A Structural Investigation of Monetary Policy Shifts*

Yoosoon Chang[†] Fei Tan[‡] and Xin Wei[§]

November 27, 2017

Abstract

This paper investigates the macroeconomic sources of the U.S. monetary policy shifts within a dynamic stochastic general equilibrium (DSGE) model. We introduce regime switching into the model that links the current regime of monetary policy to the historical fundamental shocks by an autoregressive regime factor. This generates an endogenous feedback mechanism between measured economic behavior and the monetary policy stance. We develop a particle-free variant of the mixture Kalman filter, in conjunction with a solution method that accounts for the endogeneity of switching regimes, to estimate the underlying nonlinear state space model. Our key findings are threefold. First, non-policy shocks, most notably the markup shock, have played a predominant role in driving future regime changes during the post-World War II period. Second, the estimated regime factor identifies monetary policy as slowly fluctuating between more aggressive and less aggressive regimes, in ways that are largely consistent with the conventional view on time variation in the Federal Reserve's policy behavior. Lastly, the Bayes factor strongly favors the endogenous switching version of the model over the exogenous case. We conclude that endogenizing regime changes in monetary DSGE models provides both a theoretically and empirically promising venue for understanding the purposeful nature of monetary policy.

This Version: Very preliminary, please do not quote or circulate.

JEL classification: C11, C32, E31, E52

Keywords: DSGE models, endogenous regime switching, expectations formation effects, Bayesian Analysis

---

*We are grateful to Eric M. Leeper and Joon Y. Park for many insightful comments and constructive suggestions, and also to Boreum Kwak and Bing Li for helpful discussions and feedback. We thank the participants of macro brown bag seminars and econometrics workshop at Indiana University for their comments and suggestions. All remaining errors are our own.

[†]Corresponding author. Department of Economics, Indiana University, Wylie Hall Rm 105, 100 S. Woodlawn, Bloomington IN 47401-7104. E-mail: yoosoon@indiana.edu

[‡]Department of Economics, John Cook School of Business, Saint Louis University; Interdisciplinary Center for Social Sciences, Zhejiang University. E-mail: tanf@slu.edu

[§]Department of Economics, Indiana University. E-mail: wei24@indiana.edu

1 Introduction

Monetary policy behavior is purposeful and responds endogenously to the state of the economy. The empirical work by Clarida et al. (2000), Lubik and Schorfheide (2004) and Sims and Zha (2006) find that Taylor rule describing monetary policy behavior displays important time variation in the United States. Subsequently, researchers have embedded Markov switching processes in dynamic stochastic general equilibrium (henceforth DSGE) models to explore these empirical findings. However, these works assume monetary policy regime shifts as exogenous, and therefore in an important sense the work is inconsistent with a central tenet underlying the Taylor rule: monetary policy behavior reacts endogenously to changes in the macroeconomic environment. For instance, most people who think that policy changed dramatically in the late 1970s in the United States believed this because inflation appeared to be running out of control, not because an exogenously evolving switching process happened to call for a change at that time. This belief calls for a model that makes the policy change a purposeful response of central bank to the state of the economy. Davig and Leeper (2006) took a step toward resolving this inconsistency by building a New Keynesian model with threshold switching monetary policy rule. Specifically, policy regime switches when past inflation crosses a threshold value. But, one would naturally ask: is inflation the true or the only driving force of monetary policy regime shifts? We aim to address this specific question: why have monetary policy regimes shifted and what are the driving forces?

In order to tackle this research question, we consider endogenous regime switching monetary policy, where a latent autoregressive regime factor determines regimes depending upon whether it takes a value above or below some threshold level.[1] Different regimes represent different degrees of aggressiveness of the monetary authority's response of policy rate to inflation deviation from its target. The more aggressive regime is identified as the one with regime factor being higher than the threshold level. The endogeneity is captured by the feedback mechanism in the model structure. First, federal funds rates adjust systematically to target variables consisting of past federal funds rates, inflations and output gaps. Discretionary policy interventions, e.g., surprise changes in the federal funds rate relative to what the Taylor rule mandates, represent how the monetary authority react to non-target information. These policy interventions[2] will generate cumulative impacts on the regime factor that eventually lead to a future regime change. This channel is missing in existing approaches. Second, the response to inflation switches between a more aggressive regime and a less aggressive regime. These switches alter the observed equilibrium process of macroeconomic variables in a way that captures the nonlinear features observed in

[1] The reduced-form version of our model was introduced by Chang et al. (2017).

[2] In our general model presented in Section 4.1, we allow the policy regime factor to depend not only on past monetary policy intervention but also on other non-policy fundamental shocks, and a non-fundamental regime factor shock. For ease of exposition, we use the simpler version in Introduction where only past monetary policy intervention can drive regime shift, which is motivated by Leeper and Zha (2003) and Figure 1.

the data. For example, if today the Fed sets the federal funds rate above the level implied by Taylor rule, this positive (contractionary) intervention predicts tomorrow's regime factor through the endogenous feedback mechanism. Hence, private agents who observe the policy intervention would enhance their expectations of being in a more aggressive regime tomorrow.

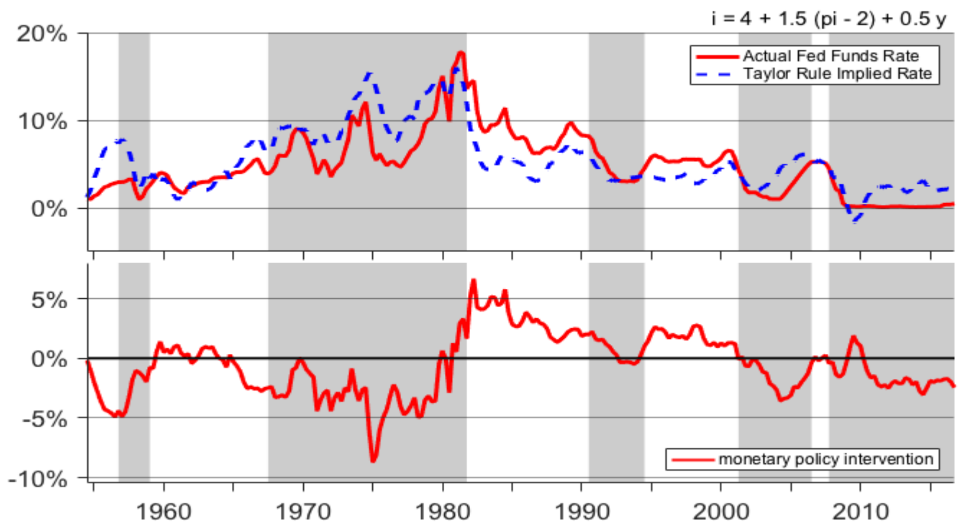
Figure 1 nicely motivates the endogenous feedback mechanism. The upper panel presents the actual federal funds rate and the Taylor rule implied rate. The Taylor rule provides a fairly accurate summary of post-World War II U.S. monetary policy.[3] However, there exists several sizable and persistent discrepancies as shown clearly in the bottom panel of Figure 1. Those are the policy interventions reflecting policy considerations of non-target information, calculated by subtracting the Taylor rule implied rate from the actual federal funds rate. The shaded areas indicate the less aggressive regime identified from the estimation of our DSGE model with endogenous feedback mechanism. Intriguingly, we see that in the early 1980s, monetary policy was too tight relative to the Taylor rule. It means that the Fed was implementing contractionary policy intervention, i.e. surprise increases in the federal funds rate in that period. Private agents observing the unanticipated persistent contractionary intervention would infer that inflation control has become the Fed's top priority and gradually reinforce their belief of a switch from less aggressive regime to more aggressive regime. The adjustment of agents' belief affects their behaviors and induces the expectations formation effects on the economy, which might have helped stabilize the price level in the 1980s.

In our model, the endogenous regime switching is captured by the correlation between current regime factor and past policy intervention. When the correlation is zero, the endogenous feedback mechanism is turned off and our model reduces to the conventional exogenous regime switching DSGE model. The correlation parameter determines to what extent regime change is driven by past monetary policy intervention. Our estimation shows that monetary policy regime change is 76% triggered by past monetary policy interventions, while 24% by contemporaneous non-fundamental regime factor shock. Our numerical experiments also show that the more it is triggered by past monetary policy interventions, the more sizable the expectations formation effects are. This is because agents can utilize more information to form their expectations on future regime evolutions. These findings reinforce the claim by Leeper and Zha (2003) that small but sustained policy intervention may significantly shift agents' beliefs about policy regime and induce the changes in behavior that Lucas (1976) emphasizes.

Besides this substantive finding, we also make two methodological contributions. First, we bring the advanced reduced-form endogenous regime switching approach to the widely used DSGE models. Prior to this paper, endogenous regime switching was studied separately by econometricians and macroeconomists. From an econometric standpoint, Filardo (1994) and Filardo and Gordon

[3] Data and parameters used to construct the Taylor rule are the same as in Taylor (1993), which are shown on the top-right of the plot: i denotes net federal funds rate, pi inflation and y output gap.

Figure 1: Relationship between Monetary Policy Intervention and Policy Regime



Notes: The shaded areas on the top and bottom panels indicate the less aggressive monetary policy regime.

(1998) have estimated Markov switching regressions with time-varying transition probabilities, following the seminal paper by Hamilton (1989). Kim et al. (2008a) consider a regression model where the state dependent coefficients are determined by an i.i.d. latent random variable that is contemporaneously correlated with innovation in regression error. More recently, Chang et al. (2017) introduces a more general model with endogenous regime switching which is driven by a latent autoregressive regime factor correlated with the past innovation to the observed time series. On the other hand, from a macroeconomic standpoint, solution methods for regime switching rational expectations models with time-varying transition probabilities have only become available recently in Maih (2015), Foerster et al. (2016) and Barthélemy and Marx (2017). The time-varying transition probabilities considered in these papers can be introduced, for example, by a prescribed function of some observed endogenous variables. Barthélemy and Marx (2017) in particular apply an implicit function theorem to numerically ensure both determinacy and accuracy of the approximate solution in a regime switching DSGE model with time-varying transition probabilities given as a linear function of past inflation deviation from target inflation. However, the aforementioned significant progress on reduced-form endogenous regime switching literature has not been introduced yet to construct and estimate the regime switching rational expectations models. Our paper contributes to advance in such a direction. We introduce a threshold-type endogenous regime switching into the model and derive from it smooth TVTPs, rather than simply imposing a functional form for the TVTPs. This derivation bridges the two literature streams and makes those solution algorithms applicable to our model.

Second, we develop a particle-free variant of the mixture Kalman filter to estimate our endogenous regime switching DSGE model, either by classical or Bayesian approach. Calculations are simplified by augmenting the model solution with the regime factor and exploiting the conditionally linear and Gaussian structure. Compared to a bootstrap particle filter, the algorithm is shown to be both more accurate in approximating the likelihood function and computationally more efficient. The latent regime factor can be readily extracted. The estimation methodology is completed by comparing the model fit of the endogenous regime switching DSGE model with its exogenous counterpart using Geweke (1999)'s harmonic mean estimator.

The remainder of the paper is organized as follows. Section 2 summarizes the relation to the literature. An analytical solution to a regime-switching Fisherian model with endogenous feedback mechanism is derived in Section 3. Section 4 embeds the endogenous feedback mechanism into a prototypical monetary DSGE model and solves the model using a first-order perturbation method. Section 5 discusses estimation strategy and shows empirical results. Section 6 discusses several potentially important extensions and concludes.

2 Relation to the Literature

The original Taylor rule is state-contingent in the sense that the policy instrument, i.e., federal funds rate, adjusts to the state of the economy, where a set of fixed parameters govern the degree of adjustment. In an environment with endogenous regime switching, the policy rule is state-contingent in this conventional sense, but also in a broader sense. Namely, the parameters governing the degree of adjustment of the federal funds rate to target variables are themselves a function of the economic state. We will first review the fixed regime and the exogenous regime switching monetary policy processes. Then, based on that, we introduce our endogenous regime switching monetary policy process.

The Great Moderation which started in the mid-1980s raised the "good luck" or "good policy" debate. The evidence on this question is mixed. But the profession resolved this largely by trusting its priors that Great Moderation was due to improved policy. Among many other papers on the issue of how the monetary policy is conducted, Taylor (1993) rule has become the most well-known and accepted functional form for monetary policy. Rare is the paper now that posits an exogenous process for money growth and claims to offer practical policy advice. Taylor (1993) suggests that monetary policy behavior is purposeful and reacts systematically to changes in the macroeconomic environment. In particular, the nominal interest rate adjusts to the past interest rate, inflation gap and output gap. These policy targets are themselves functions of the underlying states of the economy. Although the Federal Reserve does not explicitly follow the Taylor rule, many analysts have argued that the rule provides a fairly accurate summary of US monetary policy under Paul

Volcker and Alan Greenspan. It has been well studied that Taylor rule in a New Keynesian model delivers a strong case for price stability, and hence output stability. It is also intended to increase the credibility of future actions by the central bank. However, there is a major limitation of the Taylor rule: the parameters governing the systematic adjustment of policy instrument are constant. All deviations of the nominal interest rate from its systematic adjustments to target variables are folded into the exogenous monetary policy shock.

Empirical works by Clarida et al. (2000), Lubik and Schorfheide (2004) and Sims and Zha (2006) [4] found important time variation of Taylor rule in the United States, which calls for a model where policy parameters of Taylor rule could potentially change. They agree that monetary policy in the United States has been relatively well managed from the time Paul Volcker took over the helm in late 1979. It is also generally agreed that monetary policy was not so well managed in the fifteen years or so prior to Volcker. Moreover, Lucas (1976) emphasized that: if policy has changed in the past, based on the rational expectations assumption, it will be expected to change again in the future. A Markov switching model allows agents to take account of this possibility. Further, in a Markov switching framework, Leeper and Zha (2003) distinguish two effects from exogenous shocks. Direct effects capture the usual impacts of shocks in a fixed regime model where agents treat the policy regime as forever fixed in the future. Expectations formation effects arise whenever agents' rational expectations of future regime change induce them to alter their expectations functions. Expectations formation effects are the difference between the impact of a shock where the regime can change and the impact where the regime is forever fixed.

The fact that expectations formation effects arise from the changes in agents' behaviors lie at the heart of Lucas's critique. The specification of Markov switching monetary policy processes consists of two different rules of the form same as the Taylor rule with different response coefficients, and an exogenous Markov chain governing the dynamic switching between the rules. This makes the policy rule rather than just the policy instrument (the federal funds rate) state-contingent. The key feature of this Markov switching monetary policy process is that it introduces a new source of exogenous disturbance (the Markov chain governing the regime change) to the economy with important implications for the expectations formation effects. But, the exogenous Markov switching would imply that the policy change, which most economists believe helped stabilize the high inflation during the 1980s, was due to an exogenously evolving switching process that happened to call for a change at that date, but not due to an endogenous reaction of the Federal Reserve to the high inflation appeared to be running out of control. The discrepancy between the implication of exogenous Markov switching framework and the common view held by most people calls for

[4] Clarida et al. (2000) and Lubik and Schorfheide (2004) estimate fixed regime monetary DSGE over subsamples of pre-Volcker and Volcker-Greenspan periods. Sims and Zha (2006) estimate a reduced-form exogenous regime-switching model with potential switching in variances of monetary policy disturbance and structural shocks and also monetary policy coefficients.

modelling the policy regime change as a purposeful response of central bank to the state of the economy.

Davig and Leeper (2006) took a first step to introduce endogeneity into monetary policy regime switches by building a New Keynesian model with threshold switching monetary policy. More specifically, policy regime switches when past inflation crosses a threshold value. Nevertheless, no empirical evidence about how and why monetary policy regime has shifted endogenously was provided in literature. To address this question, we believe it is important to investigate the macroeconomic sources of the U.S. monetary policy shifts within a DSGE model. Following Chang et al. (2017) and Chang and Kwak (2017), we introduce a threshold-type regime switching into the model that links the current regime of monetary policy to the historical fundamental shocks by an autoregressive regime factor. This generates an endogenous feedback mechanism between measured economic behavior and the monetary policy stance. For instance, the monetary policy intervention that reflects how federal funds rate reacts to non-target information carries information, through the endogenous feedback mechanism, about future realizations of the regime factor that determines the policy regime. Private agents who observe the policy intervention would adjust their expectations of future policy regime evolutions, which induces the endogenous expectations formation effects. Indeed, the new endogenous feedback mechanism creates a channel for the fundamental shocks to induce the expectations formation effects. In the exogenous Markov switching framework of Leeper and Zha (2003), such a channel is absent.

