Population Aging, Government Policy and the Postwar Japanese Economy*

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Abstract

This paper analyzes the Postwar Japanese economy with a parsimonious neoclassical growth model that incorporates the demographic transition in Japan. We find that i) productivity growth is the most important driver of the postwar economic growth, ii) the workweek reduction policy of the 1990s significantly reduced Japanese output. iii) the increase in the fraction of the population aged above 65 years old significantly reduced output relative to its potential through the decline in the employment rate and the increase in payroll tax.

1 Introduction

A key feature of the postwar Japanese economy is the rapid population aging. The share of population above 65 years old among the population above 15

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years old has increased from 8% in 1955 to 30% in 2015 which is currently the highest among all economies in the world. In this paper, we construct a parsimonious neoclassical growth model to quantitatively assess the impact of population aging and related policies over the 1975-2015 period.

The basis of our model is a representative agent neoclassical growth model with exogenous productivity growth. Hayashi and Prescott (2002) and Chen, Imrohoroglu and Imrohoroglu (2006) show that total factor productivity growth is important in accounting for the postwar Japanese economic performance. In order to assess the impact of population aging, we modified the representative agent model such that a representative household consists of young and old adults. Population aging defined as the increase in the fraction of old adults among total adult population is treated as exogenous. We assume that young and old adults have fixed employment rates where that of old adults is lower so that population aging directly affects the aggregate employment rate. This directly reduces labor input. In addition, we incorporate government fiscal policies that are related to population aging such as government consumption, capital income tax and labor income tax.

There are several related literature on population aging in Japan. Chen, Imrohoroglu and Imrohoroglu (2007) and Braun, Ikeda and Joines (2009) use over-lapping generations model to study the effects of the demographic transition on the Japanese savings rate over the 1960-2000 period. They find that total factor productivity plays a more important role than demographic transitions in accounting for the fluctuation in the savings rate. Yamada (2012) further introduces idiosyncratic labor income shocks in the over-lapping generation model in order to analyze the inter and intra-generational distribution of earning and consumption in Japan over time.

In addition to the growth path of output, we also discuss the effects of various shocks on the hours worked per worker which has declined since the 1990s. Otsu (2009) shows that productivity growth and subsistence consumption can explain the decline in hours worked during the rapid growth period but not the decline in hours during the 1990s. Hayashi and Prescott (2002) and Yamada (2012) argue that the government policy to reduce the workweek over the 1988 to 1993 period are important for the decline in hours worked. We explicitly model the workweek shortening policy as a decline in maximum available hours over the workweek and find that it is quantitatively much more important than other labor discouraging shocks such as the increase in labor income tax and the decline in total factor productivity.

The remainder of the paper is as follows. In section 2 we will discuss the postwar Japanese macroeconomy. In section 3 we describe the model. In section 4 we explain the quantitative method and the simulation results. Section 5 concludes the paper.

2 The Postwar Japanese Macroeconomy

In this section, we present macroeconomic data that characterizes the postwar Japanese economy over the 1955-2015 period. We focus on the demographic transition and the evolution of GNP per adult, its expenditure components, production factors, and government policy variables.

2.1 Output

Figure 1 plots the real GNP per adult in Japan over the 1955-2015 period. The GNP data is from the ESRI SNA database. The vertical axis is converted into a log scale so that constantly growing variables appear as a straight line. The figure clearly shows that output growth can be divided into three phases: the rapid growth period 1955-1974, the steady growth period 1975-1991, and the lost decades 1992-2015. The postwar rapid growth has been documented extensively in the literature focusing on the convergence and the productivity growth. After the economy was hit by the oil shock in 1974 the average economic growth slowed down but was still higher than the US. The output growth accelerated in the late 1980s during the economic boom known as the "bubble economy". In 1991 the bubble economy collapsed and Japan entered a long-lasting stagnation known as the "lost decade(s)". Finally, during the last decade the Great Recession in 2009 and the East Japan earthquake in 2011.

Table 1 lists the average real GNP per adult and its growth rate over the 1955-1974, 1975-1991 and 1992-2015 periods. The average GNP per adult more than doubled over the first two subperiods. This is a result of a high average per adult output growth rate over the 1955-1974 period at 6.28%. The average growth rate over the 1975-1991 period fell to 3.16%. There has been practically no growth over the 1992-2015 period.

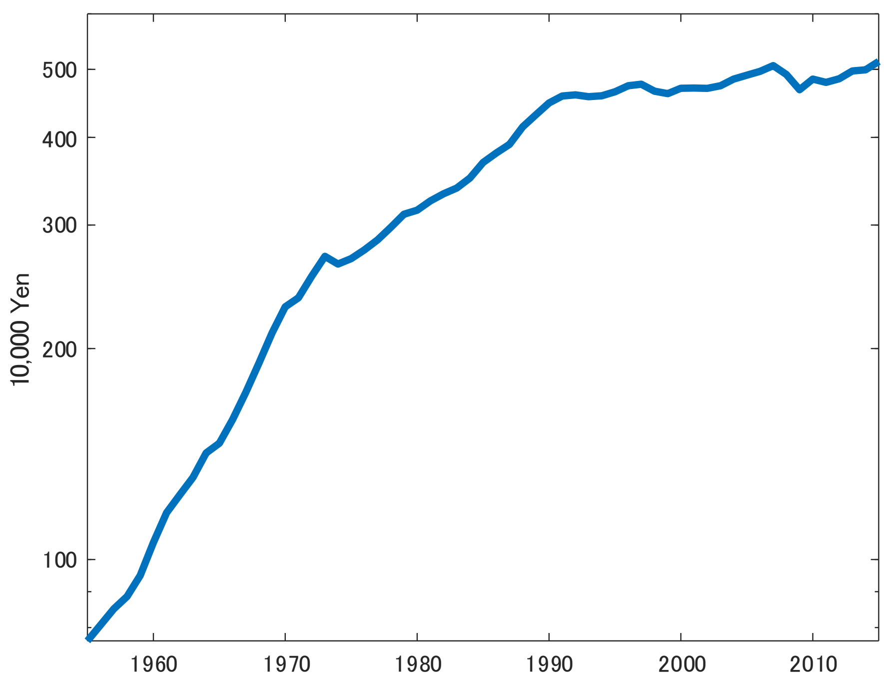


Figure 1: Japanese Real Output Per Capita: 1955-2015

Table 1. Economic Growth

	GNP per Adult in 2000 yen	GNP per Adult Growth (%)
1955 − 1974	1,588,847	6.28
1975 − 1991	3,523,622	3.16
1992 − 2015	4,791,001	0.48

2.2 Demographics

The demographic transition, namely population aging, has taken place more rapidly in Japan than any other country in the world. In this paper we consider two age groups: "Young" population defined as the population aged 15 years old to 64 years old and the "Aged" population defined as the population of those above 65 years old. The population data are from the Labor Force Survey for 1973-2015 extrapolated backwards using the census data for 1955-1972.

Figure 2 plots the demographic transition of the Japanese economy over the 1955-2015 period. The first panel presents the population of the two age groups, "Young" and "Aged", over the 1955-2015 period. "Young" population is defined as the population aged 15 years old to 64 years old and the "Aged" population is defined as the population of those above 65 years old. The population data are from the Labor Force Survey for 1973-2015 extrapolated backwards using the census data for 1955-1972. Both groups are growing until during the 1990s where the Young age group starts to shrink. This is the result of the decline in the fertility rate which has fallen below the reproductive rate. On the other hand, the Aged age group continues to grow which reflects the decline in mortality.

The second panel shows the share of the Aged population. The Aged share steadily increased from 8.0% in 1955 to 30.3% in 2015 reflecting the decline in fertility and mortality. The third panel presents the adult population growth rate. The large fluctuation in the early 1960s corresponds to the decline in fertility during the war and the subsequent baby boom. The sharp drop in 1980 reflects the decline in fertility in 1966 known as the Hinoe-Uma while the temporary rise in mid 1980s reflects the second generation baby-boomers. Overall, the population growth rate shows a declining trend from 2.1% in 1955 to 0.0% in 2015.

Table 2 reports the average population growth rates of each group during the 1955-1974, 1975-1991, and 1992-2015 periods. The growth rate of

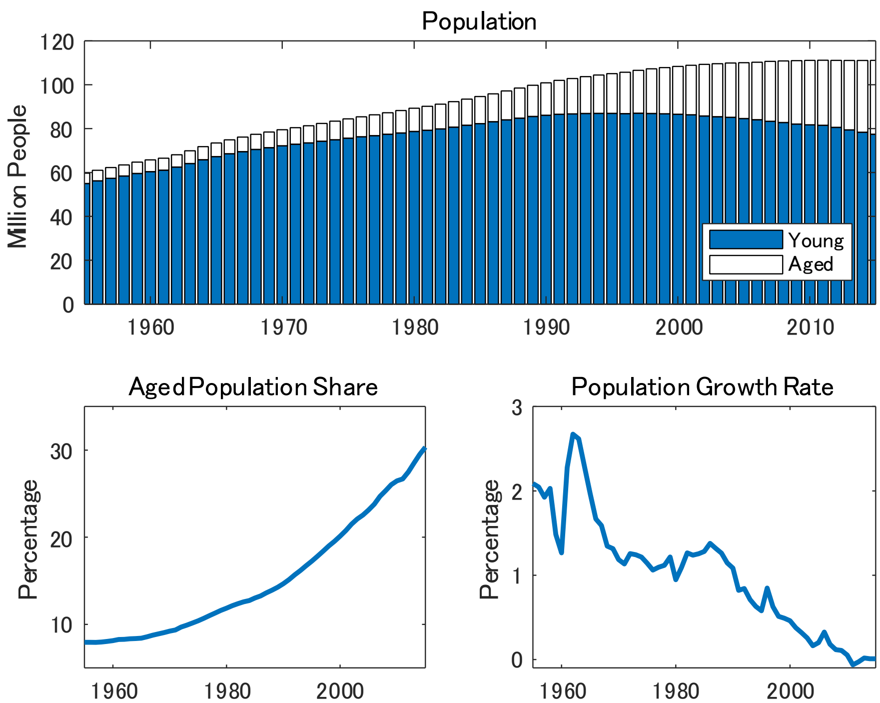


Figure 2: Demographic Transition: 1975-2015

adult population over the three subperiods are 1.73%, 1.16% and 0.32% respectively confirming the decline of the adult population growth rate over time. The growth rate of the young population over the three subperiods are 1.59%, 0.80% and -0.49% reflecting the shrinking young population group. The growth rate of the aged populaiton remains high over the same periods at 3.08%, 3.57% and 3.20% respectively. Overall, the negative effect of the decline in fertility dominates the positive effect of the decline in mortality on population growth.

