse of a pentadiagonal Toeplitz matrix, and our approach may be regarded as an application of Wang et al. (2015). The main result of the paper is summarized in Theorem 3.1 and Corollary 3.2.

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Appendix

A.1 Proof of (5)

From (4), $\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2$ is

$$\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2 = \left[\sqrt{\tfrac{1}{T}} \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(2-1)(1-0.5)\pi}{T}\right) \quad \cdots \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(T-1)(1-0.5)\pi}{T}\right)\right] - \left[\sqrt{\tfrac{1}{T}} \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(2-1)(2-0.5)\pi}{T}\right) \quad \cdots \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(T-1)(2-0.5)\pi}{T}\right)\right].$$

Let $\beta_j = \pi(j-1)/(2T)$ for $j = 2, \dots, T$. Then, it follows that

$$\sqrt{\frac{2}{T}}\cos\left(\frac{(j-1)(1-0.5)\pi}{T}\right) - \sqrt{\frac{2}{T}}\cos\left(\frac{(j-1)(2-0.5)\pi}{T}\right) = \sqrt{\frac{2}{T}}[\cos(\beta_j) - \cos(3\beta_j)] = \sqrt{\frac{32}{T}}\sin^2(\beta_j)\cos(\beta_j).$$

The last equality follows from $\cos(\beta_j) - \cos(3\beta_j) = 4\sin^2(\beta_j)\cos(\beta_j)$.

A.2 Proof of (6)

From (4), $\boldsymbol{\gamma}_T - \boldsymbol{\gamma}_{T-1}$ is

$$\boldsymbol{\gamma}_T - \boldsymbol{\gamma}_{T-1} = \left[\sqrt{\tfrac{1}{T}} \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(2-1)(T-0.5)\pi}{T}\right) \quad \cdots \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(T-1)(T-0.5)\pi}{T}\right)\right] - \left[\sqrt{\tfrac{1}{T}} \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(2-1)(T-1-0.5)\pi}{T}\right) \quad \cdots \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(T-1)(T-1-0.5)\pi}{T}\right)\right]$$

Let $\beta_j = \pi(j-1)/(2T)$ and $\kappa_j = 2T\beta_j = \pi(j-1)$ for $j = 2, \dots, T$. Then, it follows that

$$\sqrt{\frac{2}{T}}\cos\left(\frac{(j-1)(T-0.5)\pi}{T}\right) - \sqrt{\frac{2}{T}}\cos\left(\frac{(j-1)(T-1-0.5)\pi}{T}\right) = \sqrt{\frac{2}{T}}[\cos(\beta_j(2T-1)) - \cos(\beta_j(2T-3))] = \sqrt{\frac{2}{T}}[\cos(\kappa_j - \beta_j) - \cos(\kappa_j - 3\beta_j)].$$

Here, since $\sin(\kappa_j) = 0$, it follows that

$$\begin{aligned}
&\cos(\kappa_j - \beta_j) - \cos(\kappa_j - 3\beta_j) \\
&\quad = \cos(\kappa_j)\cos(\beta_j) + \sin(\kappa_j)\sin(\beta_j) - \cos(\kappa_j)\cos(3\beta_j) - \sin(\kappa_j)\sin(3\beta_j) \\
&\quad = \cos(\kappa_j)[\cos(\beta_j) - \cos(3\beta_j)] + \sin(\kappa_j)[\sin(\beta_j) - \sin(3\beta_j)] \\
&\quad = \cos(\kappa_j)[4\sin^2(\beta_j)\cos(\beta_j)] + \sin(\kappa_j)[4\sin^3(\beta_j) - 2\sin(\beta_j)] \\
&\quad = 4\sin^2(\beta_j)[\cos(\kappa_j)\cos(\beta_j) + \sin(\kappa_j)\sin(\beta_j)] - 2\sin(\kappa_j)\sin(\beta_j) \\
&\quad = 4\sin^2(\beta_j)\cos(\kappa_j - \beta_j),
\end{aligned}$$

where

$$\kappa_j - \beta_j = 2T\beta_j - \beta_j = \beta_j(2T-1) = \frac{\pi(j-1)(2T-1)}{2T} = \frac{\pi(j-1)(T-0.5)}{T}.$$

