Testing for time variation in the Natural Rate of Interest

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Abstract

This paper replicates in a wider sense the unobserved components model of Laubach and Williams (REStat 2003; 85:1063-1070) to estimate the natural rate of interest (NRI) and investigates the role of model uncertainty. A stochastic Bayesian model selection procedure is employed to test the hypothesis of time variation in the NRI against a constant NRI. The model selection confirms time variation in the NRI as a result of changes in potential output growth, but other determinants of the NRI are found constant.

JEL classification: C11, C32; E43; O40

Keywords: Natural rate of interest; Trend growth, Bayesian model selection

1 Introduction

In an influential paper, Laubach and Williams (2003, LW hereafter) use an unobserved components (UC) model to measure the natural rate of interest (NRI), defined as the real interest rate consistent with a zero output gap and stable inflation. In the model the NRI is determined by two distinct factors, the trend growth rate of output and 'other determinants', with the latter reflecting the secular pattern of (global) savings and investment. LW estimate the model using the Kalman filter and maximum likelihood (ML) and find that both factors contribute to the time variation of the NRI.[1] Holston, Laubach and Williams (2017, hereafter HLW) re-estimate the NRI with the LW model using more recent data, and report that the U.S. NRI has steadily declined over the past quarter century to historically low levels of close to zero.[2]

In this paper we replicate in a wider sense the LW analysis utilizing the HLW data for the U.S. economy. Instead of using the sequential median unbiased estimator, we estimate the model using Bayesian techniques.[3] Additionally, we employ a stochastic Bayesian model selection procedure

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[1]Specifically, LW use the median unbiased estimator proposed by Stock and Watson (1998), which is a sequential estimation procedure to account for the pile-up problem.

[2]This secular decline may be caused both by a reduced trend growth rate of output, itself potentially due to a slowdown in productivity growth, and by long-run changes in the other determinants, possibly as a consequence of the global savings glut, persistent declines in investment rates, or the increased demand for safe assets since the global financial crisis (Bernanke (2005); Blanchard, Furceri, and Pescatori (2014); IMF (2014)).

[3]Pescatori and Turunen (2015) find that a Bayesian approach yields more plausible results than maximum likelihood estimates for the unobserved variables in the LW model.

to test the hypothesis of time variation in the NRI against a constant NRI. We also account for measurement errors and postulate a stochastic process for the interest rate gap. A further extension is that we allow (but do not force) output to exhibit stochastic volatility. This way we account for potentially larger disturbances during the Great Recession of 2007-09. The model selection confirms time variation in the NRI due to changes in potential output growth, but we find no evidence of time variation arising from other determinants of the NRI. Whereas HLW identify the other determinants to account for part of the downward trend in the NRI, we find the natural rate to have fallen less strongly, particularly towards the end of the sample period.

2 The Empirical Model

LW use a semi-structural UC model of real GDP, y_t, inflation, π_t, and the real interest rate, r_t, i.e.

$$y_t = y_t^* + y_t^c + \varepsilon_t^y, \qquad \varepsilon_t^y \overset{iid}{\sim} \mathcal{N}(0, \sigma_{\varepsilon,y}^2), \tag{1}$$

$$\pi_t = \beta_\pi \pi_{t-1} + (1-\beta_\pi)\pi_{t-2,4} + \beta_y y_{t-1}^c + \varepsilon_t^\pi, \qquad \varepsilon_t^\pi \overset{iid}{\sim} \mathcal{N}(0, \sigma_{\varepsilon,\pi}^2), \tag{2}$$

$$r_t = r_t^* + r_t^c + \varepsilon_t^r, \qquad \varepsilon_t^r \overset{iid}{\sim} \mathcal{N}(0, \sigma_{\varepsilon,r}^2), \tag{3}$$

where y_t^* denotes potential output and y_t^c is the output gap. π_t is the inflation rate and $\pi_{t-2,4}$ constitutes the average of the second to fourth lags of inflation. The aggregate demand equation relates the output gap to its own lags and two lags of the real interest rate gap, i.e.

$$y_t^c = \phi_1 y_{t-1}^c + \phi_2 y_{t-2}^c + \frac{a_r}{2}\sum_{j=1}^{2} r_{t-j}^c + \exp\{h_t^c\}\eta_t^{y^c}, \qquad \eta_t^{y^c} \overset{iid}{\sim} \mathcal{N}(0,1). \tag{4}$$

