Endogenous Regime Shifts in a New Keynesian Model with a Time-Varying Natural Rate of Interest*

Kevin J. Lansing†
Federal Reserve Bank of San Francisco

June 22, 2017

Abstract

This paper develops a New Keynesian model with a time-varying natural rate of interest (r-star). The zero lower bound (ZLB) on the nominal interest rate gives rise to two local rational expectations equilibria, labeled the "targeted" and "deflation" solutions, respectively. The representative agent in the model employs forecast rules that are a weighted-average of the forecasts associated with the two equilibria, where the forecast weights are determined by recent performance. Sustained periods when the exogenous real interest rate remains below the central bank's estimate of r-star can induce the agent to place a substantially higher weight on the deflation equilibrium, causing it to occasionally become self-fulfilling. These episodes are accompanied by highly negative output gaps and a binding ZLB, reminiscent of the U.S. Great Recession. But even outside of recessions or when the ZLB is not binding, the agent may continue to assign a nontrivial weight to the deflation equilibrium forecasts, causing the central bank to consistently undershoot its inflation target, similar to the U.S. economy since mid-2012. I find that raising the central bank's inflation target to 4% from 2% can reduce, but not eliminate, the endogenous switches to the deflation equilibrium.

Keywords: *Natural rate of interest, New Keynesian, Liquidity trap, Zero lower bound, Taylor rule, Deflation.*

JEL Classification: E31, E43, E52.

*The views in this paper are my own and not necessarily those of the Federal Reserve Bank of San Francisco or the Board of Governors of the Federal Reserve System. For helpful comments and suggestions, I thank FRBSF colleagues and session participants at the 2017 AEA Meeting, the 2017 SNDE Symposium, and the 2017 Monash University Macro-Finance Workshop.

†Federal Reserve Bank of San Francisco, P.O. Box 7702, San Francisco, CA 94120-7702, email: kevin.j.lansing@sf.frb.org

1 Introduction

The period from 1988 onwards is generally viewed as an example of consistent U.S. monetary policy aimed at keeping inflation low while promoting sustainable growth and full employment. The nature of this policy is typically described in standard New Keynesian macroeconomic models by a Taylor-type rule in which movements in the federal funds rate are driven by fluctuations in inflation and a real activity variable. Amazingly, the U.S. federal funds rate has been pinned close to zero for one-fourth of the elapsed time since 1988. The U.S. economy is not alone in experiencing an extended period of zero or slightly negative nominal interest rates in recent decades.

Figure 1 plots three-month nominal Treasury bill yields in four countries, namely, the United States, Japan, Switzerland, and the United Kingdom. Nominal interest rates in the United States encountered the zero lower bound during the 1930s and from 2008.Q4 though 2015.Q4. Nominal interest rates in Japan have remained near zero since 1998.Q3, except for the period from 2006.Q4 to 2008.Q3. Nominal interest rates in Switzerland have been zero or slightly negative since 2008.Q4. Nominal interest rates in the United Kingdom have been approximately zero since 2009.Q1. Notice however, that outside of these episodes, all four countries exhibit a strong positive correlation between nominal interest rates and inflation, consistent with the Fisher relationship.

Benhabib, Schmitt-Grohé and Uribe (2001a,b) show that imposing a zero lower bound (ZLB) on the nominal interest rate in a standard New Keynesian model gives rise to two long-run endpoints (steady states).[1] The basic idea is illustrated in Figure 2, which is adapted from Bullard (2010). The two intersections of the ZLB-augmented monetary policy rule (solid red line) with the Fisher relationship (dashed black line) define two long-run endpoints. I refer to these as the "targeted equilibrium" and "deflation equilibrium," respectively. Data since 2008.Q4 lie closer to the deflation equilibrium than the targeted equilibrium.

This paper develops a New Keynesian model with a time-varying natural rate of interest (r-star), i.e., the real short-term interest rate that is consistent with full utilization of economic resources and steady inflation at the central bank's target rate. R-star is an important benchmark for monetary policy because it determines the real interest rate that policymakers should aim for once temporary shocks to the economy have dissipated and the central bank's

[1] I use the terminology "long-run endpoints" rather than "steady states" because the model developed here allows for permanent shifts in the natural rate of interest which, in turn, can shift the long-run values of some macroeconomic variables.

macroeconomic goals have been achieved. The natural rate of interest in the model is the long-run endpoint of the exogenous real interest rate process.[2] The times series process for r-star is calibrated to closely approximate the path of the U.S. natural rate series estimated by Laubach and Williams (2016).[3]

As is well known, the deflation equilibrium is locally indeterminate. I therefore consider a minimum state variable (MSV) solution that rules out sunspot variables and extra lags of fundamental state variables. The forecast rules associated with the deflation equilibrium induce more volatility in inflation and the output gap in response to real interest rate shocks. Model variables in the deflation equilibrium have distributions with lower means and higher variances than those in the targeted equilibrium. But the significant overlap in the various distributions creates a dilemma for an agent who seeks to determine the likelihood that a string of recent quarterly data observations comes from one equilibrium or the other.

The representative agent in the model contemplates the possibility of an occasionally binding ZLB that is driven by switching between the two local equilibria, implying that one set of local forecast rules might perform better than the other. This view turns out to be true in the model simulations, thus validating the agent's beliefs. Specifically, the agent employs forecast rules that are constructed as a weighted-average of the forecasts associated with each of the two local equilibria. The time-varying forecast weights are determined by recent performance, as measured by the root mean squared forecast errors for inflation and the output gap (the two variables that the agent must forecast). Sustained periods when the exogenous real interest rate remains below the central bank's estimate of r-star can induce the agent to place a substantially higher weight on the deflation equilibrium, causing it to occasionally become self-fulfilling. These episodes are accompanied by highly negative output gaps and a binding ZLB, reminiscent of the U.S. Great Recession. But even outside of recessions or when the ZLB is not binding, the agent may continue to assign a nontrivial weight to the deflation equilibrium, causing the central bank to consistently undershoot its inflation target, similar to the U.S. economy since mid-2012.

The setup considered here is similar to that of Aruoba and Schorfheide (2016) and Aruoba, Cuba-Borda, and Schorfheide (2017). These authors construct a stochastic two-regime model in which the economy may switch between a targeted-inflation regime and a deflation regime, depending on the realization of a sunspot variable. The probability of transitioning from one

[2] Willamson (2017) provides a discussion of the distinctions between the "natural," "equilibrium," and "neutral" real rates of interest—terms that are often used interchangeably in the literature.

[3] Updated data are from www.frbsf.org/economic-research/files/Laubach_Williams_updated_estimates.xlsx.

regime to the other is exogenous. In contrast, the regime switching here is not driven by a sunspot, but rather by the recent performance of forecast rules that employ observed data on macroeconomic variables. Hence, the transition probabilities that govern the regime switches are endogenous and can be influenced by a change in the monetary policy rule. Moreover, the probability weight assigned by the agent to being in one regime or the other is not restricted to be zero or one, but rather can take on intermediate values, depending on recent data.

Another closely related paper is one by Dordal-i-Carrera et al. (2016). These authors develop a New Keynesian model with volatile and persistent "risk shocks" (i.e., shocks that drive a wedge between the nominal policy rate and the short-term bond rate) to account for infrequent but long-lived ZLB episodes. A risk shock in their model is isomorphic to a real interest rate shock. Large adverse risk shocks are themselves infrequent but long-lived. As the binding ZLB episode becomes more frequent or more long-lived, the optimal inflation target increases. Unlike here, their analysis does not consider model solutions near the deflation equilibrium, but rather focuses on scenarios in which fundamental shocks are large enough to push the targeted equilibrium to a point where ZLB becomes binding.[4] In contrast, the model developed here accounts for infrequent but long-lived ZLB episodes via endogenous switching between two local equilibria, i.e., the shock process itself is not the source of the infrequent, long-lived ZLB episodes.

As part of the quantitative analysis, I examine how raising the central bank's inflation target can influence the ZLB binding frequency and the volatility of macro variables in the switching model. I find that even with an inflation target of 4%, the ZLB binding frequency in the switching model remains elevated at 9.6%, the average duration of a ZLB episode is 11 quarters, and the maximum duration of a ZLB episode is 132 quarters, or 33 years. Once the deflation equilibrium is taken into account, raising the inflation target is a less effective solution for avoiding ZLB episodes. Reducing the degree of interest rate smoothing in the monetary policy rule serves to increase the ZLB binding frequency, but the episodes exhibit shorter duration on average.

Lastly, I introduce an adaptive learning algorithm into a simplified version of the model. When the agent estimates correctly specified decision rules, the algorithm quickly converges to the vicinity of the targeted equilibrium and remains there. But when the agent estimates misspecified decision rules that fail to control for some white noise shocks, the model exhibits

[4] This is also the methodology pursued by Reifschneider and Williams (2000), Schmitt-Grohé and Uribe (2010), Chung et al. (2012), Coibion, Gorodnichenko, and Wieland (2012), Dennis (2016), and Kiley and Roberts (2017).

low frequency oscillations between the two local equilibria that are qualitatively similar to those observed in the original switching model with full-knowledge.

1.1 Related literature

A number of papers introduce backward-looking learning type mechanisms to examine the dynamics of convergence to either the targeted or the deflation equilibrium. Examples include Evans and Honkapohja (2005), Eusepi (2007), Evans, Guse, and Honkapohja (2008), Benhabib, Evans and Honkapohja (2014), and Christiano, Eichenbaum, and Johanssen (2016). Unlike here, these frameworks do not entertain the possibility of switching between equilibria. Hursey and Wolman (2010) examine the global perfect-foresight dynamics of the ZLB-augmented New Keynesian model. They conclude that "the model only tells us what equilibria exist, not how likely they are to occur" (p. 335).

Alstadheim and Henderson (2006) and Sugo and Ueda (2008) describe interest rate rules that can preclude the deflation equilibrium. Armenter (2014) considers an extension of Benhabib, Schmitt-Grohé and Uribe (2001b) in which monetary policy is governed not by a Taylor-type rule, but rather by the optimal time-consistent rule that minimizes the central bank's loss function. He shows that it may not be possible to achieve the targeted equilibrium if agents' initial inflation expectations are below the central bank's inflation target.

Numerous papers consider optimal monetary policy in response to a time-varying natural rate of interest. The models typically impose the ZLB (or effective lower bound), but the deflation equilibrium is ignored, i.e., the analysis is local to the targeted equilibrium. Examples include Eggertsson and Woodford (2003), Adam and Billi (2007), Nakov (2008), Nakata (2013), Gust, Johannsen, López-Salido (2015), Hamilton, et al. (2016), Basu and Bundick (2015), and Evans, et al. (2015). One finding of this literature is that more uncertainty about the future natural rate implies looser monetary policy today or more policy inertia.

The model developed here shares some similarities with the work of Sargent (1999) in which the model economy can endogenously switch between regimes of high versus low inflation, depending on monetary policymakers' perceptions about the slope of the long-run Phillips curve in light of recent data. Here, the endogenous regime switching depends on private-sector agents' perceptions about whether recent data are more likely to have been generated by the targeted versus the deflation equilibrium.

2 Model

The framework for the analysis is a standard New Keynesian model, augmented by a zero lower bound (ZLB) on the short-term nominal interest rate. The log-linear version of the standard New Keynesian model is taken to represent a set of global equilibrium conditions, with the only nonlinearity coming from the ZLB.[5] Private-sector behavior is governed by the following equilibrium conditions:

$$y_t = E_t\, y_{t+1} - \alpha[i_t - E_t\, \pi_{t+1} - r_t] + v_t, \qquad v_t \sim N\left(0,\, \sigma_v^2\right), \tag{1}$$

$$\pi_t = \beta E_t\, \pi_{t+1} + \kappa y_t + u_t, \qquad u_t \sim N\left(0,\, \sigma_u^2\right) \tag{2}$$

where equation (1) is the representative household's consumption Euler equation and equation (2) is the Phillips curve that is derived from the representative firm's optimal pricing decision. The variable y_t is the output gap (the log deviation of real output from potential output), π_t is the quarterly inflation rate (log difference of the price level), i_t is the short-term nominal policy interest rate, r_t is the exogenous real interest rate, and E_t is the rational expectations operator. Fluctuations in r_t can be interpreted as arising from changes in the representative agent's rate of time preference or changes in the expected growth rate of potential output.[6] The terms v_t and u_t represent an aggregate demand shock and a cost-push shock, respectively. None of the results in the paper are sensitive to the introduction of a discount factor applied to the term $E_t\, y_{t+1}$ in equation (1), along the lines of McKay, Nakamura, and Steinsson (2016).

The time series process for the real rate of interest is given by

$$r_t = \rho_r\, r_{t-1} + (1 - \rho_r)\, r_t^* + \varepsilon_t, \qquad \varepsilon_t \sim N\left(0,\, \sigma_\varepsilon^2\right), \tag{3}$$

$$r_t^* = r_{t-1}^* + \eta_t, \qquad \eta_t \sim N\left(0,\, \sigma_\eta^2\right). \tag{4}$$

Equations (3) and (4) summarize a "shifting endpoint" time series process since the long-run endpoint r_t^* can vary over time due to the permanent shock η_t. In any given period, r_t can deviate from r_t^* due to the temporary shock ε_t. The persistence of the "real interest rate gap" $r_t - r_t^*$ is governed by the parameter ρ_r, where $|\rho_r| < 1$. Kozicki and Tinsely (2012) employ this type of time series process to describe U.S. inflation. When $\rho_r = 1$, we recover the random

[5] Armenter (2016) adopts a similar approach in computing the optimal monetary policy in the presence of two steady states. Eggertsson and Sing (2016) show that the log-linear New Keynesian model behaves very similar to the true nonlinear model in the vicinity of the targeted equilibrium.

[6] Specifically, we have $r_t \equiv -\log\left[\beta \exp\left(\zeta_t\right)\right] + E_t \Delta \bar{y}_{t+1}$, where ζ_t is a shock to the agent's time discount factor β and $\bar{y}_t$ is the logarithm of potential output. For the derivation, see Hamilton, et al. (2016) or Gust, Johannsen, and Lopez-Salido (2015).

walk plus noise specification employed by Stock and Watson (2007) to describe U.S. inflation.[7]

Using equation (3) to substitute out r_t from equation (1) yields the following alternative version of the consumption Euler equation:

$$y_t = E_t\, y_{t+1} - \alpha[i_t - E_t\, \pi_{t+1} - r_t^*] + u_t + \alpha\varepsilon_t + \alpha\rho_r \left(r_{t-1} - r_{t-1}^* - \eta_t\right), \tag{5}$$

where the last three terms could be consolidated into a single aggregate demand shock. From this version, we can interpret r_t^* as the unobservable "natural rate of interest," i.e., the real interest rate that is consistent with full utilization of economic resources and steady inflation at the central bank's target rate. This interpretation is consistent with the empirical strategies of Laubach and Williams (2015), Lubik and Matthes (2015), and Kiley (2015) which view the natural rate of interest as a longer-term economic concept. In contrast, empirical strategies that employ micro-founded New Keynesian models typically view the natural (or equilibrium) rate of interest as a short-term concept, more along the lines of the variable r_t in equation (1).[8] The real interest rate gap $r_t - r_t^*$ captures a concept that has been emphasized by Fed policymakers in recent speeches, namely, a distinction between estimates of the "short-term natural of interest" and its longer-term counterpart (Yellen 2015, Dudley 2015, and Fischer 2016). Here I will refer to r_t^* as the natural rate of interest.

In the model, the agent's rational forecast for the real interest rate gap at any horizon $h \geq 1$ is given by

$$E_t \left(r_{t+h} - r_{t+h}^*\right) = (\rho_r)^h \left(r_t - E_t\, r_t^*\right), \tag{6}$$

where $E_t\, r_t^*$ represents the agent's current estimate of the natural rate computed using the Kalman filter so as to minimize the mean squared forecast error. When $|\rho_r| < 1$ as assumed here, the real interest rate gap is expected to shrink to zero as the forecast horizon h increases. In Appendix A, I show that the Kalman filter expression for $E_t\, r_t^*$ is

$$E_t\, r_t^* = \lambda \left[\frac{r_t - \rho_r\, r_{t-1}}{1 - \rho_r}\right] + (1 - \lambda)\, E_{t-1} r_{t-1}^* \tag{7}$$

$$\lambda = \frac{-(1 - \rho_r)^2\, \phi + (1 - \rho_r)\sqrt{(1 - \rho_r)^2\, \phi^2 + 4\phi}}{2}, \tag{8}$$

where λ is the Kalman gain parameter and $\phi \equiv \sigma_\eta^2 / \sigma_\varepsilon^2$. For the quantitative analysis, the values of ρ_r, σ_η^2, and σ_ε^2 are chosen so that the time path of $E_t\, r_t^*$ from equation (7) approximates the

[7] But unlike here, Stock and Watson (2007) allow for stochastic volatility in the permanent and temporary shocks.