3 Endogenous Switching in a Fisherian Model

In this section, we introduce the aforementioned endogenous feedback mechanism into a regime-switching Fisherian model. We derive an analytical solution for this simple model and show that it reduces to the solutions for the exogenous switching models and the fixed regime models as limiting cases. Based on these solutions, we demonstrate that how the newly introduced endogenous feedback mechanism modifies the macroeconomic dynamics and the expectations formation effects.

3.1 The Model

We consider the regime-switching Fisherian model studied in Davig and Leeper (2006). This simple model of inflation determination combines a Fisher equation with an interest rate rule for monetary policy. The Fisher equation can be derived from a perfectly competitive endowment economy with flexible prices and a one period nominal bond.[5] A linearized asset pricing equation for the nominal

[5] Since we derive the Fisher equation from an intertemporal optimization problem for agents, the expected real interest rate appears in the Euler equation, hence the Fisher equation. The real interest rate equals the intertemporal marginal rate of substitution and measures how expensive today's consumption is relative to tomorrow. In a conventional Fisher equation, the expected real interest rate is replaced with real interest rate at time t.

bond is given by

$$i_t = \mathbb{E}_t \pi_{t+1} + \mathbb{E}_t r_{t+1}, \tag{1}$$

where i_t denotes the nominal interest rate, $\mathbb{E}_t \pi_{t+1}$ the expected inflation and $\mathbb{E}_t r_{t+1}$ the expected real interest rate conditional on the information available at time t, which we define more explicitly below in (8). The real interest rate evolves exogenously according to

$$r_t = \rho_r r_{t-1} + \sigma_r \epsilon_t^r, \tag{2}$$

where $0 \leq \rho_r < 1$ and ϵ_t^r is an i.i.d. standard normal innovation.

We assume a monetary policy process that permits the monetary authority to vary its response to inflation, depending on the state of economy. For example, a monetary authority may systematically respond more aggressively when its policy regime factor exceeds a particular threshold. The monetary policy may then be given as

$$i_t = \alpha(s_t)\pi_t + \sigma_e \epsilon_t^e \tag{3}$$

with $\alpha_1 > \alpha_0$, where $\alpha_0 = \alpha(s_t = 0)$ and $\alpha_1 = \alpha(s_t = 1)$. The regime is determined by an index function

$$s_t = 1\{w_t \geq \tau\} \tag{4}$$

defined with the policy regime factor w_t, which evolves as an AR(1) process

$$w_{t+1} = \phi w_t + v_{t+1}. \tag{5}$$

The innovation v_{t+1} to the policy regime factor w_t is allowed to interact with the monetary policy shock in the previous period ϵ_t^e and we specify their joint distribution as

$$\begin{pmatrix} \epsilon_t^e \\ v_{t+1} \end{pmatrix} =_d \mathbb{N}\left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right), \tag{6}$$

where ρ measures the correlation between ϵ_t^e and v_{+1}.

It follows from (5) and (6) that[6]

$$w_{t+1} = \phi w_t + \rho \epsilon_t^e + \eta_{t+1}, \quad \eta_{t+1} =_d \mathbb{N}(0, 1-\rho^2). \tag{7}$$

[6] We project v_{t+1} onto the direction of ϵ_t^e and obtain

$$v_{t+1} = \frac{cov(v_{t+1}, \epsilon_t^e)}{var(\epsilon_t^e)} \epsilon_t^e + \eta_{t+1} = \rho \epsilon_t^e + \eta_{t+1},$$

where $\eta_{t+1} =_d \mathbb{N}(0, 1-\rho^2)$. Indeed, $\rho \epsilon_t^e$ is the linear least squares estimator of v_{t+1} and η_{t+1} the estimation error.

This representation highlights the endogenous feedback mechanism from current monetary policy intervention ϵ_t^e to future regime selection s_{t+1} through regime factor w_{t+1}. The endogeneity parameter ρ measures the strength of the feedback channel. When $\rho = 0$, monetary policy regimes shift completely based on the realizations of non-fundamental shock η_{t+1}. If $|\rho| = 1$, future regimes become predetermined. When they select policy regimes, policy makers do not consider any information beyond that embedded in the past policy interventions. In general, we expect $0 < |\rho| < 1$.

Further, by iterating w_{t+1} backwards in (7), we can rewrite it as

$$w_{t+1} = \rho\epsilon_t^e + \eta_{t+1} + \sum_{k=1}^{\infty} \phi^k (\rho\epsilon_{t-k}^e + \eta_{t-k+1}),$$

where $\eta_s =_d \mathbb{N}(0, 1 - \rho^2)$ for all s. We call the autoregressive coefficient ϕ policy persistency since it captures how much historical policy interventions and non-fundamental shocks would affect policy regime selection. One extreme case with $\phi = 0$ would mean that such influence on policy regime selection only lasts one period. For the other extreme case with $\phi = 1$, policy makers would put the same weight on information across all past periods. In the latter case, policy regime selection becomes most persistent.

We assume that private agents and the monetary authority can observe all current and historical values of endogenous variables (i_t), (π_t) and (r_t), exogenous shocks (ϵ_t^r) and (ϵ_t^e), and states of regime (s_t), but not the current and historical policy regime factor (w_t)[7]. Under this information structure, agents form their expectations of next period inflation by

$$\mathbb{E}_t\pi_{t+1} = \mathbb{E}(\pi_{t+1}|\mathcal{F}_t), \quad \mathcal{F}_t = \{i_k, \pi_k, r_k, \epsilon_k^r, \epsilon_k^e, s_k\}_{k=0}^t. \tag{8}$$

However, econometricians only observe realized values of endogenous variables, and therefore they would employ a filtering technique to obtain other information.

3.2 Analytical Solution

We solve the system of expectational nonlinear difference equations consisting of (1)∼(6) using the guess and verify method. Davig and Leeper (2006) show that the analytical solution for the model with fixed regime monetary policy process is given as $\pi_{t+1} = a_1 r_{t+1} + a_2\epsilon_{t+1}^e$ with some constants

[7] This assumption on information structure is critical for the subsequent analysis. Two potentially important extensions of our work can be made by modifying this information structure. First, we may allow both agents and monetary authority to observe regime factor. In this case, deriving an analytical solution is more challenging. Numerical algorithms for solving the monetary DSGE model should also be modified. It is however interesting to see that how macroeconomic dynamics and expectations formation effects vary when agents and policy makers are allowed to observe the additional information on regime strength which is determined by the level of the regime factor relative to the threshold. Second, we may assume that regime factor is observable only to monetary authority but remains latent to agents. This is more realistic but raises the issue of how agents would learn what regime factor is.

a_1 and a_2. Motivated by their solution, we start with the following guess for the true solution form with undetermined coefficients

$$\pi_{t+1} = a_1(s_{t+1}, p_{s_{t+1},0}(\epsilon^e_{t+1}))r_{t+1} + a_2(s_{t+1})\epsilon^e_{t+1}, \tag{9}$$

where s_{t+1} is the regime at period $t+1$, $p_{s_{t+1},0}(\epsilon^e_{t+1})$ the time-varying transition probability (henceforth TVTP) from regime s_{t+1} to regime 0 at period $t+2$. Accordingly, $p_{s_t+1,1} = 1 - p_{s_t+1,0}$. It is worth noting that the TVTP from regime s_{t+1} to regime 1 at $t+2$, $p_{s_{t+1},1}(\epsilon^e_{t+1})$, can be simply computed as $1 - p_{s_{t+1},0}(\epsilon^e_{t+1})$. Given the information available at time t, $\mathcal{F}_t = \{i_k, \pi_k, r_k, \epsilon^r_k, \epsilon^e_k, s_k\}^t_{k=0}$, agents solve the problem and obtain[8]

$$\pi_{t+1} = \underbrace{\frac{\rho_r}{\alpha(s_{t+1})} \frac{(\alpha_1 - \alpha_0)p_{s_{t+1},0}(\epsilon^e_{t+1}) + \alpha_1\left(\dfrac{\alpha_0}{\rho_r} - \mathbb{E}p_{00}(\epsilon^e_{t+1})\right) + \alpha_0\mathbb{E}p_{10}(\epsilon^e_{t+1})}{(\alpha_1 - \rho_r)\left(\dfrac{\alpha_0}{\rho_r} - \mathbb{E}p_{00}(\epsilon^e_{t+1})\right) + (\alpha_0 - \rho_r)\mathbb{E}p_{10}(\epsilon^e_{t+1})}}_{a_1(s_{t+1}, p_{s_{t+1},0}(\epsilon^e_{t+1}))} r_{t+1} \underbrace{- \frac{\sigma_e}{\alpha(s_{t+1})}}_{a_2(s_{t+1})} \epsilon^e_{t+1}, \tag{10}$$

where random coefficient a_1 depends on s_{t+1} and ϵ^e_{t+1}, and the constants $\mathbb{E}p_{00}(\epsilon^e_{t+1})$ and $\mathbb{E}p_{10}(\epsilon^e_{t+1})$ are to be evaluated. If we impose the restriction $\rho = 0$, (10) reduces to

$$\pi_{t+1} = \underbrace{\frac{\rho_r}{\alpha(s_{t+1})} \frac{(\alpha_1 - \alpha_0)p_{s_{t+1},0} + \alpha_1\left(\dfrac{\alpha_0}{\rho_r} - p_{00}\right) + \alpha_0 p_{10}}{(\alpha_1 - \rho_r)\left(\dfrac{\alpha_0}{\rho_r} - p_{00}\right) + (\alpha_0 - \rho_r)p_{10}}}_{a_1(s_{t+1})} r_{t+1} \underbrace{- \frac{\sigma_e}{\alpha(s_{t+1})}}_{a_2(s_{t+1})} \epsilon^e_{t+1}, \tag{11}$$

which is the solution for the exogenous regime switching model.

Two important implications can be drawn by comparing solutions (10) and (11): First, because monetary policy intervention ϵ^e_{t+1} carries with it information about regime changes in the future, its realization would affect agents' expectations on future regime evolution. This adjustment of agents' expectations alters their behaviors and, in equilibrium, causes changes in the current realization of inflation. We call these changes the *expectations formation effects*, which are the forward-looking effects identified by the endogenous feedback mechanism. The solution (10) clearly shows that the monetary policy intervention ϵ^e_{t+1} at time $t+1$ influences the distribution of the policy regime factor, and consequently also the policy regime, in the next period through the terms $p_{s_{t+1},0}(\epsilon^e_{t+1})$, $\mathbb{E}p_{00}(\epsilon^e_{t+1})$ and $\mathbb{E}p_{10}(\epsilon^e_{t+1})$. However, in solution (11) to an exogenous regime switching model where ρ equals zero, the endogenous feedback mechanism is turned off and hence there is no channel through which the monetary policy intervention could affect agents' expectations on future

[8] Detailed derivation is provided in Appendix A.

regime changes. Second, the realization of policy regime s_{t+1} in solution (10) is selected by the policy makers following the endogenous regime switching monetary policy rule. That is, the information embedded in all historical monetary policy interventions is used to determine the policy regime s_{t+1}. Then, it changes the realization of inflation. These changes are called the *selection effects*, which are the backward-looking effects raised by the endogenous feedback mechanism. In contrast, in solution (11), the policy regime is simply selected by an exogenous state variable evolving as a two-state Markov chain. In this paper, we will only study the endogenous expectations formation effects which are more relevant for the rational expectations models.

If we impose the restriction $\alpha_0 = \alpha_1 = \alpha$, solution (10) would reduce to the equilibrium inflation in the fixed regime model

$$\pi_{t+1} = \underbrace{\frac{\rho_r}{\alpha - \rho_r}}_{a_1} r_{t+1} - \underbrace{\frac{\sigma_e}{\alpha}}_{a_2} \epsilon^e_{t+1}, \tag{12}$$

where both coefficients a_1 and a_2 are deterministic constants. In this solution, monetary policy intervention only has direct effects through the second term. This is the result obtained in Davig and Leeper (2006). Therefore, our solution generalizes their result in a meaningful way.

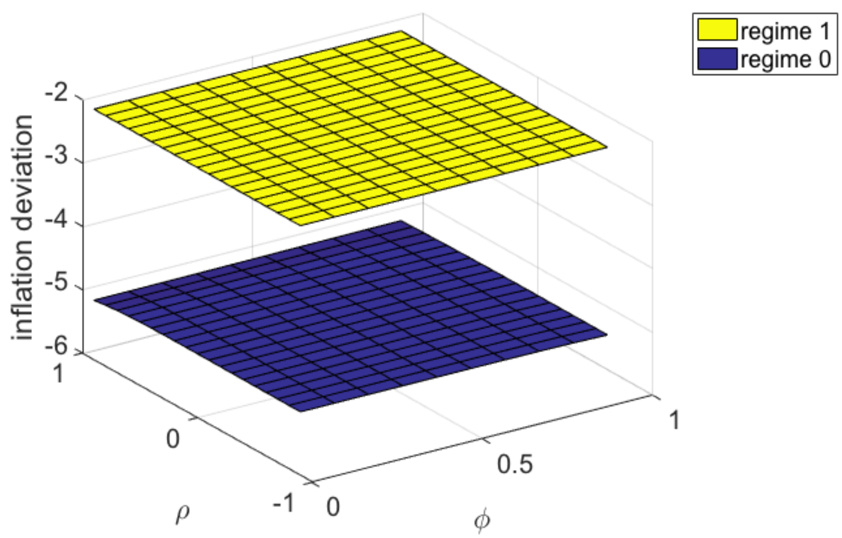
3.3 Macroeconomic Dynamics

This subsection employs the simple Fisherian model with endogenous feedback mechanism to revisit the Great Inflation periods. We show the importance of endogenous expectations formation effects, generated by the feedback mechanism, in explaining the effective price stabilization in the early 1980s. In contrast, without taking the endogeneity into account, exogenous expectations formation effects fail to generate significant price stabilization.

3.3.1 Price Stabilization

As in early 1980s, we suppose the economy is under less aggressive regime (henceforth regime 0) at period T and there will be a contractionary monetary policy shock at period $T+1$. Using solution (10), we compute inflation rate at period $T+1$ under different realizations of regime. In this simple Fisherian model, zero inflation is regarded as the monetary authority's target. From Figure 2, it is clear that comparing to regime 0, when the monetary authority selects regime 1, the destabilizing effect of contradictionary monetary policy shock on inflation is smaller. That is, the deviation from inflation target is smaller. This means that the switch of monetary policy to regime 1 reduces the size of the impact of the shock. Hence, a switch of monetary policy to the more aggressive regime helps stabilize price levels.