The evolution of potential output and the NRI rate is linked through the trend growth rate of potential output, g, modeled as a random walk:

$$y_t^* = y_{t-1}^* + g_{t-1} + \exp\{h_t^{y^*}\}\eta_t^{y^*}, \qquad \eta_t^{y^*} \overset{iid}{\sim} \mathcal{N}(0,1) \tag{5}$$

$$g_t = g_{t-1} + \eta_t^g, \qquad \eta_t^g \overset{iid}{\sim} \mathcal{N}(0, \sigma_{\eta,g}^2) \tag{6}$$

$$r_t^* = c g_{t-1} + z_t, \quad \text{where} \quad z_t = z_{t-1} + \eta_t^z, \qquad \eta_t^z \overset{iid}{\sim} \mathcal{N}(0, \sigma_{\eta,z}^2) \tag{7}$$

$$r_t^c = \omega r_{t-1}^c + \eta_t^{r^c}, \qquad \eta_t^{r^c} \overset{iid}{\sim} \mathcal{N}(0, \sigma_{\eta,r^c}^2) \tag{8}$$

The stochastic volatility terms $\exp\{h_t^c\}$ and $\exp\{h_t^{y^*}\}$ in the innovations to the trend and the cycle are included to allow for changes in the size to shocks to output, e.g. a particularly large negative output gap during the Great Recession. The standard assumption of homoscedasticity may result in a biased estimate of potential output and its growth rate in periods of large transitory shocks. However, a biased estimate of potential output growth would lead to a biased estimate of the NRI. Specifically, if the output gap is underestimated during the Great Recession due to assuming homoscedastic shocks, potential output, potential output growth and consequently the NRI would be biased downwards. The stochastic volatilities are modeled as random walks

$$h_t^k = h_{t-1}^k + \tau_t^k, \qquad \tau_t^k \overset{iid}{\sim} \mathcal{N}(0, \sigma_{\tau,k}^2), \tag{9}$$

for $k = c, y^*$.

The NRI is a function of potential output growth and the component z_t. Eqs. (6)-(7) imply that (i) real GDP is an $I(2)$ process and (ii) that the NRI varies over time and follows a random walk.[4] Testing these two assumptions is not straightforward, as it requires testing the null hypothesis of two variance parameters being equal to zero. Specifically, testing assumption (i) implies testing $\sigma^2_{\eta,g} = 0$ against $\sigma^2_{\eta,g} > 0$ and assumption (ii) implies testing $\sigma^2_{\eta,g} = \sigma^2_{\eta,z} = 0$ against $\sigma^2_{\eta,g} > 0$ and/or $\sigma^2_{\eta,z} > 0$, which is a non-regular testing problem as the null hypothesis lies on the boundary of the parameter space. In the next section we introduce a Bayesian model selection procedure that tests assumptions (i) and (ii). Furthermore, we test whether shocks to output are subject to time variation, i.e. we test the null of $\sigma^2_{\tau,c} = 0$ and $\sigma^2_{\tau,y^*} = 0$.

3 A Bayesian model selection procedure

In order to test the hypothesis outlined above, we use a Bayesian stochastic model specification search for state space models based on Frühwirth-Schnatter and Wagner (2010). For the model selection procedure, consider a non-centered parameterization for the components $\psi = g_t, z_t, h^c_t, h^{y*}_t$, i.e.

$$\psi_t = \psi_0 + \sigma_{\eta,\psi}\widetilde{\psi}_t, \quad \text{with} \quad \widetilde{\psi}_t = \widetilde{\psi}_{t-1} + \kappa_{\psi,t}, \qquad \widetilde{\psi}_0 = 0, \quad \kappa_{\psi,t} \overset{iid}{\sim} \mathcal{N}(0,1) \tag{10}$$

where ψ_0 is the initial value of the level of ψ. A crucial aspect of the non-centered parameterization is that it is not identified, i.e. the signs of $\sigma_{\eta,\psi}$ and $\widetilde{\psi}_t$ can be changed by multiplying both with -1 without changing their product. As a result of the non-identification, the likelihood function is symmetric around 0 along the $\sigma_{\eta,\psi}$ dimension and therefore multimodal. If the component displays time variation, i.e $\sigma^2_{\eta,\psi} > 0$, then the likelihood function will concentrate around the two modes $-\sigma_{\eta,\psi}$ and $\sigma_{\eta,\psi}$. As such, allowing for non-identification of $\sigma_{\eta,\psi}$ provides useful information on whether $\sigma^2_{\eta,\psi} > 0$. A second advantage of the non-centered parameterization is that when $\sigma^2_{\eta,\psi} = 0$ the transformed component $\widetilde{\psi_t}$, in contrast to ψ_t, does not degenerate to a time-invariant variable as this is now represented by ψ_0. As such, the question whether or not ψ_t is subject to time variation can be expressed as a variable selection problem in eq. (10), i.e.