Table 2. Population Growth

	Adult	Young	Aged
1955 – 1974	1.73	1.59	3.08
1975 – 1991	1.16	0.80	3.57
1992 – 2015	0.32	−0.49	3.20

2.3 Expenditure

Figure 3 plots the real GNP expenditure components: consumption, investment and government spending over the 1955-2015 period. We follow Hayashi and Prescott (2002) and define consumption as private consumption, investment as private gross domestic capital formation plus the current account, and government spending as government consumption and government domestic capital formation. The expenditure data is from the ESRI SNA database. Each variable is deflated by the GNP deflator and divided by adult population. The figure shows that all expenditure components follow the general growth path of output. However, the paths of consumption and government spending are much more smoother than that of investment.

Table 3 summarizes the evolution of each GNP expenditure component over the 1955-1974, 1975-1991 and 1992-2015 periods. Panel A shows the growth rates of each expenditure component. During the rapid growth period, investment grows much faster than consumption and government spending. During the second subperiod, the growth of all expenditure components slow down significantly. In the third subperiod, the growth of all expenditure components further slow down where the average growth rate of investment falls below zero. Panel B shows the average GNP share of expenditure components over the three subperiods. Consumption has always been the largest component and incraesed its share over the three subperiods. Investment was the second largest component during the rapid growth period but gradually

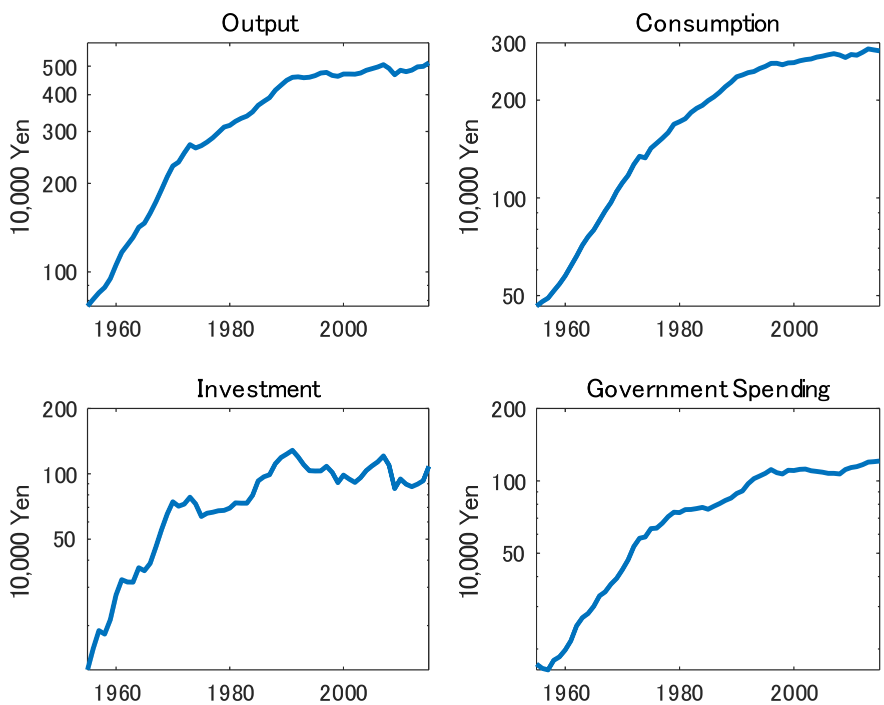


Figure 3: Japanese Expenditure Componenets: 1955-2015

lost its share and fell less than government spending in the third subperiod. Government spending consistently increased its share over the three subperiods.

Table 3. GDP Expenditure Components

A. Growth Rates (%)			
	Consumption	*Investment*	*Government*
1955 – 1974	5.60	8.14	6.49
1975 – 1991	3.15	3.72	2.52
1992 – 2015	0.67	−0.43	0.95
B. GNP Share (%)			
	Consumption	*Investment*	*Government*
1955 – 1974	53.6	25.7	20.1
1975 – 1991	53.8	24.3	22.0
1992 – 2015	55.8	21.1	23.1

2.4 Production Factors

In this section we discuss the evolution of the production factors: productivity, capital stock, employment, hours worked and productivity over the 1955-2015 period. Capital stock is defined as per adult net capital stock at the beginning of the year deflated by the GNP deflator. The sources are ESRI SNA93 dataset for 1981-2014 extrapolated backwards using the ESRI SNA68 dataset for 1970-1980 and Hayashi and Prescott (2002) for 1956-1969. The data source for hours worked after 1968 is the non-agricultural working hours data from the Labor Force Survey while for the years before that we extrapolate using the hours worked per total employment data from the Total Economy Database of the Conference Board. The data for employment is from the Labor Force Survey. We define productivity in a standard Cobb-Douglas production function:

$$Y_t = A_t K_t^{\theta} \left(E_t H_t\right)^{1-\theta}$$

where Y_t, K_t, E_t, H_t stand for aggregate output, capital, employment and hours worked per worker and θ is the capital share. The capital share $\theta = 0.381$ is calibrated to match the income data following the method of Cooley and Prescott (1995).

Figure 4 plots the production factors over the 1955-2015 period. Productivity is normalized such that the 1955 level is equal to 100. Capital stock

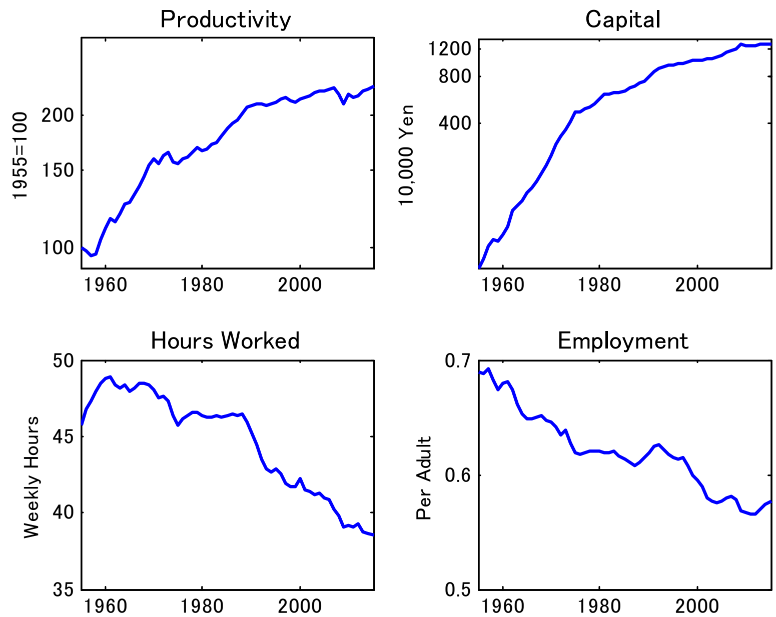


Figure 4: Japanese Production Factors: 1955-2015

is deflated by the GNP deflator and divided by the adult population. Hours worked is the hours worked per week. Employment is the ratio of total employment to adult population. Both productivity and capital stock rapidly increased during the rapid growth period and gradually slowed down over the following two subperiods. Hours worked fluctuated above 45 hours per week from 1955 to 1988. During the 1988-1993 period it dropped dramatically, and continued to fall gradually after 1993. The employment rate has been declining throughout the entire period. .

Table 4 summarizes the evolution of production factors over the 1955-2015 period. Panel A presents the growth rates of each production factor. Productivity growth gradually slowed down ove rthe three subpriods as shown in the figure. The capital stock per adult rapidly grew during the initial period and slowed down after the oil shock as output does. Hours worked per worker was practically constant over the first subperiod and gradually declined ove rthe second and third subperiods. Employment rate declined rapidly during the first subperiod, stayed constant during the second subperiod and declined again during the third subperiod. Panel B panel presents the levels of each production factors. Productivity is normalized as in Figure 4, capital stock is divided by GNP, hours is the weekly average hours worked and employment is the total employment to adult population ratio. Productivity nearly double from the first subperiod to the third subperiod. The average capital-output ratio more than doubled from 0.95 to 2.28 over the three subperiods. The average hours worked per worker declined by roughly 17 percent over the three subsperiods mainly during the 1992-2015 subperiod. The average employment rate declined by roughly 12 percent over the three subperiods. In this paper, we consider population aging as the main driver of the decline in the employment rate.[1]

Table 4. Production Factors

A. Growth Rates (%)				
	Productivity	*Capital*	*Hours*	*Employment*
1955 – 1974	2.19	11.61	0.04	−0.54
1975 – 1991	1.86	3.78	−0.29	0.07
1992 – 2015	0.43	1.58	−0.53	−0.36
B. Levels				
	Productivity	*Capital*	*Hours*	*Employment*
1955 – 1974	129.0	0.95	47.8	66.0
1975 – 1991	181.2	1.77	46.1	61.7
1992 – 2015	221.8	2.28	40.9	58.9

2.5 Government Policy

In this section, we introduce data related to government policy; labor income tax, capital income tax, government spending and working hour restricitions. We follow McDaniel (2009) and define labor income tax as the sum of "taxes on individual income, profits and capital gain" weighted by the labor income share and "social security contributions", and capital tax as the sum of the "taxes on individual income, profits and capital gain" weighted by the capital

[1] The employment rate of the young group has been rising from 68.2% in 1968 to 73.2% in 2015. Two major reasons of this trend is the increase in female participation and the extended retirement age. In contrast, the employment rate of the old group has been declining from 33.5% in 1968 to 21.7% in 2015. The main reason of this trend is the extended life expectancy where people live longer after retirement today compared to before.

income share, "taxes on corporate income, profits and capital gain" and "taxes on business property". Labor income and capital income tax rates are computed using the data from the OECD Revenue Statistics over the 1965-2015 period. We extrapolate backwards for the 1955-1964 period using the McDaniel (2009) dataset.

As long working hours was becoming a social issue during the bubble economy, the government made efforts to reduce the average hours worked. In the late 1980s the government set a target to reduce the average annual working hours to 1800 hours per worker and the 5 day workweek came into operation. In 1992 the government introduced the Act for Enforcement of Work Time Shortening which promoted shortening labor hours through firm subsidies. In 1994 the regular weekly working hours was officially reduced from 48 hours to 40 hours by the Labor Standards Act which reinforced the transition to the 5 day work week. In this paper, we interpret this policy as a gradual reduction of weekdays from 6 days to 5 days over the 1988-1994 period.