[8] See, for example, Barsky, Justiniano, and Melosi (2014), Cúrdia, et al. (2015), and Del Negro, et al. (2017).

path of the U.S. natural rate series estimated by Laubach and Williams (2016) for the sample period 1988.Q1 to 2016.Q4. Their estimation strategy assumes that the natural rate exhibits a unit root, consistent with equation (4). Hamilton, et al. (2016) present evidence that the ex-ante real rate of interest $i_t - E_t\,\pi_{t+1}$ in U.S. data is nonstationary, but they find that the gap between the ex-ante real rate and their estimate of the world long-run real rate appears to be stationary. This evidence is also consistent with equations (3) and (4) which imply that real rate gap $r_t - r_t^*$ is stationary.

The central bank's monetary policy rule is given by

$$
\begin{aligned}
i_t^* &= \rho i_{t-1}^* + (1-\rho)\left[E_t\, r_t^* + \pi^* + g_\pi\left(\overline{\pi}_t - \pi^*\right) + g_y\left(y_t - y^*\right)\right], && (9)\\
\overline{\pi}_t &= \omega\,\pi_t + (1-\omega)\,\overline{\pi}_{t-1}, && (10)\\
i_t &= \max\left\{0,\ i_t^*\right\}, && (11)
\end{aligned}
$$

where i_t^* is the desired nominal interest rate that responds to deviations of recent inflation $\overline{\pi}_t$ from the central bank's target rate π^* and to deviations of the output gap from its targeted long-run endpoint y^*. Recent inflation $\overline{\pi}_t$ is an exponentially-weighted moving average of past quarterly inflation rates so as to approximate the compound average inflation rate over the past 4 quarters—a typical central bank target variable.[9] The parameter ρ governs the degree of interest rate smoothing as i_t^* adjusts partially each period toward the value implied by the terms in square brackets.

The quantity $E_t\, r_t^* + \pi^*$ represents the targeted long-run endpoint of i_t^*. Including $E_t\, r_t^*$ in the policy rule implies that monetary policymakers continually update their estimate of the unobservable r_t^*. Support for this idea can be found in the Federal Open Market Committee's Summary of Economic Projections (SEP). Meeting participants provide their views on the projected paths of macroeconomic variables over the next three calendar years and in the longer run. Since the natural rate of interest is a longer-run concept, we can infer the median SEP projection for r_t^* by subtracting the median longer-run projection for inflation from the median longer-run projection for the nominal federal funds rate. The median SEP projection for r_t^* computed in this way has ratcheted down over time, as documented by Lansing (2016), and currently stands at 1%.

Equation (11) is the ZLB that constrains the nominal policy interest rate i_t to be non-negative. In the model simulations, I implement the occasionally binding ZLB by making

[9] Specifically, the value of ω is set to achieve $\overline{\pi}_t \simeq [\Pi_{j=0}^{3}(1+\pi_{t-j})]^{0.25} - 1$

the substitution $i_t = 0.5\, i_t^* + 0.5\sqrt{\left(i_t^*\right)^2}$ in the global equilibrium condition (1). Details are contained in the appendix.

2.1 Long-run endpoints

The Fisher relationship $i_t = r_t + E_t\, \pi_{t+1}$ is embedded in the non-stochastic version of equation (1). Consequently, when $g_\pi > 1$, the model has two long-run endpoints (steady states) as shown originally by Benhabib, Schmitt-Grohé, and Uribe (2001a,b). The novelty here is that the long-run endpoints can shift due to shifts in r_t^*. Straightforward computations using the model equations yield the following long-run endpoints that characterize the "targeted equilibrium" and the "deflation equilibrium," respectively.

Table 1. Long-run Endpoints

Targeted equilibrium	Deflation equilibrium
$\pi_t = \pi^*$	$\pi_t = -r_t^*$
$y_t = y^* \equiv \pi^* \left(1 - \beta\right) / \kappa$	$y_t = -r_t^* \left(1 - \beta\right) / \kappa$
$i_t^* = r_t^* + \pi^*$	$i_t^* = \left(r_t^* + \pi^*\right) \left[1 - g_\pi - g_y \left(1 - \beta\right) / \kappa\right]$
$i_t = r_t^* + \pi^*$	$i_t = 0$

In the targeted equilibrium, long-run inflation is at the central bank's target rate π^* and the long-run output gap y^* is slightly positive for typical calibrations with $0.99 < \beta < 1$. The long-run desired nominal interest rate i_t^* conforms to the Fisher relationship.[10] The ZLB is not binding such that $i_t = i_t^* > 0$, provided that $r_t^* > -\pi^*$. In the model simulations, I impose bounds on fluctuations in r_t^* that are based on the range of natural rate estimates obtained by Laubach and Williams (2016) for the period 1988.Q1 to 2016.Q4. In the deflation equilibrium, the long-run inflation rate, the long-run output gap, and the long-run desired nominal interest rate are all negative when $r_t^* > 0$.[11]

2.2 Local linear forecast rules

Given the linearity of the model aside from the ZLB, it is straightforward to derive the agent's rational decision rules for π_t and y_t in the vicinity of the long-run endpoints associated with each of the two equilibria. For the targeted equilibrium, the local decision rules are unique

[10] Cochrane (2015) shows that Fisherian effects appear to dominate Phillips curve effects for determining the comovement between the nominal interest rate and inflation in the standard New Keynesian model.

[11] Evans, Honkopoja, and Mitra (2016) develop a New Keynesian models that imposes a lower bound on the inflation rate that is more negative than $-r^*$ (which is assumed to be constant in their model). They show that this additional constraint gives rise to a third steady state in which the ZLB binds but the Fisher relationship does not hold.

linear functions of the state variables: r_t, $E_t\, r_t^*$, $\overline{\pi}_{t-1}$, i_{t-1}^*, u_t, and v_t. For the deflation equilibrium, I solve for the minimum state variable (MSV) solution in terms of the minimum number of fundamental state variables, i.e., abstracting from extraneous sunspot variables and extra lags of fundamental state variables.[12]

Given the local linear decision rules, we can construct the agent's conditional forecast rules for $E_t\, \pi_{t+1}$ and $E_t\, y_{t+1}$ for each of the two local equilibria. In the stochastic simulations, I substitute the local linear forecast rules into the global equilibrium conditions (1) and (2). I allow for an occasionally binding ZLB by making the substitution $i_t = 0.5\, i_t^* + 0.5\sqrt{(i_t^*)^2}$ in equation (1). Together with the monetary policy rule (9), this procedure yields a system of three nonlinear equilibrium conditions that can be solved each period to obtain the three realizations y_t, π_t and i_t^*. Details are contained in Appendices B and C.

The decision rule coefficients applied to the state variable $r_t - E_t\, r_t^*$ are much larger in magnitude in the deflation equilibrium than in the targeted equilibrium (see Appendices B and C). Consequently, the deflation equilibrium exhibits more volatility and undergoes a more severe recession in response to an adverse shock sequence that causes $r_t - E_t\, r_t^*$ to be persistently negative. The higher volatility in the deflation equilibrium is due to the binding ZLB which prevents the central bank from taking action to mitigate the consequences of the adverse shock sequence.

The local linear forecast rules for the targeted equilibrium are derived under the assumption that $i_t^* > 0$ and hence do not take into account the possibility that a shock sequence could be large enough to cause the ZLB to become binding in the future. The error induced by this assumption will depend on the frequency and duration of ZLB episodes in the targeted equilibrium. Based on model simulations, the targeted equilibrium experiences a binding ZLB in only 1.7% of the periods, with an average duration of only 4.2 quarters. Consequently, the agent's use of forecast rules that assume $i_t^* > 0$ in the targeted equilibrium seems quite reasonable.[13] The local linear forecast rules for the deflation equilibrium are derived under the assumption that $i_t^* \leq 0$ and hence do not take into account the possibility that a shock sequence could be large enough to cause the ZLB to become slack in the future. Based on model simulations, the deflation equilibrium experiences a binding ZLB in 76% of the periods, with an average duration of 29.2 quarters. The higher volatility of the deflation equilibrium causes the assumption of $i_t^* \leq 0$ to be violated in 24% of the periods. Hence, the error induced

[12] For background on MSV solutions, see McCallum (1999).

[13] Richter and Throckmorton (2016) compare linear model solutions for the targeted equilibrium in which agents ignore the possibility of future ZLB episodes to nonlinear model solutions that account for this possibility.

by the agent's use of local linear forecast rules would appear to be more significant in the deflation equilibrium. Nevertheless, as shown in the quantitative analysis of Section 4, the agent's forecast errors in the deflation equilibrium are close to white noise, giving no clear indication to the agent that the local linear forecast rules are misspecified.[14]

2.3 Endogenous regime switching in nonlinear model

Now consider a more sophisticated agent who contemplates the possibility of an occasionally binding ZLB that is driven by switching between the two local equilibria, implying that one set of local linear forecast rules might perform better than the other. The agent in the switching model can be viewed as someone thinking along the lines of Bullard (2010), i.e., the agent is aware of the two local equilibria implied by the New Keynesian framework and is concerned about the possibility of getting stuck in a deflation trap. Specifically, the agent employs forecast rules that are constructed as a weighted-average of the forecast rules associated with the two local rational expectations equilibria. The time-varying forecast rule weights are determined by recent performance, as measured by the root mean squared forecast errors for the two variables that the agent must forecast, i.e., the output gap and quarterly inflation. The forecast rules in the switching model are given by

$$\widehat{E}_t\, y_{t+1} = \mu_t E_t^{\text{targ}}\, y_{t+1} \;+\; (1-\mu_t)\, E_t^{\text{defl}}\, y_{t+1}, \tag{12}$$

$$\widehat{E}_t\, \pi_{t+1} = \mu_t E_t^{\text{targ}}\, \pi_{t+1} \;+\; (1-\mu_t)\, E_t^{\text{defl}}\, \pi_{t+1}, \tag{13}$$

where μ_t is the value that minimizes the root mean squared forecast error computed over a moving window of recent data. Specifically, μ_t is the value that minimizes:

$$\begin{aligned} RMSFE_{t-1} \;\; = \;\; \sum_{j=1}^{T_w} & \left\{ \tfrac{1}{T_w} \left[y_{t-j} - \mu_t E_{t-j-1}^{\text{targ}}\, y_{t-j} - (1-\mu_t)\, E_{t-j-1}^{\text{defl}}\, y_{t-j} \right]^2 \right. \\ & \left. + \tfrac{1}{T_w} \left[\pi_{t-j} - \mu_t E_{t-j-1}^{\text{targ}}\, \pi_{t-j} - (1-\mu_t)\, E_{t-j-1}^{\text{defl}}\, \pi_{t-j} \right]^2 \right\}^{0.5}, \end{aligned} \tag{14}$$

which shows that μ_t is computed using data dated $t-1$ or earlier. In the simulations, I impose the restriction $0 \leq \mu_t \leq 1$. Similar results are obtained if μ_t is determined by a discrete choice framework along the lines of Brock and Hommes (1998).[15]

[14] Aruoba, Cuba-Borda, and Schorfheide (2017) solve for piece-wise linear decision rules in both the targeted equilibrium and the deflation equilibrium to account for the occasionally binding nature of the ZLB constraint.

[15] In this case, $\mu_t = \{1 \;+\; \exp[\psi(RMSFE_{t-1}^{\text{targ}} - RMSFE_{t-1}^{\text{defl}})]\}^{-1}$, where $RMSFE_{t-1}^{\text{targ}}$ and $RMSFE_{t-1}^{\text{defl}}$ are the fitness measures associated with the two sets of local linear forecast rules and ψ is the "intensity of choice" parameter. As ψ becomes larger, the resulting sequence for μ_t takes on values approaching either 1 or 0, with intermediate values less likely.

Given the representative agent's conditional forecasts from equations (12) and (13), the realizations of the macroeconomic variables are determined by the following global equilibrium conditions:

$$y_t = \widehat{E}_t y_{t+1} - \alpha \left[i_t - \widehat{E}_t \pi_{t+1} - r_t \right] + v_t, \tag{15}$$

$$\pi_t = \beta \, \widehat{E}_t \pi_{t+1} + \kappa y_t + u_t, \tag{16}$$

$$i_t^* = \rho i_{t-1}^* + (1-\rho) \left[E_t \, r_t^* + \pi^* + g_\pi \left(\overline{\pi}_t - \pi^* \right) + g_y \left(y_t - y^* \right) \right], \tag{17}$$

$$i_t = 0.5 \, i_t^* + 0.5 \sqrt{\left(i_t^* \right)^2}, \tag{18}$$

where $\overline{\pi}_t = \omega \, \pi_t + (1-\omega) \, \overline{\pi}_{t-1}$.

As a check, I also compute the time-varying weight μ_t using the standard classification formula for the average conditional probability that a given series of quarterly inflation observations are drawn from one of two populations with known densities.[16] In this model, the standard formula takes the form

$$\mu_t = \frac{\mu_{t-1} \frac{1}{T_w} \sum_{j=1}^{T_w} \mathrm{f}^{\,\mathrm{targ}} \left(\pi_{t-j} \right)}{\mu_{t-1} \frac{1}{T_w} \sum_{j=1}^{T_w} \mathrm{f}^{\,\mathrm{targ}} \left(\pi_{t-j} \right) + \left(1 - \mu_{t-1} \right) \frac{1}{T_w} \sum_{j=1}^{T_w} \mathrm{f}^{\,\mathrm{defl}} \left(\pi_{t-j} \right)}, \tag{19}$$

where $\mathrm{f}^{\,\mathrm{targ}} (\pi_{t-j})$ and $\mathrm{f}^{\,\mathrm{defl}} (\pi_{t-j})$ are the probability density functions for the quarterly inflation distributions under the targeted equilibrium and the deflation equilibrium, respectively, which are assumed known to the agent.[17] For the quantitative analysis, I run a pre-simulation to compute the moments of the quarterly inflation distributions in each of the two local rational expectations equilibria. I put bounds on the prior such that $0.01 \leq \mu_{t-1} \leq 0.99$ during the simulation so that the agent never rules out the possibility of switching from one equilibrium to the other.

3 Parameter values

Table 2 shows the baseline parameter values used in the model simulations. The top group of parameters are those that appear in the private-sector equilibrium conditions (1) and (2). The middle group of parameters are those that appear in the monetary policy rule (9). The bottom group of parameters pertain to the exogenous real interest rate process and the forecast evaluation window for the switching model.

[16] See Anderson (1958), Chapter 6.

[17] Huh and Lansing (2000) employ a similar setup in a policy credibility model where the agent uses observed inflation rates to infer whether the central bank's inflation target has truly shifted to a lower mean value.

Table 2. Baseline Parameter Values

Parameter	Value	Description/Target
α	0.25	Interest rate coefficient in Euler equation.
β	0.995	Discount factor in Phillips curve.
κ	0.025	Output gap coefficient in Phillips curve.
σ_v	0.01	Std. dev. of aggregate demand shock.
σ_u	0.02	Std. dev. of cost push shock.
π^*	0.02	Central bank inflation target.
ω	0.459	$\overline{\pi}_t \simeq$ 4-quarter PCE inflation rate.
g_π	1.5	Policy rule response to inflation.
g_y	1.0	Policy rule response to output gap.
ρ	0.80	Interest rate smoothing parameter.
ρ_r	0.861	Persistence parameter for r_t.
σ_ε	0.0099	Std. dev. of temporary shock to r_t.
σ_η	0.0019	Std. dev. of permanent shock to r_t.
λ	0.0256	Optimal Kalman gain for $E_t\, r_t^*$.
T_w	8	Window length in qtrs. for forecast evaluation.

The value $\alpha = 0.25$ for the interest rate sensitivity coefficient in the consumption Euler equation (1) implies a coefficient of relative risk aversion of $1/\alpha = 4$. Such a value is consistent with the small empirical sensitivity of consumption to changes in the interest rate, as show by Campbell and Mankiw (1989). The values $\beta = 0.995$ and $\kappa = 0.025$ are identical to those employed by Evans et al. (2015) and are typical of values employed in the literature. Given the other parameter values, the standard deviations of the aggregate demand shock in equation (1) and the cost push shock in equation (2) are chosen so that the standard deviations of the output gap y_t and the 4-quarter inflation rate $\pi_{4,\,t}$ in the switching model are reasonably close to those observed in U.S. data for the period 1988.Q1 to 2016.Q4.

The inflation target of $\pi^* = 0.02$ is based on the Federal Open Market Committee's (FOMC) stated goal of 2% inflation, as measured by the 4-quarter change in the personal consumption expenditures (PCE) price index. I choose $\omega = 0.459$ to minimize the squared deviation between the 4-quarter PCE inflation rate and the exponentially-weighted moving average of quarterly PCE inflation computed using equation (10) for the period 1961.Q1 to 2016.Q4. When $\omega = 0.459$, the cumulative weight on the first four terms π_t through π_{t-3} in the exponentially-weighted moving average is 0.915. The monetary policy rule coefficients g_π, g_y and ρ are based on the Taylor (1999) rule, augmented to allow for a realistic amount of inertia in the desired nominal interest rate.