$$\psi_t = \psi_0 + \delta_\psi \sigma_{\eta,\psi}\widetilde{\psi}_t, \tag{11}$$

where δ_ψ is a binary indicator which is either 0 or 1. If $\delta_\psi = 0$, the component $\widetilde{\psi}_t$ drops from the model such that ψ_0 represents the constant slope parameter. If $\delta_\psi = 1$ then $\widetilde{\psi}_t$ is included in the model and $\sigma_{\eta,\psi}$ is estimated from the data. In this case ψ_0 is the initial value of the component.

Our Bayesian estimation approach requires choosing prior distributions for the model parameters $\rho = (\beta_\pi, \beta_y, \phi_1, \phi_2, a_r, \omega)$, the initial values for the non-centered components $\psi_0 = (g_0, z_0, h^c_0, h^{y^*}_0)$, for the binary indicators $\mathcal{M}$ and for the variances of the idiosyncratic factors $\sigma^2_\varepsilon = \left(\sigma^2_{\varepsilon,y}, \sigma^2_{\varepsilon,\pi}, \sigma^2_{\varepsilon,r}\right)$.

It is well known that when using an inverse Gamma prior distribution for the variance parameters, the choice of the shape and scale hyperparameters that define this distribution have a strong influence on the posterior when the true value of the variance is close to zero. More specifically, as the inverse Gamma does not have probability mass at zero, using it as a prior distribution tends to push the posterior density away from zero. This is of particular importance when estimating the variances of

[4] Laubach and Williams (2003) mention difficulties estimating c using ML and Holston, Laubach, and Williams (2017) assume c to be unity.

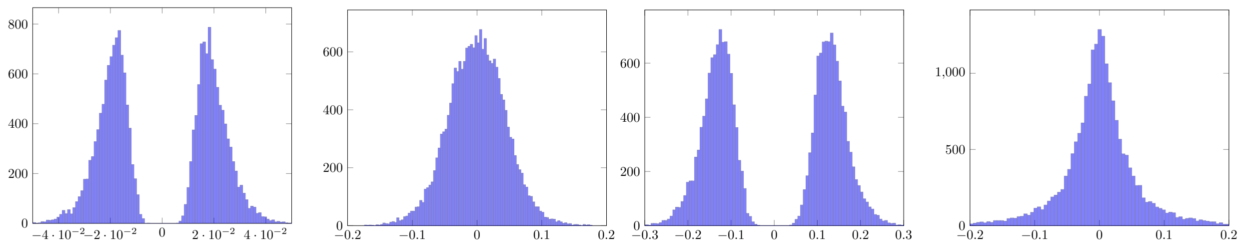
the innovations to potential output drift and the z component, which jointly determine the NRI, as for these components we want to determine their time variability. A further important advantage of the non-centered parameterization is therefore that it allows us to replace the standard inverse Gamma prior on a variance parameter σ^2 by a Gaussian prior centered at zero on σ. Centering the prior distribution at zero is appropriate as for both $\sigma^2 = 0$ and $\sigma^2 > 0$, σ is symmetric around zero. Frühwirth-Schnatter and Wagner (2010) show that, compared to using an inverse Gamma prior for σ^2, the posterior density of σ is much less sensitive to the hyperparameters of the Gaussian distribution and is not pushed away from zero when $\sigma^2 = 0$.

As such we choose a Gaussian prior distribution centered at zero with unit variance for $\sigma_{\eta,g}$, $\sigma_{\eta,z}$, $\sigma_{\tau,c}$, and σ_{τ,y^*}. For the variances of the idiosyncratic factors σ^2_ε, we choose a non-informative inverse Gamma prior distribution. For each of the model parameters in ρ, we assume a non-informative normal prior distribution. For the binary indicators $\mathcal{M}$ we choose a uniform prior distribution such that each model component has a prior probability $p_0 = 0.5$ of being included in the model.

4 Results

The model is estimated using a Gibbs sampler for state-space models. For details of the sampling scheme we refer to Frühwirth-Schnatter and Wagner (2010). We first estimate an unrestricted model with all binary indicators set to unity to generate posterior distributions for the standard deviations $(\sigma_{\eta,\psi})$ of the innovations to the 4 non-centered components of interest. If these distributions are bimodal, with low or no probability mass at zero, this can be taken as a first indication of time variation in the considered component. Results are shown in Figure 1.