Figure 5 plots the government variables. Labor tax rate rapidly rises during the 1960s through 1980s, temporarily flattens out during the 1990s and rises again after 2000. Capital tax rate dramatically drops during the 1960s rapid growth period, gradually rises peaking during the late 1980s bubble economy, and gradually declines during the 1990s lost decade(s). The government spending to GNP ratio has a slight growing trend while it tends to fall during booms such as the rapid growth period and the bubble economy and rise after recessions such as the 1974 oil shock, the 1991 bubble burst and the 2009 global financial crisis. As for the workweek reduction policy, we assume that the worker has 16 hours per weekday to possibly allocate to wrok or leisure. The reduction from 6 weekdays to 5 weekdays over the 1988-1994 period implies a reduction in weekday hours from 96 to 80. Since the transition was gradually introduced we linearly interpolate the hours between 1998 to 1994.

Table 5 summarizes the evolution of taxe rates on labor income and capital income over the 1955-2015 period. Labor income tax nearly doubles over the first two subperiods. The labor tax further increases from the second to third subperiod. As we discuss below, social security contributions is playing a key role in this increase; the average social security contribution rate is 6.96%, 13.05% and 18.48% over the three subperiods. Average capital tax rises from the first subperiod to the second subperiod and returns to the initial level in the third subperiod. Over the three subperiods, the rate of

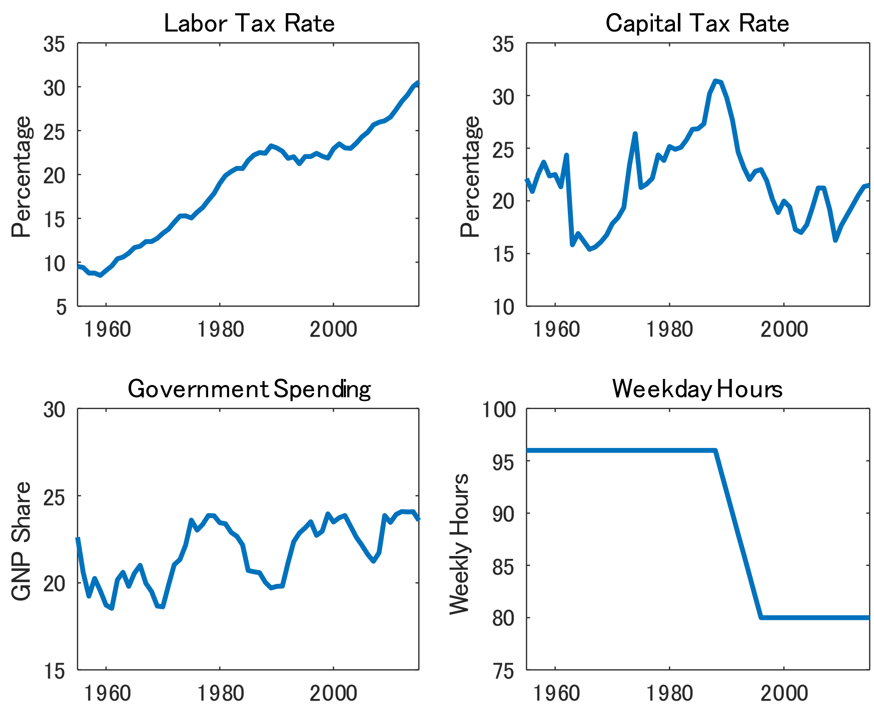


Figure 5: Japanese Government Policies: 1955-2015

labor income tax over took that of capital income tax.

Table 6. Tax Rates

	Labor	*Capital*
$1955-1974$	11.44	19.90
$1975-1991$	20.01	26.20
$1992-2015$	24.60	20.18

3 Model

The model consists of a representative household of young and aged adults, a representative firm and the government. The head of the household decides the optimal resource allocation for the whole family. The firm hires labor and capital from the household. The government taxes the households' labor and capital income to finance its expenditure.

3.1 Household

3.1.1 Individual Preference

The period utility of each member of the household i depends on consumption c and leisure l :

$$u_i = \psi \ln c_{i,t} + (1-\psi) \ln l_{i,t}.$$

Since there are only two types of household members, $i = y, o$ denotes young and old adults. We assume that the utility from leisure l is derived from the time allocated to activities during the workweek l^{ww} and the activities during the weekend l^{we}.

First, consider the case for a worker. For simplicity, we assume a separable utility function over time allocated to each leisure activity:

$$\ln l_{i,t} = \phi \ln(l_{i,t}^{ww}) + (1-\phi) \ln(l_{i,t}^{we}).$$

The time allocated to each type of activities are defined as

$$\begin{aligned} l_{i,t}^{ww} &= (\overline{\omega} - \omega_{i,t}) \times workweek_t, \\ l_{i,t}^{we} &= \overline{\omega} \times (7 - workweek_t) \end{aligned}$$

where $\overline{\omega}$ and ω_t stand for the maximum available hours to work per day and actual hours worked per day. The workweek can exogenously change over time due to government policy. We define maximum hours per week and hours worked per week as

$$\begin{aligned}\overline{h_t} &= \overline{\omega} \times workweek_t, \\ h_{i,t} &= \omega_{i,t} \times workweek_t.\end{aligned}$$

Therefore, the period preference function is

$$u_i = \psi \ln c_{i,t} + (1-\psi)\,\phi \ln\left(\overline{h_t} - h_{i,t}\right) + (1-\psi)\,\phi \ln\left(7\overline{\omega} - \overline{h_t}\right).$$

Due to homogeneity, the scaling of the preference weight parameters do not affect the maximization problem. Therefore, we rewrite the utility function of each member of the worker as

$$u_i = \Psi \ln c_{i,t} + (1-\Psi) \ln\left(\overline{h_t} - h_{i,t}\right) + \Phi_t$$

where

$$\Phi_t = (1-\Psi) \ln\left(7\overline{\omega} - \overline{h_t}\right).$$

The case of a non-worker is simply that the hours worked ω_t and thus h_t is equal to zero.

We assume that the employment rate is fixed at π so that on average the utility of each individual is

$$u_i = \Psi \ln c_{i,t} + \pi_i(1-\Psi) \ln\left(\overline{h_t} - h_{i,t}\right) + \Omega_t$$

where

$$\Omega_t = (1-\pi_i)(1-\Psi) \ln \overline{h_t} + \Phi_t.$$

Since Ω_t only includes exogenous variables, it will not affect the maximization problem given the separable utility function.

3.1.2 Household Optimization

In this model, we assume that the head of the household solves the resource allocation problem of the family which consists of young and old adults. For simplicity, we assume that the size of each family is equal to 1 and define the population share of young adults as η. The average representative family utility is

$$
\begin{aligned}
u(c,h) &= \eta_t \left[\Psi \ln c_{y,t} + \pi_y(1-\Psi)\ln\left(\overline{h_t} - h_{y,t}\right) + \Omega_{y,t}\right] \\
&\quad + (1-\eta_t)\left[\Psi \ln c_{o,t} + \pi_o(1-\Psi)\ln\left(\overline{h_t} - h_{o,t}\right) + \Omega_{o,t}\right]
\end{aligned} \tag{1}
$$

where the subscripts y, o stand for the young and old family members.

It turns out that given the separability of the utility function, both the optimal consumption level and working hours are identical across the young and the old so that $c_{y,t} = c_{o,t} = c_t$ and $h_{y,t} = h_{o,t} = h_t$. In addition, the separable term Ω_t only includes exogenous variables so that it does not affect the optimization problem. Therefore, we can imagine that the head of the household is maximizing the following family utility function:

$$
U = \max \sum_t \beta^t \left[\Psi \ln c_t + e_t(1-\Psi)\ln\left(\overline{h_t} - h_t\right)\right], \tag{2}
$$

where

$$
e_t = \eta_t \pi_y + (1-\eta_t)\,\pi_o
$$

stands for the employment rate. Since we assume constant employment shares for each age group and population aging is exogenous, the employment rate changes over time exogenously.

The household faces the following budget constraint

$$
c_t + i_t = (1-\tau_{l,t})\,w_t h_t e_t + (1-\tau_{k,t})\,r_t k_t + \pi_t + \zeta_t, \tag{3}
$$

where i_t is investment, w_t is the after tax wage rate, r_t is the rental rate on capital, k_t is the capital stock per adult, $\tau_{l,t}$ and $\tau_{k,t}$ are labor and capital income tax rates, π_t is the firm profit received as dividend per family and ζ_t is a lump sum transfer from the government.

In this model, population growth is interpreted as an increase in the number of households. We assume that existing households will support the new households by sharing their capital. Therefore, the capital stock of a representative household evolves over time according to a capital law of motion

$$
(1+n_t)k_{t+1} = i_t + (1-\delta)k_t, \tag{4}
$$

where n_t is the population growth rate.

3.2 Firm

The representative firm will produce a single good by combining capital and labor according to the following Cobb-Douglas production function:

$$Y_t = A_t K_t^{\theta} \left(h_t e_t N_t\right)^{1-\theta},$$

where Y_t is total output, A_t is the total factor productivity and N_t is the number of families in the economy.

The firm maximizes profits

$$\pi_t N_t = Y_t - w_t h_t e_t N_t - r_t K_t,$$

by choosing the optimal labor and capital level

$$\pi_t = y_t - w_t h_t e_t - r_t k_t.$$

3.3 Government

The government purchases goods and services for exogenous reasons and pays for this through labor income tax. They rebate all excess revenue to the household through lump sum transfer. Therefore, the government budget constraint is

$$G_t = \tau_{l,t} w_t h_t e_t N_t + \tau_{k,t} r_t K_t - \zeta_t N_t. \tag{5}$$

For simplicity, we assume that the government decides the amount of expenditure as a fraction of current output so that

$$G_t = g_t Y_t.$$

The government budget constraint together with the household budget constraint and firm profits, we get the per household resource constraint

$$(1 - g_t) y_t = c_t + i_t \tag{6}$$

3.4 Equilibrium

The deterministic competitive equilibrium is a set of quantities and prices

$$\{y_t, c_t, i_t, h_t, k_{t+1}, \zeta_t, w_t, r_t, e_t, n_t, \tau_{l,t}, \tau_{k,t}, g_t, A_t\}_{t=0}^{T}$$

such that;

1. The household optimizes given the series of $\{w_t, r_t, e_t, n_t, \tau_{l,t}, \tau_{k,t}, \zeta_t\}_{t=0}^{T}$ and k_0
2. The firm optimizes given $\{w_t, r_t, A_t\}$ each period
3. The government budget constraint (5) holds
4. The resource constraint (6) holds

4 Quantitative Analysis

In order to analyze the quantitative impacts of population aging and government policy we calibrate the model parameters to the Japanese data and solve the model numerically using the shooting algorithm. We further assess the impact of each channel by removing them from the model one by one and conduct counterfactual analyses.

