

# Exponential-Affine Approximations of Macro-Finance Models

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## Abstract

We generalize affine approximations commonly used in the macro-finance literature to solve a general class of discrete-time dynamic models with endogenous, state-dependent and non-Gaussian risk. We provide a mathematical foundation to heuristic affine methods by showing that they can be seen as linear perturbations around the risky steady state. We apply this technique to an endowment economy with Campbell-Cochrane habits as well as to a more complex version that features a production economy and nominal rigidities, and we provide simple analytical expressions for equilibrium term structures of asset prices and returns. In these examples the proposed generalized affine approximation performs similarly to global solution methods and outperforms alternative affine approximation methods. The approximation maintains a high quality at short as well as long horizons, and it preserves the main properties of the stochastic discount factor, including its martingale component and the maximal risk-return tradeoff.

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## 1. Introduction

Dynamic equilibrium models with state-dependent risk prices and risk exposures are necessary to capture the salient time-series properties of financial prices and to model the real effects of macroeconomic risk. Additionally, innovations of state variables can be endogenous objects in DSGE models, so their equilibrium distribution is known only after the model is solved, and can include non-Gaussian shocks. Models with these characteristics present a challenge for traditional solution techniques that resort either to accurate but computationally intensive methods or to low-order perturbation methods that can importantly misrepresent the implications of the model for

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quantities and asset prices, including the amount of time-variation in risk premia, and the effect of uncertainty on the macroeconomy.

In this paper we propose a linear approximation technique to solve and simulate DSGE models with a dynamic risk adjustment in equilibrium prices and quantities. Linear risk adjustments based on lognormality have been popular in the macro-finance literature at least since Campbell (1993) and Jermann (1998), with Malkhozov (2014) representing the most recent formalization, as they facilitate an intuitive and relatively accurate understanding of the asset pricing implications of the model.<sup>2,3</sup> We extend extant affine approximations in several ways. First, we generalize risk adjustments to non-Gaussian distributed shocks by relying on entropy as the measure of dispersion rather than variance. Second, we accommodate *dynamic* risk corrections even when risk is endogenous in the sense that innovations in the state vector contain an endogenous component whose distribution will be known only after having solved the model, as in production economies with time-varying risk aversion. Third, we preserve some nonlinearities in the characterization of innovations in the state vector to avoid spurious dynamics that can distort the implications of the approximate model, especially in terms of asset pricing.

Most importantly, we show that affine approximations have an exact connection to perturbations, and can therefore be justified on formal grounds. In particular, affine approximations coincide with a linear perturbation around the *risky* steady state.<sup>4</sup> It follows that we can provide a constructive proof of the existence of a locally unique and stable affine approximate solution by drawing a link between affine approximations and regular perturbation approximations.<sup>5</sup>

Affine approximations are particularly appealing for the purpose of asset pricing by their ability to provide analytical solutions that price easily complex dividend processes (such as those arising from a production economy) and that facilitate an intuitive understanding of the macroeconomic forces that drive asset prices. We provide explicit pricing formulas in the exponential-affine class for generalized equilibrium term structures—and hence for all related claims, including variance risk premia—as well as for some major diagnostic decompositions of the asset pricing properties of a model, including Hansen and Jagannathan (1991) bounds, Alvarez and Jermann (2005) and Hansen and Scheinkman (2009) decompositions, and Borovicka and Hansen (2014) risk-exposure and risk-price elasticities.

To test the accuracy of our method in a relevant context we consider the class of models with Campbell and Cochrane (1999) nonlinear habit formation, in which risk is endogenous and its quantity and price are nonlinear functions of the state vector. Risk price dynamics are sufficiently

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<sup>2</sup>Moreover, the affine form has a big advantage in estimation over nonlinear solution methods as it can exploit fast linear filtering methods.

<sup>3</sup>Other examples of loglinear-lognormal methods to asset pricing in production economies are Lettau and Uhlig (2000); Lettau (2003); Uhlig (2007); De Graeve, Emiris, and Wouters (2009); Bekaert, Cho, and Moreno (2010); Dew-Becker (2014); Backus, Ferriere, and Zin (2015).

<sup>4</sup>Malkhozov (2014) suggests that affine approximations compare to second-order perturbations in which dynamic second-order terms are disregarded, yet we show that the connection to perturbations can be made exact—a linearization around a risky steady state captures not only constant second-order terms but generically all higher-order terms that are constant or linear in the state vector.

<sup>5</sup>See Juillard (2011); Coeurdacier, Rey, and Winant (2011); Meyer-Gohde (2016) for recent treatments of local perturbations around the risky steady state. Our result shows how the interest in these types of perturbations can be traced back at least to the 1990s in the macro-finance literature.

nonlinear and different from the dynamics of risk exposures to prevent the existence of a closed-form equilibrium value for dividend strips, and projection methods are required to find the global solution (Campbell and Cochrane, 1999; Wachter, 2006). To compare the approximate solution with the numerical solution we report multiperiod Euler equation errors and the term structures of equity and bond yields, which capture the quality of the approximation at different time horizons. We emphasize the importance of correctly capturing the term structures of zero-coupon equities and bonds, as they are the basis for pricing other more complex claims (Lettau and Wachter, 2007; Binsbergen, Brandt, and Kojen, 2012a; Lopez, 2014) as well as they directly relate to diagnostic measures that are crucial in evaluating the pricing properties of a model of the stochastic discount factor.

We first test the performance of our approximation procedure on the endowment economies of Campbell and Cochrane (1999) and Wachter (2006). Our generalized affine approximation is accurate in solving for risk premia and volatilities of equities and bonds at the observable durations and at long durations, while it slightly distorts the correct values for medium-duration claims. The approximation method does a good overall job in capturing the level, amplitude and curvature of the term structures.

We then turn to the model of Lopez, Lopez-Salido, and Vazquez-Grande (2015) that features Campbell-Cochrane habits and a production economy in a New Keynesian DSGE model. This model is appropriate for testing the accuracy of our solution in an environment where consumption and risk are endogenously determined. In this application the full nonlinear solution is computationally expensive, while our generalized affine approximation yields a fast and tractable solution with good levels of accuracy. Our approximation outperforms substantially the alternative approximation schemes, and it similarly outperforms low-order perturbation methods.<sup>6</sup>

## 2. Generalized affine approximation

This section describes an algorithm to derive an affine approximate solution in the spirit of Jermann (1998), and that generalizes Malkhozov (2014) by capturing dynamic adjustments also for endogenous and non-Gaussian risks. Most important, we make clear the mathematical foundations of our affine approximation by showing that it can be interpreted as a first-order perturbation approximation around a suitably defined risky steady state. We then characterize the existence and uniqueness of a saddle-path affine solution by relying on the theory of regular perturbation methods of approximation (e.g., Judd, 1998; Schmitt-Grohé and Uribe, 2004).<sup>7</sup>

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<sup>6</sup>Note that to have time-varying risk premia we would need an approximation based on standard perturbations of at least third degree (Binsbergen, Fernández-Villaverde, Kojen, and Rubio-Ramírez, 2012b; Rudebusch and Swanson, 2012; see also Benigno, Benigno, and Nisticò, 2013), while our generalized affine approximation is able to accurately capture time-variation in risk premia.