Figure 2: Realized Inflation at Each Regime



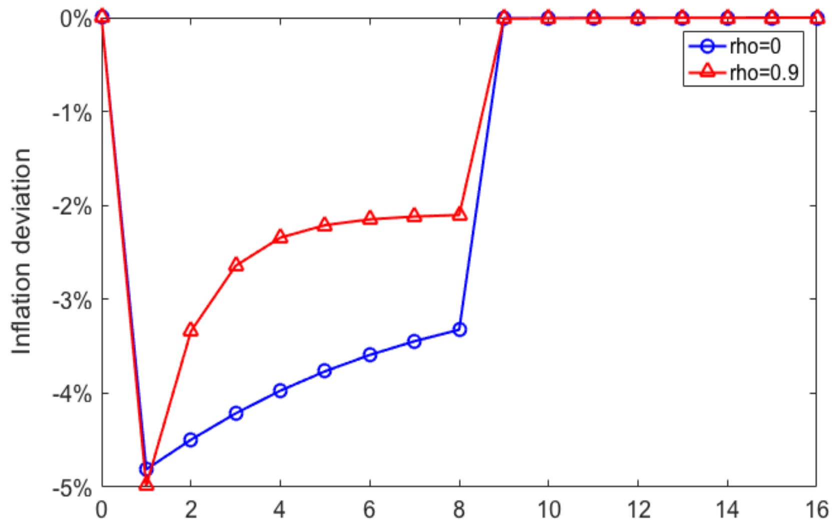
3.3.2 Expectations Formation Effects

We define direct effects and expectations formation effects as in Leeper and Zha (2003). Direct effects and expectations formation effects from a policy intervention are computed from forecasts from fixed-regime and endogenous regime switching models, respectively, conditional on a given intervention. Both of these effects are reported relative to baseline effects computed as the forecast obtained from the fixed regime model without conditioning on the given policy interventions. Suppose the economy is under regime 0 at period T. Let I_T be a hypothetical intervention at time T, specified as a K-period sequence of exogenous policy actions, $I_T = \{\tilde{\epsilon}^e_{T+1}, \ldots, \tilde{\epsilon}^e_{T+K}\}$. Although the policy advisor chooses I_T, private agents treat it as random. In Appendix B, we derive the direct effects and total effects of I_T relative to the baseline and isolate expectations formation effects as the difference between them, and present the results here:

$$\begin{aligned}
&\text{Baseline Effects} = \mathbb{E}[a_1(s_{T+K} = 0, p_{00}(\epsilon^e_{T+K}))]\rho_r^K r_T, \\
&\text{Direct Effects} = a_1(s_{T+K} = 0, p_{00}(\tilde{\epsilon}^e_{T+K}))\rho_r^K r_T + a_2(s_{T+K} = 0)\tilde{\epsilon}^e_{T+K} - \text{Baseline Effects}, \\
&\text{Total Effects} = [a_1(s_{T+K} = 0, p_{00}(\tilde{\epsilon}^e_{T+K}))\rho_r^K r_T + a_2(s_{T+K} = 0)\tilde{\epsilon}^e_{T+K}]\,\mathbb{P}(s_{T+K} = 0 | s_T = 0, I_T, \tilde{\epsilon}^e_T) \\
&\qquad + [a_1(s_{T+K} = 1, p_{10}(\tilde{\epsilon}^e_{T+K}))\rho_r^K r_T + a_2(s_{T+K} = 1)\tilde{\epsilon}^e_{T+K}]\,\mathbb{P}(s_{T+K} = 1 | s_T = 0, I_T, \tilde{\epsilon}^e_T) \\
&\qquad - \text{Baseline Effects}, \\
&\text{Expectations Formation Effects} = \text{Total Effects} - \text{Direct Effects}.
\end{aligned}$$

Suppose the monetary authority decides to implement a contractionary monetary policy in-

Figure 3: Impulse Response of Contractionary Policy Intervention



tervention over the next eight periods, and they wish to know what are the effects of this policy intervention throughout the next sixteen periods[9] . We may write this intervention as

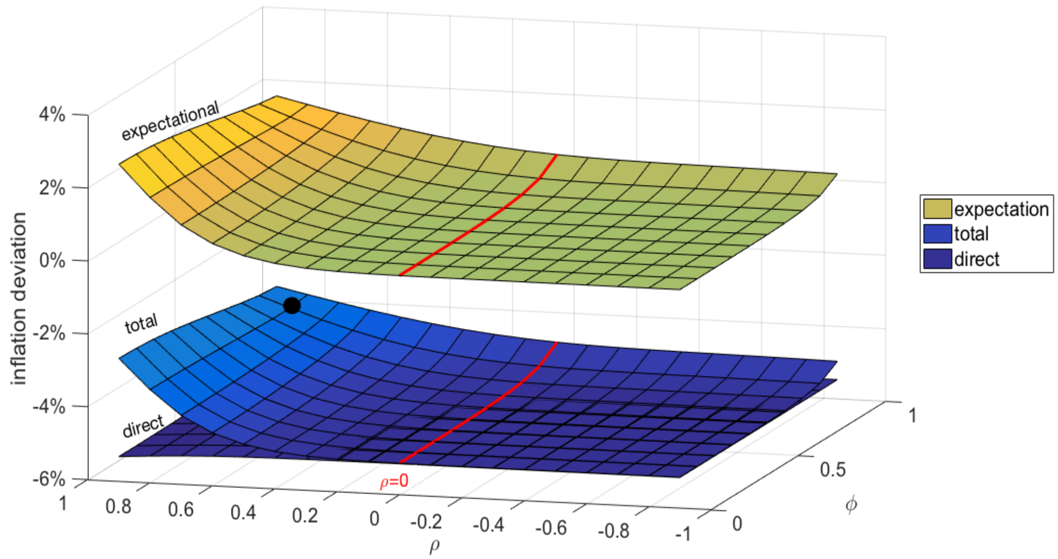
$$I_T = \{\underbrace{4\%, \ldots, 4\%}_{8 \text{ periods}}, \underbrace{0, \ldots, 0}_{8 \text{ periods}}\}.$$

In order to analyze how direct, total and expectations formation effects depend on the level of endogeneity ρ and policy persistency ϕ, we set threshold τ at 3 and consider the period $T+2$. The results are summarized in Figure 3. The black dot indicates the value of total effect at $T+2$ when $\rho = 0.9$ and $\phi = 0.9$.

Figure 4 shows that: (i) for ρ positive with ϕ fixed, policy intervention at period $T+1$ conveys more information about policy regime at period $T+2$ as ρ increases, or more precisely, agents believe that positive realization of policy intervention at period $T+1$ will significantly drive up (since $\rho > 0$) the policy regime factor at period $T+2$, and consequently the policy regime will be more likely to switch. As a result, the expectations formation effect increases in magnitude. When ρ is negative, the results are reversed but insignificant as shown in the figure. In the exogenous regime switching case with $\rho = 0$, we have insignificant expectations formation effects. Clearly, price level is more stabilized in the endogenous regime switching model especially with positive ρ, because agents adjust their beliefs on future regimes through the endogenous feedback mechanism, compared to the exogenous case. The strength of this stabilization effect is measured by the expectations

[9] Parameters are calibrated according to quarterly frequency. We choose a 4-year forecast horizon to coincide with a typical horizon the Federal Reserve considers in their routine policy making.

Figure 4: Effects of Contractionary Policy Intervention



formation effects. In other words, if one fails to properly take into account the endogenous feedback mechanism, the computation of the expectations formation effects can be significantly biased. (ii) Overall, the influence of policy persistency ϕ on the expectations formation effects is indeterminate. When ρ is positive and large, policy persistency and the expectations formation effects are negatively related. In other cases, it seems that they become positively related but insignificantly.

We also compute and compare the impulse response functions from the endogenous and exogenous regime switching models throughout the sixteen periods. We set parameters (ϕ, ρ, τ) at $(0.9, 0.9, 3)$. Figure 3 shows clearly that endogenous feedback mechanism helps explain the price stabilization in the 1980s. As already shown in Figure 4, when ρ is positive and large, monetary policy is more likely to switch to the more aggressive regime and the expectations formation effects become larger. Therefore, price level is more stabilized in the strong presence of positive endogenous feedback. There are jumps in inflation because prices are flexible in Fisherian model and hence adjust immediately after policy shock realizes.

In this section, by solving analytically the simple Fisherian model, we show that our endogenous feedback mechanism creates a channel for the monetary policy intervention to induce the expectations formation effects. Moreover, using a calibrated version of the model, we also demonstrate that the endogenous feedback mechanism need to be taken into account properly to be able to correctly compute the expectations formation effects. In Section 5, we will conduct the same exercise within a more prototypical regime switching monetary DSGE model with endogenous feedback.

4 Endogenous Switching in a New Keynesian Model

This section first presents a regime-switching monetary DSGE model, which is a New Keynesian model with a relatively standard private sector specification as in Ireland (2004), An and Schorfheide (2007) and Davig and Doh (2014). Indeed, this model has become a benchmark specification for the analysis of monetary policy and is analyzed in detail, for instance, in Woodford (2011). The primary difference relative to these specifications is that the monetary policy rule is subject to a threshold-type switching that links the current regime of monetary policy to the historical fundamental shocks by an autoregressive regime factor. This generates an endogenous feedback mechanism between the monetary policy stance and measured economic behavior. Then, we solve the endogenous regime switching monetary DSGE model by first-order perturbation method used in Barthélemy and Marx (2017).

4.1 The Model

The economy includes a representative household, a representative firm that produces a final good, a continuum of monopolistically competitive firms that each produce an intermediate good indexed by $j \in [0, 1]$, and a monetary authority. For convenience, we abstract from investment and capital accumulation. This abstraction, however, does not affect any qualitative conclusions, as Clarida et al. (1999) discussed.

The Representative Household The representative household chooses consumption C_t of a composite good relative to a habit stock[10], hours worked L_t and debt holding B_t to maximize lifetime utility

$$\mathbb{E}_t \sum_{s=0}^{\infty} \xi_{t+s} \beta^s \left(\frac{(C_{t+s}/A_{t+s})^{1-\epsilon}}{1-\epsilon} - L_{t+s} \right),$$

where $\beta \in (0, 1)$ is the discount factor, $\epsilon > 0$ the coefficient of relative risk aversion, and ξ_t a preference shock affecting the discount factor, subject to the intertemporal budget constraint

$$P_t C_t + Q_t B_t = B_{t-1} + W_t L_t + P_t D_t - P_t T_t,$$

where P_t is the aggregate price level, Q_t the price of a zero-coupon bond at time t yielding 1 in period $t + 1$, W_t the nominal wage per hour, D_t the real profits from ownership of firms, and T_t lump-sum taxes. Note that the scale factor for L_t is implicitly assumed to be 1 as in An and Schorfheide (2007).

The Intermediate-Goods-Producing Firms Intermediate-goods-producing firm j produces

[10] We assume that the habit stock is given by the level of technology A_t. This assumption ensures that the economy evolves along a balanced growth path even if the utility function is additively separable in consumption and leisure.

output Y_{jt} using labor L_{jt} as only input. The production technology is linear in labor for simplicity and given by

$$Y_{jt} = A_t L_{jt},$$

where A_t is common to all firms. The labor market is perfectly competitive, and firms are able to hire as many as demanded at the real wage.

To allow for a real effect of monetary policy, we introduce nominal rigidities *à la* Rotemberg (1982). The monopolistic intermediate-goods-producing firms pay a real adjustment cost[11]

$$AC_{jt} = \frac{\varphi}{2}\left(\frac{P_{jt}}{\Pi^* P_{jt-1}} - 1\right)^2 Y_t$$

when they adjust their prices, where $\varphi \geq 0$ governs the price stickiness in the economy, Π^* denotes the steady-state gross inflation that coincides with the central bank's inflation target, and P_{jt} denotes the nominal price set by firm $j \in [0,1]$ at time t. The price adjustment cost is in terms of the final good Y_t. Each intermediate-goods-producing firm maximizes the expected present value of the current and future real profits

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \lambda_{t+s} D_{jt+s}.$$

We use MU_t to denote the representative household's marginal utility of a unit of the consumption good at time t. Then, the stochastic discount factor for real profits can be written as

$$\beta^s \lambda_{t+s} = \beta^s \frac{MU_{t+s}}{MU_t} = \beta^s \left(\frac{C_{t+s}}{C_t}\right)^{-\epsilon} \left(\frac{A_t}{A_{t+s}}\right)^{1-\epsilon}.$$

It measures the time t real value of a unit of the consumption good in period $t+s$ to the household, which is treated as exogenous by the firm. Real profit D_{jt} at time t is given by

$$D_{jt} = \frac{P_{jt}Y_{jt}}{P_t} - \frac{W_t}{P_t} L_{jt} - \frac{\varphi}{2}\left(\frac{P_{jt}}{\Pi^* P_{jt-1}} - 1\right)^2 Y_t.$$

The Representative Final-Goods-Producing Firm The representative final-goods-producing firm purchases a continuum of intermediate goods indexed by $j \in [0,1]$ at prices P_{jt} and combines them into a final good using the following constant-returns-to-scale technology in the familiar Dixit-Stiglitz (1977) form as

$$Y_t = \left[\int_0^1 Y_{jt}^{\frac{\theta_t - 1}{\theta_t}} dj\right]^{\frac{\theta_t}{\theta_t - 1}},$$

[11]In our paper, we assume the inflation target Π^* is regime-independent which leads to regime-independent steady states for all variables. This result is summarized formally as Proposition 5 in Liu et al. (2011).

where $\theta_t > 1$ is the time-varying elasticity of substitution between goods. Therefore, the monopoly power of each intermediate-goods-producing firm and its desired markup u_t over marginal costs is also constantly changing. The steady-state markup can be solved as

$$u = \frac{\theta}{\theta - 1}$$

where θ is the steady-state elasticity of substitution. The profit maximization problem for the final-goods-producing firm yields a demand for each intermediate good given by

$$Y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\theta_t} Y_t.$$

Then, the zero-profit condition for the perfectly competitive final-goods-producing firm implies that the aggregate price level is

$$P_t = \left[\int_0^1 P_{jt}^{1-\theta_t} dj\right]^{\frac{1}{1-\theta_t}}.$$

Monetary Policy The monetary authority sets the short-term nominal rate using the following modified Taylor rule:

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*}\right)^{\rho_R} \left[\left(\frac{\Pi_t}{\Pi^*}\right)^{\alpha(s_t)} \left(\frac{Y_t}{Y_t^*}\right)^{\gamma}\right]^{1-\rho_R} e_t, \tag{13}$$

where R_t is the gross nominal interest rate, Π_t is the gross inflation which equals P_t/P_{t-1}. The parameters $\alpha(s_t)$ and γ measure the reaction to the inflation gap and output gap. The reaction to the inflation gap switches between two values depending on the regime s_t. The parameter ρ_R captures the smoothing motive of the policy maker who sets the interest rate. Steady-state nominal gross interest rate R^* equals to $r^*\Pi^*$ where r^* is the steady-state real gross interest rate. Y_t^* is the steady state level of output that would prevail in the absence of nominal rigidities. Finally, the monetary policy shock e_t stands for the unsystematic monetary policy component, following a lognormal distribution, i.e., $\ln e_t \sim i.i.d.\mathbb{N}(0, \sigma_e^2)$. Taking logarithm of equation (13) yields

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1 - \rho_R)[\alpha(s_t)\hat{\Pi}_t + \gamma \hat{Y}_t] + \hat{e}_t,$$

where $\hat{e}_t = \ln e_t$ is an iid normal shock. The hat denotes the log-deviation from the steady state or, in the case of output, from a trend path. The regime s_t is governed by an autoregressive regime factor w_t as

$$s_t = 1\{w_t \geq \tau\}, \tag{14}$$

and the regime factor w_t evolves as a first order autoregressive process specified as

$$w_{t+1} = \phi w_t + v_{t+1}. \tag{15}$$

Market Clearing The goods market clears when production equals consumption plus a real adjustment cost as

$$Y_t = C_t + \frac{\varphi}{2}\left[\frac{\Pi_t}{\Pi^*} - 1\right]^2 Y_t$$

Shocks The economy is perturbed by four exogenous fundamental shocks. Aggregate productivity A_t evolves according to

$$\ln A_{t+1} = \ln\psi + \ln A_t + \ln a_{t+1},$$

where $\ln\psi$ is the average technology growth rate. The exogenous fluctuations a_t of the technology growth rate are given by

$$\ln a_{t+1} = \rho_a \ln a_t + \sigma_a \varepsilon^a_{t+1},$$

where $|\rho_a| < 1$ and the innovation ε^a_{t+1} follows a standard normal distribution. The preference shock ξ_t follows a first order autoregressive process

$$\ln\xi_{t+1} = \rho_\xi \ln\xi_t + \sigma_\xi \varepsilon^\xi_{t+1},$$

where $|\rho_\xi| < 1$ and the innovation ε^ξ_{t+1} follows a standard normal distribution. The markup shock u_t follows

$$\ln u_{t+1} = (1-\rho_u)\ln u + \rho_u \ln u_t + \sigma_u \varepsilon^u_{t+1},$$

where $|\rho_u| < 1$ and the innovation ε^u_{t+1} follows a standard normal distribution, and as discussed earlier, the monetary policy shock is given by

$$\ln e_{t+1} = \sigma_e \varepsilon^e_{t+1},$$

where the innovation ε^e_{t+1} follows a standard normal distribution. The four innovations are independent of each other.