4.1 Solution Method

The model leads to the following equilibrium conditions.

$$\frac{\Psi}{c_t} = \mu_t \tag{7a}$$

$$\frac{1-\Psi}{\overline{h_t} - h_t} = \mu_t(1-\tau_{l,t})w_t \tag{7b}$$

$$(1+n_t)\mu_t = \beta\mu_{t+1}\left\{(1-\tau_{k,t+1})r_{t+1} + 1 - \delta\right\} \tag{7c}$$

$$r_t = \theta\frac{y_t}{k_t} \tag{7d}$$

$$w_t = (1-\theta)\frac{y_t}{h_t e_t} \tag{7e}$$

$$(1+n_t)k_{t+1} = i_t + (1-\delta)k_t, \tag{7f}$$

$$y_t = A_t k_t^{\theta}\left(h_t e_t\right)^{1-\theta} \tag{7g}$$

$$(1-g_t)y_t = c_t + i_t \tag{7h}$$

For each t, there are 8 equations 8 endogenous variables, $\{k_{t+1}, \mu_t, h_t, y_t, c_t, i_t, r_t, w_t\}$ as well as $\left\{k_1, \mu_{T+1}\right\}$, where note that k_t is the capital at the beginning of period t. Hence, if there are T time periods, we have $8T$ equations and $8T+2$ endogenous variables. Adding the initial condition for k_1 and a terminal condition (TVC), we can regard the equilibrium condition as a system of equations with $8T+2$ equations and $8T+2$ variables. Our equilibrium is defined as $\{k_{t+1}, \mu_t, h_t, y_t, c_t, i_t, r_t, w_t\}_{t=1}^{T}$ as well as $\{k_1, \mu_{T+1}\}$ that satisfies the above equilibrium equations and initial and terminal conditions, given exogenous processes.

The most straightforward strategy to solve this model is to solve a system of $8T+2$ equations for $8T+2$ variables by using a numerical solver (e.g., "fsolve" of Matlab). This is easy to programme, but it has a lot of equations because T tends to be large, it requires very good initial guess and it does not exploit some specific features of this system of equitations, which we discuss shortly.

Instead, we can use the method called the "shooting algorithm", which numerically solves the system of ordinary differential (difference) equations with boundary conditions. In our case, we conduct the computation as follows. Suppose that we know k_1 (the initial condition), and the terminal condition is given by $k_{T+1} = \overline{k}$ (a certain value exogenously specified). Then, pick a certain value for μ_0.[2] Given $\{k_1, \mu_0\}$, we have 8 equations and 8 remaining variables for time 1; $\{k_2, \mu_1, h_1, y_1, c_1, i_1, r_1, w_1\}$, which means we can solve time $t=1$ equations for $t=1$ variables.[3] Now, having $\{k_2, \mu_1\}$ on hand, we can solve for time $t=2$ equations for $\{k_3, \mu_2, h_2, y_2, c_2, i_2, r_2, w_2\}$. We repeat this until $t=T$. Of course, in general, k_{T+1} obtained in this way does not match to the terminal value $\overline{k}$. Hence, we try different values of μ_0 until k_{T+1} gets close enough to $\overline{k}$. For each trial of μ_0, k_t evolves to k_{T+1} and we keep trying different μ_0 "shooting" at target value $\overline{k}$.

There are a couple of practical issues. First, it is often not clear what kind of terminal condition we should use. In this paper we compute a sort of steady state value of k_{T+1}, assuming that total factor productivity grows at the same rate as that in period T and the other exogenous variables stay at the values in T forever. Alternatively, we could have imposed a terminal condition such as $\mu_{T+1} = \overline{\mu}$. Algebraically speaking, any condition would work as long as we can reduce one degree of freedom, but it should depend on our economic intuition; i.e., the terminal condition itself is part of the model. Moreover, the quantitative result might depend on the choice of the terminal condition.[4]

[2] Note that μ_0 is not included in our solution.

[3] One of the caveats of solver type algorithms is that it fails to exploit this property; that is, time t equations have only time t variables and time $t+1$ variables, but not, say, $t+2$ variables.

[4] The choice of the terminal condition affects the solution near the terminal date T,

Second, whether using a numerical solver or the shooting algorithm, we solve the same system of equations and hence theoretically the results should be the same. In actual computation, while the shooting algorithm is stable, it tends to accumulate numerical errors toward the terminal date. In this regard, we could say a numerical solver is more accurate. However, practically speaking, the maximum Euler equation error of the shooting algorithm is negligible. We employ both algorithms but it is rather to ensure error-proof results than numerical accuracy. We first run the shooting algorithm and then use its results as the initial guess for a standard non-linear equation solver algorithm.

4.2 Calibration

In order to carry out the numerical simulation, we calibrate the structural parameters of the model to match data. The parameter values are listed in Table 7. The parameter levels are consistent with literature.

Table 7. Parameter Values

θ	Capital Income Share	0.381
δ	Capital Depreciation Rate	0.080
β	Subjective Discount Factor	0.963
Ψ	Preference Weight	0.406
π_y	Young Employment Rate	0.687
π_o	Aged Employment Rate	0.229

The capital income share θ is calibrated following Cooley and Prescott (1995) using income data from ESRI SNA data base over the 1975-2015 period. In specific, we compute the average of

$$\theta_t = \frac{OS + DEP}{Y - (MI + IBT - SUB)}$$

where OS, DEP, MI, IBT and SUB stand for operating surplus, capital depreciation, mixed income, indirect business tax and subsidies respectively.

The capital depreciation rate is calibrated to match the capital law of motion

$$(1 + n_t)k_{t+1} = i_t + (1 - \delta_t)k_t$$

while it has relatively little impact for the rest. This is exactly what the Turnpike theorem suggest. Therefore, this sort of solution method best fits to the models that study the transition from one steady state to the other.

to investment and capital data over the 1975-2015 period.

The subjective discpunt factor is calibrated to match the capital Euler equation

$$\frac{1+n_t}{c_t} = \frac{\beta}{c_{t+1}} \left\{ (1-\tau_{k,t+1})\theta_{t+1}\frac{y_{t+1}}{k_{t+1}} + 1 - \delta_t \right\}$$

to consumption, population growth, capital tax, output and capital stock data as well as the capital share and depreciation rate over the 1975-2015 period.

The preferecne weight is calibrated to match the labor first order condition

$$\frac{1-\Psi_t}{\overline{h}_t - h_t} = \frac{\Psi_t}{c_t}(1-\tau_{l,t})(1-\theta_t)\frac{y_t}{h_t e_t}$$

to hours worked, consumption, labor tax, output, employment data as well as the capital share over the 1975-2015 period.

The employment rates of each age group are calculated directly from data over the 1975-2015 period.

4.3 Benchmark Simulation

Figure 6 presents the benchmark results which incorporates the effects of the changes in all exogenous variables: population aging, population growth, productivity growth, labor income tax, capital income tax, government expenditure and workweek reduction. The simulated output replicates the boom during the bubble economy, the stagnation after the bubble burst and the recession in 2009 pretty well. The simulated hours worked shows some discreparcy with data during the 1970s and shows a larger drop during the 1988-1994 period. The simulated output replicates the data well except that the model predicts a smoother path than the data. The simulated investment series replicates the data pretty well but exaggerates the fluctuation.

Figure 7 plots the simulated employent rate of the model economy. The only reason why the employment rate changes over time in the model is because the population share of each age group changes over time. The simulated employment rate matches the data quite well especially during the 1990s and 2000s. This shows that population aging can account for a large part of the decline in the employment rate through the employment composition effect.