<sup>7</sup>Note that the local saddle-point stability of the deterministic steady state does not in general imply locally unique and stable dynamics of the system around the risky steady state. An affine risk adjustment may therefore induce spurious dynamics relative to a standard perturbation around the deterministic steady state, thereby complicating the application of the implicit function theorem that is necessary to justify our approximation of system (1). In the online appendix, we provide a version of our algorithm that is isomorphic to a pruned third-order perturbation (e.g., Lombardo and Uhlig, 2014) and that preserves the saddle-point stability properties of the deterministic steady state.

An important difference between our method and perturbations remains, as to characterize innovations to the state vector  $(E_{t+1} - E_t)z_{t+1}$  we do not approximate maps  $\lambda$  and  $\sigma$ . We find this strategy compelling when simulating the solution of DSGE models because the maps  $\lambda$  and  $\sigma$  in leading examples have Taylor series with a small radius of convergence. Approximations of those maps can result in spurious dynamics, with an inaccurate representation of tail regions of the state space that matter most for pricing, as low-probability regions under the physical probability ( $\mathbb{P}$ ) can acquire a much larger probability mass under the risk-neutral measure ( $\mathbb{Q}$ ).

We are interested in approximating with an affine map the solution for jump variables  $y_t \in \mathbb{R}^{n_y}$  and states  $z_t \in \mathbb{R}^{n_z}$  of the dynamic system of equilibrium conditions with generic form:<sup>8</sup>

$$\begin{aligned} 0 &= \ln E_t \exp[f(y_t, z_t) + f_3 y_{t+1} + f_4 z_{t+1}] \\ z_{t+1} &= g(y_t, z_t) + e_{t+1}^y + e_{t+1}^z \end{aligned} \quad (1)$$

where  $e_{t+1}^y \doteq \lambda(z_t)(E_{t+1} - E_t)y_{t+1}$  and  $e_{t+1}^z \doteq \sigma(z_t)\varepsilon_{t+1}$  describe heteroskedastic endogenous risk that depends on innovations in jump variables in addition to exogenous risk. Operator  $\ln E_t \exp[\cdot]$  is applied elementwise to a vector-valued map, with  $E_t$  the expectations operator conditioned on time- $t$  information. Functions  $f : \mathbb{R}^{n_y+n_z} \rightarrow \mathbb{R}^{n_y}$ ,  $g : \mathbb{R}^{n_y+n_z} \rightarrow \mathbb{R}^{n_z}$ ,  $\lambda : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_z \times n_y}$  and  $\sigma : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_z \times n_\varepsilon}$  are differentiable. We denote  $f_i$  the derivative of  $f$  with respect to its  $i$ th argument,  $f_{ij}$  the derivative with respect to its  $i$ th and  $j$ th arguments, and so on. Exogenous shocks  $\varepsilon_t \in \mathbb{R}^{n_\varepsilon}$  form a martingale difference sequence with conditional cumulant generating function (ccgf),  $\kappa[\alpha(z_t); z_t] \doteq \ln E_t \exp[\alpha(z_t)^\top \varepsilon_{t+1}]$ , with map  $\alpha : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_\varepsilon}$ . Jumps and states are expressed in deviations from the respective deterministic steady-state values.

### 2.1. Heuristic derivation

To obtain our affine approximation we rewrite without loss of generality dynamic system (1) as:

$$\begin{aligned} 0 &= f(y_t, z_t) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + \mathcal{V}_t[\exp(f_3 y_{t+1} + f_4 z_{t+1})] \\ z_{t+1} &= g(y_t, z_t) + e_{t+1}^y + e_{t+1}^z \end{aligned}$$

where  $\mathcal{V}_t[\exp(x_{t+1})] \doteq \ln E_t \exp(x_{t+1}) - E_t x_{t+1}$  denotes a conditional relative entropy measure. Entropy represents a risk correction and is a function of the state vector  $\tilde{\mathcal{V}} : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_y}$  such that<sup>9</sup>

$$\tilde{\mathcal{V}}(z_t) \doteq \mathcal{V}_t[\exp(f_3 y_{t+1} + f_4 z_{t+1})]$$

<sup>8</sup>Most DSGE models can be cast into this form after suitable redefinition of variables. The main loss of generality in the representation comes from the requirement that the expectation be strictly positive.

<sup>9</sup>For example, under Gaussian shocks  $\varepsilon_t$  and a constant function  $\lambda$ , the risk correction equals the one in Malkhozov (2014). When  $\lambda(z_t)$  is a time-varying quantity and exogenous shocks are non-Gaussian, however,  $e_t^y$  and  $\varepsilon_t$  cannot be subsumed as shocks into Malkhozov's framework. Accordingly, we extend his method to any shock distribution with finite ccgf and to heteroskedastic endogenous risk  $e_t^y$ .

We are looking for an affine solution  $y_t = \bar{y} + \tilde{\Psi}(z_t - \bar{z})$ , so we linearize the system around the point  $[z_t; y_t; e_{t+1}^y; e_{t+1}^z] = [\bar{z}; \bar{y}; 0; 0]$ ,  $t \geq 0$ , as follows:

$$\begin{aligned} 0 &= f(\bar{y}, \bar{z}) + \tilde{f}_1(y_t - \bar{y}) + \tilde{f}_2(z_t - \bar{z}) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + \tilde{\mathcal{V}}(\bar{z}) + \tilde{\mathcal{V}}_1(\bar{z}) z_t \\ z_{t+1} &= g(\bar{y}, \bar{z}) + \tilde{g}_1(y_t - \bar{y}) + \tilde{g}_2(z_t - \bar{z}) + \lambda(z_t)(E_{t+1} - E_t)y_{t+1} + \sigma(z_t)\varepsilon_{t+1} \end{aligned}$$

where  $\tilde{\mathcal{V}}(z_t) = \mathcal{V}_t[\exp((f_3\tilde{\Psi} + f_4)z_{t+1})]$ , with  $\tilde{f}_i \doteq f_i(\bar{y}, \bar{z})$  and  $\tilde{g}_i \doteq g_i(\bar{y}, \bar{z})$ .<sup>10</sup>