In addition, the endogenous regime factor brings in a regime factor innovation v_{t+1}, which is distributed as multivariate normal together with the past fundamental shocks $\varepsilon_t = (\varepsilon^a_t, \varepsilon^\xi_t, \varepsilon^u_t, \varepsilon^e_t)'$, viz.,

$$\begin{pmatrix} \varepsilon_t^a \\ \varepsilon_t^\xi \\ \varepsilon_t^u \\ \varepsilon_t^e \\ v_{t+1} \end{pmatrix} \sim \mathbb{N}\left(0, \begin{pmatrix} 1 & 0 & 0 & 0 & \rho_{av} \\ 0 & 1 & 0 & 0 & \rho_{\xi v} \\ 0 & 0 & 1 & 0 & \rho_{uv} \\ 0 & 0 & 0 & 1 & \rho_{ev} \\ \rho_{av} & \rho_{\xi v} & \rho_{uv} & \rho_{ev} & 1 \end{pmatrix} \right), \tag{16}$$

where $\rho = (\rho_{av}, \rho_{\xi v}, \rho_{uv}, \rho_{ev})'$, with the normalization $\rho'\rho < 1$. Full characterization of the model is summarized in appendix C.

By projecting v_t onto ε_{t-1}, we can rewrite v_t as

$$v_t = \rho'\varepsilon_{t-1} + \sqrt{1-\rho'\rho}\,\eta_t, \quad \eta_t \sim \mathbb{N}(0,1),$$

where η_t is an iid standard normal innovation, ε_{t-1}^e and η_t are orthogonal, and each has unit variance. Hence, equation (15) is further specified as

$$w_{t+1} = \phi w_t + \rho'\varepsilon_t + \sqrt{1-\rho'\rho}\,\eta_{t+1}. \tag{17}$$

Now, note that

$$w_{t+h} = \phi^h w_t + \sum_{j=1}^{h} \phi^{h-j} v_{t+j}$$

for $h \geq 1$, and hence forecast error variance decomposition of w_{t+h} can be derived as

$$\begin{aligned} FEV(w_{t,h}) &= Var(w_{t+h} - \mathbb{E}(w_{t+h}|\mathcal{F}_t)) \\ &= \sum_{j=1}^{h} \phi^{2(h-j)} = \sum_{k=1}^{4} \underbrace{\sum_{j=1}^{h} \rho_k^2 \phi^{2(h-j)}}_{k\text{-th fundamental}} + \underbrace{\sum_{j=1}^{h} \left(1 - \sum_{k=1}^{4} \rho_k^2\right) \phi^{2(h-j)}}_{\text{non-fundamental}} \end{aligned} \tag{18}$$

for $h \geq 1$. It follows that the percent of the h-step ahead forecast error variance due to the k-th fundamental shock is given by ρ_k^2, which is independent of h. Hence, we interpret the magnitude of ρ_k^2 as "the contribution of the k-th fundamental shock to the monetary policy shifts".

4.2 Solving Endogenous Regime-Switching Monetary DSGE Model

We solve our endogenous regime-switching monetary DSGE model using a first-order perturbation method, which largely facilitates the estimation. Maih (2015), Foerster et al. (2016) and Barthélemy and Marx (2017) all solve non-linear regime-switching rational expectations models using a perturbation method. Maih (2015) and Foerster et al. (2016) propose algorithms based on successive differentiations in the vein of Kim et al. (2008b). Barthélemy and Marx (2017), following Cho (2016), use a functional iteration technique to find the solution of a fixed-point problem. Moreover, they complement other papers by showing how to apply the implicit function theorem to numerically ensure both determinacy and accuracy of the approximate solution. Because we are interested in eventually evaluating how switching parameters would affect the determinacy condition, we choose to use Barthélemy and Marx (2017)'s algorithm in this paper.

In order to apply Barthélemy and Marx (2017)'s perturbation method, we must first compute the time-varying transition probabilities. It is important to notice that our threshold-type switching consisting of equations specified in (14) - (16) can be represented as the TVTP-type switching employed in Barthélemy and Marx (2017). In our model, the transition probability from regime 0 to regime 0 can be computed as

$$\mathbb{P}\{s_{t+1}=0|s_t=0,\varepsilon_t\}=\frac{\displaystyle\int_{-\infty}^{\tau\sqrt{1-\phi^2}}\Phi_\rho\left(\tau-\frac{\phi x}{\sqrt{1-\phi^2}}-\sum_{k=1}^{4}\rho_k\varepsilon_t^k\right)\varphi(x)dx}{\Phi(\tau\sqrt{1-\phi^2})} \tag{19}$$

Analogously, the transition probability from regime 1 to regime 0 is computed as

$$\mathbb{P}\{s_{t+1}=0|s_t=1,\varepsilon_t\}=\frac{\displaystyle\int_{\tau\sqrt{1-\phi^2}}^{\infty}\Phi_\rho\left(\tau-\frac{\phi x}{\sqrt{1-\phi^2}}-\sum_{k=1}^{4}\rho_k\varepsilon_t^k\right)\varphi(x)dx}{1-\Phi(\tau\sqrt{1-\phi^2})} \tag{20}$$

Then, the transition probabilities from regime 0 to regime 1 and from regime 1 to regime 1 can be readily obtained as

$$\mathbb{P}\{s_{t+1}=1|s_t=0,\varepsilon_t\}=1-\mathbb{P}\{s_{t+1}=0|s_t=0,\varepsilon_t\} \tag{21}$$

$$\mathbb{P}\{s_{t+1}=1|s_t=1,\varepsilon_t\}=1-\mathbb{P}\{s_{t+1}=0|s_t=1,\varepsilon_t\} \tag{22}$$

The above TVTPs are computed following the steps summarized in Appendix A.2. With these TVTPs, we can rewrite the full characterization as a set of equations consisting of four equilibrium conditions (42) - (45), technology process (48) and four exogenous shocks (49)-(52), and the TVTPs (19)-(22).

Since the nonstationary technology process A_t induces a stochastic trend in output and consumption, it is convenient to express the model in terms of detrended variables $c_t = C_t/A_t$ and $y_t = Y_t/A_t$. A non-stochastic steady state is defined as an equilibrium when shocks are turned off and inflation is at its target rate. In terms of detrended variables, the non-stochastic steady state can be computed as[12]

$$\overline{Z} = (\overline{y}, \overline{\Pi}, \overline{R}, \overline{a}, \overline{\xi}, \overline{u}, \overline{e})' = \left(\left(\frac{\theta - 1}{\theta} \right)^{1/\epsilon}, \Pi^*, \frac{\psi \Pi^*}{\beta}, 1, 1, u, 1 \right)'.$$

We rewrite all variables in terms of log-deviations from their steady states which are signified by 'hat' and collected in a (7×1) vector Z_t as

$$Z_t = (\hat{y}_t, \hat{\Pi}_t, \hat{R}_t, \hat{a}_t, \hat{\xi}_t, \hat{u}_t, \hat{e}_t)'.$$

Then, our model can be represented as

$$\mathbb{E}_t[f_{s_t}(Z_{t+1}, Z_t, Z_{t-1}, \chi \varepsilon_t)] = 0, \tag{23}$$

where ε_t is a 4-dimensional vector of innovations, i.e., $\varepsilon_t = (\varepsilon_t^a, \varepsilon_t^\xi, \varepsilon_t^u, \varepsilon_t^e),'$ and χ is a positive scalar. The transition probabilities of s_t from regime i to regime j are given in (19)-(22). Let the set U^∞ represent an infinite sequence including the current and entire past states and shocks, i.e., $(s^t, \varepsilon^t) = ((s_t, \varepsilon_t), (s_{t-1}, \varepsilon_{t-1}), \cdots)$. We describe a solution for our model as a continuous and bounded function $g : U^\infty \to \mathcal{R}^7$ of all the current and past states[13] and shocks, satisfying the condition (23) specified by the full characterization provided in Appendix C. We denote by $\mathcal{B}$ the set of all such continuous and bounded functions g. The left-hand side of (23) can be denoted as an operator $\mathcal{N}$ acting on a bounded continuous function g in $\mathcal{B}$ and the scale parameter χ. We may therefore rewrite (23) as

$$\mathcal{N}(g, \chi)(s^t, \varepsilon^t) = \sum_j \int p_{s_t, j}(\varepsilon_t) f_{s_t}(g(js^t, \varepsilon\varepsilon^t), g(s^t, \varepsilon^t), g(s^{t-1}, \varepsilon^{t-1}), \chi \varepsilon_t) p(\varepsilon) d\varepsilon = 0,$$

where $p(\varepsilon)$ denotes the Gaussian probability density function for ε. In this functional framework, the problem can be reinterpreted as finding the zeros of the operator $\mathcal{N}$. This can be done by applying the implicit function theorem to the operator $\mathcal{N}(g, \chi)$ as shown in Barthélemy and Marx (2017).

For the variables in log-deviations, the steady state for model (23) given in (23) is simply given by $Z_{ss} = 0$. By application of the implicit function theorem around the steady state $Z_{ss} = 0$ and

[12] c_t is substituted using market clearing condition.

[13] Essentially, regime change should be regarded as an additional source of shock. But, this shock is very different from the i.i.d. Gaussian fundamental shocks ε_t. This is because the regime state s_t is a time-inhomogeneous Markov process.

with χ set at 1, we obtain the following first-order perturbation solution for our model

$$Z_t = \underbrace{A_1(s_t)}_{7\times7} Z_{t-1} + \underbrace{A_2(s_t)}_{7\times4} \varepsilon_t \tag{24}$$

where the four coefficient matrices $A_1(s_t = j)$ and $A_2(s_t = j)$ for $j = 0, 1$ are obtained numerically by using Barthélemy and Marx (2017)'s solution algorithm.[14] Due to our assumption that the inflation target Π^* is invariant across regimes, the steady-state of our model given in (23) is state-invariant. This, together with the fact that the steady-state values $\bar{\epsilon}$ of the structural shocks are zero, implies that the channel that provides endogenous feedback from structural shocks to the regime factor vanishes in the steady-state. This is why the coefficient matrix A_1 in the above solution depends only on s_t. The other parameters defining the our TVTP's, the persistency of the regime factor ϕ and the threshold τ, are still present in the coefficient A_1. We may obtain a solution which explicitly reflects the presence of endogenous feedback by letting the inflation target Π^* be regime specific, which yields a regime-dependent steady-state.[15]

5 Estimation of Monetary DSGE Models

5.1 Contact with the Literature

The Great Moderation which started in the mid-1980s raised the famous "good luck" or "good policy" debate. To answer this question, literature has used the workhorse New Keynesian model with a simple monetary policy rule – the Taylor rule. The nominal rigidity (price stickiness) in the New Keynesian model allows for a real effect of monetary policy. Lubik and Schorfheide (2004) use Bayesian approach (with Kalman filter) to estimate a fixed-regime monetary DSGE model over three subsample periods. Marginal likelihood comparison suggests that the pre-Volcker posterior concentrates almost all of its probability mass in the indeterminacy region, while the post-1982 posterior concentrates in the determinacy region. According to this result, they conclude that the U.S. monetary policy post-1982 is consistent with determinacy, whereas the pre-Volcker policy is not, which supports the "good policy" argument. The fixed-regime monetary DSGE model is solved by the approach developed by Sims (2002), and extended in Lubik and Schorfheide (2003).

In the spirit of the Lucas (1976) critique, the empirical observations of time variation in the

[14] Barthélemy and Marx (2017)'s solution algorithm requires the TVTPs to be smooth functions (at least C^2). In our model, TVTPs are indeed smooth functions as shown in Figure 6. However, it is worth noting that for a simple threshold-type switching model where regime is determined by an observed variable crossing a certain threshold, its TVTPs are step functions which makes Barthélemy and Marx (2017)'s perturbation method not applicable. This type of discontinuity does not occur in our model since the transition is determined by a continuous latent regime factor.

[15] In fact, Barthélemy and Marx (2017) allows for the existence of a regime-dependent steady-state. See their Assumption 2.

Taylor rule call for regime switching monetary policy: If policy has switched in the past, it might be expected to switch again in the future. Davig and Doh (2014) use a Bayesian approach (with approximate Kalman filter of Kim and Nelson (1999)) to estimate a Markov-switching monetary DSGE model that allows shifts in the monetary policy reaction coefficients and structural shock volatilities. They find that a more aggressive monetary policy regime was in place after the Volcker disinflation and before 1970 than during the Great Inflation of the 1970s. Their estimates also indicate that a low-volatility regime has been in place during most of the sample period after 1984. They then connect the timing of the different regimes to a measure of inflation persistence. Bianchi (2012) uses a Bayesian approach (with Gibbs sampling algorithm) to estimate a similar model as Davig and Doh (2014) but with an additional interest rate smoothing parameter. His model also allows switching in both policy coefficients and shock variances. His estimates capture better features for the recent crisis. To explore the role of agents' beliefs he applies counterfactual simulations and finds that: If, in the 1970s, agents had anticipated the appointment of an extremely conservative Chairman, inflation would have been lower. The large drop in inflation and output at the end of 2008 would have been mitigated if agents had expected the Federal Reserve to be exceptionally active in the near future. These two papers both use MSLRE method (Farmer et al., 2011) to solve the exogenous regime switching monetary DSGE models.

While the exogenous regime switching setup allows researchers to determine the timing and study the macroeconomic implications of U.S. monetary policy regime shifts, it remains silent about the origin of this policy regime change. The attempt to tackle this research question calls for modeling monetary policy regime change as a purposeful response of central bank to the state of economy, rather than following an exogenously evolving switching process. Only recently, Maih (2015), Foerster et al. (2016) and Barthélemy and Marx (2017) developed perturbation methods for solving regime switching DSGE models with time-varying transition probabilities. Instead of imposing a functional form on TVTP, we introduce a threshold-type switching that links the current monetary policy stance to the historical fundamental shocks by a latent autoregressive regime factor which generates TVTPs through the aforementioned endogenous feedback mechanism. Nevertheless, their solution algorithms are applicable to our model because our threshold-type switching has a TVTP representation as shown in Section 4.2.

5.2 Estimation Strategy

5.2.1 Data

To construct the system of measurement equations specified below in (25), we need three series of U.S. quarterly data. For the per capita real output growth rate (in percentage) YGR_t, we divide the level of real gross domestic product (FRED mnenomic "$GDPC1$") by the quarterly average of the Civilian Non-institutional Population (FRED mnenomic "$CNP16OV$") to obtain per capita real output, then take its log difference and scale by 100. For the annualized net inflation rate (in percentage) INF_t, we take log difference of quarterly consumer price index (FRED mnenomic "$CPIAUCSL$") and scale by 400. For the annualized net nominal interest rate (in percentage) INT_t, we use the effective federal funds rate (FRED mnenomic "$FEDFUNDS$"). All data come from Federal Reserve Bank of St. Louis, Economic Data-FRED. The sample spans 1954:Q3 to 2007:Q4. We do not include post-crisis data because the unconventional monetary policy carried out at the zero lower bound can not be well explained by the prototypical New Keynesian model.