Figure 1: Posterior distributions of the standard deviations



(a) $\sigma_{\eta,g}$ **(b)** $\sigma_{\eta,z}$ **(c)** $\sigma_{\tau,c}$ **(d)** σ_{τ,y^*}

Note: Posterior distributions of the standard deviations for potential output growth ($\sigma_{\eta,g}$), component z_t ($\sigma_{\eta,z}$), and the stochastic volatility processes to the output gap ($\sigma_{\tau,c}$) and potential output (σ_{τ,y^*}) in the unrestricted model (all binary indicators set to 1).

Clear-cut bimodality is found in the posterior distribution of the standard deviation of the innovations to potential output growth ($\sigma_{\eta,g}$) and to the volatility of the output gap ($\sigma_{\tau,c}$). Regarding the innovations to the component z_t and the volatility to potential output, the posterior distributions of $\sigma_{\eta,z}$ and σ_{τ,y^*} are clearly unimodal around zero.

As a more formal test for time variation, we next sample the stochastic binary indicators together with the other parameters in the model. Table 1 displays the individual posterior probabilities for the binary indicators being unity. These probabilities are calculated as the average selection frequencies over all iterations of the Gibbs sampler. The second row shows results for our benchmark case of a unit variance for the binary indicators, i.e. $A_0 = 1$. This implies a relatively loose prior on the degree of time variation σ. To check for robustness, the other rows show results for alternative values of A_0. The first row shows results for the case where $A_0 = 0.1$. This corresponds to a relatively stronger prior that allows for less time variation. The third and fourth rows show results for diffuse prior distributions that allow for large variances on the time-varying components.

Table 1: Posterior inclusion probabilities for the binary indicators over different prior variances A_0

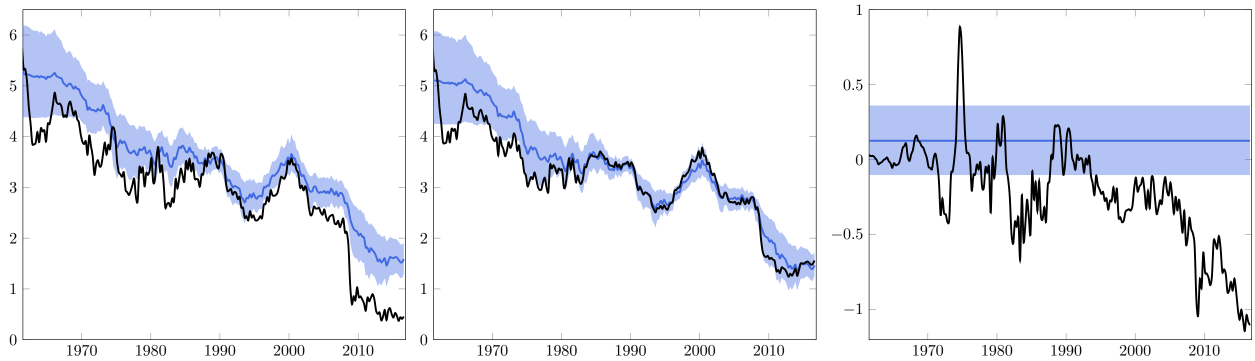
Prior		Posterior			
p_0	A_0	δ_g	δ_z	δ_{τ^c}	$\delta_{\tau_{y^*}}$
0.5	0.1	1	0.62	1	0.37
0.5	1	1	0.33	1	0.10
0.5	10	1	0.14	1	0.02
0.5	100	1	0.06	1	0.002

Table 2 reports the posterior parameter estimates of the model. All parameters are correctly signed and significant with the exception of the (time invariant) component z_0, which turns out to be insignificant.

Table 2: Posterior parameter estimates

	Parameter	Mean	5%	95%
Phillips curve	β_π	0.57	0.48	0.65
	β_y	0.06	0.02	0.10
IS equation	ϕ_1	1.27	1.14	1.39
	ϕ_2	-0.32	-0.44	-0.19
	a_r	-0.05	-0.08	-0.02
	ω	0.93	0.89	0.97
Non-centered components	g_0	1.28	1.06	1.52
	z_0	0.13	-0.10	0.36
	h_0^c	0.51	0.43	0.59
	$h_0^{y^*}$	0.18	0.16	0.21
	$\sigma_{\eta,g}$	0.02	0.01	0.03
	σ_{η,h^c}	0.13	0.07	0.20
Measurement errors	$\sigma^2_{\varepsilon,y}$	0.06	0.00	0.15
	$\sigma^2_{\varepsilon,\pi}$	0.92	0.85	0.98
	$\sigma^2_{\varepsilon,r}$	0.49	0.39	0.61

Figure 2: NRI and its components



(a) NRI **(b)** Potential growth **(c)** z

Note: Reported are the mean, the 5th and the 95th percentile of the posterior distribution of the NRI and its components from our model (blue) and from HLW, 2017 (black).