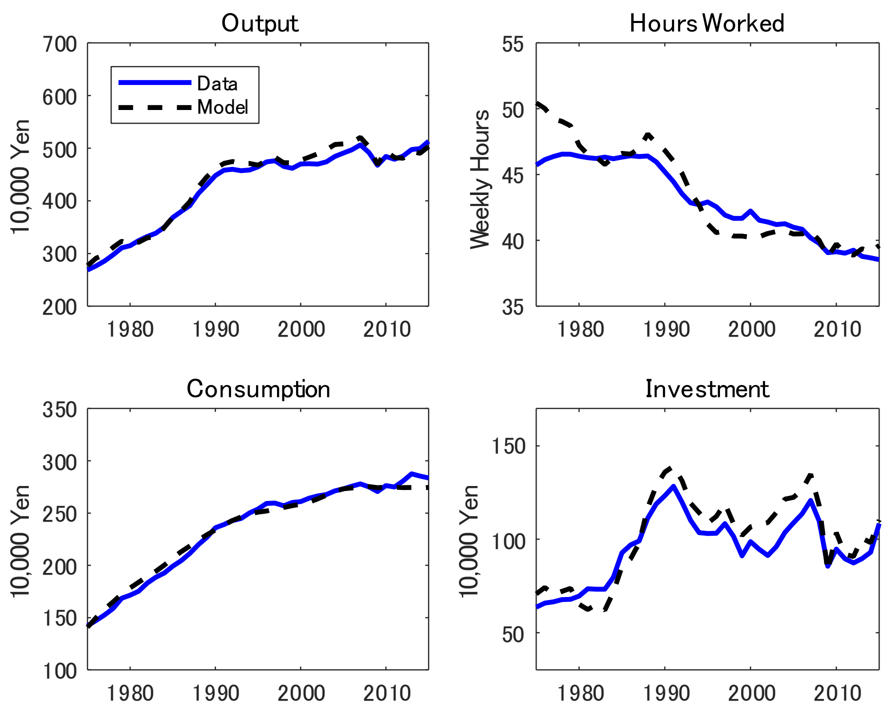


Figure 6: Benchmark Simulation

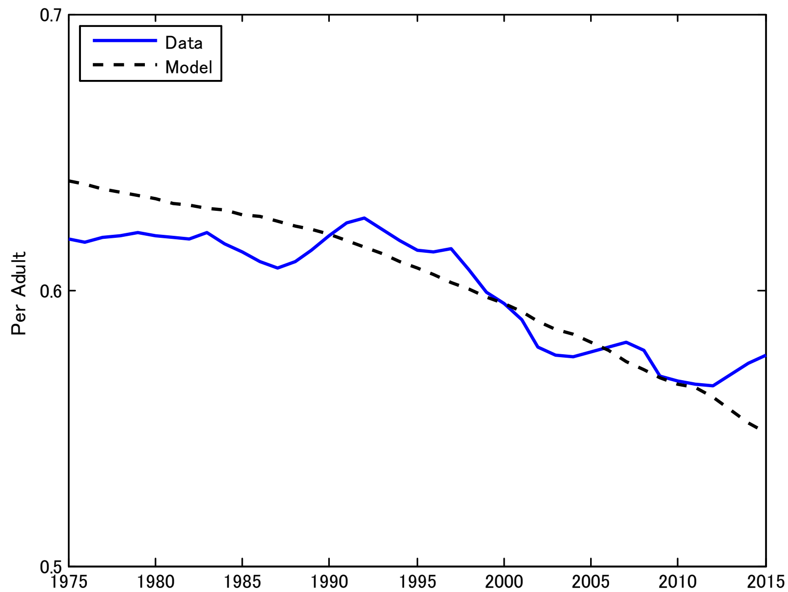


Figure 7: Simulated Employment Rate: 1975-2015

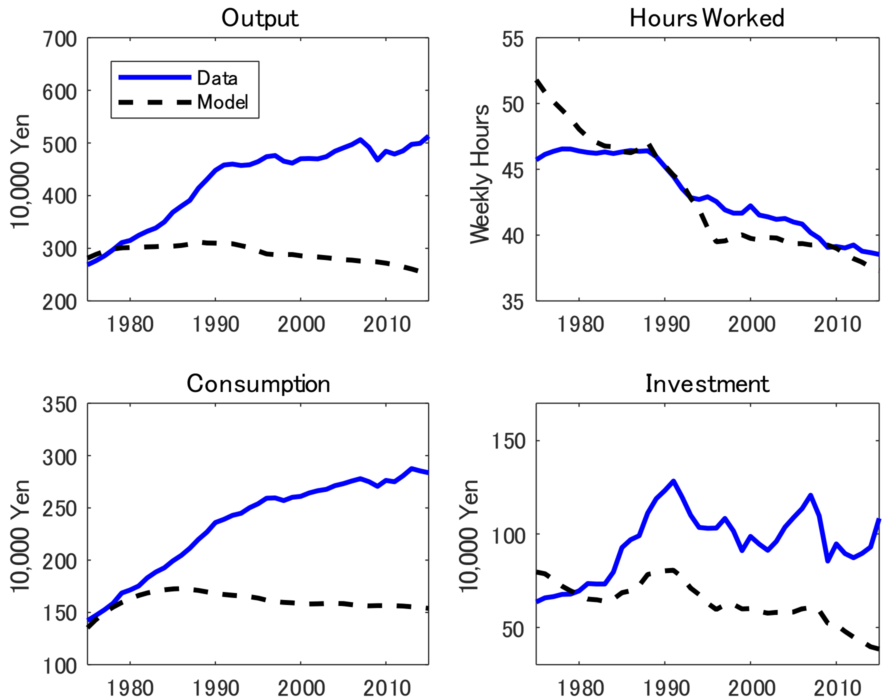


Figure 8: Counterfactual Simulation A: Constant Productivity

4.4 Counterfactual Analyses

Next we run simulations with counterfactual models turning off the fluctuation of one exogenous variable at a time. For each simulation we reset the terminal condition and set the final period capital as that implied by each counterfactual model. The difference between the benchmark and the counterfactual simulation represents the effect of the selected exogenous variable.

Figure 8 plots the simulation results of the model without productivity growth. It is clear that without productivity growth, the model cannot explain the growth in output, consumption and investment after the oil shock. Interestingly, the simulated hours worked is virtually identical to that of the benchmark model. This implies that productivity growth is not important in accounting for the evolution of hours worked.

Figure 9 presents the simulation result from the model shutting down the

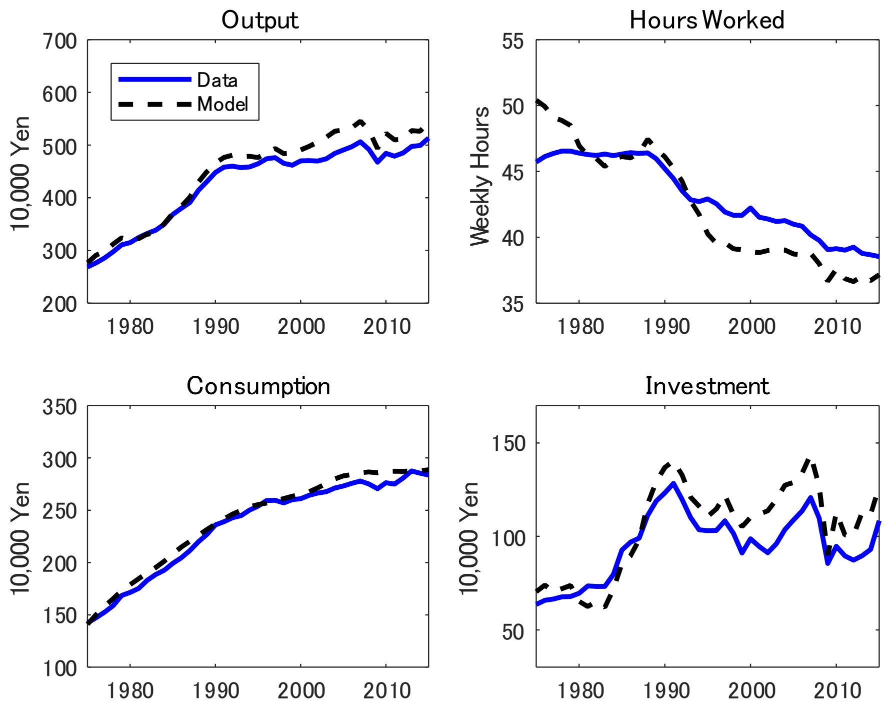


Figure 9: Counterfactual Simulation B: Constant Aged Ratio

increase in the aged population share. The model without population aging has higher output than the benchmark model implying that population aging reduced output due to the decline in employment. Interestingly, population aging has a positive effect on hours worked per worker as the firms want substitute employment by hours. However, overall total hours worked declines. This reduces the marginal product of capital and hence investment. The reduction of labor and capital stock results in lower output. Consequently consumption is lower than the benchmark.

Figure 10 presents the simulation result from the model with constant population growth. Population growth in our model works like capital depreciation rate as the aggregate capital stock has to be spread out among more families each period, which is known as the capital dilution effect. The model with constant population growth has lower output than the benchmark

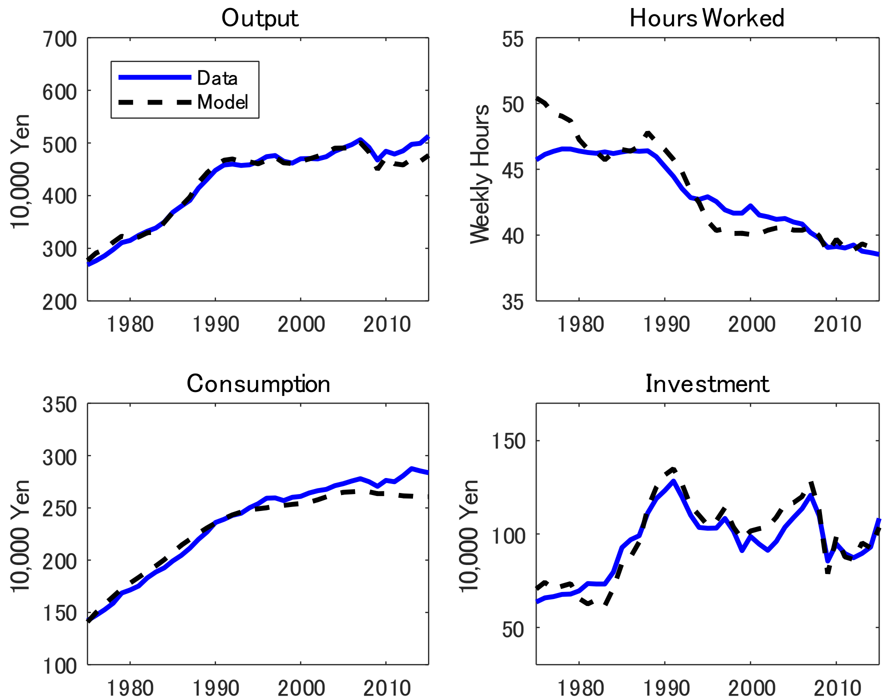


Figure 10: Counterfactual Simulation C: Constant Population Growth

model implying that the decline in the population growth rate increased output due to a decrease in capital dilution.

Figure 11 shows the simulation results for the model with constant labor taxes. The model with constant labor income tax has higher output than the benchmark model which implies that the increase in labor income tax reduced output. The main reason is the depressing effect of labor income tax on hours worked per worker. Consumption and investment are also depressed due to the reduced income.

Figure 12 presents the simulation results from the model with constant capital income tax. The simulation shows that the rise in capital income tax reduced output during the bubble period. The main reason is the depressing effect on capital accumulation. Investment is particularly affected during the 1980s when the tax rate increased dramatically and less so onwards as