The parameter values that govern the evolution of r_t and r_t^* in equations (3) and (4) are calibrated so that the Kalman filter estimate $E_t\, r_t^*$ computed from equation (7) approximates the one-sided estimate of the U.S. natural rate series from Laubach and Williams (2016) for the period 1988.Q1 to 2016.Q4. The time series for r_t in the data is constructed as the nominal federal funds rate minus expected quarterly inflation computed from a rolling 40-quarter, 4-lag vector autoregression that includes the nominal funds rate, quarterly PCE inflation (annualized), and the CBO output gap. Equation (3) implies $E_t\, r_{t+1} = \rho_r\, r_t + (1 - \rho_r)\, E_t\, r_t^*$. I choose $\rho_r = 0.861$ to minimize the squared forecast error $[r_{t+1} - \rho_r\, r_t\, - (1 - \rho_r)\, E_t\, r_t^*]^2$ over the period 1988.Q1 to 2016.Q4, where $E_t\, r_t^*$ is given by the Laubach-Williams estimate. Given the value of ρ_r, I choose $\lambda = 0.0256$ to minimize the squared deviations between the model-implied estimate $E_t\, r_t^*$ from equation (7) and the Laubach-Williams estimate. Given these values for ρ_r and λ, I solve for the value $\phi \equiv \sigma_\eta^2/\sigma_\varepsilon^2 = 0.0349$ to satisfy the optimal Kalman gain formula (8). Given ϕ, I solve for the value of σ_ε that allows the model-predicted standard deviation of Δr_t to match the corresponding value in the data for the period 1988.Q1 to 2016.Q4. Finally, given ϕ and σ_ε, we have $\sigma_\eta = \sigma_\varepsilon \sqrt{\phi}$.

The window length in quarters for computing the agent's forecast fitness measure from equation (14) is set to $T_w = 8$. Each period, the agent chooses the weight μ_t on the targeted forecast rules so as to minimize the root mean squared forecast errors over the past 2 years. In simulations, this choice produces a ZLB binding frequency that is close to that observed in U.S. data since 1988. I also examine the sensitivity of the results to higher values of T_w. Higher values of T_w serve to reduce the ZLB binding frequency by reducing the likelihood of switches to the deflation equilibrium.

Figure 3 plots the one-sided estimate of the U.S. natural rate series from Laubach and Williams (2016), updated through 2016.Q4. The series (dashed red line) shows a downward-sloping trend. This pattern is consistent with the declines in global real interest rates observed over the same period (International Monetary Fund 2014, Rachel and Smith 2015). The time series process for the natural rate in the model (dotted green line) provides a good approximation of the Laubach-Williams series from 1988 onwards. Table 3 compares the properties of the U.S. real interest rate to those implied by the model.

Table 3. Properties of Real Interest Rate: Data versus Model

Statistic	U.S. Data 1988.Q1 to 2016.Q4	Model
Std. Dev. Δr_t	0.0103	0.0103
Std. Dev. $\Delta^2 r_t$	0.0151	0.0161
Std. Dev. $r_t - E_t\, r_t^*$	0.0173	0.0164
Corr. Lag 1 Δr_t	-0.071	-0.067
Corr. Lag 2 Δr_t	-0.199	-0.058

Notes: $\Delta r_t \equiv r_t - r_{t-1}$. $\Delta^2 r_t \equiv \Delta r_t - \Delta r_{t-1}$. The real interest rate r_t in U.S. data is defined as the nominal federal funds rate minus expected quarterly inflation computed from a rolling 40-quarter, 4-lag vector autoregression that includes the nominal funds rate, quarterly PCE. inflation, and the CBO output gap. The Kalman filter estimate $E_t\, r_t^*$ in U.S. data corresponds to the Laubach-Williams one-sided estimate. Model statistics are computed analytically from the laws of motion (3), (4), (7) and (8).

For the baseline simulation, I impose the bounds $-0.0041 \leq r_t^* \leq 0.0371$, which corresponds to the range of values for the Laubach-Williams one-sided estimates for the period 1988.Q1 to 2016.Q4. I also consider an alternative simulation that imposes the wider bounds $-0.015 \leq r_t^* \leq 0.0371$, where the lower bound of -1.5% is the long-run value of the natural rate of interest computed by Eggertsson, Mehrotra, and Robbins (2017) using a life cycle model calibrated to the U.S. economy in 2015. In a representative agent model, the long-run natural rate pins down the mean risk free rate of return. The mean risk free rate can be negative if the product of the coefficient of relative risk aversion and the variance of consumption growth are sufficiently high, implying a very strong precautionary saving motive.[18]

4 Quantitative analysis

4.1 U.S. data

The top left panel of Figure 4 shows that the U.S. real interest rate has remained mostly below the Laubach-Williams estimate of r_t^* since early 2009. The bottom left panel shows that the nominal federal funds rate was approximately zero from 2008.Q4 through 2015.Q4. In the same panel, I plot the nominal federal funds predicted by a Taylor-type rule of the form (9) using the parameter values in Table 2 with $E_t\, r_t^*$ given by Laubach-Williams one-sided estimate, $\overline{\pi}_t$ given by the 4-quarter PCE inflation rate, and y_t given by the CBO output

[18] In a representative agent model, $\log(R_{t+1}^f) = -\log(E_t M_{t+1})$, where R_{t+1}^f is the gross risk free rate and M_{t+1} is the agent's stochastic discount factor. Assuming *iid* consumption growth and power utility, the mean risk free rate is given by $E[\log(R_{t+1}^f)] = -\log(\beta) + \gamma\overline{x} - \gamma^2\sigma_x^2/2$, where β is the agent's time discount factor, γ is the coefficient of relative risk aversion, $\overline{x}$ is the mean consumption growth rate and σ_x^2 is the variance of consumption growth. Assuming $\beta \simeq 1$ such that $\log(\beta) \simeq 0$, the condition $\gamma\sigma_x^2 > 2\,\overline{x}$ implies $E[\log(R_{t+1}^f)] < 0$. For details of the derivation, see Lansing and LeRoy (2014).

gap. The desired nominal funds rate predicted by the Taylor-type rule is negative starting in 2009.Q1 and remains negative through the end of the data sample in 2016.Q4.[19]

The top right panel of Figure 4 shows that the 4-quarter PCE inflation rate was briefly negative in 2009 and has remained below the Fed's 2% inflation target since 2012.Q2. The same panel shows that expected inflation, as measured by the 5-year breakeven inflation rate derived from inflation-indexed Treasury securities, dropped sharply during the Great Recession and remains below its pre-recession level. The bottom right panel shows that the Great Recession was very severe, pushing the CBO output gap down to -6.3% at the business cycle trough in 2009.Q2. The output gap remains negative at -0.9% in 2016.Q4, more than seven years after the Great Recession ended. The various endpoints plotted in Figure 4 are computed using the expressions in Table 1, with r_t^* given by the Laubach-Williams one-sided estimate. Although not shown, the wide confidence intervals surrounding the Laubach-Williams estimate of r_t^* would not rule out values for the true natural rate that lie deeper into negative territory. As r_t^* approaches zero or becomes negative, the "deflation" equilibrium is characterized by zero or low inflation, allowing this equilibrium to provide a better fit of recent U.S. inflation data.

4.2 Switching model simulations

Figure 5 plots some key variables from simulations of the switching model. When the exogenous real interest rate gap $r_t - E_t\, r_t^*$ is negative for a sustained interval (top panel), the resulting downward pressure on π_t and y_t serves to reduce the recent $RMSFE$ of the deflation forecast rules and increase the recent $RMSFE$ of the targeted forecast rules (middle panel). Around period 1725, the shift in relative forecast performance induces the agent to place a substantially lower weight on the targeted equilibrium forecast rules, causing the deflation equilibrium to become temporarily self-fulfilling (bottom panel). Then around period 1800, the real rate gap once again becomes positive, causing the $RMSFE$ of the deflation forecast rules to once again exceed the $RMSFE$ of the targeted forecast rules. The agent increases the weight on the targeted forecast rules, causing the targeted equilibrium to be restored.

Qualitatively similar results are obtained if the agent employs Bayes law (19) to compute the likelihood that a string of recent π_t observations is drawn from one equilibrium distribution or the other. Interestingly, it is the agent's subjective belief that the deflation equilibrium is possible that allows it to become a reality. If the agent could somehow commit to employing

[19] Augmenting the Taylor-type rule to allow for a response to other variables (such as 4-quarter real GDP growth and an index of macroeconomoic uncertainty) can produce a predicted path for the nominal funds rate that turns slightly positive by the end of the data sample. See Lansing (2017).

the forecast rule weight $\mu_t = 1$ for all t, then the economy would always remain in the targeted equilibrium.

Figure 6 plots the distributions of macro variables in each of the three model versions. The macro variables in the deflation equilibrium have distributions with lower means but higher variances than those in the targeted equilibrium. But the significant overlap in the various distributions creates a dilemma for an agent who seeks to determine the likelihood that a string of recent quarterly inflation observations are drawn from one equilibrium or the other. Variables in the switching model have means that are somewhat lower and variances that are somewhat higher than those in the targeted equilibrium. Consequently, the central bank in the switching model undershoots its inflation target and the volatilities of the output gap and inflation are both higher relative to the targeted equilibrium.

Hills, Nakata, and Schmidt (2016) show that the risk of encountering the zero lower bound in the future can shift agents' expectations such that the central bank undershoots its inflation target in the present. Something similar is at work here. When the agent increases the weight $1-\mu_t$ on the deflation forecast rules, this can cause realized inflation to undershoot the central bank's target for a sustained interval, even when the ZLB is not binding. The switching model allows for low-frequency swings in the level of inflation that are driven solely by expectational feedback, not by changes in the monetary policy rule.[20]

Table 4 provides a quantitative comparison between the U.S. data and the results of model simulations. Overall, the statistics generated by the switching model compare favorably to those in U.S. data from 1988.Q1 to 2016.Q4. For example, the switching model predicts a ZLB binding frequency of 17.6% versus 25.0% in the data. However, the mean 4-quarter inflation rate in the switching model is only 0.93% versus 2.16% in the data going back to 1988.Q1. This switching model prediction is more in line with data from Japan than the United States. But going forward, any continued undershooting of the Federal Reserve's 2% inflation target (as has been the case since mid-2012) would serve to push down the mean 4-quarter inflation rate in U.S. data, bringing it closer to the switching model prediction. It's worth noting that the persistent undershooting of the Fed's inflation target since mid-2012 appears to have partially worked its way into inflation expectations. The Federal Reserve Bank of Atlanta's Business Inflation Expectation (BIE) shows that while most survey respondents understand that the Fed's inflation target is 2%, about two-fifths of respondents believe that the Fed is more likely

[20] Lansing (2009) achieves a similar result in a model where agents employ a Stock and Watson (2007) time series framework to forecast quarterly inflation.

to accept an inflation rate below target than to accept an inflation rate above target (Altig, Parker, and Meyer, 2017).

When $\omega = 0.459$, the exponentially-weighted moving average of quarterly inflation $\overline{\pi}_t$ computed using equation (11) provides a very good approximation of the 4-quarter inflation rate $\pi_{4,t}$. Although not shown in Table 4, the mean, standard deviation, and first-order autocorrelation statistics for $\overline{\pi}_t$ in the switching model are 0.94%, 1.61%, and 0.80, respectively. These values are close to the corresponding statistics for $\pi_{4,t}$ of 0.93%, 1.56%, and 0.90.

Using data from all advanced economies since 1950, Dordal-i-Carrera et al. (2016) estimate an average ZLB binding frequency of 7.5% and an average ZLB episode duration of 14 quarters. Excluding the period from 1968 to 1984 (when inflation and nominal interest rates were too high for the ZLB to be practically reached) raises the average ZLB binding frequency and average ZLB episode duration to 10% and 18 quarters, respectively. For the period of consistent U.S. monetary policy starting in 1988.Q1, the single ZLB episode lasted 29 quarters.

Figure 7 plots the distribution of ZLB durations in each model version. Unlike the targeted equilibrium, the switching model can produce infrequent, but long-lived ZLB episodes in response to small, normally distributed shocks. The average duration of a ZLB episode in the switching model is 11.3 quarters, with a maximum of duration of 138 quarters (Table 4). From Figure 7, we see that a 29 quarter ZLB episode is essentially impossible in the targeted equilibrium but can occur with a 5% frequency in the switching model. To account for infrequent but long-lived ZLB episodes within the targeted equilibrium, Dordal-i-Carreras, et al. (2016) develop a model with large but infrequent shocks.[21]

[21] In a New Keynesian model with physical capital, Dennis (2016) shows that the introduction of capital adjustment costs can help to generate infrequent, long-lived ZLB episodes in the targeted equilibrium.

Table 4. Unconditional Moments: Data versus Model

Statistic	U.S. Data 1988.Q1-2016.Q4	Model Simulations Targeted	Deflation	Switching
Mean y_t	-1.51%	0.40%	-0.45%	0.48%
Std. Dev.	1.75%	1.55%	3.79%	2.25%
Corr. Lag 1	0.95	0.48	0.79	0.67
Mean $\pi_{4,\,t}$	2.16%	1.98%	-1.71%	0.93%
Std. Dev.	1.09%	0.98%	1.67%	1.56%
Corr. Lag 1	0.89	0.74	0.91	0.90
Mean i_t^*	2.84%	3.68%	-2.70%	2.18%
Std. Dev.	3.47%	1.76%	3.65%	2.89%
Corr. Lag 1	0.99	0.98	0.98	0.98
% periods $i_t = 0$	25.0%	1.66%	75.9%	17.6%
Mean ZLB duration	29 qtrs.	4.3 qtrs.	29.2 qtrs.	11.3 qtrs.
Max. ZLB duration	29 qtrs.	34 qtrs.	260 qtrs.	138 qtrs.

Notes: The ZLB episode in U.S. data is from 2008.Q4 through 2015.Q4. Model results are computed from a 300,000 period simulation. $\pi_{4,\,t} \equiv [\Pi_{j=0}^{3}(1+\pi_{t-j})]^{0.25} - 1$.

The mean value of μ_t (weight on targeted forecast rules) in the switching model is 0.70 with a standard deviation of 0.29. Larger values for the window length T_w that is used to compute the forecast fitness measure from equation (14) serve to reduce the frequency of regime switches and thereby raise the mean 4-quarter inflation rate. For example, when T_w is increased to 16 quarters, the mean value of μ_t is higher at 0.80 and the standard deviation is lower at 0.21. With $T_w = 16$, the ZLB binding frequency in the switching model drops to 9.6% and the average ZLB duration is lower at 8.0 quarters. The mean value of $\pi_{4,\,t}$ increases to 1.29% from 0.93%.

Figure 8 plots simulations from each of the three model versions: targeted, deflation, and switching. All three versions employ the same sequence of stochastic shocks. When the weight on the targeted forecast rules starts dropping towards zero around period 1725, the switching model generates a negative desired nominal interest rate, brief deflation followed by below-target inflation, and a highly negative output gap, reminiscent of the U.S. Great Recession and its aftermath (Figure 4). The severity of the recession in the switching model is due to the larger response coefficient on the state variable $r_t - E_t\, r_t^*$ in the deflation equilibrium decision rule for y_t. Specifically, the response coefficient in the deflation equilibrium is 2.47 versus 0.83 in the targeted equilibrium. The deflation equilibrium response coefficient receives more weight as $\mu_t \to 0$, causing the effects of an adverse real rate shock to be transmitted more forcefully to the output gap. Evans, Honkapohja, and Mitra (2016) argue that the deflation equilibrium is not a convincing explanation of the U.S. Great Recession since the steady state level of real activity in the deflation equilibrium is not much below the steady state level of real activity in the targeted equilibrium. However, their analysis does not consider the significant differences in the dynamic responses of the two equilibria to an adverse real interest rate shock, as revealed by the two sets of local decision rules.

Table 5 summarizes the properties of the agent's forecast errors in each of the three model versions. Equations (15) through (18) show that there are two endogenous variables that the agent must forecast: y_{t+1} and π_{t+1}. The forecast error is given by $err^{x}_{t+1} = x_{t+1} - F_t\, x_{t+1}$ for $x_{t+1} \in \{y_{t+1},\ \pi_{t+1}\}$, where $F_t\, x_{t+1}$ is the value predicted by the local linear forecast rule or, in the case of the switching model, the weighted average forecast rule, (12) or (13). As noted earlier in Section 2.3, the agent's use of linear forecast rules in a nonlinear environment that is subject to an occasionally binding ZLB would be expected to introduce errors, particularly in the more-volatile deflation equilibrium. Nevertheless, Table 5 shows that the agent's forecast errors in all three model versions are close to white noise, giving no clear indication to the agent that the forecast rules are misspecified.