5.2.2 State Space Representation

The first-order perturbation solution (24) has a conditionally linear Gaussian structure which makes feasible the application of our newly developed Kalman filter with Markov switching[16]. This new filter will be used to estimate our state space model. We combine the first-order perturbation solution with the measurement equations specified below in (25) to form a regime-dependent state space model. For filtering convenience, we augment the state vector by including one-period-ahead latent endogenous regime factor w_{t+1}. In addition, we also include the past output gap $\hat{y}_{t-1}$ as we need it to construct our first measurement equation in (25). Therefore, our state vector X_t is given by

$$X_t = (\hat{y}_t, \hat{\Pi}_t, \hat{R}_t, \hat{a}_t, \hat{\xi}_t, \hat{u}_t, \hat{e}_t, w_{t+1}, \hat{y}_{t-1})'.$$

Correspondingly, we augment the shock vector ε_t with the shock η_{t+1} to the added regime factor w_{t+1}. Then, it is given by

$$\varepsilon_t = (\varepsilon_t^a, \varepsilon_t^\xi, \varepsilon_t^u, \varepsilon_t^e, \eta_{t+1})'.$$

Therefore, our measurement equation is given as

$$y_t = D(s_t, \Theta) + Z(s_t, \Theta)X_t + \nu_t,$$

which can be more explicitly written as

$$\underbrace{\begin{pmatrix} YGR_t \\ INF_t \\ INT_t \end{pmatrix}}_{y_t} = \underbrace{\begin{pmatrix} \psi^{(Q)} \\ \pi^{(A)} \\ 4\psi^{(Q)} + r^{(A)} + \pi^{(A)} \end{pmatrix}}_{D(s_t,\Theta)} + \underbrace{100 \begin{pmatrix} \hat{y}_t - \hat{y}_{t-1} + \hat{a}_t \\ 4\hat{\Pi}_t \\ 4\hat{R}_t \end{pmatrix}}_{Z(s_t,\Theta)X_t} + \nu_t, \tag{25}$$

where $\psi^{(Q)}$ is the quarterly net growth rate of technology in percentage, $r^{(A)}$ the annualized net real interest rate in percentage, and $\pi^{(A)}$ the annualized net inflation target in percentage, and with $\nu_t \sim \mathbb{N}(0, \Sigma_\nu)$, where ν_t are measurement errors and their standard deviations are set to be 20% of

[16] Detailed algorithm is provided in Appendix D.

the sample standard deviations of real data, i.e., $\Sigma_\nu = 0.2^2 \times \text{diag}(\text{var}(y_t))$. Transition equations consist of solution (24), the law of motion (17) for the autoregressive regime factor w_{t+1}, and the identity $\hat{y}_{t-1} = \hat{y}_{t-1}$. It can be simply represented as

$$X_t = \underbrace{G(s_t, \Theta)}_{9\times 9} X_{t-1} + \underbrace{M(s_t, \Theta)}_{9\times 5} \varepsilon_t, \tag{26}$$

with $\varepsilon_t \sim \mathbb{N}(0, I_5)$, where

$$G(s_t, \Theta) = \begin{pmatrix} \underbrace{A_1(s_t)}_{7\times 7} & \underbrace{0}_{7\times 2} \\ \underbrace{0}_{1\times 7} & \phi \quad 0 \\ 1 \quad \underbrace{0}_{1\times 6} & 0 \quad 0 \end{pmatrix}, \quad M(s_t, \Theta) = \begin{pmatrix} \underbrace{A_2(s_t)}_{7\times 4} & \underbrace{0}_{7\times 1} \\ \underbrace{\rho'}_{1\times 4} & \sqrt{1-\rho'\rho} \\ \multicolumn{2}{c}{\underbrace{0}_{1\times 5}} \end{pmatrix}$$

For the above state space representation (25) and (26), likelihood function can be easily evaluated using our newly-developed Kalman filter with Markov switching.

5.2.3 Quasi-Bayesian MLE

As is well known, implementation of MLE can be particularly challenging in applications involving DSGE models. Several classes of problems can arise in minimizing the negative log-likelihood function: discontinuities, multiple local minima, and identification.

Quasi-Bayesian methods have been widely used to help induce some desired curvature in the likelihood function surface. Assuming that we have already obtained a log-likelihood function $\log L(Y_{1:T}|\Theta)$ where $Y_{1:T}$ denotes real data from time 1 to time T and Θ denotes all parameters to be estimated, a quasi-Bayesian method augments the log-likelihood function with a prior distribution specified over Θ, denoted by $p(\Theta)$. Then, the quasi-Bayesian ML estimator is defined as

$$\hat{\Theta} = \underset{\Theta \in R(\Theta)}{\arg\min} - \log L(Y_{1:T}|\Theta) - \log p(\Theta)$$

Notice that the estimator corresponds to the mode of the log of the posterior distribution $p(\Theta|Y_{1:T})$. In the special case where the prior is diffuse or uninformative, the estimator $\hat{\Theta}$ converges to the classical ML estimator. When the prior is proper, the quasi-Bayesian estimate of Θ may be interpreted as the one obtained by maximization of a penalized log-likelihood function. The penalty $\log p(\Theta)$ depends on the strength of the researcher's prior about Θ and has the effect of pulling the estimator towards the mode of the prior density.

5.2.4 Bayesian Approach

In order to get more robust estimates, we employ a Markov Chain Monte Carlo (MCMC) technique based on our newly developed filter, which is a modified Kalman filter designed to deal with the endogenous feedback mechanism within our monetary DSGE model. The posterior samples drawn using this Bayesian approach not only helps us to find the more robust posterior mean and mode, but also provides the empirical joint distribution of all estimated parameters which are used for inference. A standard random walk Metropolis-Hastings algorithm is applied to implement the MCMC technique. We now summarize the algorithm for drawing Markov chain samples $x^{(j)}$ for $j = 1, \ldots, N$ of Θ which will converge to its posterior kernel

$$p(\Theta|Y_{1:T}) = L(Y_{1:T}|\Theta)p(\Theta).$$

In Step 2.1, c denotes a tuning scalar and Σ the inverse of negative Hessian obtained from the quasi-Bayesian MLE.

Step 1. Initialize the Markov chain with the quasi-Bayesian ML estimates: $x^{(0)} = \hat{\Theta}$.

Step 2. Repeat Steps 2.1-2.3 for $j = 1, 2, \ldots, N$.

Step 2.1. Generate y from $q(x^{(j-1)}, \cdot) =_d \mathbb{N}(x^{(j-1)}, c\Sigma)$ and u from $\mathcal{U}(0, 1)$.

Step 2.2. Compute the probability of move

$$\alpha(x^{(j-1)}, y) = \min\left[\frac{p(y|Y_{1:T})q(y, x^{(j)})}{p(x^{(j)}|Y_{1:T})q(x^{(j)}, y)}, 1\right]$$

Step 2.3. If $u \leq \alpha(x^{(j-1)}, y)$
− Set $x^{(j)} = y$.

Else
− Set $x^{(j)} = x^{(j-1)}$.

Step 3. Return the values $\{x^{(1)}, x^{(2)}, \ldots, x^{(N)}\}$.

As N gets large, the Markov chain converges to the posterior kernel $p(\Theta|Y_{1:T})$.

5.3 Empirical Analysis

5.3.1 Prior Distribution

The last three columns in Table 1 report prior distributions for all parameters. The priors for all but those specific to our endogenous feedback mechanism are in line with previous results in the literature and are relatively loose. For example, the parameters α_0, α_1 of coefficients to inflation and annualized net inflation target $\pi^{(A)}$ are from Bianchi (2012), risk aversion ϵ is from Lubik and Schorfheide (2004), and quarterly growth rate of technology $\psi^{(Q)}$ is from Davig and Doh (2014). Other structural parameters are also chosen to match closely those reported in the literature. For instance, coefficient to output gap γ has the same prior distribution and mean as Bianchi (2012) but with a smaller standard deviation from Davig and Doh (2014). Annualized net real interest rate $r^{(A)}$ has the same prior distribution and standard deviation but with a mean closer to the estimation result of Bianchi (2012). Steady-state elasticity of substitution between intermediate goods θ is set such that its implied markup of intermediate-good price over marginal cost is close to the estimate in Bianchi (2012). Level of price stickiness φ is set such that it together with ϵ, θ and inflation target Π^* would imply the slope of NKPC that is close to its estimate reported in Davig and Doh (2014). Autocorrelation coefficients and standard deviations for shocks are set to be relatively uninformative and loose as Beta(0.5,0.2) and Inverse Gamma(2,0.1). Finally, the most important new parameters (ρ, ϕ, τ) in our endogenous feedback mechanism are set to be relatively uninformative.

5.3.2 Empirical Findings

Concerning the parameters of the Taylor rule, coefficients to inflation are estimated to be 0.88 and 1.55 for less aggressive and more aggresive regimes, respectively. Other parameter estimates are in line with the results reported in the previous literature. For instance, annualized inflation target is estimated to be 2.06% and annualized real interest rate is 0.67%. We take a more serious look at the estimates of the parameters defining our endogenous feedback mechanism. Autocorrelation coefficient ϕ of regime factor is estimated to be 0.88, which means the regime factor is relatively persistent. More importantly, endogeneity parameter ρ_{uv} is estimated to be -0.48, which is of the largest magnitude among the four candidates driving monetary policy regime change. As we pointed out earlier below (18), ρ_k^2 captures the contribution of the k-th fundamental shock to the monetary policy shifts. Therefore, we may conclude that monetary policy shifts are 23.1% driven by past markup shocks. The contribution from preference shock, monetary policy intervention and technology shock are 13.2%, 2.0% and 0.3%, respectively.

Figure 5 shows the extracted policy regime factor. Shaded areas indicate the periods of the less aggressive regimes when the regime factor lies below the estimated threshold level. Monetary policy turns out to be more aggressive during the early years of the sample, from 1954 to 1967, except a short period on the second half of 1950s where inflation suddenly went up but soon controlled. During that time, Fed Chairman Martin had a clear goal in mind that the Fed would raise interest rates in response to an overheated economy.

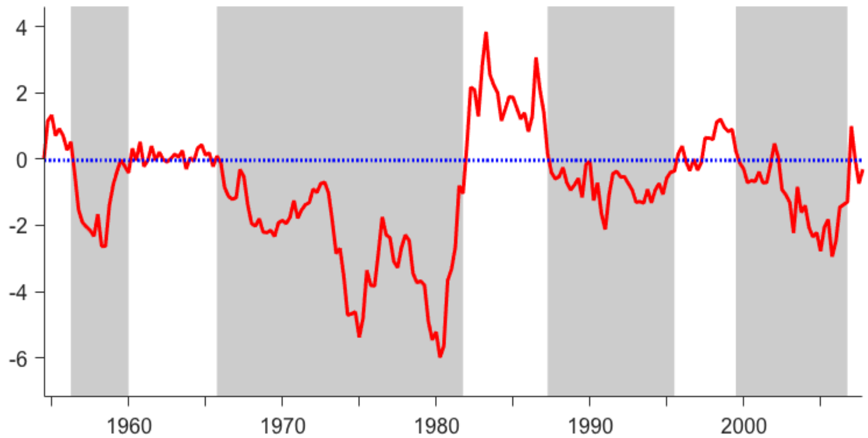
Towards the end of the 1960s, the policy regime factor started to fall below the threshold as can

Table 1: Posterior Estimates and Priors of the Parameters of the Regime-Switching DSGE Model

	Prior			Posterior		
Parameter	Density	π_1	π_2	Mean	5%	95%
α_0	G	1	0.4	0.8823	0.5680	1.2126
α_1	G	1.8	0.4	1.5500	1.0260	2.1531
γ	G	0.25	0.05	0.3576	0.2737	0.4528
ρ_R	B	0.5	0.2	0.7188	0.6175	0.8015
ϵ	G	2	0.5	2.5854	1.8818	3.4272
θ	G	8	3	8.6788	4.3752	14.2601
φ	G	67	10	73.7395	57.8012	91.3200
$\pi^{(A)}$	G	2	0.2	2.0573	1.7328	2.4030
$r^{(A)}$	G	0.8	0.2	0.6665	0.4297	0.9366
$\psi^{(Q)}$	G	0.5	0.1	0.4391	0.3519	0.5290
ρ_a	B	0.5	0.2	0.6217	0.4437	0.7737
ρ_ξ	B	0.5	0.2	0.9160	0.8790	0.9480
ρ_u	B	0.5	0.2	0.9481	0.9173	0.9739
$100\sigma_a$	I	2	0.1	0.3700	0.2500	0.4900
$100\sigma_\xi$	I	2	0.1	2.9900	2.2500	3.8900
$100\sigma_u$	I	2	0.1	1.6600	0.7300	2.9300
$100\sigma_e$	I	2	0.1	0.1400	0.1100	0.1700
ϕ	B	0.5	0.3	0.8816	0.7847	0.9746
τ	N	0	1	-0.0331	-1.7541	1.6790
ρ_{av}	U	-1	1	-0.0514	-0.3395	0.2826
$\rho_{\xi v}$	U	-1	1	0.3627	-0.2873	0.8357
ρ_{uv}	U	-1	1	-0.4804	-0.9425	0.2458
ρ_{ev}	U	-1	1	0.1426	-0.1552	0.5591

Notes: Parameter estimates are obtained from 300,000 posterior draws after 40,000 burn-in. π_1 and π_2 are the means and standard deviations for Gamma, Beta and Normal distributions, the left side and right side for Uniform distribution, and the parameters ν and s for Inverse Gamma distribution with density $p(\sigma|\nu, s) \propto \sigma^{-\nu-1}\exp(-\nu s^2/2\sigma^2)$.

Figure 5: Extracted Monetary Policy Regime Factor



Notes: The shaded areas indicate the less aggressive monetary policy regime.

be seen clearly in Figure 5 and monetary policy moved into the first significant and persistent less aggressive regime. As the inflation seemed to decrease in the early 1970s, the regime factor climbed upwards to the threshold. However, as inflations rose again to a higher level in the mid-1970s, the regime factor fell sharply which implies that the strength of the less aggressive regime had been dramatically reinforced. The Fed Chairman Arthur F. Burns is often thought to be responsible for the high and variable inflation that prevailed during the 1970s. It is commonly accepted that on several occasions he had to succumb to the requests of the White House. In fact, the estimates show that for almost entire duration of his chairmanship, the Fed followed a less aggressive regime. During these years, the more aggressive regime would have required a much higher federal funds rate, which must have been politically very costly.

This long period of less aggressive monetary policy ended around the second half of 1981, shortly after Paul Volcker took office in August 1979. Volcker was appointed with the precise goal of ending the high inflation. The jump in the regime factor which placed it significantly above the threshold confirms the widespread belief that he delivered on his commitment.

The estimation results on the second half of our sample periods coincide very well with the results in Chang and Kwak (2017). Since 1981, monetary policy was more aggressive until the end of 2007, right before the Great Recession, except for two relatively short periods around the recessions in 1991 and 2001.

To summarize, our results strongly support the idea that the appointment of Volcker marked a change in the Fed's inflation stance compared to the 1970s. And, the evolution of regime factor captures repeated fluctuations between a more aggressive and a less aggressive regime for Post-World War II U.S. monetary policy.

5.4 Model Comparison

This section compares the model fit between the endogenous regime switching model and its exogenous counterpart by both Bayesian and classical approaches. Under the Bayesian approach, we choose to use Geweke (1999)'s harmonic mean estimator to estimate the marginal data density. The marginal likelihood can be estimated by

$$\hat{p}(Y) = \left[\frac{1}{M} \sum_{k=1}^{M} \frac{f(\Theta^{(k)})}{p(Y|\Theta^{(k)})p(\Theta^{(k)})} \right]^{-1}$$

where $\Theta^{(k)}$ is drawn from $p(\Theta|Y)$ and $p(Y|\Theta^{(k)})$ can be evaluated from our Kalman filter with Markov switching.

Thus, the marginal data density for the endogenous regime switching model can be readily computed using the posterior samples obtained in Section 5.2.4. For the exogenous regime switching model, we impose $\rho = 0$ and reestimate the model. Its marginal data density can be computed accordingly. Table 2 shows that the difference in the log-marginal likelihood between the endogenous and exogenous regime switching models is roughly 12, significantly larger than 4.74 (as in Kass and Raftery 1995). Based on the Jeffrey criterion (1998), the endogenous regime switching model is strongly preferred over the exogenous case.

Table 2: Log Marginal Likelihood Estimates for the Endogenous and Exogenous Models

	endogenous	exogenous
$\ln \hat{p}(Y)$	-1289.45	-1301.37

5.5 Macro Dynamics

This section will present the macroeconomic dynamics for both estimated endogenous and exogenous regime switching models. With the estimates of parameters for both models, we plot the impulse response functions to determine how the new endogenous feedback mechanism would significantly modify the macroeconomic dynamics comparing to the conventional exogenous case. Further, we compute the endogenous and exogenous expectations formation effects to analyze how these macroeconomic dynamics are induced by the changes of agents' beliefs about future policy regimes evolutions.