We verify the conjectured affine solution and compute its coefficients as to solution to the following system of equations:

$$\begin{aligned} y_t &= \bar{y} + \tilde{\Psi}(z_t - \bar{z}) \\ z_{t+1} &= \bar{z} + (\tilde{g}_1\tilde{\Psi} + \tilde{g}_2)(z_t - \bar{z}) + \tilde{\sigma}_z(z_t)\varepsilon_{t+1}, & \bar{z} &= g(\bar{y}, \bar{z}), \quad \tilde{\sigma}_z(z_t) \doteq [I_{n_z} - \lambda(z_t)\tilde{\Psi}]^{-1}\sigma(z_t) \\ 0 &= f(\bar{y}, \bar{z}) + f_3\bar{y} + f_4\bar{z} + \tilde{\mathcal{V}}(\bar{z}), & \tilde{\mathcal{V}}(z_t) &= \kappa[(f_3\tilde{\Psi} + f_4)\tilde{\sigma}_z(z_t); z_t] \\ 0 &= \tilde{f}_1\tilde{\Psi} + \tilde{f}_2 + (f_3\tilde{\Psi} + f_4)(\tilde{g}_1\tilde{\Psi} + \tilde{g}_2) + \tilde{\mathcal{V}}_1(\bar{z}) \end{aligned} \quad (2)$$

where the link with the ccgf follows from the properties of entropy. For example, under Gaussianity,  $\varepsilon_t \sim Niid(0, I_{n_\varepsilon})$ , one has  $\kappa[(f_3\tilde{\Psi} + f_4)\tilde{\sigma}_z(z_t); z_t] = \frac{1}{2}diag[(f_3\tilde{\Psi} + f_4)\tilde{\sigma}_z(z_t)\tilde{\sigma}_z(z_t)^\top(f_3\tilde{\Psi} + f_4)^\top]$ .

Heuristically, affine approximations can be summarized as follows: we start by a linearization of system (1); we rewrite expectational equations in terms of a certainty-equivalent and an entropy terms; we use the outcome of the linearization to characterize entropy; and we linearize this term to derive an affine risk adjustment that consists of a constant and a time-varying component.

## 2.2. Formal derivation

Expression (2) includes nonlinear matrix equations in the unknown coefficients that are amenable to Newton-type numerical solution methods. However, the matrix equations are sufficiently nonlinear to allow for multiple solutions and to complicate the characterization of the local uniqueness and stability of the affine solution. In this context, we provide a constructive proof of the existence of a saddle-path solution to system (2) by drawing a link between affine approximations and perturbations around the *risky steady state*, defined by Juillard (2011) and Coeurdacier, Rey, and Winant (2011) as “the point where agents choose to stay at a given date if they expect future risk and if the realization of shocks is 0 at this date” (Coeurdacier et al., 2011).

Proposition 1 provides the mathematical foundation for the affine approximations common in the finance literature. Part a. shows how affine approximations coincide with linear perturbations around the risky steady state, thereby inheriting the local stability and uniqueness properties of the perturbed approximation. Part b. provides an algorithm to check the saddle-path stability for any solution to the nonlinear equations (2).

<sup>10</sup>By considering entropy rather than the variance of the state vector we reach the exact solution in linear models whenever the exogenous shock has an affine ccgf (in addition to Gaussianity, an example is the Poisson-normal mixture popular in the disaster-risk literature—e.g., Backus, Chernov, and Martin, 2011).

To set the ground for perturbation methods, we consider the parametrized family of system (1):

$$\begin{aligned} 0 &= E_t x_{t+1}^q + v_t^q, \quad v_t^q \doteq \mathcal{V}_t[\exp(x_{t+1}^q)] \\ z_{t+1}^q &= g[y(z_t, q), z_t] + \lambda(z_t)(E_{t+1} - E_t)y[(z_t, q, q\varepsilon_{t+1}), q] + q\sigma(z_t)\varepsilon_{t+1} \\ x_{t+1}^q &\doteq f[y(z_t, q), z_t] + f_3 y[z(z_t, q, q\varepsilon_{t+1}), q] + f_4 z(z_t, q, q\varepsilon_{t+1}) \end{aligned} \quad (3)$$

where the perturbation scalar  $q \in [0, 1]$  indexes the system's sensitivity to shocks, so that under  $q = 1$  the dynamics coincide with the original model (1). Let  $y(z_t, q)$  be the unknown solution for equilibrium jump variables.

**Proposition 1. (a)** *Suppose that  $(y_t, z_t, q) = (\bar{y}, \bar{z}, 1)$  is a saddle point. Then, the affine approximate solution  $y_t = \bar{y} + \bar{\Psi}(z_t - \bar{z})$  and  $E_t z_{t+1} = \bar{z} + (\bar{g}_1 \bar{\Psi} + \bar{g}_2)(z_t - \bar{z})$  is locally unique and stable and coincides with the linear perturbation around the risky steady state  $y_t = y(\bar{z}, 1) + y_1(\bar{z}, 1)(z_t - \bar{z})$  and  $E_t z_{t+1} = g[y(\bar{z}, 1), \bar{z}] + \{g_1[y(\bar{z}, 1), \bar{z}]y_1(\bar{z}, 1) + g_2[y(\bar{z}, 1), \bar{z}]\}(z_t - \bar{z})$ .*

**(b)** *The point  $(y_t, z_t, q) = (\bar{y}, \bar{z}, 1)$  is a saddle point if and only if the generalized eigenvalues  $\alpha(\Gamma, \Xi) \doteq \{\alpha \in \mathbb{C} : \det(\Gamma\alpha - \Xi) = 0\} = \{\alpha_i, i = 1, \dots, n_y + n_z\}$  of the square matrices:*

$$\Gamma \doteq \begin{bmatrix} f_4 & f_3 \\ I_{n_z} & 0 \end{bmatrix} \in \mathbb{C}^{n_y+n_z \times n_y+n_z}, \quad \Xi \doteq \begin{bmatrix} -f_2(\bar{y}, \bar{z}) - \bar{\mathcal{V}}_1(\bar{z}) & -f_1(\bar{y}, \bar{z}) \\ g_2(\bar{y}, \bar{z}) & g_1(\bar{y}, \bar{z}) \end{bmatrix} \in \mathbb{C}^{n_y+n_z \times n_y+n_z}$$

are such that there are  $n_z$  generalized eigenvalues with modulus within the unit circle and  $n_y$  with modulus larger than unity (Blanchard and Kahn, 1980).

Appendix A provides a proof of the proposition.