Table 5. Properties of Forecast Errors

	Model Simulations		
Statistic	Targeted	Deflation	Switching
$Corr(err^{y}_{t+1},\, err^{y}_{t})$	0.001	-0.007	0.022
$Corr(err^{\pi}_{t+1},\, err^{\pi}_{t})$	0.003	0.002	0.058
$E\left(err^{y}_{t+1}\right)$	-0.001%	-0.114%	0.002%
$E\left(err^{\pi}_{t+1}\right)$	-0.004%	-0.007%	0.004%
$\sqrt{E[(err^{y}_{t+1})^2]}$	1.217%	2.216%	1.513%
$\sqrt{E[\left(err^{\pi}_{t+1}\right)^2]}$	1.972%	2.013%	1.994%

Notes: Model results are computed from a 300,000 period simulation.

Recall that the long-run endpoint of π_t in the deflation equilibrium is $-r^*_t$. Therefore, allowing for negative values of r^*_t in the simulation will serve to increase the mean 4-quarter inflation rate in both the deflation equilibrium and the switching model. For example, Figure 5 shows that the real interest rate gap $r_t - E_t\, r^*_t$ becomes positive around period 1800 because $E_t\, r^*_t$ becomes negative. At the same time, Figure 8 shows that the 4-quarter inflation rate in the deflation equilibrium becomes positive around period 1800. The wide confidence intervals around the Laubach-Williams estimate of the U.S. natural rate would not rule out values that are more negative at the end of the data sample. To explore this idea further, the simulations are repeated while imposing the wider bounds $-0.015 \leq r^*_t \leq 0.0371$, where the lower bound of -1.5% is the long-run value of the natural rate of interest computed by Eggertsson, Mehrotra, and Robbins (2017) using a life cycle model calibrated to the U.S. economy in 2015.

Table 6 compares the results of the original switching model simulation to the alternative simulation with $-0.015 \leq r_t^* \leq 0.0371$. The mean 4-quarter inflation rate in the alternative simulation increases to 1.09% from 0.93% in the original simulation. Moreover, the ZLB binding frequency increases to 20.8%, which is closer to the U.S. data value of 25.0% shown in Table 4 for the period 1988.Q1 to 2016.Q4.

Table 6. Effect of Natural Rate Range in Switching Model

Statistic	$-0.0041 \leq r_t^* \leq 0.0371$	$-0.015 \leq r_t^* \leq 0.0371$
Mean y_t	0.48%	0.43%
Std. Dev.	2.25%	2.28%
Corr. Lag 1	0.67	0.68
Mean $\pi_{4,t}$	0.93%	1.09%
Std. Dev.	1.56%	1.50%
Corr. Lag 1	0.90	0.89
Mean i_t^*	2.18%	1.83%
Std. Dev.	2.89%	2.88%
Corr. Lag 1	0.98	0.98
% periods $i_t = 0$	17.6%	20.8%
Mean ZLB duration	11.3 qtrs.	11.1 qtrs.
Max. ZLB duration	138 qtrs.	140 qtrs.

Notes: Model results are computed from a 300,000 period simulation $\pi_{4,t} \equiv [\Pi_{j=0}^{3}(1+\pi_{t-j})]^{0.25} - 1$.

4.3 Effect of raising the inflation target

In a press conference held on June 14, 2017, Fed Chair Janet Yellen stated that the determining the appropriate level of the inflation target "is one of the most important questions facing monetary policy around the world in the future." Numerous authors make the case for a higher inflation target using frameworks that ignore the deflation equilibrium.[22] This methodology likely understates the benefits of a higher inflation target because the analysis does not take into account the idea that a higher target can prevent switching to the volatile deflation equilibrium where recessions are more severe. Aruoba and Schorfheide (2016) consider the welfare implications of a 4% inflation target in a framework that does consider the possibility of switching to the deflation equilibrium via an exogenous sunspot shock. They conclude (p. 395) that "the case for a higher inflation target is not particularly strong." It's worth noting, however, that the probability of switching to the deflation equilibrium in the Aruoba-Schorfheide model is governed by an exogenous sunspot shock that is invariant to changes in the inflation target. In contrast, the framework developed here has the potential to reduce the

[22] See, for example, Blanchard, Dell'Ariccia, and Mauro (2010), Ball and Mazumder (2011), and Ball (2013).

probability of switching to the deflation equilibrium. This is a potentially important benefit that should be considered as part of any cost-benefit analysis of a higher inflation target.

Table 7 shows the effects of raising the central bank's inflation target. As π^* increases, the ZLB binding frequency declines as does the mean duration of ZLB episodes. Higher values of π^* serve to reduce the volatility of y_t because the volatile deflation equilibrium is now avoided more often. But at the same time, higher values of π^* serve to *increase* the volatility of $\pi_{4,t}$. This is because higher values of π^* widen the spread between the two inflation distributions implied by the the two local equilibria. This causes the mean inflation rate to shift by a larger amount whenever a regime switch does occur. The table shows that an increase in π^* can reduce, but not eliminate, the endogenous switches to the deflation equilibrium. Even with an inflation target of 4%, the ZLB binding frequency remains relatively high at 9.6%, the average duration of a ZLB episode is 11.0 quarters, and the maximum duration of a ZLB episode is 132 quarters, or 33 years. Once the possibility of the deflation equilibrium is taken into account, raising the inflation target is less effective solution for avoiding ZLB episodes.

Table 7. Effect of Raising the Inflation Target in Switching Model

Statistic	$\pi^* = 0.02$	$\pi^* = 0.03$	$\pi^* = 0.035$	$\pi^* = 0.04$	$\pi^* = 0.05$
% periods $i_t = 0$	17.6%	13.3%	11.4%	9.6%	6.5%
Mean ZLB duration	11.3 qtrs.	11.3 qtrs.	11.2 qtrs.	11.0 qtrs.	10.8 qtrs.
Max. ZLB duration	138 qtrs.	138 qtrs.	131 qtrs.	132 qtrs.	133 qtrs.
Std. Dev. y_t	2.25%	2.18%	2.15%	2.11%	2.04%
Std. Dev. $\pi_{4,t}$	1.56%	1.68%	1.74%	1.79%	1.85%
Std. Dev. i_t^*	2.89%	3.00%	3.04%	3.06%	3.04%
Loss value, $\gamma = 1$	2.94%	2.80%	2.82%	2.90%	3.25%
Loss value, $\gamma = 0.25$	2.20%	2.04%	2.06%	2.17%	2.60%

Note: Model results computed from a 300,000 period simulation. $\pi_{4,t} \equiv [\Pi_{j=0}^{3}(1+\pi_{t-j})]^{0.25} - 1$.

Coibion, Gorodnichenko, and Wieland (2012) calibrate their model to deliver a ZLB binding frequency equal to that observed in U.S. data going back to the year 1950. From the start of 1950 to the end of 2015, the ZLB was binding in 29 out of 260 quarters, or 11.2% of the time. The average U.S. CPI inflation rate since 1950 is about 4% . Table 7 shows that the switching model with $\pi^* = 0.04$ delivers a ZLB binding frequency of 9.6%—close to the U.S. value of 11.2% going back to 1950. Taking into account the micro-founded welfare costs of positive and variable inflation, Coibion, Gorodnichenko, and Wieland (2012) compute an optimal inflation rate for their model which is relatively low, less than 2% per year. Their analysis is extended by Dordal-i-Carrera et al. (2016), who modify the shock process in the model to match the now-higher ZLB binding frequency implied by additional years of data for the U.S. and other advanced economies. The optimal inflation rate for the modified model lies in the range of 2.5% to 4%.

Kiley and Roberts (2017) perform stochastic simulations using the DSGE model of Lindé, Smets, and Wouters (2016) which is estimated over the period 1965.Q1 to 2014.Q2. They consider constant values of the natural rate of interest as low as 1% and draws shocks from the estimated distributions of the model. When monetary policy follows a simple Taylor (1999) rule with no inertia, they find that the ZLB binding frequency can be as high as 32.6% with an mean ZLB duration of 12 quarters (p. 22). The very high ZLB binding frequency obtains even though the model solution considers only the targeted equilibrium. In contrast, the simulations here deliver a baseline ZLB binding frequency in the switching model of 17.6%, despite allowing for a natural rate of interest as low as -0.41% and further allowing for the possibility of switches to the deflation equilibrium. The much higher ZLB binding frequency obtained by Kiley and Roberts (2017) appears to be partly due to the shock distributions which are based on the more-volatile U.S. data sample going back to 1965. Here, in contrast, the shock distributions are based on a more-recent U.S. data sample going back to 1988—a period of consistent monetary policy.

In addition to their use of a more-volatile shock distribution, Kiley and Roberts (2017) employ a Taylor-type rule with no interest rate smoothing component. This feature of their model would also appear to contribute to a higher ZLB binding frequency. Table 8 shows the effect of changing the interest rate smoothing parameter ρ in the switching model. The baseline case with $\rho = 0.8$ is compared to the no smoothing case with $\rho = 0$. For all values of π^*, the $\rho = 0$ case implies a higher frequency of hitting the ZLB, but the episodes exhibit shorter duration on average. Hence, from a ZLB perspective, there appears to be no clear advantage to reducing the degree of interest rate smoothing in the monetary policy rule.

Table 8. Effect of Interest Rate Smoothing in Switching Model

Statistic	$\pi^* = 0.02$	$\pi^* = 0.03$	$\pi^* = 0.04$	$\pi^* = 0.05$
$\rho = 0.8$				
% periods $i_t = 0$	17.6%	13.3%	9.6%	6.5%
Mean ZLB duration	11.3 qtrs.	11.3 qtrs.	11.0 qtrs.	10.8 qtrs.
Max. ZLB duration	138 qtrs.	138 qtrs.	132 qtrs.	133 qtrs.
$\rho = 0$				
% periods $i_t = 0$	24.9%	20.8%	16.8%	13.2%
Mean ZLB duration	3.9 qtrs.	4.0 qtrs.	4.0 qtrs.	4.0 qtrs.
Max. ZLB duration	102 qtrs.	108 qtrs.	113 qtrs.	114 qtrs.

Note: Model results computed from a 300,000 period simulation.

Following Kiley and Roberts (2017), I use a simple loss function approach to quantify the various trade-offs that are involved in raising the inflation target. The trade-offs here include: (1) reducing the likelihood of switches to the volatile deflation equilibrium and thereby lowering the ZLB binding frequency, (2) endogenous changes in the volatilities of inflation and the output gap, and (3) economic distortions that come from a higher average inflation. The loss function employed here takes the following form

$$Loss \;=\; E\left\{\left[\pi_{4,\,t}-0.02\right]^{2} \;+\; \gamma\left[y_{t}-0.02\left(1-\beta\right)/\kappa\right]^{2}\right\}, \tag{20}$$

where 0.02 and $0.02\left(1-\beta\right)/\kappa$ are the long-run endpoints in the targeted equilibrium when $\pi^{*}=0.02$, as shown in Table 1. The presumption is that the central bank in the baseline calibration with $\pi^{*}=0.02$ has chosen to target the "optimal" levels of $\pi_{4,\,t}$ and y_t. Hence, any shift away from the original target values when adopting $\pi^{*}>0.02$ would introduce economic distortions, thus raising the value of the loss function. Also following Kiley and Roberts (2017), I consider two values for the relative weight γ on the second term that captures the loss from output gap deviations. The bottom rows of Table 7 show that the simple loss function approach would favor a modest increase in the central bank's inflation target. Specifically, the loss function is minimized at $\pi^{*}=0.03$ for $\gamma=1$ or $\gamma=0.25$.

4.4 Adaptive learning in a simplified model

Up to this point, I have assumed that the representative agent has full knowledge of the forecast rules associated with each of the two local rational expectations equilibria. While the full-knowledge assumption is standard in models with a unique equilibrium, the computational burden on the agent is considerably higher in the present context. To relax the full-knowledge assumption, I introduce an adaptive learning algorithm into a simplified version of the model. Starting from the original model, the simplified model imposes $\rho=0$, $\omega=1$, and $\sigma_{\eta}=0$. These settings eliminate i^{*}_{t-1} and $\overline{\pi}_{t-1}$ as state variables and cause the natural rate of interest r^{*} to be constant. I set $r^{*}=0.0187$, corresponding to the average value of the Laubach-Williams estimate from 1988.Q1 to 2016.Q4. Other parameter values are identical to those in the original model. The theoretical decision rules for each of the two local rational expectations equilibria are shown in Appendix D.

The learning agent employs decision rules that are estimated in real-time using model-generated data. The resulting dynamics are self-referential because the agent's current decision rules influence the data generated by the model and thereby influence the estimated decision rules in subsequent periods. An equilibrium is said to be "learnable" if the estimated decision rules converge to the theoretical decision rules implied by the rational expectations solution.[23]

I consider two versions of the learning algorithm. In the first version, the agent estimates a set of correctly specified decision rules that take the form:

$$y_t = c_{0,t} + c_{1,t}(r_t - r^*) + c_{2,t}\, v_t + c_{3,t}\, u_t, \tag{21}$$

$$\pi_t = d_{0,t} + d_{1,t}(r_t - r^*) + d_{2,t}\, v_t + d_{3,t}\, u_t, \tag{22}$$

where the coefficients $c_{i,t}$ and $d_{i,t}$ for $i = 0, 1, 2, 3$ are estimated each period using an ordinary least squares regression applied to a rolling window of the most recent 32 quarters (8 years) of model-generated data. For the first 32 quarters of the simulation, the coefficient values are set halfway between the two values implied by each of the two local rational expectations equilibria. I assume that the agent can observe r_t, v_t, and u_t and knows the laws of motion for all state variables. In the second version, I assume that the agent estimates a set of misspecified decision rules that omit the terms involving the two white noise shocks v_t and u_t. In both versions, the agent uses the previous period's estimated decision rules to construct the subjective forecasts $\widehat{E}_t\, y_{t+1} = c_{0,t-1} + c_{1,t-1}\rho_r(r_t - r^*)$ and $\widehat{E}_t\, \pi_{t+1} = d_{0,t-1} + d_{1,t-1}\rho_r(r_t - r^*)$, where the values of ρ_r and r^* are known. Use of the previous period's decision rules to construct the forecasts avoids simultaneity in the realized and forecasted values of the output gap and inflation.

The learning agent's subjective forecasts are substituted into the global equilibrium conditions (1) and (2). I allow for an occasionally binding ZLB by making the substitution $i_t = 0.5\, i_t^* + 0.5\sqrt{(i_t^*)^2}$ in the equilibrium condition (1). When combined with the simplified policy rule (11) that imposes $\rho = 0$ and $\omega = 1$, this procedure yields a system of three equations that can be solved each period to obtain the three realizations y_t, π_t, and i_t^*. Given these realizations, the decision rule coefficients are then updated.[24]

Figure 9 plots the estimated coefficients $c_{0,t}$, $c_{1,t}$, $d_{0,t}$, and $d_{1,t}$ from simulations of the learning algorithm. When the agent estimates correctly specified decision rules (black lines), the coefficients quickly converge to the vicinity of the targeted equilibrium values and remain there. This result shows that the targeted equilibrium is learnable. In contrast, when the agent estimates misspecified decision rules (blue lines), the coefficients exhibit low frequency

[23] Christiano, Eichenbaum, and Johanssen (2016) consider a much more specialized learning experiment in which the ZLB is already binding due to a fundamental shock. Given this condition, they investigate whether various learning algorithms converge to either a high inflation or a low inflation equilibrium.

[24] To ensure stability of the learning algorithm, I impose the following projection facility: If the estimated values of $c_{0,t}$ or $d_{0,t}$ fall outside the range of two times the targeted equilibrium value on the upside or two times the deflation equilibrium value on the downside, then the agent does not update the affected decision rule for that period only. Intuitively, the agent is assumed to apply a common sense economic restriction on the magnitude of the decision rule coefficients.

oscillations that occasionally approach, reach, or go beyond the deflation equilibrium values. The agent's failure to control for the shocks v_t and u_t in the regressions causes the estimated values of the other coefficients to fluctuate, preventing the algorithm from converging to either of the two equilibria.[25] The low frequency oscillations in the decision rule coefficients induce movements in the macroeconomic variables that are qualitatively similar to those observed in Figure 8 for the original switching model with full-knowledge. Hence, the switching model dynamics are reasonably robust to departures from the full-knowledge assumption.

5 Conclusion

Standard New Keynesian models subject to a ZLB exhibit two long-run endpoints (steady states) associated with two local rational expectations equilibria. Most studies employing New Keynesian models focus on the targeted equilibrium and ignore the possibility of switching to the deflation equilibrium. But there appears to be no clear theoretical or empirical reason why the deflation equilibrium should be ruled out. Indeed, Bullard (2010) concludes that by "promising to remain at zero for a long time," central banks may inadvertently coordinate private-sector expectations to select the deflation equilibrium.