Fig. 2 plots our estimates of the NRI, potential output growth and the other determinants comprised in component z_t (in blue), together with the results from HLW (in black). We are able to replicate the secular declines of potential growth and the NRI, as indicated by the substantial overlap of the confidence bands of our estimation with the point estimates reported by HLW. However, due to the constancy of z_t, we identify the downward trend of the NRI as solely determined by the receding growth rate of potential output, whereas both LW and HLW also find z_t to fall over time. Whereas this reinforces the decline of the NRI in their analysis, we find the natural rate to have fallen less strongly, particularly towards the end of the sample period.[5]

5 Conclusion

Laubach and Williams (2003) use a small semi-structural unobserved components model to jointly estimate the natural rate of interest (NRI), the output gap and the trend growth rate of output from U.S. data on GDP, inflation and the nominal short-term interest rate. The NRI is modeled as a time-varying process determined by the trend growth rate of output as well as other determinants reflecting (global) patterns of savings and investment. LW estimate the model using the Kalman filter and maximum likelihood (ML) and find that both factors contribute to the time variation of the NRI. In this paper we replicate the analysis with more recent data for the U.S. taken from Holston, Laubach and Williams (2017). However, instead of using the sequential median unbiased estimator, we estimate the model using Bayesian techniques. Moreover, rather than assuming time variability of the natural rate, we use a stochastic Bayesian model selection procedure to explicitly test the hypothesis of time variation in the NRI against a constant NRI. We also account for measurement errors and postulate a stochastic process for the interest rate gap. A further extension is that we allow

[5]The difference in the behavior of the NRI in the aftermath of the Great Recession is also due to the higher growth rate of potential output in our model, as can be seen from Figure 2, although the impact is quantitatively rather small. We also find a stronger and more drawn-out reaction of the output gap after the crisis (not reported here), which stays negative for longer in comparison to the results of HLW. As a consequence, the level of potential output does not recede as strongly in our model, thus accounting for its higher growth rate. In contrast, HLW find a more muted reaction of the output gap such that the movements of actual output are explained to a larger extent by potential output, thereby lowering their estimated growth rate. Our model therefore identifies a somewhat stronger business-cycle component to the decline of the NRI in the aftermath of the Great Recession.

(but do not force) output to exhibit stochastic volatility to account for potentially larger disturbances during the Great Recession of 2007-09.

The model selection confirms time variation in the NRI due to changes in potential output growth, but we find no evidence of time variation arising from other determinants of the NRI. We are able to replicate the secular decline of the NRI, where our results are rather close to those reported by Holston, Laubach and Williams (2017). However, whereas these authors identify the other determinants to account for part of the downward trend in the NRI, we find the natural rate to have fallen less strongly, particularly towards the end of the sample period.

References

Bernanke, B. S. (2005): "The global saving glut and the U.S. current account deficit," Speech 77, Board of Governors of the Federal Reserve System (U.S.).

Blanchard, O., D. Furceri, and A. Pescatori (2014): "A prolonged period of low real interest rates?," in *Secular Stagnation: Facts, Causes and Cures*, ed. by C. Teulings, and R. Baldwin, pp. 101–110. CEPR Press.

Frühwirth-Schnatter, S., and H. Wagner (2010): "Stochastic model specification search for Gaussian and partial non-Gaussian state space models," *Journal of Econometrics*, 154(1), 85–100.

Holston, K., T. Laubach, and J. C. Williams (2017): "Measuring the natural rate of interest: International trends and determinants," *Journal of International Economics*, 108(Supplement 1), 59–75.

IMF (2014): "Perspectives on global real interest rates," in *World Economic Outlook: Recovery Strengthens, Remains Uneven.*, pp. 83–112. International Monetary Fund, Washington, DC.

Laubach, T., and J. C. Williams (2003): "Measuring the Natural Rate of Interest," *The Review of Economics and Statistics*, 85(4), 1063–1070.

Pescatori, A., and J. Turunen (2015): "Lower for Longer; Neutral Rates in the United States," IMF Working Papers 15/135, International Monetary Fund.

Stock, J. H., and M. W. Watson (1998): "Median Unbiased Estimation of Coefficient Variance in a Time-Varying Parameter Model," *Journal of the American Statistical Association*, 93, 349–358.