Proposition 2 offers a way to construct one particular solution to the nonlinear equations (2) using output from regular perturbations around the deterministic steady state  $(y_t, z_t, q) = 0$  of order  $N \rightarrow \infty$ . This solution is instructive in that it shows how our generalized affine approximation picks up *all* higher-order terms in perturbations around the deterministic steady state that are either constant or linear in the state vector. The proposition offers an exact result whenever  $\bar{z} = 0$ . In practice, and even when  $\bar{z} \neq 0$ , we exploit proposition 2 by choosing a tractable, finite  $N$  to find an initial guess for coefficients  $\bar{y}$  and  $\bar{\Psi}$  to be used as a starting point in a Newton-type numerical solution method for expression (2). Even in the presence of multiple saddle-point risky steady states  $(\bar{y}, \bar{z}, 1)$ , a  $N$  as high as two or three narrows the solution down to a unique one.

**Proposition 2.** *Suppose that  $\bar{z} = 0$  is a fixed point of equation  $\bar{z} = g(\bar{y}, \bar{z})$ . Suppose that functions  $y(0, q)$  and  $y_1(0, q)$  have convergent Taylor series at  $q = 0$  with radius of convergence larger than unity. Then, we have:*

$$\bar{y} = \lim_{q \uparrow 1} \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{1}{i!} \left. \frac{\partial^i y(0, q)}{\partial q^i} \right|_{q=0} q^i, \quad \bar{\Psi} = \lim_{q \uparrow 1} \lim_{N \rightarrow \infty} \sum_{i=0}^N \frac{1}{i!} \left. \frac{\partial^i y_1(0, q)}{\partial q^i} \right|_{q=0} q^i$$

using output from perturbations around the deterministic steady state  $(z_t, q) = (0, 0)$ .

Appendix B provides a proof of the proposition.

### 3. Approximate equilibrium asset pricing

After having solved for the equilibrium allocation and prices, we can price assets in zero net supply by relying on no-arbitrage relations  $0 = \ln E_t[\exp(m_{t+1} + r_{t+1}^j)]$ , where  $m$  is the stochastic discount factor and  $r^j$  the return paid off by the  $j$ th claim in zero net supply. While these no-arbitrage relations can be included in system (1), we illustrate these pricing implications separately to make clear how affine approximations help inspecting equilibrium asset prices. In doing so, we will apply the simple algorithm described in section 2: split expectational equations into a certainty equivalent  $E_t(m_{t+1} + r_{t+1}^j)$  and an entropy term  $\mathcal{V}_t[\exp(m_{t+1} + r_{t+1}^j)]$ , conjecture an affine approximate solution to characterize entropy, linearize it, and identify the unknown coefficients.

For concreteness, suppose that the approximate equilibrium allocation using the method of section 2 results in an approximate stable Gaussian Markov process  $\widehat{z}_t \doteq z_t - \widetilde{z} \in \mathbb{R}^{n_z}$  and a log cashflow process  $d_t \in \mathbb{R}$  that is an element of the vector of jump variables  $y_t \in \mathbb{R}^{n_y}$ , whose approximate joint distribution under the physical probabilities ( $\mathbb{P}$ ) is: with distribution

$$\begin{bmatrix} \widehat{z}_{t+1} \\ \Delta d_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_d \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix} \widehat{z}_t + \begin{bmatrix} B(\widehat{z}_t) \\ D(\widehat{z}_t) \end{bmatrix} \varepsilon_{t+1}, \quad \varepsilon_t \stackrel{\mathbb{P}}{\sim} \text{Niid}(0, I_{n_\varepsilon}) \quad (4)$$

with coefficients  $\mu_d \in \mathbb{R}$ ,  $A \in \mathbb{R}^{n_z \times n_z}$  and  $C' \in \mathbb{R}^{n_z}$ , and where the random matrices  $B(\widehat{z}_t) \doteq \widetilde{\sigma}_z(z_t) \in \mathbb{R}^{n_z \times n_\varepsilon}$  and  $D(\widehat{z}_t)' \in \mathbb{R}^{n_\varepsilon}$  allow for heteroskedasticity in the process. The assumed structure implies the joint ccgf:

$$\ln E_t^{\mathbb{P}} [e^{u_z' \widehat{z}_{t+1} + u_d \Delta d_{t+1}}] = u' \mu + \frac{1}{2} u' \Sigma(\widehat{z}_t) \Sigma(\widehat{z}_t)' u + u' \Phi \widehat{z}_t,$$

for  $u = [u_z; u_d] \in \mathbb{R}^{n_z+1}$ , with parameters  $\mu \doteq [0_{n_z}; \mu_d] \in \mathbb{R}^{n_z+1}$ ,  $\Phi \doteq [A; C] \in \mathbb{R}^{n_z+1 \times n_z}$  and the random matrix  $\Sigma(\widehat{z}_t) \doteq [B(\widehat{z}_t); D(\widehat{z}_t)] \in \mathbb{R}^{n_z+1 \times n_\varepsilon}$ .

The equilibrium risk-free rate is described by the Euler equation,  $0 = \ln E_t^{\mathbb{P}}[\exp(m_{t+1} + r_t)]$ , and hence we can write the one-period, conditionally linear, log stochastic discount factor as

$$m_{t+1} = -r(\widehat{z}_t) - \frac{1}{2} \gamma(\widehat{z}_t)' \gamma(\widehat{z}_t) - \gamma(\widehat{z}_t)' \varepsilon_{t+1} \quad (5)$$

where  $\gamma(\widehat{z}_t) \in \mathbb{R}^{n_\varepsilon}$  is the possibly time-varying price of risk and  $r(\widehat{z}_t) \in \mathbb{R}$  is the risk-free log return. Accordingly, we define the multiplicative martingale,

$$Q_{t+1} = Q_t e^{-\frac{1}{2} \gamma(\widehat{z}_t)' \gamma(\widehat{z}_t) - \gamma(\widehat{z}_t)' \varepsilon_{t+1}},$$

to construct the change of measure from physical to risk-neutral probabilities,  $d\mathbb{Q}/d\mathbb{P}$ .

Let the risk premium:

$$\pi(\widehat{z}_t) \doteq \Sigma(\widehat{z}_t) \gamma(\widehat{z}_t)$$

be the product of the quantity of risk in the joint process  $[z; \Delta d]$  and the price of risk.

Under this structure, we are able to characterize in proposition 3 the dependence of generalized equilibrium term-structure components on the states. By proposition 1, proposition 3 describes equilibrium asset prices linearized around the risky steady state.