I examine a version of the New Keynesian model with a time-varying natural rate of interest and endogenous forecast rule switching based on past performance. The model can produce severe recessions when the real interest rate gap is negative, causing the representative agent to place a significant weight on the forecast rules associated with the deflation equilibrium. Escape from the deflation equilibrium occurs endogenously when the real interest rate gap eventually starts rising. But even in normal times, a non-trivial weight on the deflation forecast rules may cause central bank to undershoot its inflation target and raise the volatilities of macro variables relative to the targeted equilibrium.

A simple loss function approach favors a modest increase in the central bank's inflation target to around 3%. But even with an inflation target of 4%, the ZLB binding frequency remains relatively high at 9.6% and the average duration of a ZLB episode is 11 quarters. These results suggest that concerns about getting stuck in a deflation trap for an extended period are legitimate if one believes that the standard New Keynesian model provides a good description of advanced economies with inflation targeting central banks.

[25] This result shares some similarity with the findings of Bullard and Cho (2005) who employ a monetary policy model with a unique steady state. They show that "certain types of seemingly minor misspecifications along with agent learning might combine to change the global dynamics of the economy in unexpected ways." (p. 1842).

A Appendix: Kalman filter estimate of r-star

Straightforward computations using the laws of motion (3) and (4) yield the following unconditional moments for the first difference of the exogenous real interest rate:

$$Var\left(\Delta r_t\right) = \sigma_\varepsilon^2 \left[\frac{(1-\rho_r)\,\phi + 2}{1+\rho_r}\right], \tag{A.1}$$

$$Var\left(\Delta^2 r_t\right) = \sigma_\varepsilon^2 \left[\frac{2\,(1-\rho_r)^2\,\phi + 6}{1+\rho_r}\right], \tag{A.2}$$

$$Corr\left(\Delta r_t,\ \Delta r_{t-1-k}\right) = \frac{(\rho_r)^k\,(\rho_r\,\phi - 1)}{\phi + 2/\,(1-\rho_r)}, \tag{A.3}$$

where $\Delta r_t \equiv r_t - r_{t-1}$, $\Delta^2 r_t \equiv \Delta r_t - \Delta r_{t-1}$, and $\phi \equiv \sigma_\eta^2/\sigma_\varepsilon^2$. Equation (A.3) shows that the agent can infer the values of ρ_r and ϕ from the autocorrelation structure of Δr_t, which is observable.

Solving equation (3) for r_t^* yields:

$$r_t^* = \underbrace{\frac{r_t - \rho_r r_{t-1}}{1-\rho_r}}_{\text{Signal}} - \underbrace{\varepsilon_t}_{\text{Noise}}, \tag{A.4}$$

where the first term represents the signal and the second term represents the noise. Equation (4) shows that the Kalman filter estimate of r_t^*, denoted by $E_t r_t^*$, is a weighted average of the signal and the previous period's estimate $E_{t-1}\,r_{t-1}^*$, where the weight assigned to the signal is the Kalman gain parameter λ.

The agent's one step ahead forecast error is given by

$$\begin{aligned} err_{t+1} &= r_{t+1} - E_t\,r_{t+1}, \\ &= r_{t+1} - \left[\rho_r r_t + (1-\rho_r)\,E_t\,r_t^*\right], \\ &= \varepsilon_{t+1} + (1-\rho_r)\,\eta_{t+1} + (1-\rho_r)\left(r_t^* - E_t\,r_t^*\right), \end{aligned} \tag{A.5}$$

where the last term in (A.5) represents the estimation error for r_t^*. The optimal value of λ minimizes the mean squared forecast error, as given by

$$E\left(err_{t+1}^2\right) = \sigma_\varepsilon^2\left[1 + (1-\rho_r)^2\,\phi\right] + (1-\rho_r)^2\,Var\left(r_t^* - E_t\,r_t^*\right). \tag{A.6}$$

where $Var\left(r_t^* - E_t\,r_t^*\right)$ is the unconditional variance of the estimation error.

The law of motion for the estimation error follows directly from equation (7) and can be written as

$$r_t^* - E_t\,r_t^* = \lambda\left[\frac{z_t - \rho_r\,z_{t-1}}{1-\rho_r}\right] + (1-\lambda)\left(r_{t-1}^* - E_{t-1}\,r_{t-1}^*\right) - \left[\frac{1-\lambda-\rho_r}{1-\rho_r}\right]\eta_t, \tag{A.7}$$

where $z_t \equiv r_t - r_t^*$ is the actual real rate gap. The law of motion for z_t follows directly from equations (3) and (4) and can be written as

$$z_t = \rho_r \, z_{t-1} - \rho_r \, \eta_t + \varepsilon_t. \tag{A.8}$$

Starting from equations (A.7) and (A.8), we can compute the following expression for the unconditional variance of the estimation error

$$Var\left(r_t^* - E_t \, r_t^*\right) = \sigma_\varepsilon^2 \left\{ \frac{\lambda\left(\rho_r^2 \phi + 1\right) + (1-\lambda-\rho_r)\left[(1-\lambda)(1-\rho_r)/\lambda + \rho_r\right]\phi}{(2-\lambda)(1-\rho_r)^2} \right\},$$

which can be substituted into equation (A.6) to obtain a complicated expression for $E\left(err_{t+1}^2\right)$ in terms of λ. From this expression, we can compute the gradient

$$\frac{\partial E\left(err_{t+1}^2\right)}{\partial \lambda} = \frac{2\left[\lambda^2 - (1-\lambda)(1-\rho_r)^2 \phi\right]}{(2-\lambda)^2 \lambda^2}. \tag{A.9}$$

Setting the gradient equal to zero yields a quadratic equation in λ. The root that minimizes $E\left(err_{t+1}^2\right)$ is given by equation (8).

B Appendix: Targeted equilibrium

To solve for the local linear forecast rules associated with the targeted equilibrium, I assume that $i_t^* = i_t > 0$ for all t, i.e., the ZLB is never binding. Starting from equation (9) we have:

$$i_t^* = \rho i_{t-1}^* + (1-\rho)\left[E_t \, r_t^* + \pi^* + g_\pi \, \omega\left(\pi_t - \pi^*\right) + g_\pi \, (1-\omega)\left(\overline{\pi}_{t-1} - \pi^*\right) + g_y\left(y_t - y^*\right)\right] \tag{B.1}$$

where I have used equation (10) to eliminate $\overline{\pi}_t$.

Equation (B.1) together with the Euler equation (1) and the Phillips curve (2) form a linear system of three equations in the three unknown decision rules for π_t, y_t, and i_t^*. The state variables are: r_t, $E_t r_t^*$, $\overline{\pi}_{t-1}$, i_{t-1}^*, v_t, and u_t. Standard techniques yield a set of linear decision rules of the form

$$\begin{bmatrix} y_t - \pi^*(1-\beta)/\kappa \\ \pi_t - \pi^* \\ i_t^* - \left(E_t \, r_t^* + \pi^*\right) \end{bmatrix} = \mathbf{A} \begin{bmatrix} r_t - E_t \, r_t^* \\ \overline{\pi}_{t-1} - \pi^* \\ i_{t-1}^* - E_t \, r_t^* - \pi^* \\ v_t \\ u_t \end{bmatrix}, \tag{B.2}$$

where $\mathbf{A}$ is a 3×5 matrix of decision rule coefficients. For the parameter values shown in Table 2, the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 0.832 & -0.209 & -0.551 & 0.858 & -0.178 \\ 0.087 & -0.021 & -0.042 & 0.014 & 0.982 \\ 0.178 & 0.117 & 0.684 & 0.174 & 0.100 \end{bmatrix}. \tag{B.3}$$

Iterating the linear decision rules in (B.2) ahead one period and then taking the conditional expectation of both sides yields the following set of linear forecast rules associated with the targeted equilibrium:

$$E_t y_{t+1} = y^* + \mathbf{A}_{11}\rho_r \left(r_t - E_t\, r_t^*\right) + \mathbf{A}_{12}\left(\overline{\pi}_t - \pi^*\right) + \mathbf{A}_{13}\left(i_t^* - E_t\, r_t^* - \pi^*\right), \tag{B.4}$$

$$E_t \pi_{t+1} = \pi^* + \mathbf{A}_{21}\rho_r \left(r_t - E_t\, r_t^*\right) + \mathbf{A}_{22}\left(\overline{\pi}_t - \pi^*\right) + \mathbf{A}_{23}\left(i_t^* - E_t\, r_t^* - \pi^*\right), \tag{B.5}$$

where $\mathbf{A}_{i\,j}$ represents the corresponding element of the matrix $\mathbf{A}$ and I have substituted in $E_t\left(r_{t+1} - E_{t+1} r_{t+1}^*\right) = \rho_r\left(r_t - E_t\, r_t^*\right)$. Notice that the forecast rules depend on the realization of π_t because $\overline{\pi}_t$ depends on π_t via equation (10). Also, the forecast rules depend on the realization of i_t^* due to the interest rate smoothing term in (B.1). Hence, the model allows for simultaneity between the forecasted and realized values of π_t and i_t^*. Neither the agent or the central bank are required to forecast i_{t+1}^*.

The linear forecast rules (B.4) and (B.5) are derived under the assumption that the ZLB is never binding. However, in the stochastic simulation of the targeted equilibrium, I allow for an occasionally binding ZLB. When simulating the model, I substitute the local linear forecast rules given by (B.4) and (B.5) into the global equilibrium conditions (1) and (2). I allow for an occasionally binding ZLB by making the substitution $i_t = 0.5\, i_t^* + 0.5\sqrt{\left(i_t^*\right)^2}$ in the equilibrium condition (1). Together with the monetary policy rule (B.1), this procedure yields a system of three equations that are solved each period to obtain the three realizations y_t, π_t and i_t^*.

C Appendix: Deflation equilibrium

To solve for the local linear forecast rules associated with the deflation equilibrium, I assume $i_t^* \leq 0$ such that $i_t = 0$ for all t, i.e., the ZLB is always binding. Equation (B.1) applies unchanged to the deflation equilibrium, as does the Phillips curve (2). However, due to the binding ZLB, the Euler equation (1) now becomes

$$y_t = E_t\, y_{t+1} + \alpha[E_t\, \pi_{t+1} + r_t] + v_t. \tag{C.1}$$

Equation (C.1) together with equations (B.1) and (2) form a linear system of three equations in the three unknown decision rules for π_t, y_t, and i_t^*. The state variables are: r_t, $E_t r_t^*$, $\overline{\pi}_{t-1}$, i_{t-1}^*, v_t, and u_t. The minimum state variable (MSV) solution yields a set of linear decision rules of the form

$$\begin{bmatrix} y_t - (-E_t\, r_t^*)\,(1-\beta)\,/\kappa \\ \pi_t - (-E_t\, r_t^*) \\ i_t^* - (E_t\, r_t^* + \pi^*)\,[1 - g_\pi - g_y\,(1-\beta)\,/\kappa] \end{bmatrix} = \mathbf{B} \begin{bmatrix} r_t - E_t\, r_t^* \\ \overline{\pi}_{t-1} - (-E_t\, r_t^*) \\ i_{t-1}^* - (E_t\, r_t^* + \pi^*)\left[1 - g_\pi - \frac{g_y(1-\beta)}{\kappa}\right] \\ v_t \\ u_t \end{bmatrix}, \tag{C.2}$$

where $\mathbf{B}$ is a 3×5 matrix of constant coefficients. The MSV solution implies $\mathbf{B}_{12} = \mathbf{B}_{22} = 0$ and $\mathbf{B}_{13} = \mathbf{B}_{23} = 0$. For the parameter values shown in Table 2, the matrix $\mathbf{B}$ is

$$\mathbf{B} = \begin{bmatrix} 2.470 & 0 & 0 & 1 & 0 \\ 0.431 & 0 & 0 & 0.025 & 1 \\ 0.553 & 0.162 & 0.8 & 0.203 & 0.138 \end{bmatrix}. \tag{C.3}$$

Comparing the first column of matrix $\mathbf{B}$ in equation (C.3) to the first column of matrix $\mathbf{A}$ in equation (B.3) shows that a shock to $r_t - E_t r_t^*$ will be transmitted more forcefully to macro variables in the deflation equilibrium than in targeted equilibrium. Specifically, we have

$$\frac{\mathbf{B}_{11}}{\mathbf{A}_{11}} = 3.0, \qquad \frac{\mathbf{B}_{21}}{\mathbf{A}_{21}} = 4.9, \qquad \frac{\mathbf{B}_{31}}{\mathbf{A}_{31}} = 3.1. \tag{C.4}$$

For the special case when $\rho = 0$ and $\omega = 1$, it is straightforward to derive the following analytical relationship between the decision rule coefficients for the two local equilibria:

$$\frac{\mathbf{B}_{11}}{\mathbf{A}_{11}} = \frac{\mathbf{B}_{21}}{\mathbf{A}_{21}} = \frac{\mathbf{B}_{31}}{\mathbf{A}_{31}} = 1 + \frac{\alpha\,[\kappa g_\pi + (1-\beta\rho_r)\,g_y]}{(1-\beta\rho_r)\,(1-\rho_r) - \alpha\kappa\rho_r} > 1. \tag{C.5}$$

Iterating the linear decision rules in (C.2) ahead one period and then taking the conditional expectation of both sides yields the following set of local linear forecast rules for the deflation equilibrium:

$$E_t\, y_{t+1} = -E_t\, r_t^*\,(1-\beta)\,/\kappa + \mathbf{B}_{11}\rho_r\,(r_t - E_t\, r_t^*)\,, \tag{C.6}$$

$$E_t\pi_{t+1} = -E_t\, r_t^* + \mathbf{B}_{21}\rho_r\,(r_t - E_t\, r_t^*)\,, \tag{C.7}$$

where the MSV solution implies $\mathbf{B}_{12} = \mathbf{B}_{22} = 0$ and $\mathbf{B}_{13} = \mathbf{B}_{23} = 0$ and I have substituted in $E_t\left(r_{t+1} - E_{t+1} r_{t+1}^*\right) = \rho_r\,(r_t - E_t\, r_t^*)$. Neither the agent or the central bank are required to forecast i_{t+1}^*.

The linear forecast rules (C.6) and (C.7) are derived under the assumption that the ZLB is always binding. However, in the stochastic simulation of the deflation equilibrium, I allow for an occasionally binding ZLB. When simulating the model, I substitute the local linear forecast rules given by (C.6) and (C.7) into the global equilibrium conditions (1) and (2). I allow for an occasionally binding ZLB by making the substitution $i_t = 0.5\, i_t^* + 0.5\sqrt{(i_t^*)^2}$ in the equilibrium condition (1). Together with the monetary policy rule (B.1), this procedure yields a system of three equations that are solved each period to obtain the three realizations y_t, π_t, and i_t^*.

D Appendix: Simplified model

This appendix provides the theoretical equilibrium decision rules for the simplified model that is used in the adaptive learning algorithm described in Section 4.4. Starting from the original model, the simplified model imposes $\rho = 0$, $\omega = 1$, and $\sigma_\eta = 0$. These settings eliminate i_{t-1}^* and $\overline{\pi}_{t-1}$ as state variables and cause the natural rate of interest r^* to be constant.

The targeted equilibrium decision rules are

$$\begin{bmatrix} y_t - \pi^*\left(1-\beta\right)/\kappa \\ \pi_t - \pi^* \\ i_t^* - \left(r^* + \pi^*\right) \end{bmatrix} = \mathbf{A} \begin{bmatrix} r_t - r^* \\ v_t \\ u_t \end{bmatrix}, \tag{D.1}$$

where the elements of the matrix $\mathbf{A}$ are given by

$$\mathbf{A} = \begin{bmatrix} \frac{\alpha(1-\beta\rho_r)}{(1-\rho_r+\alpha g_y)(1-\beta\rho_r)+\alpha\kappa(g_\pi-\rho_r)} & \frac{1}{1+\alpha\kappa g_\pi+\alpha g_y} & \frac{-\alpha g_\pi}{1+\alpha\kappa g_\pi+\alpha g_y} \\ \frac{\alpha\kappa}{(1-\rho_r+\alpha g_y)(1-\beta\rho_r)+\alpha\kappa(g_\pi-\rho_r)} & \frac{\kappa}{1+\alpha\kappa g_\pi+\alpha g_y} & \frac{1+\alpha g_y}{1+\alpha\kappa g_\pi+\alpha g_y} \\ \frac{\alpha\kappa g_\pi+\alpha g_y(1-\beta\rho_r)}{(1-\rho_r+\alpha g_y)(1-\beta\rho_r)+\alpha\kappa(g_\pi-\rho_r)} & \frac{\alpha\kappa g_\pi+g_y}{1+\alpha\kappa g_\pi+\alpha g_y} & \frac{g_\pi}{1+\alpha\kappa g_\pi+\alpha g_y} \end{bmatrix}. \tag{D.2}$$

The deflation equilibrium decision rules are

$$\begin{bmatrix} y_t - \left(-r^*\right)\left(1-\beta\right)/\kappa \\ \pi_t - \left(-r^*\right) \\ i_t^* - \left(r^* + \pi^*\right)\left[1 - g_\pi - g_y\left(1-\beta\right)/\kappa\right] \end{bmatrix} = \mathbf{B} \begin{bmatrix} r_t - r^* \\ v_t \\ u_t \end{bmatrix}, \tag{D.3}$$

where the elements of the matrix $\mathbf{B}$ for the MSV solution are given by

$$\mathbf{B} = \begin{bmatrix} \frac{\alpha(1-\beta\rho_r)}{(1-\rho_r)(1-\beta\rho_r)-\alpha\kappa\rho_r} & 1 & 0 \\ \frac{\alpha\kappa}{(1-\rho_r)(1-\beta\rho_r)-\alpha\kappa\rho_r} & \kappa & 1 \\ \frac{\alpha(1-\beta\rho_r)}{(1-\rho_r)(1-\beta\rho_r)-\alpha\kappa\rho_r} & \kappa g_\pi + g_y & g_\pi \end{bmatrix}. \tag{D.4}$$

References

Adam. K. and R. Billi 2007 Discretionary Monetary Policy and the Zero Lower Bound on Nominal Interest Rates, *Journal of Monetary Economics* 54, 728-752.