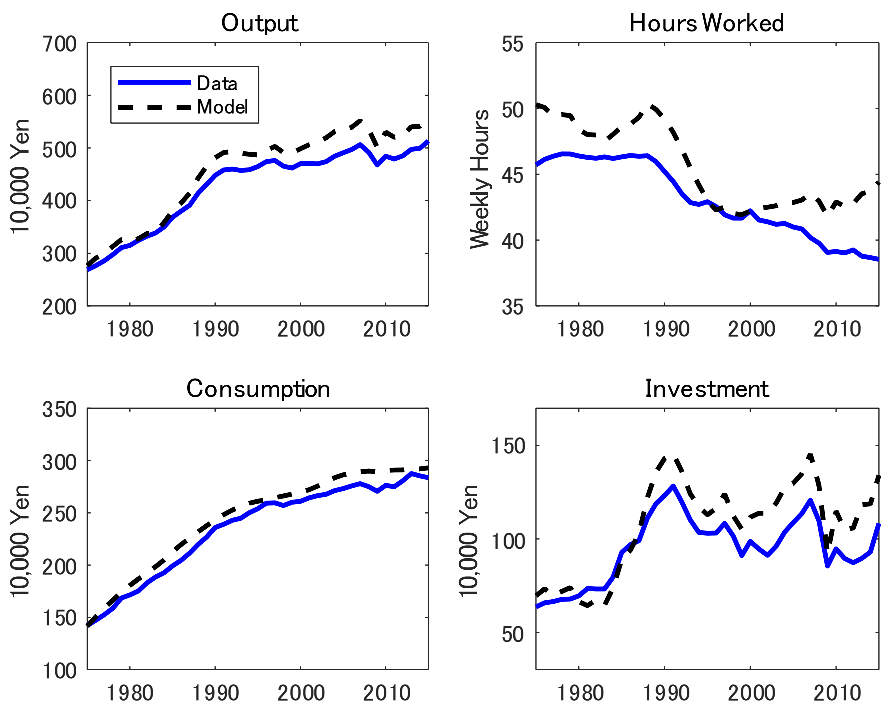


Figure 11: Counterfactual Simulation D: Constant Labor Taxes

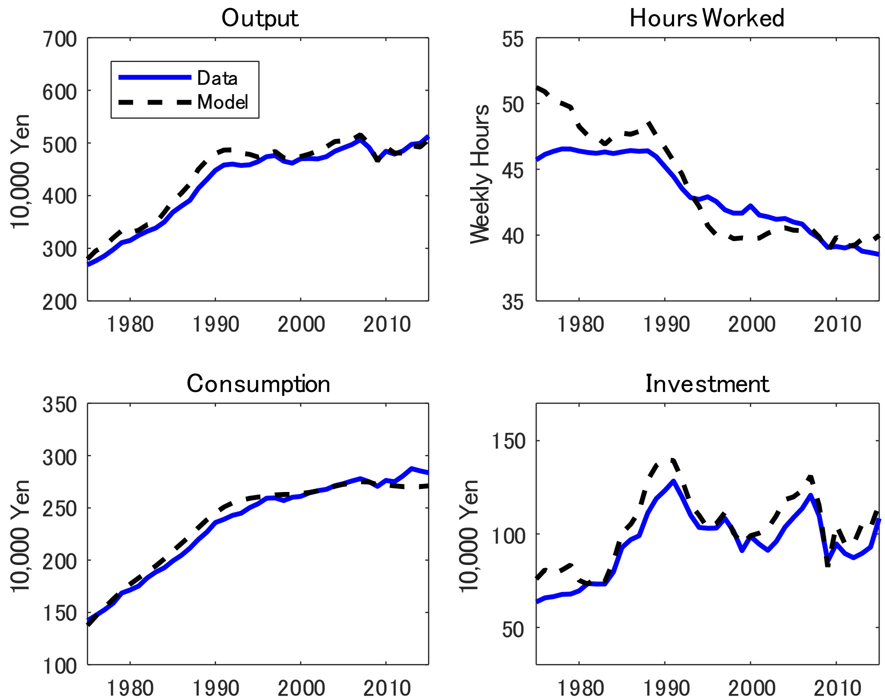


Figure 12: Counterfactual Simulation E: Constant Capital Tax

it declined. The rise in capital income tax also depresses hours worked as the decline in capital accumulation decreases the marginal product of labor. Consumption is depressed due to the reduced income.

Figure 13 presents the simulation results from the model with constant government expenditure. The simulation shows that the fluctuation in government spending had little effect on the economy.

Figure 14 shows the simulation result for the model with constant workweek. The results show that if the workweek remained constant output would have been higher than the benchmark model which implies that the reduction in workweek reduced output. In fact, all of the decline in hours worked during the 1990s can be attributed to the workweek reduction policy. Consumption and investment is also depressed by this policy due to the decline in income.

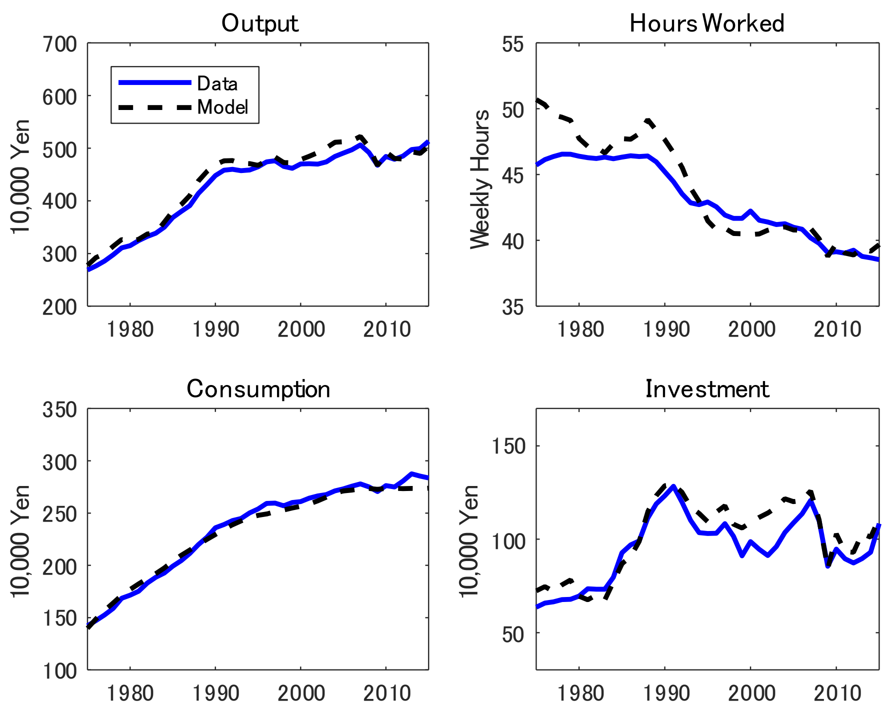


Figure 13: Counterfactual Simulation F: Government Expenditure

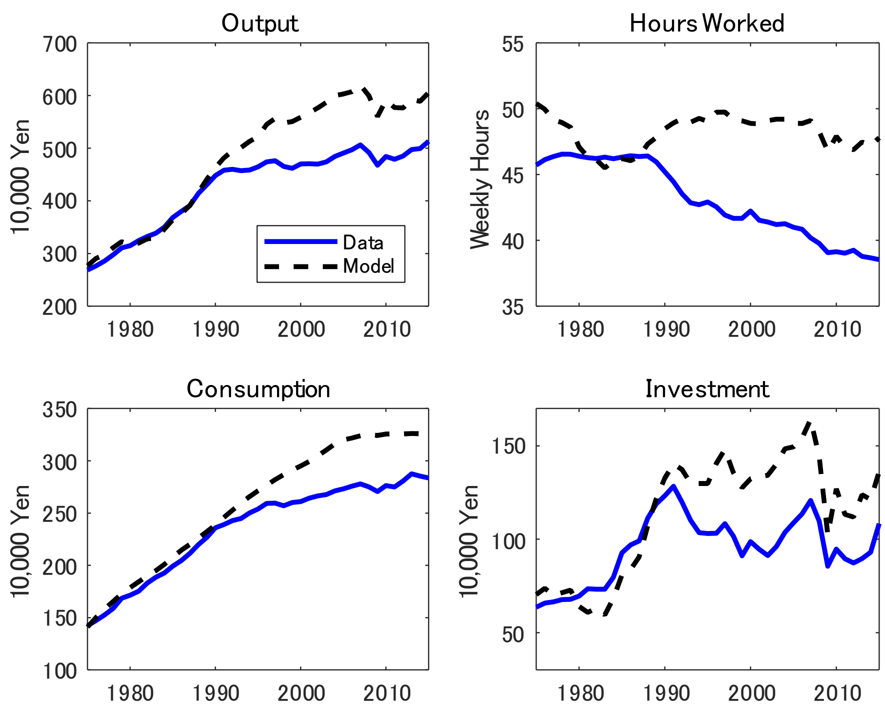


Figure 14: Counterfactual Simulation G: Constant Workweek

Table 8 summarizes the long run effects of each exogenous variable. The first row provides the output level in 2015 from the benchmark simulation while the rows below provide that from the counterfactual simulations. We also provide the difference between the benchmark simulation and the counterfactual simulations which shows the effect of the exogenous variable we left out in each simulation. For instance, the simulation with constant productivity predicts 2015 output per capita to be 50 percent below the benchmark level. Therefore, productivity growth is responsible for 50 percent of the output per capita in 2015. Population aging reduces the per capita output in 2015 by 8.0 percent while population shrinking increases it by 5.4 percent. Furthermore, labor tax reduces the per capita output in 2015 by 11.0 percent while capital tax and government spending had little effect on per capita output. Finally, the workweek shortening reduced per capita output in 2015 by 20.3 percent.

Table 8. Long run Effects of Exogenous Variables

	2015 Output per capita mil. Yen	Difference from Benchmark Percentage
Benchmark	503.6	—
Productivity	251.7	-50.0
Aged Ratio	543.8	8.0
Population Growth	476.6	-5.4
Labor Tax	559.0	11.0
Capital Tax	506.6	0.6
Government Expenditure	504.0	0.1
Workweek	605.7	20.3

5 Population Aging and Structural Transformation

Up to now we have treated the changes in government spending as exogenous. We believe that popualtion aging could have affected the composition of governemnt spending by raising the demand of health care services relative to other goods. Figure 15 plots the share of government medical service expenditure among total government consumption. In this section, we explicitly model how the rise in the aged population share can affect government

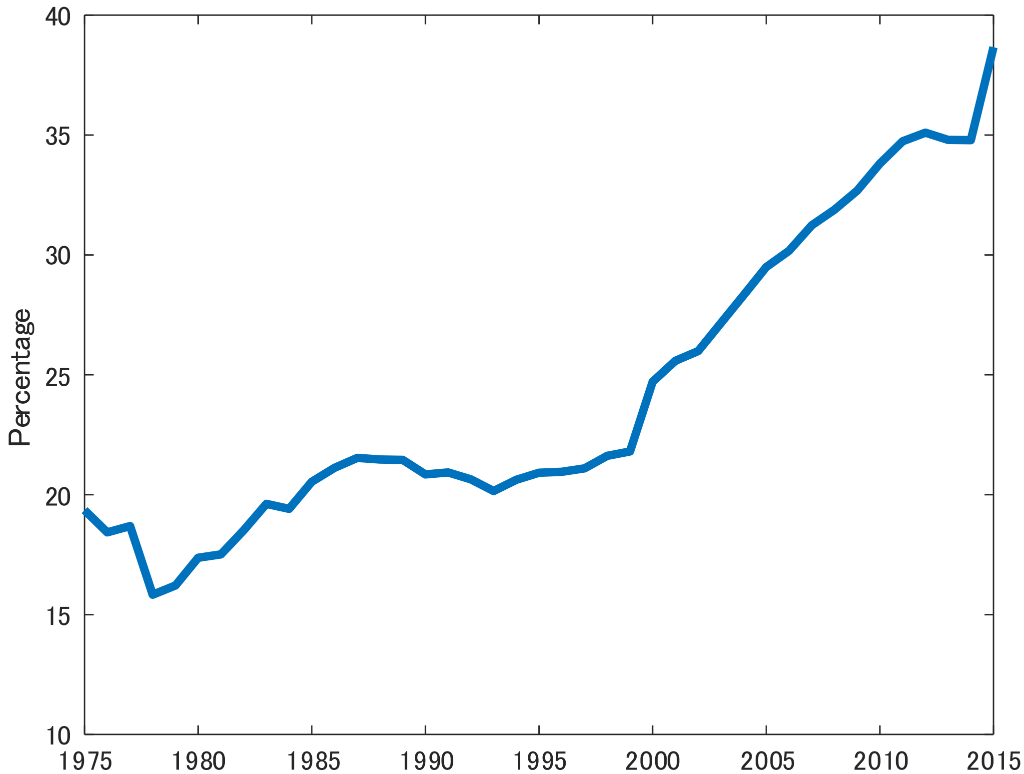


Figure 15: Government Spending Share of Medical Services

spending and productivity through structural transformation.