**Proposition 3** (Pricing strips). *Under assumptions (4) and (5), we obtain the following linear approximations of yields and risk premia:*

- (a) *The  $n$ th cashflow strip yield,  $y_{d,t}^{(n)} \doteq -\frac{1}{n} \ln(P_{d,t}^{(n)}/D_t)$ , with  $P_{d,t}^{(n)} = E_t^{\mathbb{P}}(M_{t,t+n}D_{t+n})$  the no-arbitrage price of the  $n$ th strip of cashflow process  $d$ , has the approximate affine form*

$$y_{d,t}^{(n)} = -\frac{1}{n}A^{(n)} - \frac{1}{n}B_z^{(n)}\widehat{z}_t$$

*with term structure coefficients determined by the quadratic matrix difference equations*

$$\begin{aligned} A^{(n)} &= A^{(n-1)} - r(0) + \begin{bmatrix} B_z^{(n-1)} \\ 1 \end{bmatrix}' [\mu - \pi(0)] + \frac{1}{2} \begin{bmatrix} B_z^{(n-1)} \\ 1 \end{bmatrix}' \Sigma(0)\Sigma(0)' \begin{bmatrix} B_z^{(n-1)} \\ 1 \end{bmatrix} \\ B_z^{(n)}e_i &= -r_1(0)e_i + \begin{bmatrix} B_z^{(n-1)} \\ 1 \end{bmatrix}' \left( [\Phi - \pi_1(0)]e_i + \Sigma(0)\Sigma_1(0)'(I_{n_s+1} \otimes e_i) \begin{bmatrix} B_s^{(n-1)} \\ 1 \end{bmatrix} \right) \end{aligned} \quad (6)$$

*where  $e_i \in \mathbb{R}^{n_z}$  is the unit vector in the  $i$ th direction,  $i = 1, \dots, n_z$ , and with boundary condition  $[A^{(0)}; B_z^{(0)}] = 0$ .*

- (b) *The holding-period risk premium commanded by the  $n$ -period ahead cashflow strip is*

$$\ln E_t^{\mathbb{P}} R_{d,t+1}^{e,(n)} \doteq \ln E_t^{\mathbb{P}} \left( \frac{P_{0,t}^{(1)} P_{d,t+1}^{(n-1)}}{P_{d,t}^{(n)}} \right) = V_{n-1,t} \gamma_t$$

*where coefficient  $V_{n,t} \doteq D(\widehat{z}_t) + B_z^{(n)} B(\widehat{z}_t)$  controls the loading of the unexpected component of the  $n$ th holding period log return on the shock, which in turn implies:*

$$r_{d,t+1}^{e,(n)} \doteq \ln R_{d,t+1}^{e,(n)} = V_{n-1,t} \gamma(\widehat{z}_t) - \frac{1}{2} V_{n-1,t} V'_{n-1,t} + V_{n-1,t} \varepsilon_{t+1}$$

- (c) *The per-period hold-to-maturity risk premium commanded by the  $n$ th cashflow strip is<sup>11</sup>*

$$\frac{1}{n} \ln E_t^{\mathbb{P}} \left( \frac{P_{0,t}^{(n)} D_{t+n}}{P_{d,t}^{(n)}} \right) = \frac{1}{n} [A_g^{(n)} + A_0^{(n)} - A_d^{(n)}] + \frac{1}{n} [B_{g,z}^{(n)} + B_{0,n}^{(n)} - B_{d,z}^{(n)}] \widehat{z}_t$$

*where subscripts 0 and  $d$  index the term structure coefficients associated with real bonds and the relevant cashflow process respectively, and where coefficients  $\{A_g^{(n)}, B_g^{(n)}\}$  determine the term*

<sup>11</sup>Objects of interest that take the form of hold-to-maturity expected excess returns include, for example, the term structure of the inflation risk premium (when the cashflow process is the inverse of inflation—see Ang et al., 2008) and the term structure of the welfare cost of uncertainty (when the cashflow process is consumption—see Lopez, 2014).



structure of anticipated cashflow growth

$$\begin{aligned}\frac{1}{n} \ln E_t^{\mathbb{P}} \left( \frac{D_{t+n}}{D_t} \right) &= \frac{1}{n} A_g^{(n)} + \frac{1}{n} B_g^{(n)} \widehat{z}_t \\ A_g^{(n)} &= \mu_d + A_g^{(n-1)} + \frac{1}{2} \begin{bmatrix} B_g^{(n-1)'} \\ 1 \end{bmatrix}' \Sigma(0) \Sigma(0)' \begin{bmatrix} B_g^{(n-1)} \\ 1 \end{bmatrix} \\ B_g^{(n)} e_i &= \begin{bmatrix} B_g^{(n-1)'} \\ 1 \end{bmatrix}' \Sigma(0) \Sigma_1(0)' (I_{n_s+1} \otimes e_i) \begin{bmatrix} B_g^{(n-1)} \\ 1 \end{bmatrix}\end{aligned}$$

where  $e_i \in \mathbb{R}^{n_z}$  is the unit vector in the  $i$ th direction,  $i = 1, \dots, n_z$ , with boundary condition  $[A_g^{(0)}; B_g^{(0)}] = 0$ .

Appendix C provides a proof of the proposition.

We can now use the approximate equilibrium prices of generalized strips to find expressions for the price of spanned payoffs. For example, given equilibrium strip prices,  $\{E_t M_{t,t+n} D_{t+n}\}$ , the return on the market portfolio is

$$E_t R_{t+1}^m = \sum_{n=1}^{\infty} \omega_{n,t} E_t R_{d,t+1}^{(n)}$$

where  $\omega_{n,t} \doteq E_t M_{t,t+n} D_{t+n} / \sum_{n=1}^{\infty} E_t M_{t,t+n} D_{t+n}$  with  $\sum_{n=1}^{\infty} \omega_{n,t} = 1$ , whose approximate distribution can be constructed using simulated moments of strip prices and returns, and that can be further approximated analytically to gain additional insight into the determinants of the equity premium.

**Corollary** (Pricing portfolio returns and variance). *The holding-period risk premium commanded by the market portfolio and its realized return have the approximate forms*

$$\begin{aligned}\ln E_t R_{t+1}^{e,m} &= \sum_{n=1}^{\infty} \omega_n V_{n-1,t} \gamma(\widehat{z}_t) \\ E_t r_{t+1}^{e,m} &= \sum_{n=1}^{\infty} \omega_n \left( V_{n-1,t} \gamma(\widehat{z}_t) - \frac{1}{2} \|V_{n-1,t}\|^2 \right)\end{aligned}$$

where  $\omega_n$  denotes the deterministic steady-state value of the  $n$ th portfolio weight.

Appendix D proves this corollary.

### 3.1. Approximate asset pricing diagnostics

We are interested in evaluating the accuracy of our solution in pricing financial claims. The literature provides a set of diagnostic tools that can be used as a first round of tests of a model of the stochastic discount factor. An important criterion to evaluate the accuracy of an approximation method is that it correctly captures these implications because they are crucial in evaluating a model of the discount factor.