[To be added: impulse response functions]

6 Extensions and Conclusions

In this section, we propose several interesting and potentially important extensions based on our current work and conclude.

6.1 Extensions

Monetary-fiscal interaction In our current model, there are four structural shocks as candidates of driving forces to monetary policy regime change. Based on various theoretical and empirical findings in previous literature, it is natural and interesting to include fiscal policy into the DSGE model and consider it as another important candidate that drives choice of monetary policy regime.

6.2 Conclusions

We introduce a novel endogenous regime switching approach to a prototype monetary DSGE model. The endogeneity brings about a new feedback mechanism from past monetary policy interventions to central bank's current regime selection. To better illustrate how this mechanism works, we derive an analytical solution to a regime switching Fisherian model with the endogenous feedback mechanism. Numerical experiments show that this feedback mechanism would significantly affect the macroeconomic dynamics and expectations formation effects. In the empirical part, estimation shows that Post-World War II U.S. monetary policy regime change is 23.1% driven by past markup shocks, which has played a predominant role among the four fundamental shocks. The extracted latent regime factor indicate that monetary policy is identified by repeated fluctuations between a more aggressive and a less aggressive regime, with the latter prevailing in the 1970s and during the recent crisis. Lastly, the Bayes factor strongly favors the endogenous switching version of the model over the exogenous case.

References

S. An and F. Schorfheide. Bayesian analysis of DSGE models. *Econometric Reviews*, 26(2-4): 113–172, 2007.

J. Barthélemy and M. Marx. Solving endogenous regime switching models. *Journal of Economic Dynamics and Control*, 77:1–25, 2017.

F. Bianchi. Regime switches, agents' beliefs, and post-World War II US macroeconomic dynamics. *Review of Economic Studies*, 80(2):463–490, 2012.

Y. Chang and B. Kwak. Endogenous monetary-fiscal regime change in the United States. *Working Paper, Indiana University*, 2017.

Y. Chang, Y. Choi, and J. Y. Park. A new approach to model regime switching. *Journal of Econometrics*, 196(1):127–143, 2017.

S. Cho. Sufficient conditions for determinacy in a class of Markov-switching rational expectations models. *Review of Economic Dynamics*, 21:182–200, 2016.

R. Clarida, J. Galí, and M. Gertler. The science of monetary policy: A New Keynesian perspective. *Journal of Economic Literature*, 37:1661–1707, 1999.

R. Clarida, J. Galí, and M. Gertler. Monetary policy rules and macroeconomic stability: evidence and some theory. *The Quarterly Journal of Economics*, 115(1):147–180, 2000.

T. Davig and T. Doh. Monetary policy regime shifts and inflation persistence. *Review of Economics and Statistics*, 96(5):862–875, 2014.

T. Davig and E. M. Leeper. Endogenous monetary policy regime change. In *NBER International Seminar on Macroeconomics 2006*, pages 345–377. MIT Press, Cambridge, MA, 2006.

R. E. Farmer, D. F. Waggoner, and T. Zha. Minimal state variable solutions to Markov-switching rational expectations models. *Journal of Economic Dynamics and Control*, 35(12):2150–2166, 2011.

A. J. Filardo. Business-cycle phases and their transitional dynamics. *Journal of Business & Economic Statistics*, 12(3):299–308, 1994.

A. J. Filardo and S. F. Gordon. Business cycle durations. *Journal of Econometrics*, 85(1):99–123, 1998.

A. Foerster, J. F. Rubio-Ramrez, D. F. Waggoner, and T. Zha. Perturbation methods for Markov-switching dynamic stochastic general equilibrium models. *Quantitative Economics*, 7(2):637–669, 2016.

J. Geweke. Using simulation methods for Bayesian econometric models: inference, development, and communication. *Econometric Reviews*, 18(1):1–73, 1999.

J. D. Hamilton. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica*, 57(2):357–384, 1989.

P. N. Ireland. Technology shocks in the new Keynesian model. *Review of Economics and Statistics*, 86(4):923–936, 2004.

R. E. Kass and A. E. Raftery. Bayes factors. *Journal of the American Statistical Association*, 90 (430):773–795, 1995.

C.-J. Kim and C. R. Nelson. *State-space models with regime switching: Classical and Gibbs-sampling approaches with applications*, volume 2. MIT Press, Cambridge, MA, 1999.

C.-J. Kim, J. Piger, and R. Startz. Estimation of Markov regime-switching regression models with endogenous switching. *Journal of Econometrics*, 143(2):263–273, 2008a.

J. Kim, S. Kim, E. Schaumburg, and C. A. Sims. Calculating and using second-order accurate solutions of discrete time dynamic equilibrium models. *Journal of Economic Dynamics and Control*, 32(11):3397–3414, 2008b.

E. M. Leeper and T. Zha. Modest policy interventions. *Journal of Monetary Economics*, 50(8): 1673–1700, 2003.

Z. Liu, D. F. Waggoner, and T. Zha. Sources of macroeconomic fluctuations: A regime-switching DSGE approach. *Quantitative Economics*, 2(2):251–301, 2011.

T. A. Lubik and F. Schorfheide. Computing sunspot equilibria in linear rational expectations models. *Journal of Economic Dynamics and Control*, 28(2):273–285, 2003.

T. A. Lubik and F. Schorfheide. Testing for indeterminacy: an application to US monetary policy. *The American Economic Review*, 94(1):190–217, 2004.

R. E. Lucas. Econometric policy evaluation: A critique. In *Carnegie-Rochester conference series on public policy*, volume 1, pages 19–46. North-Holland, Amsterdam, 1976.

J. Maih. Efficient perturbation methods for solving regime-switching DSGE models. *Working Paper, Norges Bank*, 2015.

J. J. Rotemberg. Sticky prices in the United States. *Journal of Political Economy*, 90(6):1187–1211, 1982.

C. A. Sims. Solving linear rational expectations models. *Computational Economics*, 20(1):1–20, 2002.

C. A. Sims and T. Zha. Were there regime switches in US monetary policy? *The American Economic Review*, 96(1):54–81, 2006.

J. B. Taylor. Discretion versus policy rules in practice. In *Carnegie-Rochester conference series on public policy*, volume 39, pages 195–214. North-Holland, Amsterdam, 1993.

M. Woodford. *Interest and prices: Foundations of a theory of monetary policy*. Princeton University Press, Princeton, NJ, 2011.

Appendices

A Derivation of Analytical Solution to Fisherian Model

A.1 Analytical Derivation

The equilibrium conditions (1)∼(6) form a system of expectational nonlinear difference equations. Solving this system is tantamount to solving a fixed point problem. Suppose that agents come across a decision rule to evaluate π_{t+1} given all past and current information. Mathematically, it means that π_{t+1} will be integrated out with respect to its conditional distribution conditioning on current information set. Then, agents can solve π_{t+1}. If the resulting solution shares the same form with the initial guess (i.e., initial decision rule), then it is regarded the true solution to the original system. Otherwise, agents will use the resulting solution as an updated guess and follow the same procedure until convergence to a fixed point. This solution approach is often called guess and verify (or undetermined coefficients).

We start with the following guess of true solution with undetermined coefficients[17]

$$\pi_{t+1} = a_1(s_{t+1}, p_{s_{t+1},0}(\epsilon^e_{t+1}))r_{t+1} + a_2(s_{t+1})\epsilon^e_{t+1}, \tag{27}$$

where s_{t+1} is regime at period $t+1$, $p_{s_{t+1},0}(\epsilon^e_{t+1})$ the TVTP from regime s_{t+1} to regime 0 at period $t+2$. The TVTP from regime s_{t+1} to regime 1 at $t+2$, $p_{s_{t+1},1}(\epsilon^e_{t+1})$, can be simply computed as $1 - p_{s_{t+1},0}(\epsilon^e_{t+1})$. In this guess, monetary policy shock ϵ^e_{t+1} appears not only in the second term which has a direct effect on π_{t+1}, but also changes agents' beliefs, in the first term, on the distribution of next period regime through the endogenous feedback mechanism.

Plugging (27) into (8), the second term of (27) vanishes because random variables s_{t+1} and ϵ^e_{t+1} are independent. Also notice that r_{t+1} is independent of s_{t+1} and ϵ^e_{t+1}. Therefore, we obtain

$$\begin{aligned}
\mathbb{E}_t\pi_{t+1} &= \mathbb{E}(a_1(s_{t+1}, p_{s_{t+1},0}(\epsilon^e_{t+1}))r_{t+1}|\mathcal{F}_t) + 0 \\
&= \mathbb{E}(a_1(s_{t+1}, p_{s_{t+1},0}(\epsilon^e_{t+1}))|\mathcal{F}_t)\mathbb{E}(r_{t+1}|\mathcal{F}_t) \\
&= [\mathbb{E}(a_1(s_{t+1}=0, p_{00}(\epsilon^e_{t+1}))|\mathcal{F}_t)\cdot p_{s_t,0}(\epsilon^e_t) \\
&+ \mathbb{E}(a_1(s_{t+1}=1, p_{10}(\epsilon^e_{t+1}))|\mathcal{F}_t)\cdot p_{s_t,1}(\epsilon^e_t)]\cdot\rho_r r_t \\
&= [\mathbb{E}(a_1(s_{t+1}=0, p_{00}(\epsilon^e_{t+1})))\cdot p_{s_t,0}(\epsilon^e_t) \\
&+ \mathbb{E}(a_1(s_{t+1}=1, p_{10}(\epsilon^e_{t+1})))\cdot p_{s_t,1}(\epsilon^e_t)]\cdot\rho_r r_t.
\end{aligned} \tag{28}$$

[17] This is a correct guess we obtained after a few failed trials. We believe the true solution should permit the additivity between r_{t+1} and ϵ^e_{t+1} because as a limiting case, it must reduces to the known result of equation (12) in literature. Also, it reflects that regime switching is the only nonlinear feature in the model. Conditional on regime, the model would be fully linear. It is worth noting that solution is unique under the given structure of initial guess. Other forms of solutions might or might not exist.

The last equality holds by the independence of $\mathcal{F}_t$ and ϵ^e_{t+1}. Plugging (28) and (2) into (1), then combining (3) yields

$$\begin{aligned} i_t &= [\mathbb{E}(a_1(s_{t+1}=0, p_{00}(\epsilon^e_{t+1}))) \cdot p_{s_t,0}(\epsilon^e_t) + \mathbb{E}(a_1(s_{t+1}=1, p_{10}(\epsilon^e_{t+1}))) \cdot p_{s_t,1}(\epsilon^e_t) + 1] \cdot \rho_r r_t \\ &= \alpha(s_t)\pi_t + \sigma_e \epsilon^e_t. \end{aligned}$$

Taking one period forward, we can solve π_{t+1} as

$$\begin{aligned} \pi_{t+1} = \frac{\rho_r}{\alpha(s_{t+1})} &[\mathbb{E}(a_1(s_{t+2}=0, p_{00}(\epsilon^e_{t+2}))) \cdot p_{s_{t+1},0}(\epsilon^e_{t+1}) \\ &+ \mathbb{E}(a_1(s_{t+2}=1, p_{10}(\epsilon^e_{t+2}))) \cdot p_{s_{t+1},1}(\epsilon^e_{t+1}) + 1] r_{t+1} - \frac{\sigma_e}{\alpha(s_{t+1})}\epsilon^e_{t+1}. \end{aligned} \tag{29}$$

Comparing (29) with (27), we match the unknown coefficients

$$\begin{aligned} a_1(s_{t+1}, p_{s_{t+1},0}(\epsilon^e_{t+1})) \; = \frac{\rho_r}{\alpha(s_{t+1})} &[\mathbb{E}(a_1(s_{t+2}=0, p_{00}(\epsilon^e_{t+2}))) \cdot p_{s_{t+1},0}(\epsilon^e_{t+1}) \\ &+ \mathbb{E}(a_1(s_{t+2}=1, p_{10}(\epsilon^e_{t+2}))) \cdot p_{s_{t+1},1}(\epsilon^e_{t+1}) + 1], \end{aligned} \tag{30}$$

and

$$a_2(s_{t+1}) = -\frac{\sigma_e}{\alpha(s_{t+1})}. \tag{31}$$

To determine a_1, we first denote the two expectational terms in the right-hand-side of (30) by

$$C_0 = \mathbb{E}(a_1(s_{t+2}=0, p_{00}(\epsilon^e_{t+2}))), \quad \text{and} \quad C_1 = \mathbb{E}(a_1(s_{t+2}=1, p_{10}(\epsilon^e_{t+2}))), \tag{32}$$

which are two undetermined constants since ϵ^e_{t+2} is iid and hence the unconditional expectation of any function of ϵ^e_{t+2} would not depend on time index. Then, we consider $s_{t+1} = 0$ and take expectation with respect to ϵ^e_{t+1} in both sides of (30). Notice that the left-hand-side of (30) becomes C_0 because of the reason just mentioned. Therefore, we obtain from (30) and (32) that

$$C_0 = \mathbb{E}(a_1(s_{t+1}=0, p_{00}(\epsilon^e_{t+1}))) = \frac{\rho_r}{\alpha_0}[C_0 \mathbb{E} p_{00}(\epsilon^e_{t+1}) + C_1 \mathbb{E} p_{01}(\epsilon^e_{t+1}) + 1], \tag{33}$$

where $\alpha_0 = \alpha(s_{t+1} = 0)$. Similarly, considering $s_{t+1} = 1$ in (30) would yield

$$C_1 = \mathbb{E}(a_1(s_{t+1}=1, p_{10}(\epsilon^e_{t+1}))) = \frac{\rho_r}{\alpha_1}[C_0 \mathbb{E} p_{10}(\epsilon^e_{t+1}) + C_1 \mathbb{E} p_{11}(\epsilon^e_{t+1}) + 1], \tag{34}$$

where $\alpha_1 = \alpha(s_{t+1} = 1)$.

Now, we can write C_1 as a function of C_0 by (34):

$$\begin{aligned} C_1 &= \frac{\frac{\rho_r}{\alpha_1}\mathbb{E}p_{10}(\epsilon^e_{t+1})}{1-\frac{\rho_r}{\alpha_1}\mathbb{E}p_{11}(\epsilon^e_{t+1})}C_0 + \frac{\frac{\rho_r}{\alpha_1}}{1-\frac{\rho_r}{\alpha_1}\mathbb{E}p_{11}(\epsilon^e_{t+1})} \\ &= \frac{\rho_r\mathbb{E}p_{10}(\epsilon^e_{t+1})}{\alpha_1-\rho_r+\rho_r\mathbb{E}p_{10}(\epsilon^e_{t+1})}C_0 + \frac{\rho_r}{\alpha_1-\rho_r+\rho_r\mathbb{E}p_{10}(\epsilon^e_{t+1})}, \end{aligned} \tag{35}$$

where in the denominator we replaced $\mathbb{E}p_{11}(\epsilon^e_{t+1})$ with $1-\mathbb{E}p_{10}(\epsilon^e_{t+1})$. Plugging (35) into (33), we solve

$$\begin{aligned} C_0 &= \frac{\alpha_1+\rho_r(\mathbb{E}p_{10}(\epsilon^e_{t+1})-\mathbb{E}p_{00}(\epsilon^e_{t+1}))}{\alpha_0\left(\frac{\alpha_1}{\rho_r}-1+\mathbb{E}p_{10}(\epsilon^e_{t+1})\right)-\alpha_1\mathbb{E}p_{00}(\epsilon^e_{t+1})-\rho_r(\mathbb{E}p_{10}(\epsilon^e_{t+1})-\mathbb{E}p_{00}(\epsilon^e_{t+1}))} \\ &= \frac{\alpha_1+\rho_r(\mathbb{E}p_{10}(\epsilon^e_{t+1})-\mathbb{E}p_{00}(\epsilon^e_{t+1}))}{(\alpha_1-\rho_r)\left(\frac{\alpha_0}{\rho_r}-\mathbb{E}p_{00}(\epsilon^e_{t+1})\right)+(\alpha_0-\rho_r)\mathbb{E}p_{10}(\epsilon^e_{t+1})} > 0, \end{aligned} \tag{36}$$

where $\mathbb{E}p_{00}(\epsilon^e_{t+1})$ and $\mathbb{E}p_{10}(\epsilon^e_{t+1})$ can be obtained by taking expectation of TVTP with respect to ϵ^e_{t+1}. Details will be provided in Section A.2. C_1 is then given by (35) and (36). Finally, substituting C_0 and C_1 into (30), we may determine the random coefficient a_1.