5.1 Household's Problem

Imagine that consumption consists of consumption expenditure of goods and medical services. Then, we can modify the definition of consumption of the young and old adults in (1) as follows

$$
\begin{aligned}
c_{y,t} &= c_{yg,t}^{\omega_y} c_{ys,t}^{1-\omega_y}, \\
c_{o,t} &= c_{og,t}^{\omega_o} c_{os,t}^{1-\omega_o},
\end{aligned}
$$

where $c_{ij,t}$ stands for the consumption of age group i for consumption type j.

The budget constraint (3) becomes

$$\begin{aligned} & \eta_t(c_{yg,t} + (1 - s_y)p_t c_{ys,t}) + (1 - \eta_t)(c_{og,t} + (1 - s_o)p_t c_{os,t}) + i_t \\ = & (1 - \tau_{l,t})\, w_t h_t e_t + (1 - \tau_{k,t})\, r_t k_t + \zeta_t, \end{aligned}$$

where everything is denominated at the price of goods and p_t is the price of medical services relative to goods. We also assume that the government subsidizes the purchase of medical services which are considered as government consumption in the national accounts. We assume that consumption goods and investment goods are identical manufactured goods.

Household optimality implies

$$\begin{aligned} \frac{c_{yg,t}}{c_{ys,t}} &= \frac{\omega_y}{1 - \omega_y}(1 - s_y)p_t, \\ \frac{c_{og,t}}{c_{os,t}} &= \frac{\omega_o}{1 - \omega_o}(1 - s_o)p_t, \\ \frac{c_{yg,t}}{c_{og,t}} &= \frac{\omega_y}{\omega_o}, \\ \frac{c_{ys,t}}{c_{os,t}} &= \frac{1 - \omega_y}{1 - \omega_o}\frac{1 - s_o}{1 - s_y}. \end{aligned}$$

The total expenditure of the two types of consumtion can be aggregated as

$$\begin{aligned} c_{g,t} &= \eta_t c_{yg,t} + (1 - \eta_t)c_{og,t}, \\ c_{s,t} &= \eta_t c_{ys,t} + (1 - \eta_t)c_{os,t}. \end{aligned}$$

From the household equilibirum conditions, we can derive the nominal consumption expenditure ratio

$$\frac{p_t c_{s,t}}{c_{g,t}} = \frac{\eta_t \frac{1-\omega_y}{1-s_y} + (1 - \eta_t)\frac{1-\omega_o}{1-s_o}}{\eta_t \omega_y + (1 - \eta_t)\omega_o}.$$

It is straightforward to show that when $\omega_y > \omega_o$ and/or $s_y < s_o$ population aging tends to increase the government expenditure share on medical services.[5]

[5] We present a more general case with CES preferences in the appendix.

5.2 Government Expenditure

In this section we asume that the government allocates total expenditure between medical subsidies S_t, other consumption $C_{g,t}$ and investment $I_{g,t}$. The government budget constraint can be modified to

$$\begin{aligned} G_t &= S_t + C_{g,t} + I_{g,t} \\ &= \tau_{l,t} w_t h_t e_t N_t + \tau_{k,t} r_t K_t - \zeta_t N_t. \end{aligned}$$

where total medical subsidies is defined as

$$S_t = \eta_t s_y p_t c_{ys,t} + (1-\eta_t) s_o p_t c_{os,t}.$$

In order to understand how population aging affects government expenditure, we first derive the ratio of government expenditure on health care services to total household consumption expenditure:

$$\frac{S_t}{C_t} = \eta_t \frac{s_y}{1-s_y}(1-\omega_y) + (1-\eta_t)\frac{s_o}{1-s_o}(1-\omega_o). \tag{8}$$

Next, we assume that the government exogenously determines the ratio of total government expenditure to private consumption $\frac{G_t}{C_t}$ instead of that to output $\frac{G_t}{Y_t}$. Therefore, the share of subsidies among total government expenditures is directly affected by population aging through its effect on $\frac{S_t}{C_t}$:

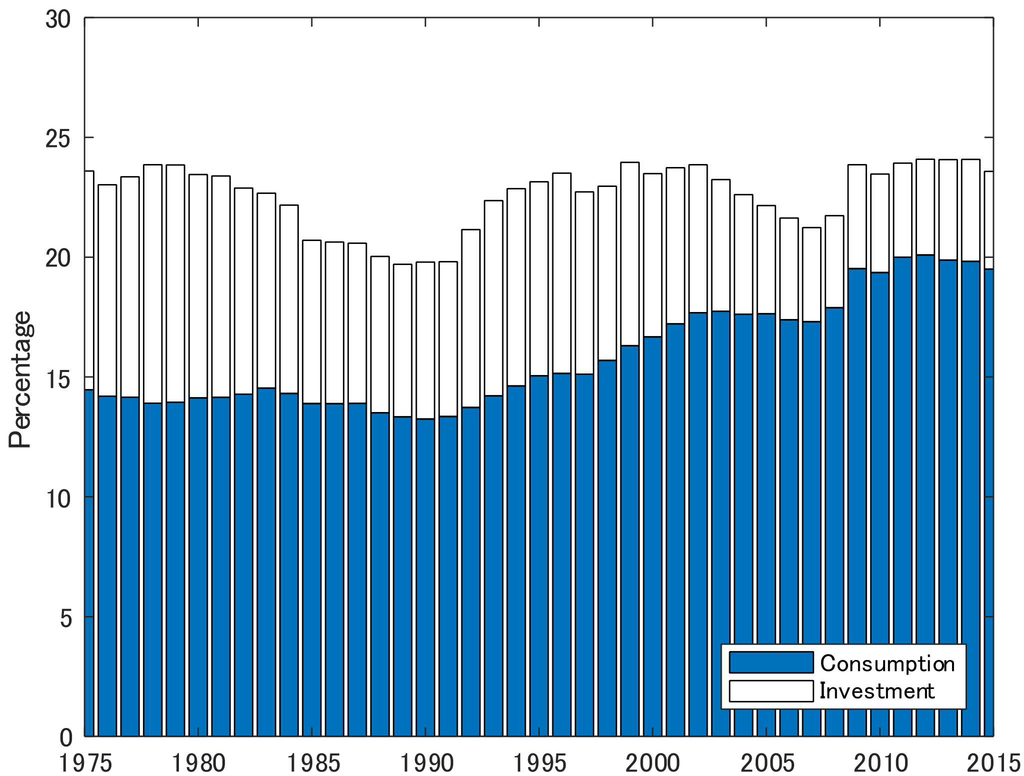
$$\frac{S_t}{G_t} = \frac{S_t}{C_t} \times \frac{C_t}{G_t}.$$

In other words, population aging increases the share of medical subsidies among total government expenditure when $\omega_y > \omega_o$ and/or $s_y < s_o$.

Figure 16 presents the breakdowns of total government expenditure to GNP ratio. The top white area represents the government investment to GNP ratio while the bottom colored area represents the government consumption to GNP ratio over the 1975-2015 period. It is clear that government investment has shrunk relative to GNP as the share of government consumption has grown. This is potentially harmful for the economy if public capital contributes to aggregate productivity.

Consider a production function

$$Y_t = A_{g,t} K_{g,t}^{\phi} K_{p,t}^{\theta} L_t^{1-\theta}$$


30
25
20
15
10
5
0
Percentage
1975
1980
1985
1990
1995
2000
2005
2010
2015
Consumption
Investment

where $K_{g,t}$ stands for government capital and $K_{p,t}$ stands for private capital. We denote $A_{g,t}$ as the productivity adjusted for government capital where the unadjusted productivity is

$$A_t = A_{g,t} K_{g,t}^{\phi}.$$

Therefore, the slow down in governemnt capital accumulation leads to a slow down in the growth of unadjusted productivity A_t when $\phi > 0$.

We further argue that the increase in the government consumption share of medical services is raising pressure on other government consumption components that are related to productivity growth. The first channel is through government capital accumulation. The second channel is through human capital accumulation. Table 9 presents the evolution of government spending components: general public services, national defense, public safety, economic services, environment preservation, public housing, medical services, culture, education, and welfare, over the 1980-2015 period.[6] It is clear that, among the ten components, medical services is the only component that shows a clear growing trend. In contrast, consumption on general services and education services has clearly declined. The later is potentially harmful for the economy if governemnt expenditure on education services leads to human capital accumulation.

	Gen.	Def.	Safety	Econ.	Env.	Hous.	Med.	Cul.	Edu.	Welf.
1980	14.5	5.4	7.6	11.5	3.0	1.3	22.6	1.3	20.9	12.0
1985	13.5	5.9	7.1	11.5	3.0	1.4	24.3	1.4	20.4	11.7
1990	14.0	5.9	6.8	11.8	3.0	1.5	24.5	1.5	19.4	11.5
1995	13.2	5.0	7.1	11.9	3.2	1.8	25.5	1.8	18.3	12.1
2000	12.7	4.7	6.7	12.5	3.3	1.8	27.9	1.9	17.2	11.2
2005	11.9	4.6	6.4	13.6	3.5	1.9	30.4	1.7	15.8	10.2
2010	10.4	4.2	6.1	13.3	3.0	2.0	33.3	1.5	15.1	11.3
2015	8.4	4.3	5.9	12.2	3.2	1.8	36.9	1.4	13.2	12.6

[6] The classification of government spending components have changed from SNA68 to SNA93 so we cannot go beyond 1980.

6 Conclusion

In this paper we constructed a dynamic general equilibrium model to quantitative analyze the impacts of demographics, productivity and government policy on the Japanese economy during the 1975-2014 period. We find that total factor productivity is necessary to account for the growth, population aging, the increase in social security contribution and the workweek shortening policy significantly dampened economic growth. We further show that population aging can account for a significant portion of the structural transformation from goods to services and the increase in government consumption. This could potentially account for part of the recent slow down in productivity growth.