A first diagnostic tool to assess a model of the discount factor is the Hansen and Jagannathan (1991) bound, which shows how no-arbitrage pricing implies that the volatility of the discount factor must dominate empirical measures of the maximal risk-return tradeoff. In this context our generalized affine approximation is appropriate to correctly represent the bound. In fact, the Gaussian property allows for an easy expression for the bound,

$$\left| \frac{\ln E_t R_{t+1}^e}{\sqrt{\text{var}_t(r_{t+1}^e)}} \right| \leq \sqrt{\text{var}_t(m_{t+1})} = \sqrt{\gamma(\bar{z}_t)' \gamma(\bar{z}_t)}$$

for all available excess returns, whose dynamics we characterize correctly because we rely on the exact nonlinear map from the  $\mathbb{P}$ -dynamics of the state vector into the  $\mathbb{P}$ -dynamics of the Hansen-Jagannathan bound.

Furthermore, our generalized affine approximation provides tractable expressions for diagnostic decompositions in the spirit of Alvarez and Jermann (2005) and Hansen and Scheinkman (2009). They show how, under appropriate regularity conditions, the stochastic discount factor can be decomposed as

$$M_{t+1} = M_{t+1}^P M_{t+1}^T$$

where a so-called transient component  $M_{t+1}^T$  controls the pricing of long-duration bonds and the martingale component  $M_{t+1}^P$  with  $E_t M_{t+1}^P = 1$  controls the maximum risk premium in the complete-market economy. In absence of a transient component the properties of the martingale component imply a flat term structure of real interest rates. In absence of a martingale component the long-run real bond premium is the highest premium available. The data reject both extreme cases (Alvarez and Jermann, 2005), so a first way to use the decomposition is to check that a model of the stochastic discount factor produces two nontrivial components.

Moreover, two main properties of the decomposition that rest on the no-arbitrage pricing formula and Jensen's inequality are the relationship between the transient component of the discount factor and the holding-period return on a infinite-maturity zero-coupon bond,

$$m_{t+1}^T = - \lim_{n \rightarrow \infty} r_{0,t+1}^{(n)}$$

and the property of the entropy ratio

$$\frac{\mathcal{V}_t(M_{t+1}^P)}{\mathcal{V}_t(M_{t+1})} = 1 - \frac{E_t r_{0,t+1}^{e,(\infty)}}{\frac{1}{2} \mathcal{V}_t(M_{t+1})} \geq 1 - \frac{E_t r_{0,t+1}^{e,(\infty)}}{\max E_t r_{t+1}^e}$$

where the maximum is taken over all available excess returns and where relative entropy is defined as  $\mathcal{V}_t(X_{t+1}) \doteq 2[\ln(E_t X_{t+1}) - E_t \ln(X_{t+1})]$ . Alvarez and Jermann advocate for using such a property as a diagnostic tool for a model of the discount factor. Namely, they combine equity and bond returns to estimate the right-hand side of the inequality as being close to unity on average; additionally, the imperfect correlation between equity and bond expected excess returns suggests time-variation in the ratio (see also Koijen, Lustig, and Nieuwerburgh, 2010; Lettau and Ludvigson, 2010).

These properties make clear how an approximation method that correctly captures the term structures of equities and bonds, and especially their long-run properties, is a method that correctly

captures the diagnostic decomposition of the discount factor. Our emphasis on evaluating the quality of the approximation via the term structures of claims to different cashflow processes in section 4 rests therefore on this observation.

Following Hansen and Scheinkman (2009), the decomposition can be understood starting from the solution  $[\delta; f]$  to the eigenfunction problem

$$E_t^{\mathbb{P}}[M_{t+1}f(\widehat{z}_{t+1})] = \delta f(\widehat{z}_t)$$

for some function  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  and scalar  $\delta \in \mathbb{R}$ , with transient and martingale components constructed as  $M_{t+1}^T = \delta f(\widehat{z}_t)/f(\widehat{z}_{t+1})$  and  $M_{t+1}^P = M_{t+1}f(\widehat{z}_{t+1})/\delta f(\widehat{z}_t)$ .

**Proposition 4** (Asset pricing diagnostics). *In the context of the general model of section 2, the approximate generalized affine solution of the eigenfunction problem,*

$$f(\widehat{z}_t) = e^{u_z' \widehat{z}_t}$$

where the eigenfunction parameters  $[\delta; u_z]$  solve the nonlinear system of  $n_z + 1$  equations

$$\begin{aligned} 0 &= \ln(\delta) + r(0) - \begin{bmatrix} u_z \\ 0 \end{bmatrix}' [\mu - \pi(0)] - \frac{1}{2} \begin{bmatrix} u_z \\ 0 \end{bmatrix}' \Sigma(0)\Sigma(0)' \begin{bmatrix} u_z \\ 0 \end{bmatrix} \\ u_z' e_i &= \begin{bmatrix} u_z \\ 0 \end{bmatrix}' \left( [\Phi - \pi_1(0)]e_i + \Sigma(0)B_1(0)'(I_{n_s} \otimes e_i)u_s \right) - r_1(0)e_i \end{aligned}$$

where  $e_i \in \mathbb{R}^{n_z}$  is the unit vector in the  $i$ th direction,  $i = 1, \dots, n_z$ .

The system can be solved by iterating to convergence the difference equation

$$u_z^{(n)'} e_i = \begin{bmatrix} u_z^{(n-1)'} \\ 0 \end{bmatrix}' \left( [\Phi - \pi_1(0)]e_i + \Sigma(0)B_1(0)'(I_{n_s} \otimes e_i)u_s^{(n-1)} \right) - r_1(0)e_i$$

If the iterations converge the solution exists and is  $u_z = u_z^{(\infty)}$ , which in turn allows for recovering  $\delta$ .

Appendix E provides a proof of the proposition.

The iterative procedure described in proposition 4 is revealing in that it makes clear how  $u_z^{(n)}$  is the equilibrium coefficient in the approximate generalized affine pricing of real bond yields, i.e., of claims to the cashflow process  $d = 0$  in equation (6). Accordingly, in this affine context

$$\begin{aligned} \ln(\delta) &= \lim_{n \rightarrow \infty} [A_0^{(n)} - A_0^{(n-1)}] \\ \ln[f(x_t)] &= B_{0,z}^{(\infty)'} \widehat{z}_t \end{aligned}$$

characterize a solution to the eigenfunction problem, where  $\{A_0^{(n)}; B_{0,z}^{(n)}\}$  are the coefficients in the equilibrium expression of real bond yields. It follows that

$$\begin{aligned} m_{t+1}^T &= \ln(\delta) + B_{0,z}^{(\infty)'} \widehat{z}_t - B_{0,z}^{(\infty)'} \widehat{z}_{t+1} \\ &= -r_{0,t+1}^{(\infty)} \end{aligned}$$

which verifies the equivalence between Hansen-Scheinkman and Alvarez-Jermann decompositions in this context of affine pricing (see also Kojien et al., 2010).