We may now derive our solution for π_{t+1} by plugging the expressions (30), (31) of the random coefficients a_1, a_2 into our initial guess given in (27) as follows:

$$\begin{aligned} \pi_{t+1} =& \frac{\rho_r}{\alpha(s_{t+1})}\left[C_0 p_{s_{t+1},0}(\epsilon^e_{t+1}) + C_1 p_{s_{t+1},1}(\epsilon^e_{t+1}) + 1\right] r_{t+1} - \frac{\sigma_e}{\alpha(s_{t+1})}\epsilon^e_{t+1} \\ =& \frac{\rho_r}{\alpha(s_{t+1})}\left[\frac{(\alpha_1-\rho_r)p_{s_{t+1},0}(\epsilon^e_{t+1})+\rho_r\mathbb{E}p_{10}(\epsilon^e_{t+1})}{\alpha_1-\rho_r+\rho_r\mathbb{E}p_{10}(\epsilon^e_{t+1})}C_0 + \frac{\alpha_1-\rho_r p_{s_{t+1},0}(\epsilon^e_{t+1})+\rho_r\mathbb{E}p_{10}(\epsilon^e_{t+1})}{\alpha_1-\rho_r+\rho_r\mathbb{E}p_{10}(\epsilon^e_{t+1})}\right] \\ & \cdot r_{t+1} - \frac{\sigma_e}{\alpha(s_{t+1})}\epsilon^e_{t+1} \\ =& \underbrace{\frac{\rho_r}{\alpha(s_{t+1})}\frac{(\alpha_1-\alpha_0)p_{s_{t+1},0}(\epsilon^e_{t+1})+\alpha_1\left(\frac{\alpha_0}{\rho_r}-\mathbb{E}p_{00}(\epsilon^e_{t+1})\right)+\alpha_0\mathbb{E}p_{10}(\epsilon^e_{t+1})}{(\alpha_1-\rho_r)\left(\frac{\alpha_0}{\rho_r}-\mathbb{E}p_{00}(\epsilon^e_{t+1})\right)+(\alpha_0-\rho_r)\mathbb{E}p_{10}(\epsilon^e_{t+1})}}_{a_1(s_{t+1},p_{s_{t+1},0}(\epsilon^e_{t+1}))} r_{t+1} - \underbrace{\frac{\sigma_e}{\alpha(s_{t+1})}}_{a_2(s_{t+1})}\epsilon^e_{t+1}, \end{aligned} \tag{37}$$

where the random coefficient a_1 depends on both s_{t+1} and ϵ^e_{t+1}, while a_2 only on s_{t+1}. $\mathbb{E}p_{00}(\epsilon^e_{t+1})$ and $\mathbb{E}p_{10}(\epsilon^e_{t+1})$ are two constants which need be evaluated.

A.2 Numerical Evaluation

To compute coefficient a_1 in (37), we need to evaluate $p_{s_{t+1},0}(\epsilon^e_{t+1}), \mathbb{E}p_{00}(\epsilon^e_{t+1})$ and $\mathbb{E}p_{10}(\epsilon^e_{t+1})$. This can be done as follows.

Suppose $|\rho| < 1$ and $|\phi| < 1$, we compute the transition probability for regime s_t following theorem 3.1 in Chang et al. (2017). The bivariate normality of ϵ^e_t and v_{t+1} implies

$$v_{t+1}|\epsilon^e_t \;=_d\; \mathbb{N}(\rho\epsilon^e_t, 1-\rho^2).$$

Define

$$z_{t+1} = \frac{v_{t+1} - \rho\epsilon^e_t}{\sqrt{1-\rho^2}} = \frac{w_{t+1} - \phi w_t}{\sqrt{1-\rho^2}} - \rho\frac{\epsilon^e_t}{\sqrt{1-\rho^2}},$$

where $\epsilon^e_t = (i_t - \alpha(s_t)\pi_t)/\sigma_e$. It follows that

$$p(z_{t+1}|w_t, \epsilon^e_t) = p(z_{t+1}|w_t, i_t, \pi_t) =_d \mathbb{N}(0, 1).$$

To simplify notation, we use ϵ^e_t instead of the set of endogenous variables in what follows. Then,

$$\begin{aligned} \mathbb{P}\{w_{t+1} < \tau|w_t, \epsilon^e_t\} =&\; \mathbb{P}\left\{z_{t+1} < \frac{\tau - \phi w_t}{\sqrt{1-\rho^2}} - \rho\frac{\epsilon^e_t}{\sqrt{1-\rho^2}}\middle| w_t, \epsilon^e_t\right\} \\ =&\; \Phi_\rho\left(\tau - \phi w_t - \rho\epsilon^e_t\right), \end{aligned}$$

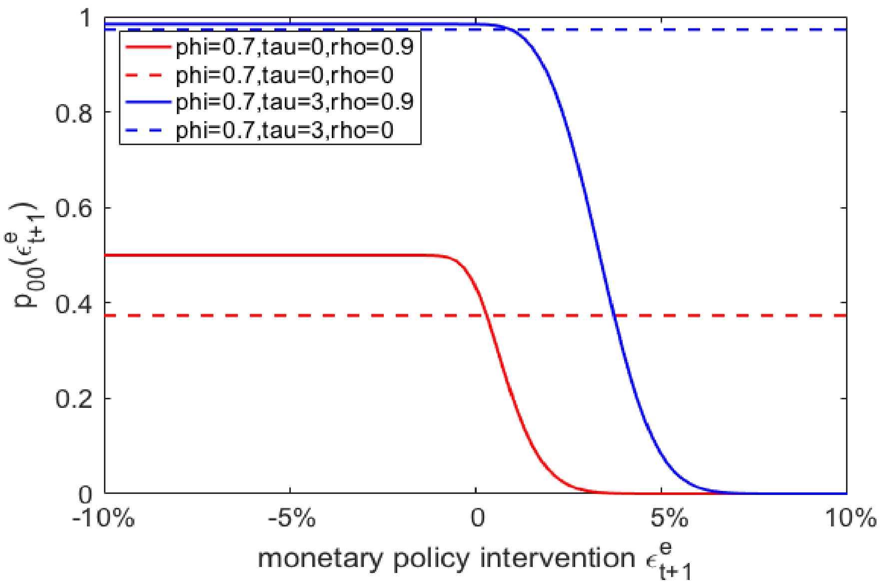
where we use $\Phi_\rho(x) = \Phi(x/\sqrt{1-\rho^2})$ to simplify notation. Now, note that the $AR(1)$ process w_t has the unconditional density $w_t =_d \mathbb{N}\left(0, 1/(1-\phi^2)\right)$, which implies $w_t\sqrt{1-\phi^2} =_d \mathbb{N}(0, 1)$. Consequently, we have

$$\begin{aligned} \mathbb{P}\{s_{t+1} = 0|s_t = 0, \epsilon^e_t\} =& \mathbb{P}\{w_{t+1} < \tau|w_t < \tau, \epsilon^e_t\} \\ =& \mathbb{P}\{w_{t+1} < \tau|w_t\sqrt{1-\phi^2} < \tau\sqrt{1-\phi^2}, \epsilon^e_t\} \\ =& \frac{\mathbb{P}\{w_{t+1} < \tau, w_t\sqrt{1-\phi^2} < \tau\sqrt{1-\phi^2} \mid \epsilon^e_t\}}{\mathbb{P}(w_t\sqrt{1-\phi^2} < \tau\sqrt{1-\phi^2})} \\ =& \frac{\int_{-\infty}^{\tau\sqrt{1-\phi^2}} \Phi_\rho\left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho\epsilon^e_t\right)\varphi(x)dx}{\Phi(\tau\sqrt{1-\phi^2})}, \end{aligned}$$

Therefore, we can obtain

$$p_{00}(\epsilon^e_{t+1}) = \mathbb{P}(s_{t+2} = 0|s_{t+1} = 0, \epsilon^e_{t+1}) = \frac{\int_{-\infty}^{\tau\sqrt{1-\phi^2}} \Phi_\rho\left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho\epsilon^e_{t+1}\right)\varphi(x)dx}{\Phi(\tau\sqrt{1-\phi^2})}. \quad (38)$$

Figure 6: Time-Varying Transition Probabilites from Regime 0 to Regime 0



Analogously

$$p_{10}(\epsilon^e_{t+1}) = \mathbb{P}(s_{t+2} = 0 | s_{t+1} = 1, \epsilon^e_{t+1}) = \frac{\int_{\tau\sqrt{1-\phi^2}}^{\infty} \Phi_\rho\left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho\epsilon^e_{t+1}\right)\varphi(x)dx}{1 - \Phi(\tau\sqrt{1-\phi^2})}. \quad (39)$$

Then, it follows from (38) and (39) that

$$p_{01}(\epsilon^e_{t+1}) = \mathbb{P}(s_{t+2} = 1 | s_{t+1} = 0, \epsilon^e_{t+1}) = 1 - p_{00}(\epsilon^e_{t+1}),$$

$$p_{11}(\epsilon^e_{t+1}) = \mathbb{P}(s_{t+2} = 1 | s_{t+1} = 1, \epsilon^e_{t+1}) = 1 - p_{10}(\epsilon^e_{t+1}).$$

Figure 6 displays how TVTP is determined by parameters (ϕ, τ, ρ). If $\rho = 0$, (ϕ, τ) has one-to-one correspondence to the parameters in exogenous switching model, i.e., constant transition probabilities $(\bar{p}_{00}, \bar{p}_{11})$. So, they determine the magnitude of transition probabilities, while the endogeneity parameter ρ governs the fluctuation of transition probabilities.

Next, as in Chang et al. (2017), we may rewrite (38) and (39) as bivariate normal distribution functions[18]

$$p_{00}(\epsilon^e_{t+1}) = \frac{\int_{-\infty}^{\tau\sqrt{1-\phi^2}} \int_{-\infty}^{\frac{\tau - \rho\epsilon^e_{t+1}}{\sqrt{1-\rho^2}}} f_1(x, y)dydx}{\Phi(\tau\sqrt{1-\phi^2})},$$

[18] Detailed derivations can be provided upon request.

with

$$f_1(x,y) = \mathbb{N}\left(0, \begin{pmatrix} 1 & \dfrac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} \\ \dfrac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & 1 + \dfrac{\phi^2}{(1-\rho^2)(1-\phi^2)} \end{pmatrix}\right),$$

and,

$$p_{10}(\epsilon^e_{t+1}) = \frac{\displaystyle\int_{-\infty}^{-\tau\sqrt{1-\phi^2}} \int_{-\infty}^{\frac{\tau-\rho\epsilon^e_{t+1}}{\sqrt{1-\rho^2}}} f_2(x,y)dydx}{1-\Phi(\tau\sqrt{1-\phi^2})},$$

with

$$f_2(x,y) = \mathbb{N}\left(0, \begin{pmatrix} 1 & -\dfrac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} \\ -\dfrac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & 1 + \dfrac{\phi^2}{(1-\rho^2)(1-\phi^2)} \end{pmatrix}\right).$$

Similarly, we may rewrite $\mathbb{E}p_{00}(\epsilon^e_{t+1})$ and $\mathbb{E}p_{10}(\epsilon^e_{t+1})$ as trivariate normal distribution functions

$$\begin{aligned}\mathbb{E}p_{00}(\epsilon^e_{t+1}) &= \frac{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\tau\sqrt{1-\phi^2}} \Phi_\rho\left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho\epsilon^e_{t+1}\right)\varphi(x)\varphi(\epsilon^e_{t+1})dxd\epsilon^e_{t+1}}{\Phi(\tau\sqrt{1-\phi^2})} \\ &= \frac{\displaystyle\int_{-\infty}^{\tau\sqrt{1-\phi^2}} \int_{-\infty}^{\tau/\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} f_3(x,y,\epsilon)d\epsilon dydx}{\Phi(\tau\sqrt{1-\phi^2})},\end{aligned}$$

with

$$f_3(x,y,\epsilon) = \mathbb{N}\left(0, \begin{pmatrix} 1 & \dfrac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & 0 \\ \dfrac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & \dfrac{1}{(1-\rho^2)(1-\phi^2)} & \dfrac{\rho}{\sqrt{1-\rho^2}} \\ 0 & \dfrac{\rho}{\sqrt{1-\rho^2}} & 1 \end{pmatrix}\right),$$

and,

$$\begin{aligned}\mathbb{E}p_{10}(\epsilon^e_{t+1}) &= \frac{\displaystyle\int_{-\infty}^{\infty}\int_{\tau\sqrt{1-\phi^2}}^{\infty} \Phi_\rho\left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho\epsilon^e_{t+1}\right)\varphi(x)\varphi(\epsilon^e_{t+1})dxd\epsilon^e_{t+1}}{1-\Phi(\tau\sqrt{1-\phi^2})} \\ &= \frac{\displaystyle\int_{-\infty}^{-\tau\sqrt{1-\phi^2}} \int_{-\infty}^{\tau/\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} f_4(x,y,\epsilon)d\epsilon dydx}{1-\Phi(\tau\sqrt{1-\phi^2})},\end{aligned}$$

with

$$f_4(x,y,\epsilon) = \mathbb{N}\left(0, \begin{pmatrix} 1 & -\frac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & 0 \\ -\frac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & \frac{1}{(1-\rho^2)(1-\phi^2)} & \frac{\rho}{\sqrt{1-\rho^2}} \\ 0 & \frac{\rho}{\sqrt{1-\rho^2}} & 1 \end{pmatrix}\right).$$

Finally, using *mvncdf* function in Matlab, we are able to evaluate (37).

B Computations of Expectations Formation Effects

We define direct and expectations formation effects as in Leeper and Zha (2003). The K-period forecast using solution (10), given information at T, is

$$\begin{aligned}\pi_{T+K} =& a_1(s_{T+K}, p_{s_{T+K},0}(\epsilon^e_{T+K}))\rho_r^K r_T + a_1(s_{T+K}, p_{s_{T+K},0}(\epsilon^e_{T+K}))\sum_{u=0}^{K-1}\rho_r^u \sigma_r \epsilon^r_{T+K-u} \\ &+ a_2(s_{T+K})\epsilon^e_{T+K}.\end{aligned}$$

Direct effects and expectations formation effects from a policy intervention are computed from forecasts from fixed-regime and endogenous regime switching models, respectively, conditional on a given intervention. Suppose the economy is under regime 0 at period T. Both of these forecasts are reported relative to a baseline forecast denoted by

$$\begin{aligned}\text{Baseline effects} &= \mathbb{E}(\pi_{T+K}|\mathcal{F}_T, s_t = 0, t = T+1, \ldots, T+K) \\ &= \mathbb{E}[a_1(s_{T+K} = 0, p_{00}(\epsilon^e_{T+K}))]\rho_r^K r_T.\end{aligned}$$

Let I_T be a hypothetical intervention at time T, specified as a K-period sequence of exogenous policy actions, $I_T = \{\tilde{\epsilon}^e_{T+1}, \ldots, \tilde{\epsilon}^e_{T+K}\}$. Although the policy advisor chooses I_T, private agents treat it as random. Denote a forecast of $\{\pi_{T+K}\}$ conditional on I_T by $\mathbb{E}(\pi_{T+K}|I_T, \mathcal{F}_T, s_t = 0, t = T+1, \cdots, T+K)$. Direct effects of I_T relative to baseline are

$$\begin{aligned}\text{Direct effects} &= \mathbb{E}(\pi_{T+K}|I_T, \mathcal{F}_T, s_t = 0, t = T+1, \cdots, T+K) - \text{Baseline effects} \\ &= a_1(s_{T+K} = 0, p_{00}(\tilde{\epsilon}^e_{T+K}))\rho_r^K r_T + a_2(s_{T+K} = 0)\tilde{\epsilon}^e_{T+K} \\ &\quad - \mathbb{E}[a_1(s_{T+K} = 0, p_{00}(\epsilon^e_{T+K}))]\rho_r^K r_T. \end{aligned} \tag{40}$$

In an exogenous regime switching model, when the regime index is fixed, the model becomes linear. Hence, direct effect captures the linear effect from policy intervention. In our endogenous regime switching model, however, that is not true in general. Even if regime index is fixed for future periods $T+1$ to $T+K$, agents' belief in the strength of that prevailing regime can still be affected by the monetary policy intervention. It is clearly shown in the first term of (40) that monetary policy intervention $\tilde{\epsilon}^e_{T+K}$ causes changes in direct effects through TVTP $p_{00}(\tilde{\epsilon}^e_{T+K})$. This additional channel makes the direct effect include some nonlinear feature.