A.3 Application of the SMW formula to $(\boldsymbol{B} - \alpha \boldsymbol{V}\boldsymbol{V}')^{-1}$

As in Cornea-Madeira (2017), by applying the SMW formula to $(\boldsymbol{B} - \alpha \boldsymbol{q}_1\boldsymbol{q}_1' - \alpha \boldsymbol{q}_T\boldsymbol{q}_T')^{-1}$, we obtain the following results:

$$\begin{aligned}
&(\boldsymbol{B} - \alpha \boldsymbol{q}_1\boldsymbol{q}_1' - \alpha \boldsymbol{q}_T\boldsymbol{q}_T')^{-1} \\
&\qquad = (\boldsymbol{B} - \alpha \boldsymbol{q}_1\boldsymbol{q}_1')^{-1} + \alpha \frac{(\boldsymbol{B} - \alpha \boldsymbol{q}_1\boldsymbol{q}_1')^{-1}\boldsymbol{q}_T\boldsymbol{q}_T'(\boldsymbol{B} - \alpha \boldsymbol{q}_1\boldsymbol{q}_1')^{-1}}{1 - \alpha \boldsymbol{q}_T'(\boldsymbol{B} - \alpha \boldsymbol{q}_1\boldsymbol{q}_1')^{-1}\boldsymbol{q}_T},
\end{aligned} \tag{A.1}$$

where

$$(\boldsymbol{B} - \alpha \boldsymbol{q}_1\boldsymbol{q}_1')^{-1} = \boldsymbol{B}^{-1} + \alpha \frac{\boldsymbol{B}^{-1}\boldsymbol{q}_1\boldsymbol{q}_1'\boldsymbol{B}^{-1}}{1 - \alpha \boldsymbol{q}_1'\boldsymbol{B}^{-1}\boldsymbol{q}_1}. \tag{A.2}$$

On the other hand, by applying the SMW formula to $(\boldsymbol{B} - \alpha \boldsymbol{V}\boldsymbol{V}')^{-1}$, we obtain

$$\begin{aligned}
&(\boldsymbol{B} - \alpha \boldsymbol{V}\boldsymbol{V}')^{-1} \\
&\quad = \boldsymbol{B}^{-1} - \boldsymbol{B}^{-1}\boldsymbol{V}(\boldsymbol{V}'\boldsymbol{B}^{-1}\boldsymbol{V} - \alpha^{-1}\boldsymbol{I}_2)^{-1}\boldsymbol{V}'\boldsymbol{B}^{-1} \\
&\quad = \boldsymbol{B}^{-1} - [\boldsymbol{B}^{-1}\boldsymbol{q}_1, \boldsymbol{B}^{-1}\boldsymbol{q}_T] \begin{bmatrix} \boldsymbol{q}_1'\boldsymbol{B}^{-1}\boldsymbol{q}_1 - \alpha^{-1} & \boldsymbol{q}_1'\boldsymbol{B}^{-1}\boldsymbol{q}_T \\ \boldsymbol{q}_T'\boldsymbol{B}^{-1}\boldsymbol{q}_1 & \boldsymbol{q}_T'\boldsymbol{B}^{-1}\boldsymbol{q}_T - \alpha^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{q}_1'\boldsymbol{B}^{-1} \\ \boldsymbol{q}_T'\boldsymbol{B}^{-1} \end{bmatrix}. \qquad \text{(A.3)}
\end{aligned}$$

By comparing (A.3) with (A.1) and (A.2), it is observable that (A.3) is preferable to (A.1), mainly because (A.3) is symmetric with respect to $\boldsymbol{q}_1$ and $\boldsymbol{q}_T$.