Altig, D., N. Parker, and B. Meyer 2017 The Fed's Inflation Goal: What Does the Public Know? Federal Reserve Bank of Atlanta, Macroblog (April 19).

Anderson, T.W. 1958. *An Introduction to Multivariate Statistical Analysis*. New York: John Wiley & Sons Inc.

Aruoba, S.B., P. Cuba-Borda, and F. Schorfheide 2017 Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries, *Review of Economic Studies*, forthcoming.

Aruoba, S.B. and F. Schorfheide 2016 Inflation During and After the Zero Lower, in Federal Reserve Bank of Kansas City 2015 Economic Policy Symposium Proceedings, *Inflation Dynamics and Monetary Policy,* pp. 359-436.

Ball, L., and S. Mazumder 2011 Inflation Dynamics and the Great Recession, *Brookings Papers on Economic Activity*, Spring, , 337-402.

Ball L. 2013 The Case for Four Percent Inflation, *Central Bank Review* 13, 17-31.

Barsky, R., A. Justiniano, and L. Melosi 2014 The Natural Rate of Interest and Its Usefulness for Monetary Policy, *American Economic Review: Papers & Proceedings* 104(5), 37-43.

Basu, S. and B. Bundick 2015 Endogenous Volatility at the Zero Lower Bound: Implications for Stabilization Policy, NBER Working Paper 21838.

Benhabib, J., S. Schmitt-Grohé, and M. Uribe 2001a Monetary Policy and Multiple Equilibria, *American Economic Review* 91, 167-186.

Benhabib, J., S. Schmitt-Grohé, and M. Uribe 2001b The Perils of Taylor Rules, *Journal of Economic Theory* 96, 40-69.

Benhabib, J. G.W. Evans and S. Honkapohja (2014) Liquidity Traps and Expectation Dynamics: Fiscal Stimulus or Fiscal Austerity? *Journal of Economic Dynamics and Control* 45, 220-238.

Blanchard, O., G. Dell'Ariccia, and P. Mauro 2010 Rethinking Macroeconomic Policy, *Journal of Money, Credit and Banking* 42(s1), 199-215.

Bullard, J. 2010 Seven Faces of "The Peril," Federal Reserve Bank of St. Louis, *Review*, September/October, 92, 339-352.

Bullard, J. and I.-K. Cho,I.-K. 2005 Escapist Policy Rules, *Journal of Economic Dynamics and Control* 29,1841-1866.

Brock, W.A. and C.H. Hommes 1998 Heterogenous Beliefs and Routes to Chaos in a Simple Asset Pricing Model *Journal of Economic Dynamics and Control* 22, 1235-1274.

Campbell, J.Y. and N.G. Mankiw 1989 Consumption, Income, and Interest Rates: Reinterpreting the Time Series Evidence, in O.J. Blanchard and S. Fischer (eds.), *NBER Macroeconomics Annual 1989, Volume 4.* Cambridge MA: MIT Press, pp. 185-246.

Christiano, L.J., M. Eichenbaum, and B.K. Johannsen 2016 Does the New Keynesian Model Have a Uniqueness Problem?" Working Paper.

Chung, H., J.-P. Laforte, D. Reifschneider, and J.C. Williams 2012 Have We Underestimated the Likelihood and Severity of Zero Lower Bound Events? *Journal of Money, Credit and Banking* 44, 47-82.

Cochrane, J.H. 2015 Do Higher Interest Rates Raise or Lower Inflation? Working Paper.

Coibion, O., Y. Gorodnichenko, and J. Wieland 2012 The Optimal Inflation Rate in New Keynesian models: Should Central Banks Raise Their inflation Targets in Light of the Zero Lower Bound? *Review of Economic Studies* 79, 1371-1406.

Cúrdia, V., A. Ferrero, G.C. Ng, and A. Tambalotti 2015 Has U.S. Monetary Policy Tracked the Efficient Interest Rate? *Journal of Monetary Economics* 70, 72-83.

Dennis, R. 2016 Durations at the Zero Lower Bound, University of Glasgow, Working Paper.

Del Negro, M., D. Giannone, M.P. Giannoni, and A. Tambalotti 2017 Safety, Liquidity, and the Natural Rate of Interest, Brookings Institution Conference Paper (March 23-24).

Dordal-i-Carreras, M., O. Coibion, Y. Gorodnichenko, J. Wieland 2016 Infrequent but Long-Lived Zero-Bound Episodes and the Optimal Rate of Inflation, *Annual Review of Economics* 8, 497-520.

Dudley, W.C. 2015 The U.S. Economic Outlook and Monetary Policy, Remarks at the Economic Club of New York, New York City (November 12).

Eggertsson, G.B. and M. Woodford 2003 The Zero Bound on Interest Rates and Optimal Monetary Policy, *Brookings Papers on Economic Activity* 2003(1), 139-211.

Eggertsson, G.B. and S.R. Sing 2016 Log-linear Approximation versus an Exact Solution at the ZLB in the New Keynesian Model, NBER Working Paper 22784.

Eggertsson, G.B., N.R. Mehrotra, and J.A. Robbins 2017 A Model of Secular Stagnation: Theory and Quantitative Evaluation, NBER Working Paper 23093.

Eusepi, S. 2007 Learnability and Monetary Policy: A Global Perspective, *Journal of Monetary Economics* 54, 1115-1131.

Evans, G.W. and S. Honkapohja 2005 Policy Interaction, Expectations and the Liquidity Trap, *Review of Economic Dynamics* 8, 303–323.

Evans, G.W., E. Guse, and, S. Honkapohja 2008 Liquidity Traps, Learning, and Stagnation, *European Economic Review* 52, 1438-1463.

Evans, G.W., S. Honkapohja, and K. Mitra 2016 Expectations, Stagnation and Fiscal Policy, Working Paper.

Evans, C., J. Fisher, F. Gourio, and S. Krane 2015 Risk Management for Monetary Policy Near the Zero Lower Bound, *Brookings Papers on Economic Activity* (Spring), 141-196

Fischer, S. 2016 Monetary Policy, Financial Stability, and the Zero Lower Bound, Remarks at the Annual Meeting of the American Economic Association, San Francisco, California (January 3).

Gust, C.J., B.K. Johannsen, and D. López-Salido 2015 Monetary Policy, Incomplete Information, and the Zero Lower Bound, Board of Governors of the Federal Reserve System, Finance and Economics Discussion Series Paper 2015-099.

Hamilton, J.D., E.S. Harris, J. Hatzius, and K.D. West 2016 The Equilibrium Real Funds Rate: Past, Present and Future, *IMF Economic Review* , forthcoming.

Hills, T.S., T. Nakata, and S. Schmidt 2016 The Risky Steady State and the Interest Rate Lower Bound, Federal Reserve Board, Finance and Economics Discussion Series Paper 2016-009.

Hursey, T. and A.L. Wolman 2010 Monetary Policy and Global Equilibria in a Production Economy, Federal Reserve Bank of Richmond, *Economic Quarterly* 96 (4), 317-337.

Huh, C.G. and K.J. Lansing 2000 Expectations, credibility, and disinflation in a small macroeconomic model, *Journal of Economics and Business* 51, 51-86.

International Monetary Fund 2014 Perspectives on Global Real Interest Rates, Chapter 3 of World Economic Outlook (WEO), Recovery Strengthens, Remains Uneven (April).

Kiley, M.T. 2015 What Can the Data Tell Us About the Equilibrium Real Interest Rate? Federal Reserve Board, Finance and Economics Discussion Series Paper 2015-077.

Kiley, M.T. and J.M. Roberts 2017 Monetary policy in a Low Interest Rate World, *Brookings Papers on Economic Activity* (Spring), forthcoming.

Kozicki, S. and P.A. Tinsley 2012 Effective Use of Survey Information in Estimating the Evolution of Expected Inflation *Journal of Money Credit and Banking* 44, 145-169.

Lansing, K.J. 2009 Time-Varying U.S. Inflation Dynamics and the New Keynesian Phillips Curve, *Review of Economic Dynamics* 12, 304-326.

Lansing, K.J. 2016 Projecting the Long-Run Natural Rate of Interest, Federal Reserve Bank of San Francisco *Economic Letter* 2016-25 (August 29).

Lansing, K.J. 2017 R-star, Uncertainty, and Monetary Policy, Federal Reserve Bank of San Francisco *Economic Letter* 2017-16 (May 30).

Lansing, K.J. and S.F. LeRoy 2014 Risk Aversion, Investor Information, and Stock Market Volatility, *European Economic Review* 70, 88-107.

Laubach, T. and J.C. Williams 2016 Measuring the Natural Rate of Interest Redux, *Business Economics* 51, 257-267.

Lindé, J., F. Smets, and R. Wouters 2016 Challenges for Central Bank Macro Models, in J.B. Taylor and H. Uhlig (eds.), *Handbook of Macroeconomics Volume 2.* Amsterdam: Elsevier, pp. 2185-2262.

Lubik, T.A. and C. Matthes 2015 Calculating the Natural Rate of Interest: A Comparison of Two Alternative Approaches, Federal Reserve Bank of Richmond, *Economic Brief* (October 15).

McCallum, B. 1999 Role of the Minimal State Variable Criterion in Rational Expectations Models, *International Tax and Public Finance* 6, 621–639.

McKay, A., E. Nakamura, and J. Steinsson 2016 The Discounted Euler Equation: A Note, NBER Working Paper 22129.

Nakata, T. 2013 Optimal Fiscal and Monetary Policy with Occasionally Binding Zero Bound Constraints, Federal Reserve Board, Finance and Economics Discussion Series Paper 2013-40.

Nakov, A. 2008 Optimal and Simple Monetary Policy Rules with Zero Floor on the Nominal Interest Rate, *International Journal of Central Banking* 4, 73–127.

Rachel, L. and T.D. Smith 2015 Secular Drivers of the Global Real Interest Rate, Bank of England, Staff Working Paper No. 571.

Reifschneider, D. and J.C. Williams 2000 Three Lessons for Monetary Policy in a Low-inflation Era, *Journal of Money, Credit, and Banking* 32, 936-966.

Richter, A.W. and N.A. Throckmorton 2016 Are Nonlinear Methods Necessary at the Zero Lower Bound? Federal Reserve Bank of Dallas, Working Paper 1606.

Sargent, Thomas J. 1999 *The Conquest of American Inflation.* Princeton: Princeton University Press.

Schmitt-Grohé, S. and M. Uribe 2010 The Optimal Rate of Inflation, in: B.M. Friedman and

M. Woodford (eds.), *Handbook of Monetary Economics, 1st Edition, Volume 3.* San Diego CA: Elsevier, pp. 653-722.

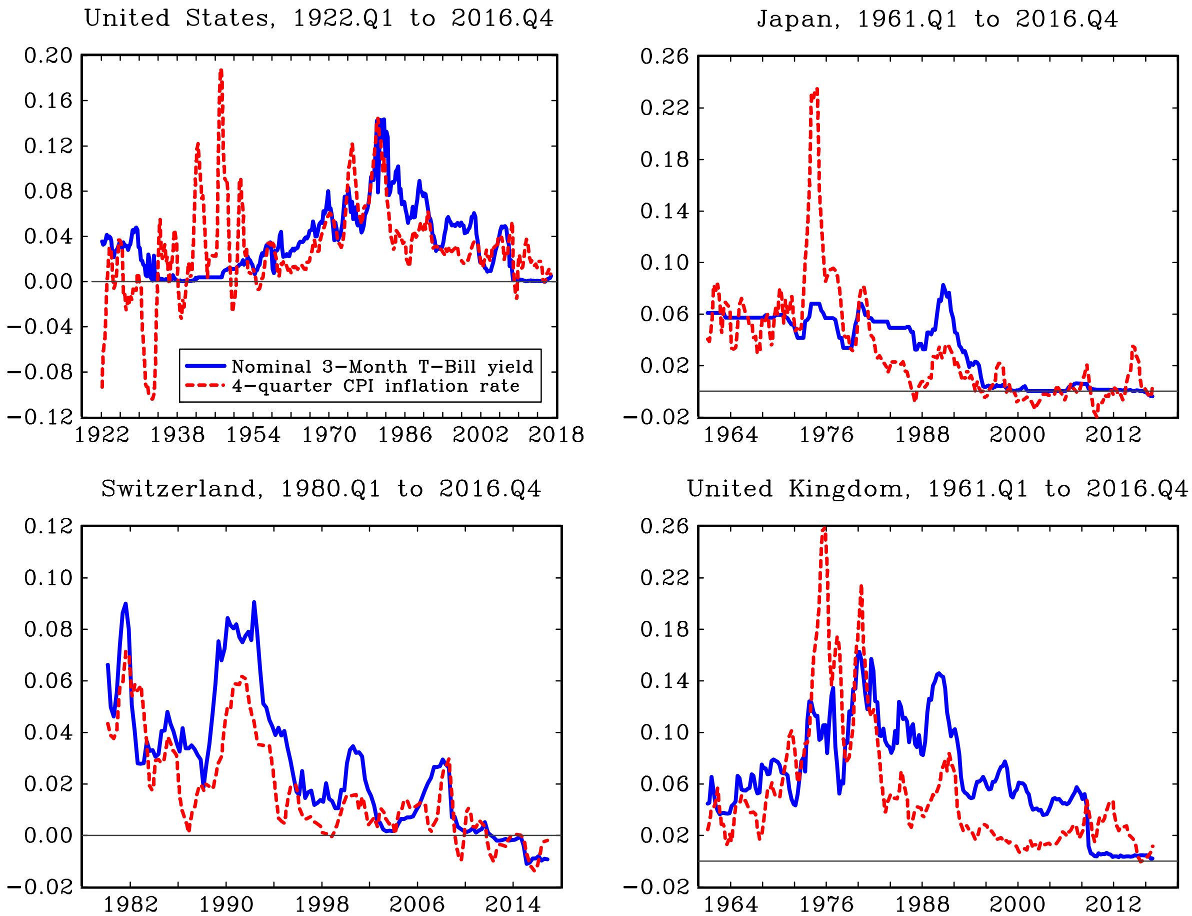
Stock, J.H. and M.W. Watson 2007 Why has Inflation Become Harder to Forecast? *Journal of Money, Credit, and Banking*, 39, 3-34.

Taylor, J.B. 1999. A Historical Analysis of Monetary Policy Rules, in J.B. Taylor (ed.), *Monetary Policy Rules.* Chicago: University of Chicago Press, pp. 319-348.

Williamson., S. 2017. Low Real Interest Rates and Monetary Policy, New Monetarist Economics (March 29).