### 3.2. Inspecting the mechanism

The generalized affine approximation is particularly useful to provide an intuitive understanding of the macroeconomic forces that drive the prices of financial claims and the risk premia they command. Using equation  $r_{t+1}^{e,(n)} = E_t r_{t+1}^{e,(n)} + V_{n-1,t} \varepsilon_{t+1}$ , we can decompose the reaction of the quantity of risk in the  $n$ th strip return to a shock along direction  $\alpha_t$  into three components,

$$V_{n-1,t} \alpha'_t = \left[ \underbrace{D(\widehat{z}_t)}_{\text{short-run cashflow risk}} + \underbrace{B_z^{(n-1)} B(\widehat{z}_t)}_{\text{long-run cashflow and discount-rate risk}} \right] \alpha'_t \quad (7)$$

which are the basis to understand the shape of the term structure of holding-period risk premia,  $\ln E_t R_{t+1}^{e,(n)} = \gamma'_t V'_{n-1,t}$ . The first element on the right-hand side of the equation controls the cashflow effect due to contemporaneous shocks to dividends. The second element captures the income effect as well as the substitution effects of past shocks.

Another way the generalized affine approximation facilitates an intuitive understanding of the asset-pricing implications of the macro-finance model is by providing simple expressions for the dynamic value decomposition proposed by Borovicka and Hansen (2014) as measures to quantify the exposures of cashflows to shocks over alternative horizons and the corresponding compensations commanded by investors. In particular, for any increase in one-step ahead uncertainty along dimension  $\alpha_t$  we can define cashflow and discount-rate elasticities as

$$\begin{aligned} \varepsilon_{g,t}^{(n)} &\doteq \frac{d}{dr} \ln E_t^{\mathbb{P}} \left[ \frac{D_{t+n}}{D_t} e^{r\alpha_t \varepsilon_{t+1} - \frac{r^2}{2} \|\alpha_t\|^2} \right]_{r=0} = \left[ D(\widehat{z}_t) + B_g^{(n-1)} B(\widehat{z}_t) \right] \alpha'_t \\ \varepsilon_{p,t}^{(n)} &\doteq \frac{d}{dr} \ln E_t^{\mathbb{P}} \left[ \frac{D_{t+n}}{D_t} e^{r\alpha_t \varepsilon_{t+1} - \frac{r^2}{2} \|\alpha_t\|^2} \right]_{r=0} - \frac{d}{dr} \ln E_t^{\mathbb{P}} \left[ M_{t,t+n} \frac{D_{t+n}}{D_t} e^{r\alpha_t \varepsilon_{t+1} - \frac{r^2}{2} \|\alpha_t\|^2} \right]_{r=0} \\ &= \left[ \gamma(\widehat{z}_t)' + (B_g^{(n-1)} - B_z^{(n-1)}) B(\widehat{z}_t) \right] \alpha'_t \end{aligned} \quad (8)$$

Appendix F derives expression (8). These elasticities capture the impact of current shocks on future cashflows ( $\varepsilon_{g,t}^{(n)}$ ) and on future expected returns ( $\varepsilon_{p,t}^{(n)}$ ), while the impact on valuations can be recovered as the value elasticity  $\varepsilon_{g,t}^{(n)} - \varepsilon_{p,t}^{(n)}$ .

The decompositions (7) and (8) are particularly useful to gain insight into the drivers of asset prices, and they are deeply linked. In fact, when  $\alpha_t = \gamma(\widehat{z}_t)'$  (a discount-rate shock), we have

$$\ln E_t R_{d,t+1}^{e,(n)} = \varepsilon_{g,t}^{(n)} - \varepsilon_{p,t}^{(n)} + \text{var}_t(m_{t+1})$$

and so holding-period risk premia are equivalent to a strictly positive level factor (a precautionary motive, as investors require some compensation when future marginal utility is uncertain) plus the cashflow effect of positive consumption news on future dividends minus the discount-rate effect of the shock on investors' required compensation.

#### 4. Accuracy of the approximation

We compare the performance of our generalized affine approximation to global solutions as well as to the exponential-affine approximation of Malkhozov (2014), by applying them to three macro-finance models with nonlinear habits à la Campbell-Cochrane. These type of habits is particularly suited to test our approximation as they display endogenous risk (surplus consumption is a state of the economy and is driven by consumption news, which are endogenous objects outside an endowment economy) and strong heteroskedasticity. In particular, preferences are captured by the function:

$$U_t = (C_t - H_t)^{1-\gamma} + \chi_t(N_t) + \beta E_t U_{t+1}$$

with  $C_t$  denoting real consumption and  $H_t$  an external habit that is a nonlinear function of past consumption, and where  $\chi_t(N_t)$  allows for a dependence of preferences on labor  $N_t$ . Parameter  $\beta$  is the subjective discount rate and  $\gamma$  controls the curvature of the utility function.

In these models a large and time-varying price of risk,  $x_t$ , matches a high and volatile equity premium while the nonlinearity in the external habit is calibrated to ensure that the precautionary savings effect largely offsets the intertemporal substitution effect in the determination of the risk-free rate, thus avoiding a risk-free rate puzzle. Note that precautionary savings are captured by entropy, which therefore contains a dynamic term that conventional linearizations would disregard.

Specifically, the scalar price of risk is a nonlinear function  $x_t \doteq \gamma[1 + \Lambda(\widehat{s}_t)]$  of the surplus-consumption ratio,  $s_t \doteq \ln[(C_t - H_t)/C_t]$ , whose dynamics have the heteroskedastic form

$$\widehat{s}_{t+1} = \phi \widehat{s}_t + \Lambda(\widehat{s}_t) \varepsilon_{t+1}^c \quad (9)$$

where  $\widehat{s}_t \doteq s_t - s$ ,  $|\phi| < 1$  is a parameter,  $\Lambda(\widehat{s}_t) = S^{-1} \sqrt{1 - 2\widehat{s}_t} - 1$  is a nonlinear function with the calibration  $S = \sqrt{\gamma \text{var}(\varepsilon^c) / (1 - \phi - \xi_1/\gamma)}$ , and  $\varepsilon_{t+1}^c \doteq c_{t+1} - E_t^{\mathbb{P}} c_{t+1}$  represents consumption news. The stochastic discount factor is

$$\begin{aligned} m_{t+1} &= \ln(\beta) - \gamma \Delta c_{t+1} - \gamma \Delta s_{t+1} \\ &\approx \ln(\beta) - \gamma E_t \Delta c_{t+1} + \gamma(1 - \phi) \widehat{s}_t - x_t \varepsilon_{t+1}^c \end{aligned}$$

The nonlinear dynamics of the time-varying price of risk,  $x(\widehat{s}_t)$ —a square-root process in Campbell and Cochrane (1999)—are responsible for the absence of a closed-form solution.