A.4 A MATLAB/GNU Octave function to calculate $(\boldsymbol{I}_T + \alpha \boldsymbol{D}'\boldsymbol{D})^{-1}$ in (1) based on (23)

```
function HP_hat_matrix = calc_HP_hat_matrix(T, alpha)


% T: sample size
% alpha: smoothing parameter


Lam = diag( 4*(sin((1:T-2)*pi/(2*(T-1))).^2) );
G = zeros(T-2,T-2);
for i = 1:T-2
   for j = 1:T-2
      G(i,j) = sqrt(2/(T-1))*sin(i*j*pi/(T-1));
   end
end


invC = zeros(T-2,T-2);
for i = 1:T-2
   for j = 1:T-2
      s = 0;
```

```
      for k = 1:T-2
         s = s + G(i,k)*G(j,k)/( (1/alpha)+Lam(k,k)^2 );
      end
      invC(i,j) = s;
   end
end

Xi = zeros(T,T);
DG = diff([zeros(2,T-2);G;zeros(2,T-2)],2);
for i = 1:T
   for j = 1:T
      s = 0;
      for k = 1:T-2
         s = s+DG(i,k)*DG(j,k)/((1/alpha)+Lam(k,k)^2);
      end
      Xi(i,j) = s;
   end
end

Up1 = zeros(T,1); Up2 = zeros(T,1);
for i=1:T
   s1 = 0; s2 = 0;
   for k = 1:T-2
     s1 = s1+DG(i,k)*G(1,k)/((1/alpha)+Lam(k,k)^2);
     s2 = s2+DG(i,k)*G(end,k)/((1/alpha)+Lam(k,k)^2);
   end
   Up1(i) = s1; Up2(i) = s2;
end
```

```
c11 = invC(1,1); c22 = invC(end,end); c12 = invC(1,end); c21 = invC(end,1);
den = (1+c11)*(1+c22)-c12*c21;
Tau = zeros(T,T);
for i=1:T
   for j=1:T
      num = (1+c22)*Up1(i)*Up1(j)-c12*Up1(i)*Up2(j)-c21*Up2(i)*Up1(j) ...
            +(1+c11)*Up2(i)*Up2(j);
      Tau(i,j) = num/den;
   end
end

Z = zeros(T,T);
I = eye(T);
for i=1:T
   for j=1:T
      Z(i,j) = I(i,j)-Xi(i,j)+Tau(i,j);
   end
end

HP_hat_matrix = Z;

end
```