Without fixing the future regime indices, now, an intervention may trigger changes in agents' beliefs about future's prevailing policy regime. The changes in agents' expectations of future policy evolution alter their optimal behavior and hence induce the expectations formation effects. Total effects of the monetary policy intervention combine direct effects with expectations formation effects. Total effects relative to the baseline projection are

$$\begin{aligned}\text{Total effects} =& \mathbb{E}(\pi_{T+K}|I_T,\mathcal{F}_T) - \text{Baseline effects}\\ =&[a_1(s_{T+K}=0,p_{00}(\tilde{\epsilon}^e_{T+K}))\rho_r^K r_T + a_2(s_{T+K}=0)\tilde{\epsilon}^e_{T+K}]\cdot\mathbb{P}(s_{T+K}=0|s_T=0,I_T,\tilde{\epsilon}^e_T)\\ &+[a_1(s_{T+K}=1,p_{10}(\tilde{\epsilon}^e_{T+K}))\rho_r^K r_T + a_2(s_{T+K}=1)\tilde{\epsilon}^e_{T+K}]\cdot\mathbb{P}(s_{T+K}=1|s_T=0,I_T,\tilde{\epsilon}^e_T)\\ &-\mathbb{E}[a_1(s_{T+K}=0,p_{00}(\epsilon^e_{T+K}))]\rho_r^K r_T. \end{aligned} \tag{41}$$

Total effects are simply equal to direct effects in the extreme case that agents always believe future regimes will remain the same, i.e., $\mathbb{P}(s_{T+K}=0|s_T=0,I_T,\tilde{\epsilon}^e_T)=1$. It is obvious that (41) reduces to (40) in such case. Finally, with direct effects and total effects computed relative to the same baseline, we isolate expectations formation effects, defined as the difference between (40) and (41):

$$\text{Expectations formation effect} = \text{Total effects - Direct effects}$$

It is understood that policy interventions may trigger changes in agents' beliefs about future policy regime evolution. However, Leeper and Zha (2003) provide no channel for such changes to happen, but assume that once agents *ad-hocly* realize that policy regime may change, they would put constant weight on each regime. This is a direct implication and drawback of using exogenous Markov chain for regime switching. In our paper, we endogenize the expectations formation effects by introducing the endogenous feedback mechanism from current monetary policy intervention to next period's policy regime change. Upon observing the current policy regime and policy intervention, agents will use these information to alter their beliefs about next period's policy regime and put time-varying weight on each regime index. This continuous adjustment, rather than the sharp discrete change, of agents' beliefs in future regime evolution generates much richer macroeconomic dynamics of monetary policy intervention as shown in our paper.

C Full Characterization of Endogenous Monetary DSGE

The first-order conditions, market clearing condition, monetary policy, states of regimes and law of motions of shocks form the following expectational nonlinear difference equations:

$$\begin{aligned}\frac{\theta}{\theta-1} =& \theta u_t \left(\frac{C_t}{A_t}\right)^{\epsilon} - \varphi(\theta u_t - \theta + 1)\frac{\Pi_t}{\Pi^*}\left[\frac{\Pi_t}{\Pi^*} - 1\right] \\ &+ \beta\varphi(\theta u_t - \theta + 1)\mathbb{E}_t\left[\frac{Y_{t+1}/A_{t+1}}{Y_t/A_t}\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t}\right)^{-\epsilon}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*} - 1\right)\right]\end{aligned} \tag{42}$$

$$Y_t = C_t + \frac{\varphi}{2}\left[\frac{\Pi_t}{\Pi^*} - 1\right]^2 Y_t \tag{43}$$

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*}\right)^{\rho_R}\left[\left(\frac{\Pi_t}{\Pi^*}\right)^{\alpha(s_t)}\left(\frac{Y_t}{Y_t^*}\right)^{\gamma}\right]^{1-\rho_R} e_t \tag{44}$$

$$\mathbb{E}_t\left[\frac{\beta R_t}{\Pi_{t+1}}\left(\frac{C_t/A_t}{C_{t+1}/A_{t+1}}\right)^{\epsilon}\left(\frac{A_t}{A_{t+1}}\right)\left(\frac{\xi_{t+1}}{\xi_t}\right)\right] = 1 \tag{45}$$

$$s_{t+1} = 1\{w_{t+1} \geq \tau\} \tag{46}$$

$$w_{t+1} = \phi w_t + v_{t+1}, \tag{47}$$

where the aggregate productivity A_{t+1}, technology fluctuation a_{t+1}, price markup shock u_{t+1}, monetary policy shock e_{t+1} and latent factor innovation v_{t+1} have the following laws of motion

$$\ln A_{t+1} = \ln\psi + \ln A_t + \ln a_{t+1} \tag{48}$$

$$\ln a_{t+1} = \rho_a \ln a_t + \sigma_a \varepsilon^a_{t+1} \tag{49}$$

$$\ln \xi_{t+1} = \rho_\xi \ln \xi_t + \sigma_\xi \varepsilon^\xi_{t+1} \tag{50}$$

$$\ln u_{t+1} = (1-\rho_u)\ln u + \rho_u \ln u_t + \sigma_u \varepsilon^u_{t+1} \tag{51}$$

$$\ln e_{t+1} = \sigma_e \varepsilon^e_{t+1} \tag{52}$$

$$\begin{pmatrix} \varepsilon^a_t \\ \varepsilon^\xi_t \\ \varepsilon^u_t \\ \varepsilon^e_t \\ v_{t+1} \end{pmatrix} \sim \mathbb{N}\left(0, \begin{pmatrix} 1 & 0 & 0 & 0 & \rho_{av} \\ 0 & 1 & 0 & 0 & \rho_{\xi v} \\ 0 & 0 & 1 & 0 & \rho_{uv} \\ 0 & 0 & 0 & 1 & \rho_{ev} \\ \rho_{av} & \rho_{\xi v} & \rho_{uv} & \rho_{ev} & 1 \end{pmatrix}\right) \tag{53}$$

with the normalization $\rho'\rho < 1$. Equation (42) is the non-linear New Keynesian Phillips curve reflecting the optimal price setting of intermediate firms, equation (43) reports market clearing conditions, equation (44) is the monetary policy rule and equation (45) is the Euler equation of households.

D Algorithm for State Space Model with Markov Switching

Consider the regime-dependent state space model consisting of (25) and (26), which we present here again for easy reference as

$$y_t = D(s_t, \Theta) + Z(s_t, \Theta)X_t + \nu_t, \quad \nu_t \sim \mathbb{N}(0, \Sigma_\nu) \tag{54}$$

$$X_t = G(s_t, \Theta)X_{t-1} + M(s_t, \Theta)\varepsilon_t, \quad \varepsilon_t \sim \mathbb{N}(0, I_{n_\varepsilon}), \tag{55}$$

where Θ stacks all DSGE parameters, threshold τ, persistency ϕ and correlations ρ, $s_t = 1\{w_t \geq \tau\}$ is the regime index, and w_t is the AR latent factor introduced in (15). In our model, $D(s_t, \Theta)$ and $Z(s_t, \Theta)$ are given by $= D(\Theta)$ and Z, respectively. We introduce the endogenous feedback by assuming

$$\begin{pmatrix} \varepsilon_t \\ \upsilon_{t+1} \end{pmatrix} \sim \mathbb{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I_{n_\varepsilon} & \rho' \\ \rho & 1 \end{pmatrix} \right)$$

where ε_t is the fundamental shock vector.

We augment the state vector X_t by w_{t+1}, i.e., $\varsigma_t = [X_t', w_{t+1}]'$, and the shock vector ε_t by η_{t+1}, i.e., $\xi_t = [\varepsilon_t', \eta_{t+1}]'$. Moreover, we rewrite the law of motion for the regime factor w_t in (15) as

$$w_{t+1} = \phi w_t + \rho' \varepsilon_t + \sqrt{1 - \rho'\rho}\,\eta_{t+1}, \quad \eta_{t+1} \sim \mathbb{N}(0, 1)$$

and add it to the state transition system (55). For notational convenience, define the conditional mean and variances

$$\varsigma_{t|t-1}^k = \mathbb{E}(\varsigma_t | s_t = k, Y_{1:t-1}), \quad P_{t|t-1}^k = Var(\varsigma_t | s_t = k, Y_{1:t-1})$$
$$\varsigma_{t|t}^k = \mathbb{E}(\varsigma_t | s_t = k, Y_{1:t}), \quad P_{t|t}^k = Var(\varsigma_t | s_t = k, Y_{1:t})$$

and the probabilities

$$p_{t|t-1}^k = \mathbb{P}(s_t = k | Y_{1:t-1}), \quad p_{t|t}^k = \mathbb{P}(s_t = k | Y_{1:t}),$$

for $k = 0, 1$.

The filtering algorithm is summarized below.

1. **Initialization.** For $k = 0, 1$, initialize $(\varsigma_{0|0}^k, P_{0|0}^k)$ using the invariant distribution and specify

$p^k_{0|0}$. Then,

$$\varsigma_{0|0} = p^0_{0|0}\varsigma^0_{0|0} + p^1_{0|0}\varsigma^1_{0|0},$$
$$P_{0|0} = \sum_{k=0}^{1} p^k_{0|0}\left\{P^k_{0|0} + (\varsigma_{0|0} - \varsigma^k_{0|0})(\varsigma_{0|0} - \varsigma^k_{0|0})'\right\}.$$

2. **Recursion.** For each $t = 1, \dots, T$, we go through the following steps.

A. **Forecasting.** Approximate $w_t|Y_{1:t-1}$ by normal distribution

$$p(w_t|Y_{1:t-1}) = \mathbb{N}(\mu_{t-1|t-1}, \sigma^2_{t-1|t-1})$$

where $(\mu_{t-1|t-1}, \sigma^2_{t-1|t-1})$ can be extracted from the last elements of $(\varsigma_{t-1|t-1}, P_{t-1|t-1})$. We can then forecast regime probabilities as

$$p^1_{t|t-1} = \mathbb{P}(w_t \geq \tau|Y_{1:t-1}) = 1 - \Phi\left(\frac{\tau - \mu_{t-1|t-1}}{\sigma_{t-1|t-1}}\right)$$

and $p^0_{t|t-1} = 1 - p^1_{t|t-1}$. Next, for $k = 0, 1$, apply the forecasting step of the Kalman filter to obtain

$$\varsigma^k_{t|t-1} = G(s_t = k)\varsigma_{t-1|t-1} \tag{56}$$
$$P^k_{t|t-1} = G(s_t = k)P_{t-1|t-1}G(s_t = k)' + M(s_t = k)\Sigma_\xi M(s_t = k)'$$

for the unobservables, where $(\varsigma_{t-1|t-1}, P_{t-1|t-1})$ are obtained from the previous updating step or initialization. The extracted predictive states of any order, including the autoregressive latent factor, can now be obtained from

$$\varsigma_{t|t-1} = \mathbb{E}[\varsigma_t|Y_{1:t-1}] = p^0_{t|t-1}\varsigma^0_{t|t-1} + p^1_{t|t-1}\varsigma^1_{t|t-1}.$$

by marginalization.

B. **Likelihood evaluation.** Apply the Kalman filter forecasting step for observables to obtain

$$y^k_{t|t-1} = D(s_t = k) + Z(s_t = k)\varsigma^k_{t|t-1}$$
$$F^k_{t|t-1} = Z(s_t = k)P^k_{t|t-1}Z(s_t = k)' + \Sigma_\nu.$$

Then the period-t likelihood contribution can be computed as

$$\begin{aligned}p(y_t|Y_{1:t-1},\Theta) &= \sum_{s_t} p(y_t|s_t,Y_{1:t-1})P(s_t|Y_{1:t-1}) \\ &= p_N(y_t|y^0_{t|t-1},F^0_{t|t-1})p^0_{t|t-1} + p_N(y_t|y^1_{t|t-1},F^1_{t|t-1})p^1_{t|t-1},\end{aligned}$$

where $p(y_t|s_t = k, Y_{1:t-1}) = p_N(y_t|y^k_{t|t-1}, F^k_{t|t-1})$ is a multivariate normal distribution with mean $y^k_{t|t-1}$ and covariance matrix $F^k_{t|t-1}$.

C. **Updating.** To update, use Bayes formula to deduce

$$P(s_t|Y_{1:t}) = P(s_t|y_t,Y_{1:t-1}) = \frac{p(y_t|s_t,Y_{1:t-1})P(s_t|Y_{1:t-1})}{p(y_t|Y_{1:t-1})},$$

which yields

$$p^0_{t|t} = \frac{p_N(y_t|y^0_{t|t-1},F^0_{t|t-1})p^0_{t|t-1}}{p(y_t|Y_{1:t-1})}$$

and $p^1_{t|t} = 1 - p^0_{t|t}$.

Then apply the updating step of the Kalman filter to obtain[19]

$$\begin{aligned}\varsigma^k_{t|t} &= \varsigma^k_{t|t-1} + P^k_{t|t-1}Z(s_t=k)'(F^k_{t|t-1})^{-1}(y_t - y^k_{t|t-1}) \\ P^k_{t|t} &= P^k_{t|t-1} - P^k_{t|t-1}Z(s_t=k)'(F^k_{t|t-1})^{-1}Z(s_t=k)P^k_{t|t-1},\end{aligned} \tag{57}$$

from which we can easily compute the extracted filtered states of any order, including the autoregressive latent regime factor as

$$\begin{aligned}\varsigma_{t|t} &= p^0_{t|t}\varsigma^0_{t|t} + p^1_{t|t}\varsigma^1_{t|t} \\ P_{t|t} &= \mathbb{E}\left[(\varsigma_t - \varsigma_{t|t})(\varsigma_t - \varsigma_{t|t})'|Y_{1:t}\right] \\ &= \sum_{k=0}^{1} p^k_{t|t}\left\{P^k_{t|t} + (\varsigma_{t|t} - \varsigma^k_{t|t})(\varsigma_{t|t} - \varsigma^k_{t|t})'\right\}\end{aligned} \tag{58}$$

[19] Here, it is worthwhile to clarify that our algorithm involves an approximation. Equation (57) would be exact only if, conditional on $s_t = k$, $Y_{1:t-1}$, the distribution of ς_t is Normal. However, it is a mixture of Normals as shown in equation (56) and (58). Essentially, the mixture of Normals is approximated by a Normal distribution to permit the calculation of Kalman gain in (57). One can still motivate (57) as the linear projection of ς_t on y_t and $\varsigma_{t-1|t-1}$ (taking s_t as given). Thus the algorithm is certainly calculating a sensible inference about ς_t. Notice, however, that (57) is not calculating the linear projection of ς_t on $y_t, y_{t-1}, \ldots$ since $\varsigma_{t-1|t-1}$ is a nonlinear function of $y_{t-1}, y_{t-2}, \ldots$. We will examine the performance of approximation by comparing results to those from a bootstrap particle filter. The preliminary results are striking: the approximation employed in our filter performs an excellent job, with a considerable advantage in computation time.

by marginalization.

3. **Quasi-Bayesian MLE.** The log-likelihood function can be written as

$$\log L(Y_{1:T}|\Theta) = \log p(y_1) + \sum_{t=2}^{T} \log p(y_t|Y_{1:t-1}),$$

from which the quasi-Bayesian ML estimator $\hat{\Theta}$ of Θ is computed as

$$\hat{\Theta} = \underset{\Theta \in R(\Theta)}{\arg\max} \log L(Y_{1:T}|\Theta) + \log p(\Theta).$$