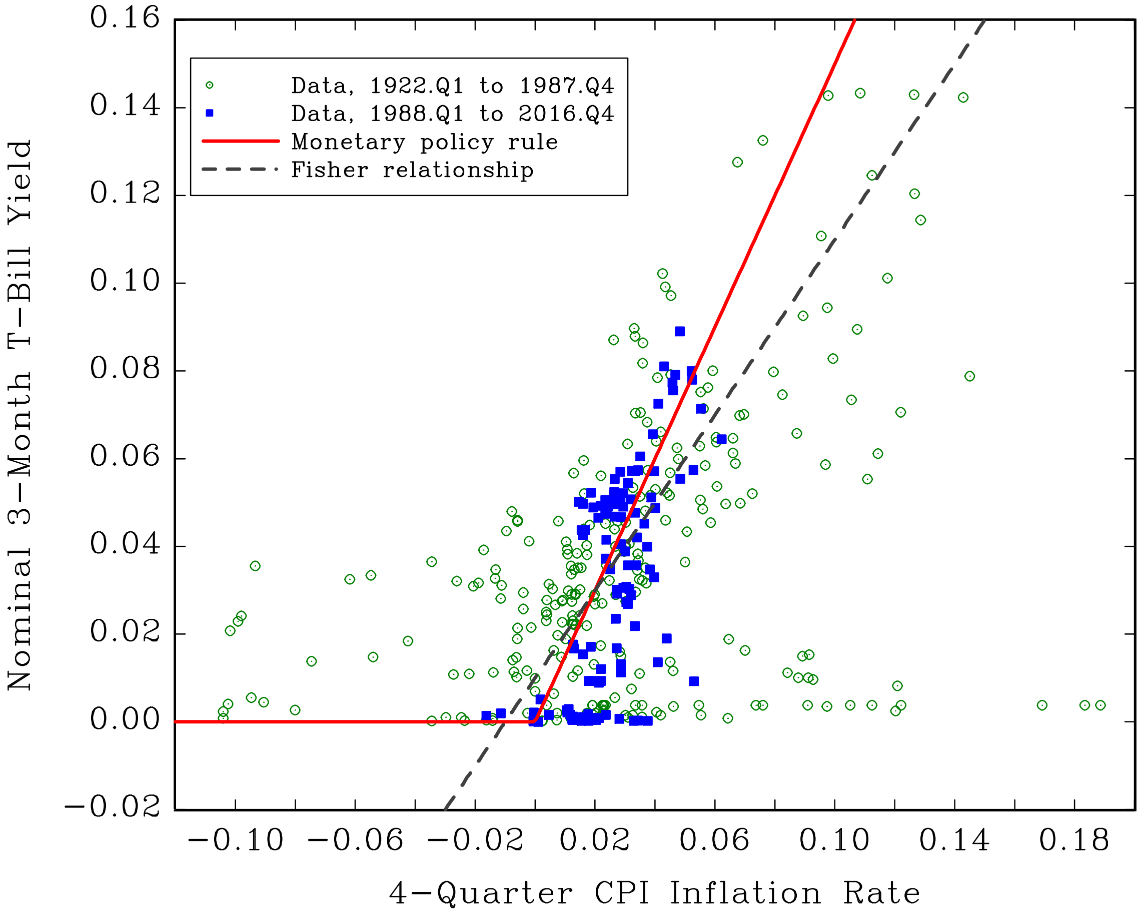
Yellen, J. 2015 Normalizing Monetary Policy: Prospects and Perspectives, Remarks at "The New Normal Monetary Policy," a research conference sponsored by the Federal Reserve Bank of San Francisco, San Francisco, California (March 27).

Figure 1: Nominal Interest Rates and Inflation in Four Countries



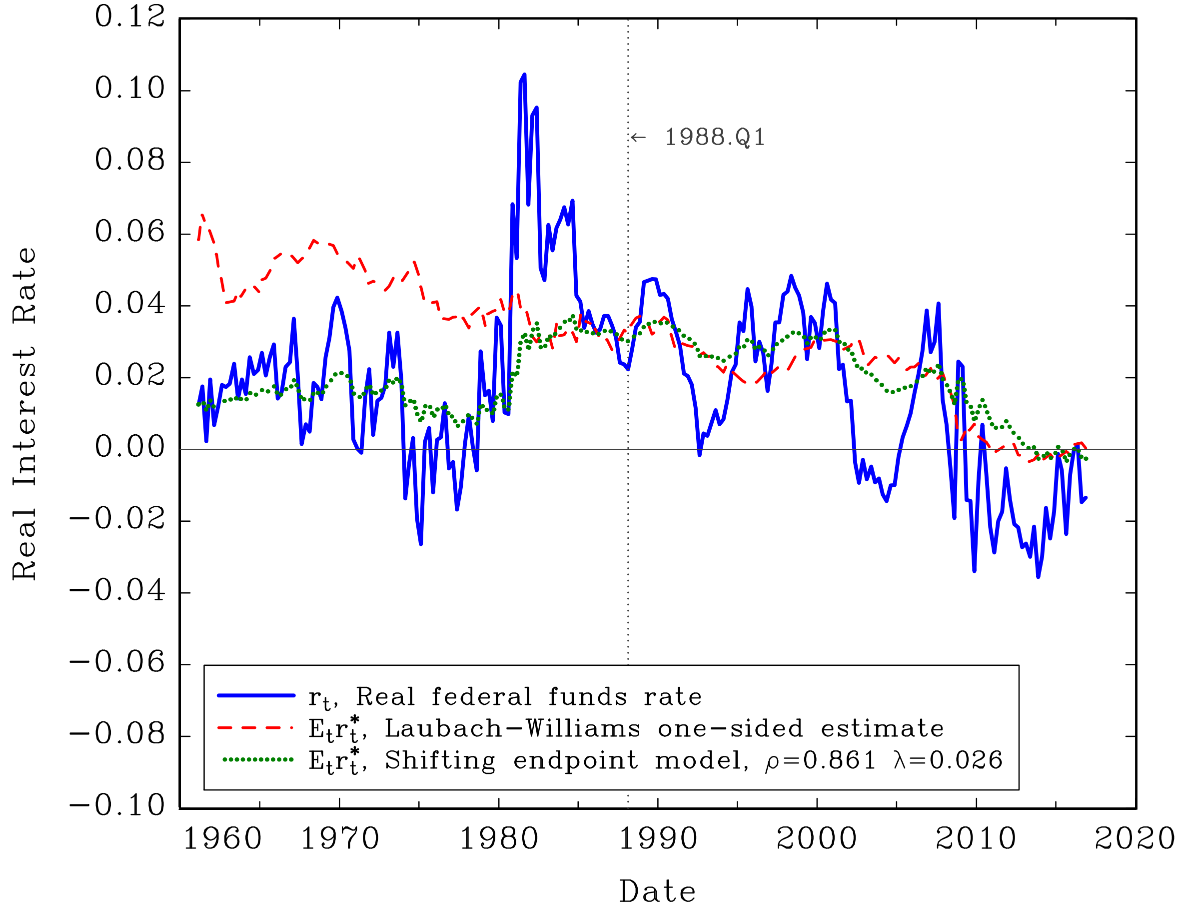
Nominal interest rates in the United States encountered the zero lower bound during the 1930s and from 2008.Q4 though 2015.Q4. Nominal interest rates in Japan have remained near zero since 1998.Q3, except for the period from 2006.Q4 to 2008.Q3. Nominal interest rates in Switzerland have been zero or slightly negative since 2008.Q4. Nominal interest rates in the United Kingdom have been approximately zero since 2009.Q1. Outside of these episodes, all four countries exhibit a strong positive correlation between nominal interest rates and inflation, consistent with the Fisher relationship.

Figure 2: U.S. Nominal Interest Rates and Inflation



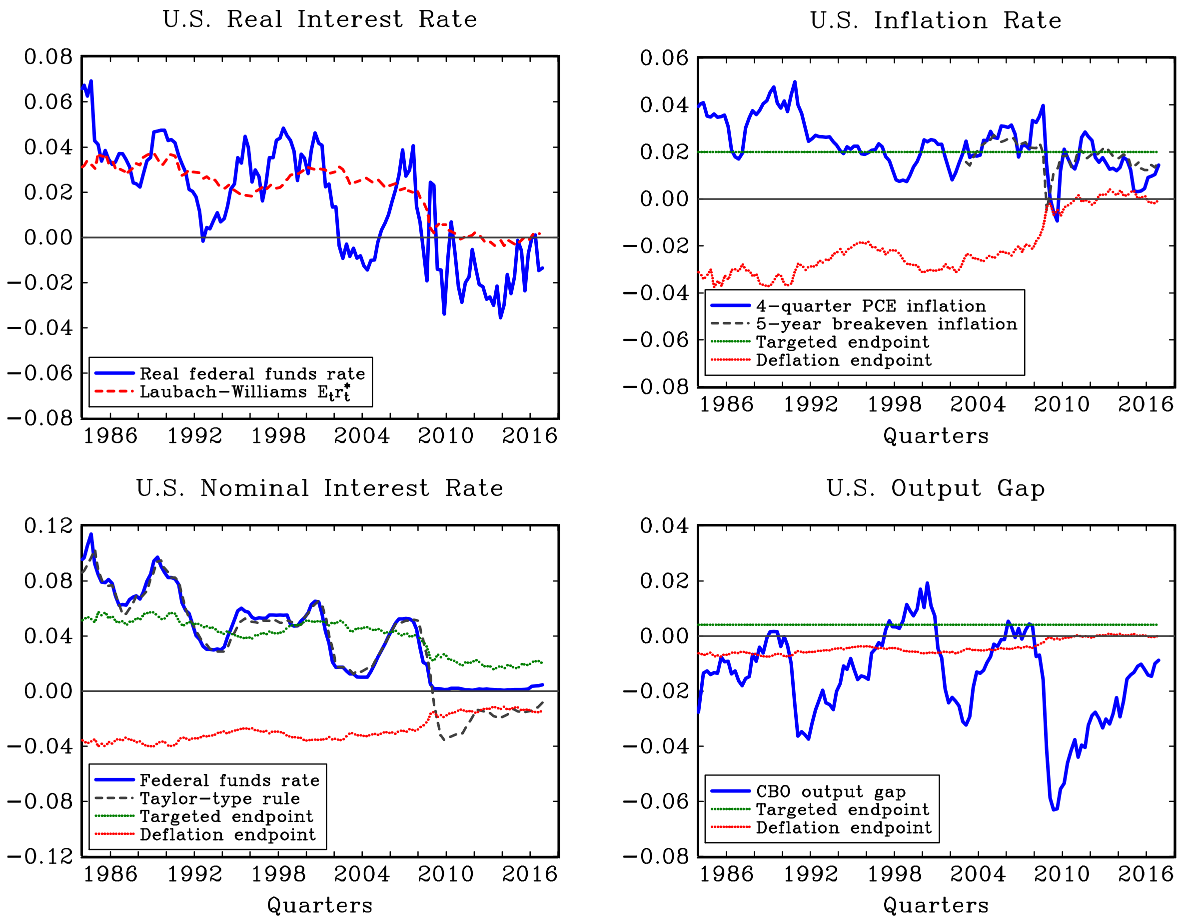
The two intersections of the ZLB-augmented monetary policy rule (solid red line) with the Fisher relationship (dashed black line) define two long-run endpoints, labeled the "targeted equilibrium" and "deflation equilibrium," respectively. The monetary policy rule is $i_t = r^* + \pi^* + g_\pi (\pi_t - \pi^*)$ with $r^* = 0.01$, $\pi^* = 0.02$ and $g_\pi = 1.5$. The Fisher relationship is $i_t = r^* + \pi_t$. Data since 2008.Q4 lie closer to the deflation equilibrium than the targeted equilibrium.

Figure 3: U.S. Real Interest Rates



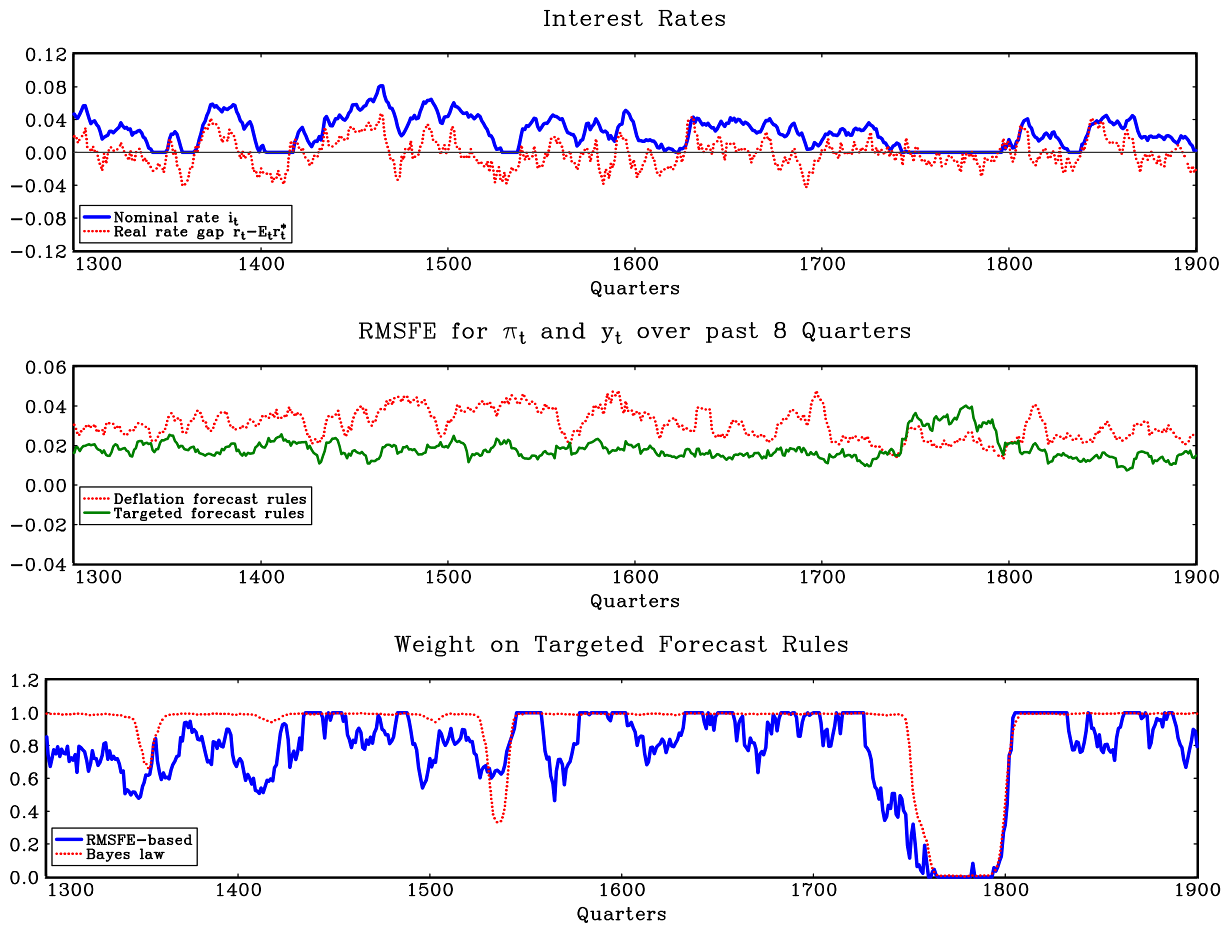
The real federal funds rate (blue line) is defined as the nominal federal funds rate minus expected quarterly inflation computed from a rolling 40-quarter, 4-lag vector autoregression that includes the nominal funds rate, quarterly PCE inflation (annualized), and the CBO output gap. The time series process for the Kalman filter estimate of the natural rate of interest in the model (dotted green line) is calibrated to approximate the one-sided estimate of the U.S. natural rate series (dashed red line) from Laubach and Williams (2016, updated) for the data sample 1988.Q1 to 2016.Q4, representing a period of consistent monetary policy.