In order to simplify the computation, we have made several assumption in the model. First, we assume a representative household with only young and old people. This is as if we assume a overlapping generation model with complete capital markets and full altruistic bequests. It is interesting to investigate whether our results hold in an overlapping generation setting as in Chen, Imrohoroglu and Imrohoroglu (2007) and Braun, Ikeda and Joines (2009). Second, we assumed log utility which led to equal consumption levels across age groups. Allowing more general utility functions can enable us to investigate issues such as inter-generational inequality. Third, we took the employment rates of each group as a constant. The data shows that the employment rate of the young is slightly increasing reflecting the increase in female labor market participation. On the other hand, the employment rate of the old has been declining over time reflecting the increase in the share of the old-old who are incapable to work. Therefore, incorporating the changes in employment rates in each group should increase the output dampening effect of population aging. While these are all interesting extensions, we will leave these for future research as they are beyond the scope of this paper.

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A Detailed Derivation of the Structural Transformation Model

$$
\begin{aligned}
c_{y,t} &= \left(\omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}, \\
c_{o,t} &= \left(\omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}},
\end{aligned}
$$

The budget constraint (3) becomes

$$
\begin{aligned}
& \eta_t(c_{yg,t} + p_t(1-s_y)c_{ys,t}) + (1-\eta_t)(c_{og,t} + (1-s_o)p_t c_{os,t}) + i_t \\
= \;& (1-\tau_{l,t})\, w_t h_t e_t + (1-\tau_{k,t})\, r_t k_t + \zeta_t,
\end{aligned}
$$

where everything is denominated at the price of goods and p_t is the price of services relative to goods. We assume that consumption goods and investment goods are identical goods. Household optimality implies

$$\begin{aligned}
\Psi \frac{\omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{-\frac{1}{\varepsilon}}}{\left(\omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)} &= \Psi \omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{-\frac{1}{\varepsilon}} c_{y,t}^{\frac{1-\varepsilon}{\varepsilon}} = \lambda_t, \\
\Psi \frac{(1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{-\frac{1}{\varepsilon}}}{\left(\omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)} &= \Psi (1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{-\frac{1}{\varepsilon}} c_{y,t}^{\frac{1-\varepsilon}{\varepsilon}} = \lambda_t (1-s_y) p_t, \\
\Psi \frac{\omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{-\frac{1}{\varepsilon}}}{\left(\omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)} &= \Psi \omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{-\frac{1}{\varepsilon}} c_{o,t}^{\frac{1-\varepsilon}{\varepsilon}} = \lambda_t, \\
\Psi \frac{(1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{-\frac{1}{\varepsilon}}}{\left(\omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)} &= \Psi (1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{-\frac{1}{\varepsilon}} c_{o,t}^{\frac{1-\varepsilon}{\varepsilon}} = \lambda_t (1-s_o) p_t,
\end{aligned}$$

so that

$$\begin{aligned}
\frac{c_{yg,t}}{c_{ys,t}} &= \frac{\omega_y}{1-\omega_y} \left((1-s_y)p_t\right)^{\varepsilon}, \\
\frac{c_{og,t}}{c_{os,t}} &= \frac{\omega_o}{1-\omega_o} \left((1-s_o)p_t\right)^{\varepsilon}, \\
\frac{c_{yg,t}}{c_{og,t}} &= \frac{\omega_y}{\omega_o} \left(\frac{c_{y,t}}{c_{o,t}}\right)^{1-\varepsilon}, \\
\frac{c_{ys,t}}{c_{os,t}} &= \frac{1-\omega_y}{1-\omega_o} \left(\frac{1-s_o}{1-s_y}\right)^{\varepsilon} \left(\frac{c_{y,t}}{c_{o,t}}\right)^{1-\varepsilon},
\end{aligned}$$

where

$$\begin{aligned}
c_{y,t} &= \left(\omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left(\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1} + 1\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_y)^{\frac{1}{\varepsilon-1}} c_{ys,t} \\
&= \left(1 + \frac{1-\omega_y}{\omega_y}\left((1-s_y)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_y{}^{\frac{1}{\varepsilon-1}} c_{yg,t}, \\
c_{o,t} &= \left(\omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left(\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1} + 1\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_o)^{\frac{1}{\varepsilon-1}} c_{os,t} \\
&= \left(1 + \frac{1-\omega_o}{\omega_o}\left((1-s_o)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_o{}^{\frac{1}{\varepsilon-1}} c_{og,t}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{c_{y,t}}{c_{o,t}} &= \frac{\left(1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_y)^{\frac{1}{\varepsilon-1}}}{\left(1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_o)^{\frac{1}{\varepsilon-1}}} \frac{c_{ys,t}}{c_{os,t}}, \\
&= \frac{\left(1+\frac{1-\omega_y}{\omega_y}\left((1-s_y)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_y{}^{\frac{1}{\varepsilon-1}}}{\left(1+\frac{1-\omega_o}{\omega_o}\left((1-s_o)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_o{}^{\frac{1}{\varepsilon-1}}} \frac{c_{yg,t}}{c_{og,t}}. \\
\frac{c_{yg,t}}{c_{og,t}} &= \frac{\omega_y}{\omega_o}\left(\frac{\left(1+\frac{1-\omega_y}{\omega_y}\left((1-s_y)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_y{}^{\frac{1}{\varepsilon-1}}}{\left(1+\frac{1-\omega_o}{\omega_o}\left((1-s_o)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_o{}^{\frac{1}{\varepsilon-1}}} \frac{c_{yg,t}}{c_{og,t}}\right)^{1-\varepsilon} \\
&= \frac{1+\frac{1-\omega_o}{\omega_o}\left((1-s_o)p_t\right)^{1-\varepsilon}}{1+\frac{1-\omega_y}{\omega_y}\left((1-s_y)p_t\right)^{1-\varepsilon}} \\
\frac{c_{ys,t}}{c_{os,t}} &= \frac{1-\omega_y}{1-\omega_o}\left(\frac{1-s_o}{1-s_y}\right)^{\varepsilon}\left(\frac{\left(1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_y)^{\frac{1}{\varepsilon-1}}}{\left(1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_o)^{\frac{1}{\varepsilon-1}}} \frac{c_{ys,t}}{c_{os,t}}\right)^{1-\varepsilon}, \\
&= \frac{1-s_o}{1-s_y} \frac{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}}.
\end{aligned}$$

Then the consumption expenditure of the household for each type of product is

$$\begin{aligned}
c_{s,t} &= \eta_t(1-s_y)c_{ys,t} + (1-\eta_t)(1-s_o)c_{os,t}, \\
&= \left(\eta_t \frac{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}} + 1 - \eta_t\right)(1-s_o)c_{os,t}, \\
c_{g,t} &= \eta_t c_{yg,t} + (1-\eta_t)c_{og,t}, \\
&= \left(\eta_t \frac{1+\frac{1-\omega_o}{\omega_o}\left((1-s_o)p_t\right)^{1-\varepsilon}}{1+\frac{1-\omega_y}{\omega_y}\left((1-s_y)p_t\right)^{1-\varepsilon}} + 1 - \eta_t\right)c_{og,t},
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{p_t c_{s,t}}{c_{g,t}} &= \frac{\left(\eta_t \frac{1-s_o}{1-s_y}\frac{1+\frac{\omega_o}{1-\omega_o}((1-s_o)p_t)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}((1-s_y)p_t)^{\varepsilon-1}} + 1 - \eta_t\right)}{\left(\eta_t \frac{1+\frac{1-\omega_o}{\omega_o}((1-s_o)p_t)^{1-\varepsilon}}{1+\frac{1-\omega_y}{\omega_y}((1-s_y)p_t)^{1-\varepsilon}} + 1 - \eta_t\right)} \frac{(1-s_o)p_t c_{os,t}}{c_{og,t}}, \\
&= \frac{\left(\eta_t \frac{1-s_o}{1-s_y}\frac{1+\frac{\omega_o}{1-\omega_o}((1-s_o)p_t)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}((1-s_y)p_t)^{\varepsilon-1}} + 1 - \eta_t\right)}{\left(\eta_t \frac{1+\frac{1-\omega_o}{\omega_o}((1-s_o)p_t)^{1-\varepsilon}}{1+\frac{1-\omega_y}{\omega_y}((1-s_y)p_t)^{1-\varepsilon}} + 1 - \eta_t\right)} \frac{(1-s_o)p_t}{\left(\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon}\right)}.
\end{aligned}$$

Next, total consumption expenditure is

$$\begin{aligned}
C_t &= \eta_t(c_{yg,t} + (1-s_y)p_t c_{ys,t}) + (1-\eta_t)(c_{og,t} + (1-s_o)p_t c_{os,t}) \\
&= .\eta_t\left(\frac{c_{yg,t}}{c_{ys,t}} + (1-s_y)p_t\right)\frac{c_{ys,t}}{c_{os,t}}c_{os,t} + (1-\eta_t)\left(\frac{c_{og,t}}{c_{os,t}} + (1-s_o)p_t\right)c_{os,t} \\
&= \eta_t\left(\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon} + p_t(1-s_y)\right)\frac{1-s_o}{1-s_y}\frac{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}}c_{os,t} \\
&\quad +(1-\eta_t)\left(\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon} + (1-s_o)p_t\right)c_{os,t} \\
&= \eta_t\left(1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}\right)(1-s_y)p_t\frac{1-s_o}{1-s_y}\frac{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}}c_{os,t} \\
&\quad +(1-\eta_t)\left(1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}\right)(1-s_o)p_t c_{os,t} \\
&= \left(1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}\right)(1-s_o)p_t c_{os,t}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
\frac{S_t}{C_t} &= \frac{\eta_t s_y p_t c_{ys,t} + (1-\eta_t) s_o p_t c_{os,t}}{\left(1 + \frac{\omega_o}{1-\omega_o} \left((1-s_o)p_t\right)^{\varepsilon-1}\right)(1-s_o)p_t c_{os,t}} \\
&= \frac{\left(\eta_t s_y \frac{c_{ys,t}}{c_{os,t}} + (1-\eta_t) s_o\right) p_t c_{os,t}}{\left(1 + \frac{\omega_o}{1-\omega_o} \left((1-s_o)p_t\right)^{\varepsilon-1}\right)(1-s_o)p_t c_{os,t}} \\
&= \frac{\left(\eta_t s_y \frac{1-s_o}{1-s_y} \frac{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}} + (1-\eta_t) s_o\right)}{\left(1 + \frac{\omega_o}{1-\omega_o} \left((1-s_o)p_t\right)^{\varepsilon-1}\right)(1-s_o)} \\
&= \eta_t \frac{s_y}{1-s_y} \frac{1}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}} + (1-\eta_t)\frac{s_o}{1-s_o}\frac{1}{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}
\end{aligned}
$$