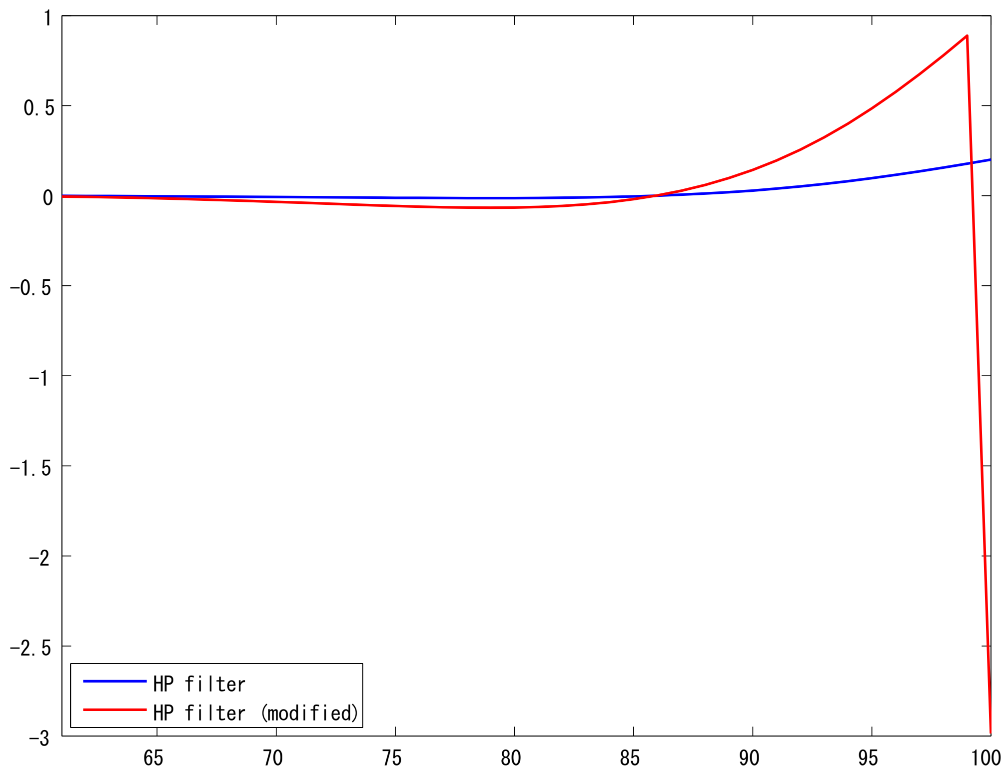

Figure 1: Plots of the T-th row of $(\boldsymbol{I}_T + \alpha \boldsymbol{D}'\boldsymbol{D})^{-1}$ (blue line) and $\boldsymbol{I}_T - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}$ (red line) when $T = 100$ and $\alpha = 1600$.

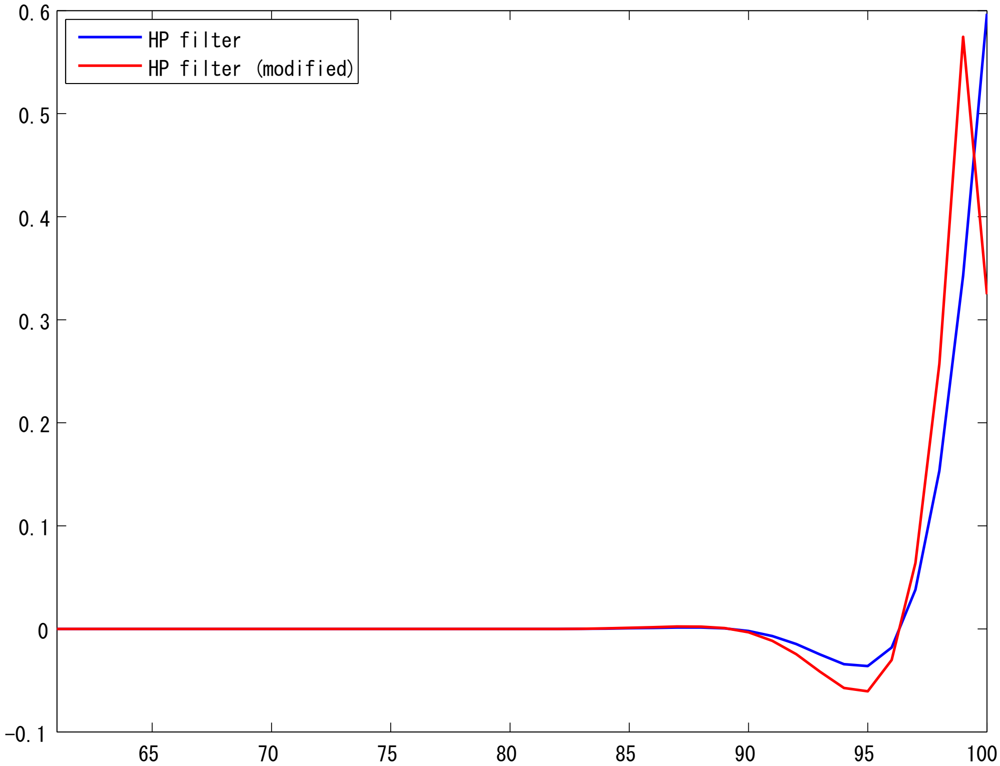

Figure 2: Plots of the T-th row of $(\boldsymbol{I}_T + \alpha \boldsymbol{D}'\boldsymbol{D})^{-1}$ (blue line) and $\boldsymbol{I}_T - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}$ (red line) when $T = 100$ and $\alpha = 6.25$.