Figure 4: U.S. Data



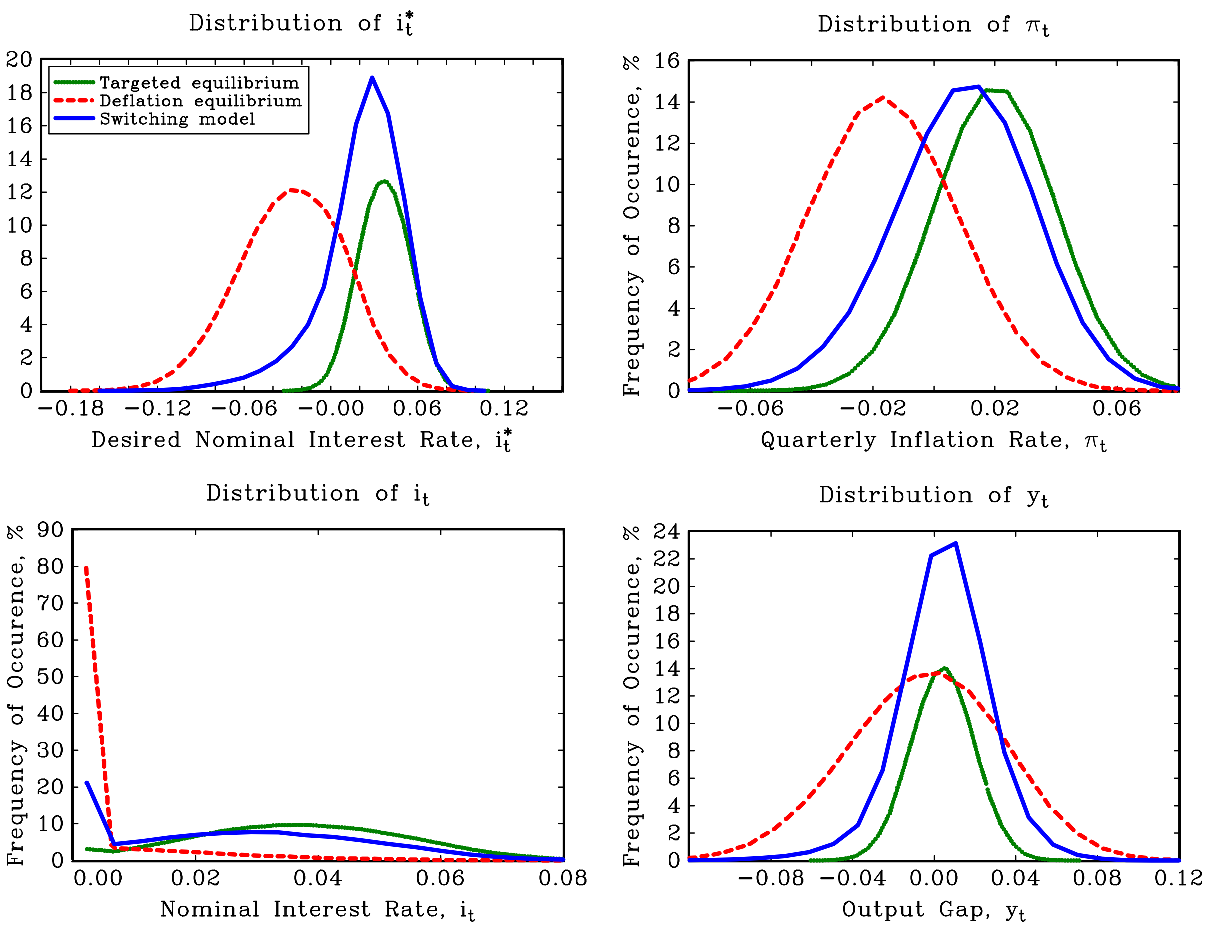
The U.S. real federal funds rate has remained mostly below the Laubach-Williams estimate of r_t^* since early 2009. The nominal federal funds rate was approximately zero from 2008.Q4 through 2015.Q4. A Taylor-type rule of the form (9) using the parameter values in Table 2, with $E_t\, r_t^*$ given by Laubach-Williams estimate, $\overline{\pi}_t$ given by the 4-quarter PCE inflation rate, and y_t given by the CBO output gap predicts that the desired nominal funds rate was negative starting in 2009.Q1 and remains negative through the end of the data sample in 2016.Q4. The 4-quarter PCE inflation rate was briefly negative in 2009 and has remained below the Fed's 2% inflation target since 2012.Q2. Expected inflation, as measured by the 5-year breakeven inflation rate derived from inflation-indexed Treasury securities, dropped sharply during the Great Recession and remains below its pre-recession level. The Great Recession was very severe, pushing the CBO output gap down to −6.3% at the business cycle trough in 2009.Q2. The output gap remains negative at −0.9% in 2016.Q4, more than seven years after the Great Recession ended. The various endpoints plotted in the figure are computed using the expressions in Table 1, with r_t^* given by the Laubach-Williams estimate. As r_t^* approaches zero or becomes negative, the "deflation" equilibrium is characterized by zero or low inflation, allowing this equilibrium to provide a better fit of recent U.S. inflation data.

Figure 5: Model Simulation: Endogenous Regime Switching



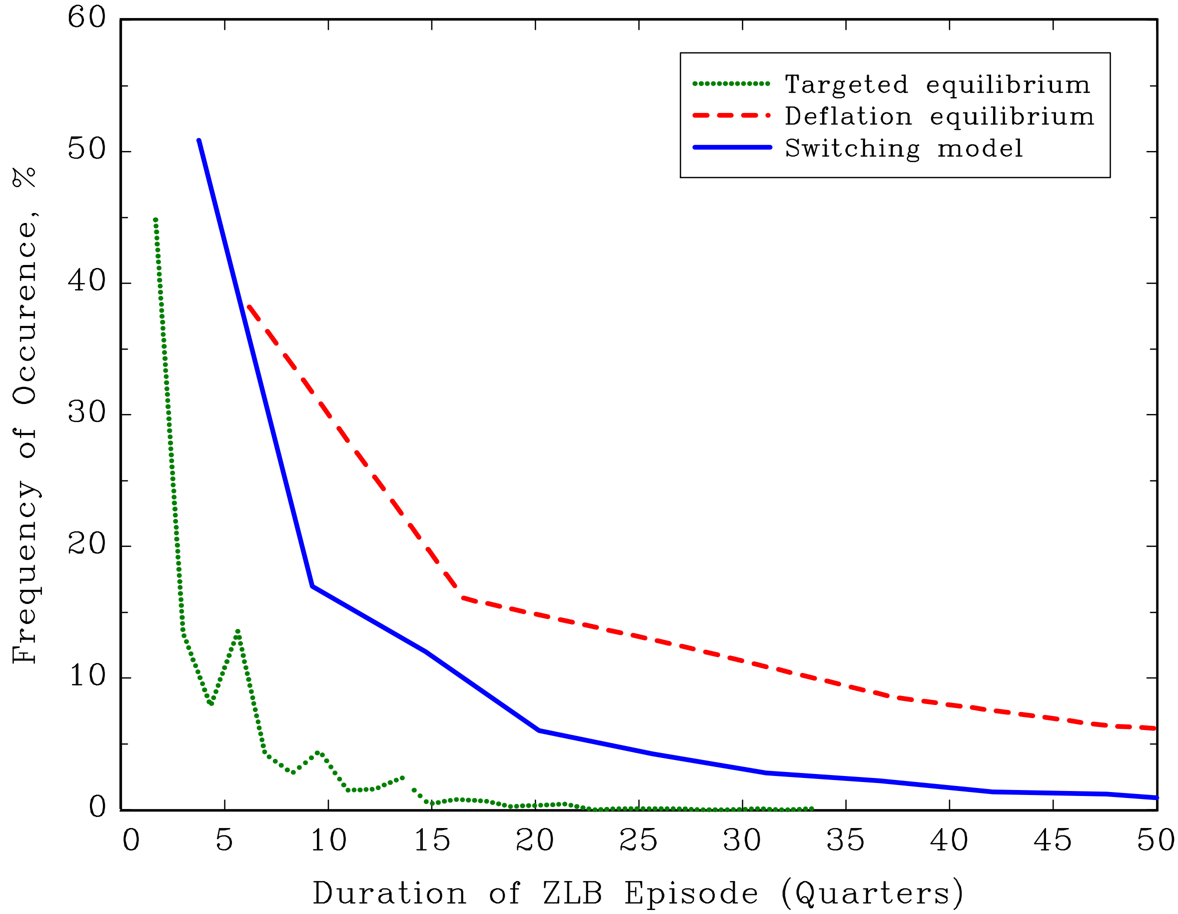
When the exogenous real interest gap $r_t - E_t r_t^*$ is negative for a sustained period (top panel), the resulting downward pressure on π_t and y_t serves to reduce the recent $RMSFE$ of the deflation forecast rules and increase the recent $RMSFE$ of the targeted forecast rules (middle panel). Around period 1725, the shift in relative forecast performance induces the agent to place a substantially lower weight on the targeted equilibrium forecast rules, causing the deflation equilibrium to become temporarily self-fulfilling (bottom panel). The shift in relative forecast performance can induce the representative agent to place a substantially higher weight on the deflation forecast rules, causing the deflation equilibrium to occasionally become self-fulfilling. Around period 1800, the real rate gap once again becomes positive, causing the $RMSFE$ of the deflation forecast rules to once again exceed the $RMSFE$ of the targeted forecast rules. The agent now increases the weight on the targeted forecast rules, causing the targeted equilibrium to be restored. Qualitatively similar results are obtained if the agent employs Bayes law to compute the likelihood that a string of recent quarterly inflation observations comes from one equilibrium or the other.

Figure 6: Model Simulations: Distributions of Endogenous Variables



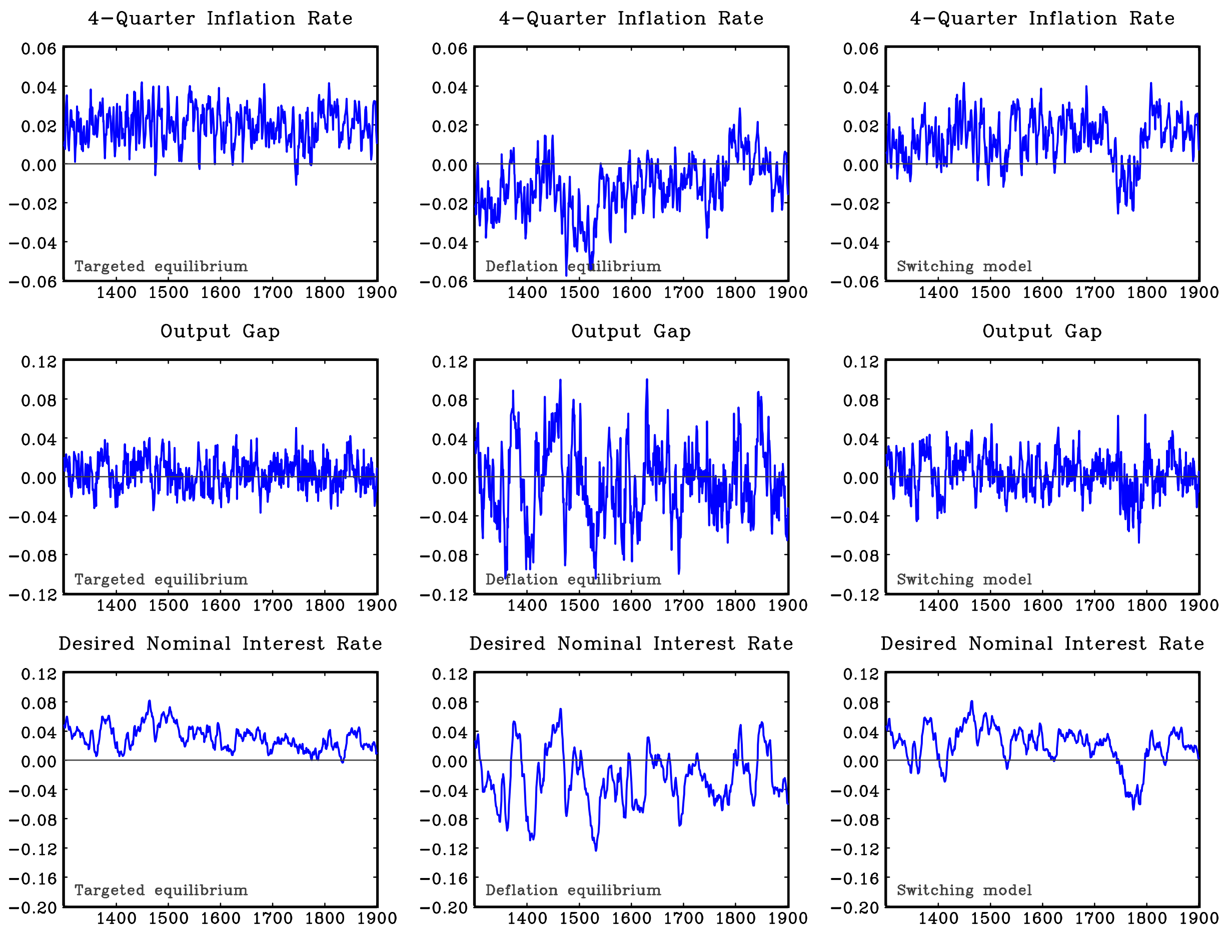
Model variables in the deflation equilibrium have distributions with lower means but higher variances than those in the targeted equilibrium. But the significant overlap in the various distributions creates a dilemma for an agent who seeks to determine the likelihood that a string of recent quarterly observations are drawn from one equilibrium or the other. Variables in the switching model have means that are somewhat lower and variances that are somewhat higher than those in the targeted equilibrium. Consequently, the central bank in the switching model undershoots its inflation target and the volatilities of the output gap and inflation are both higher relative to the targeted equilibrium.

Figure 7: Model Simulations: Distribution of ZLB Durations



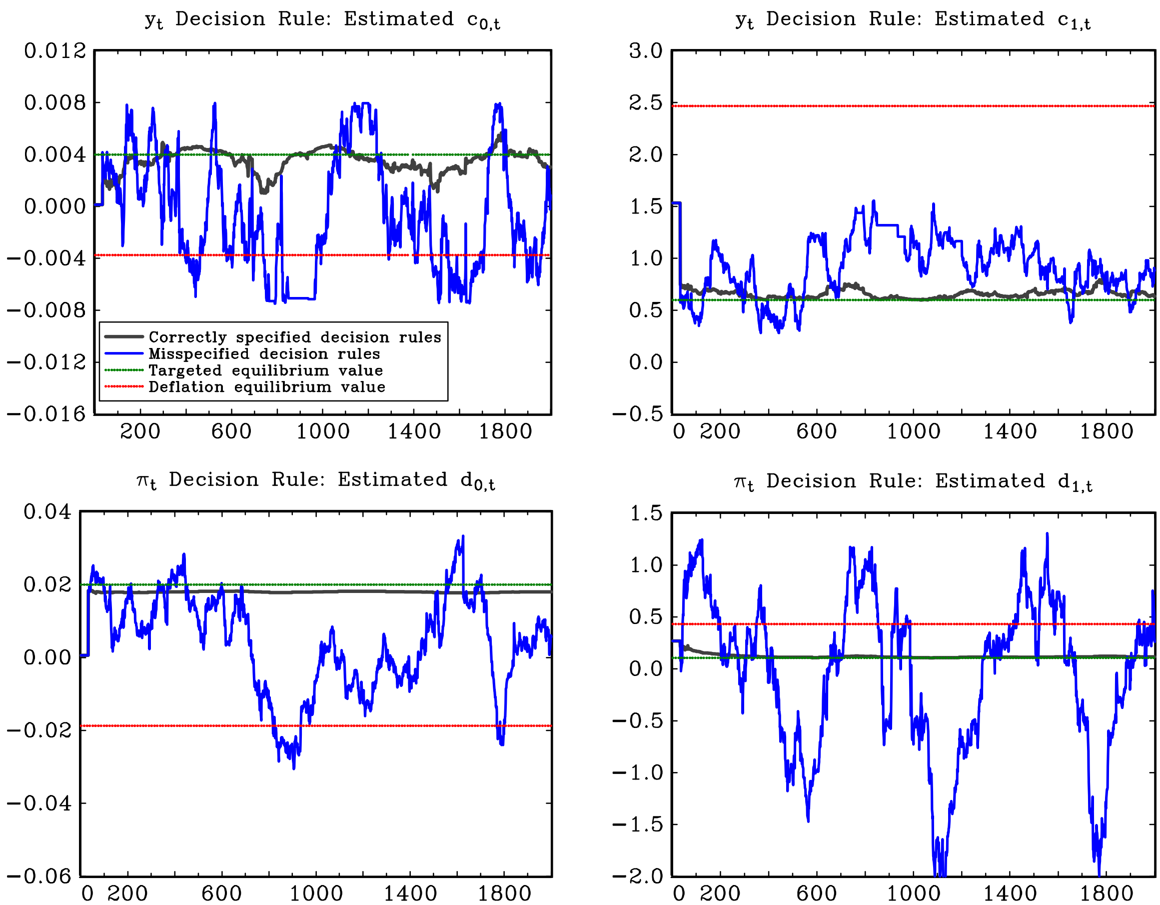
Unlike the targeted equilibrium, the switching model can produce infrequent, but long-lived ZLB episodes in response to small, normally distributed shocks. To account for infrequent but long-lived ZLB episodes, the targeted equilibrium would require large shocks that are themselves infrequent but long-lived, as in Dordal-i-Carreras, et al. (2016).

Figure 8: Model Simulations: Comparing Three Model Versions



All three model versions employ the same sequence of stochastic shocks. Around period 1725 in the switching model, the weight on the targeted forecast rules starts dropping toward zero, causing the deflation equilibrium to become temporarily self-fulfilling. The episode results in brief deflation followed by below-target inflation, a highly negative output gap, and a negative desired nominal interest rate, reminiscent of the U.S. Great Recession and its aftermath.

Figure 9: Adaptive Learning in a Simplified Model



Starting from the original model, the simplified model imposes $\rho = 0$, $\omega = 1$, and $\sigma_\eta = 0$. These settings eliminate i^*_{t-1} and $\overline{\pi}_{t-1}$ as state variables and cause the natural rate of interest r^* to be constant. When the agent estimates correctly specified decision rules (black lines), the coefficients quickly converge to the vicinity of the targeted equilibrium values and remain there. In contrast, when the agent estimates misspecified decision rules that fail to control for the white noise shocks v_t and u_t (blue lines), the coefficients exhibit low frequency oscillations that can occasionally approach, reach, or go beyond the deflation equilibrium values. The low frequency oscillations in the decision rule coefficients induce movements in macroeconomic variables that are qualitatively similar to those observed in Figure 8 for the original switching model with full-knowledge.