We evaluate the quality of our approximation by comparing the term structures of zero-coupon equities; this exercise allows for decomposing the quality of the approximation at different time horizons and for claims that are the basis for pricing other more complex assets. We consider the models of Campbell and Cochrane (1999) and Wachter (2006) as well-known examples of nonlinear habit models that fit into our framework of section 2. We then turn to the model of Lopez et al. (2015) as an example of a model that unites nonlinear habits with a New Keynesian production economy. In this context, consumption innovations are endogenous objects with unknown distribution that therefore cannot be subsumed among the shocks in the framework by Malkhozov (2014) to derive a dynamic risk adjustment reflecting time-varying precautionary motives.

For the models of Campbell and Cochrane (1999) and Wachter (2006) the global solution is calculated using cubic splines, while for Lopez et al. (2015) the global solution is projected onto the

Parameter	Campbell-Cochrane	Wachter
Frequency	monthly	quarterly
Subjective discount factor, $\beta$	$0.90^{1/12}$	0.98
Utility curvature, $\gamma$	2	2
Habit persistence, $\phi$	$0.87^{1/12}$	0.97
Short-run rate exposure to $s_t$ , $\xi_1$	0	0.011
Mean consumption growth (in %), $\mu_c$	1.89/12	0.549
Mean cashflow growth (in %), $\mu_d$	1.89/12	-0.920
Standard deviation of consumption growth (in %), $\sigma_v$	$1.50/\sqrt{12}$	0.431
Standard deviation of cashflow growth (in %), $\sigma_w$	$11.2/\sqrt{12}$	0.588
Consumption-cashflows correlation, $\rho$	0.200	0.205
State persistence, $\psi_1$	—	0.941
State volatility, $\psi_2$	—	0.297

Table 1: Deep parameters and their calibration. Cashflow growth is dividend growth in Campbell and Cochrane (1999) and inverse inflation in Wachter (2006).

subspace spanned by a basis of Chebyshev polynomials of up to degree eight, as macroeconomic quantities in this model have to be solved for.<sup>12</sup> Since the equilibrium allocation is not exogenously given, we consider multiperiod Euler equation errors as an additional metric to contrast the quality of the approximation.

Our generalized affine approximation is accurate in solving for risk premia and volatilities of short- and long-duration equities and bonds—although it seems to scale up the correct values for medium-duration bonds and equities—and outperforms the alternative approximation schemes.<sup>13</sup>

To clarify the importance of our extension of extant exponential-affine approximations we compare four methods. First, projected solutions reflect accurate numerical procedures that we take as the exact solution. Second, our generalized affine approximation that coincides with a linear perturbation around the risky steady state, except in its treatment of innovations of the state vector—in particular, it does not linearize the map  $\Lambda(\widehat{s}_t)$  to represent  $(E_{t+1} - E_t)z_{t+1}$ , which has a Taylor series at  $\widehat{s}_t = 0$  with radius of convergence of just .5, excluding a good approximation of tail regions of the state space that matter most for pricing. Third, we plot a linear perturbation around the risky steady state to highlight the consequences of the limited radius of convergence of map  $\Lambda(\widehat{s}_t)$  and the spurious dynamics that result from approximating it to simulate the states. Finally, to highlight the importance of our dynamic correction for endogenous risk, we show an exponential-affine approximation that linearizes the map  $\Lambda$  and that reserves no special treatment to endogenous innovations in the state vector. This approximation coincides with the procedure in Malkhozov (2014).

<sup>12</sup>See the online appendix for details on the construction of the global solution.

<sup>13</sup>Moreover, the affine approximation offers an excellent starting point for numerical optimization algorithms that construct the nonlinear projected solution. For example, iterations on the Fredholm pricing operator to project entire term structures can explode; by projecting only the difference between the projected term structure components and their affine approximations we can avoid explosive numerical behavior.

#### 4.1. Example 1: Campbell and Cochrane (1999)

In the benchmark model by Campbell and Cochrane (1999), consumption and market dividends are random walks with the structure:

$$\begin{bmatrix} \Delta c_{t+1} \\ \Delta d_{t+1} \end{bmatrix} = \begin{bmatrix} \mu_c \\ \mu_d \end{bmatrix} + \begin{bmatrix} \sigma_c & 0 \\ \rho\sigma_d & \sqrt{1-\rho^2}\sigma_d \end{bmatrix} \varepsilon_{t+1}$$

where  $\varepsilon_t \sim Niid(0, I_2)$ . Combined with state equation (9), the model fits the framework (4), so we can invoke the results of section 3 to price financial claims via no-arbitrage relations.

Table 1 reports the calibration of the parameter values. Figure 1 compares the global solution with our proposed generalized affine solution and the alternative affine approximations. The figures report the term structure of equilibrium risk premia and realized return volatilities of zero-coupon equities and bonds. Since all risk is exogenous in this example, the standard affine approximation coincides with a linear perturbation around the risky steady state. However, the generalized affine approximation is unambiguously preferable in practice because it does not induce spurious dynamics in the state vector by its approximation of function  $\Lambda$ . The approximations that simulate using linearized conditional volatilities severely overstate the asymptotic risk premium. Relative to the projected solution, the fit of the generalized affine approximation manages to capture the level, amplitude and shape of the term structures.

#### 4.2. Example 2: Wachter (2006)

In the model of Wachter (2006), consumption and nominal bond cashflows,  $\Delta d = -\pi$ , have structure

$$\begin{bmatrix} \Delta c_{t+1} \\ \Delta d_{t+1} \end{bmatrix} = \begin{bmatrix} \mu_c \\ \mu_d \end{bmatrix} + \begin{bmatrix} \sigma_c & 0 \\ \rho\sigma_d & \sqrt{1-\rho^2}\sigma_d \end{bmatrix} \varepsilon_{t+1}$$

$$\zeta_{t+1} = \psi_1 \zeta_t + \psi_2 \begin{bmatrix} \sigma_c & 0 \end{bmatrix} \varepsilon_{t+1}$$

where  $\varepsilon_t \sim Niid(0, I_2)$ , where  $\zeta_t$  represents an exogenous state driving inflation.

We calibrate the parameters with the values reported in table 1. Figure 2 contrasts the global solution with our generalized affine approximation and the benchmarks. The figures report the term structure of equilibrium risk premia and volatilities of equities and of real and nominal interest rates. The plots confirm the overall good performance of our generalized affine approximation which captures well the level, amplitude and shape of the term structures. The approximation errors of the alternative affine methods are larger in comparison and owe entirely to the linearization of the conditional volatility of the state vector.

#### 4.3. Example 3: Lopez et al. (2015)

We turn to a version of the macro-financially separate New Keynesian model with nonlinear habits of Lopez et al. (2015). The model is a production economy, so we have first to solve for the equilibrium allocation as shown in section 2 before exploiting no-arbitrage relations to price all financial claims in zero net supply.

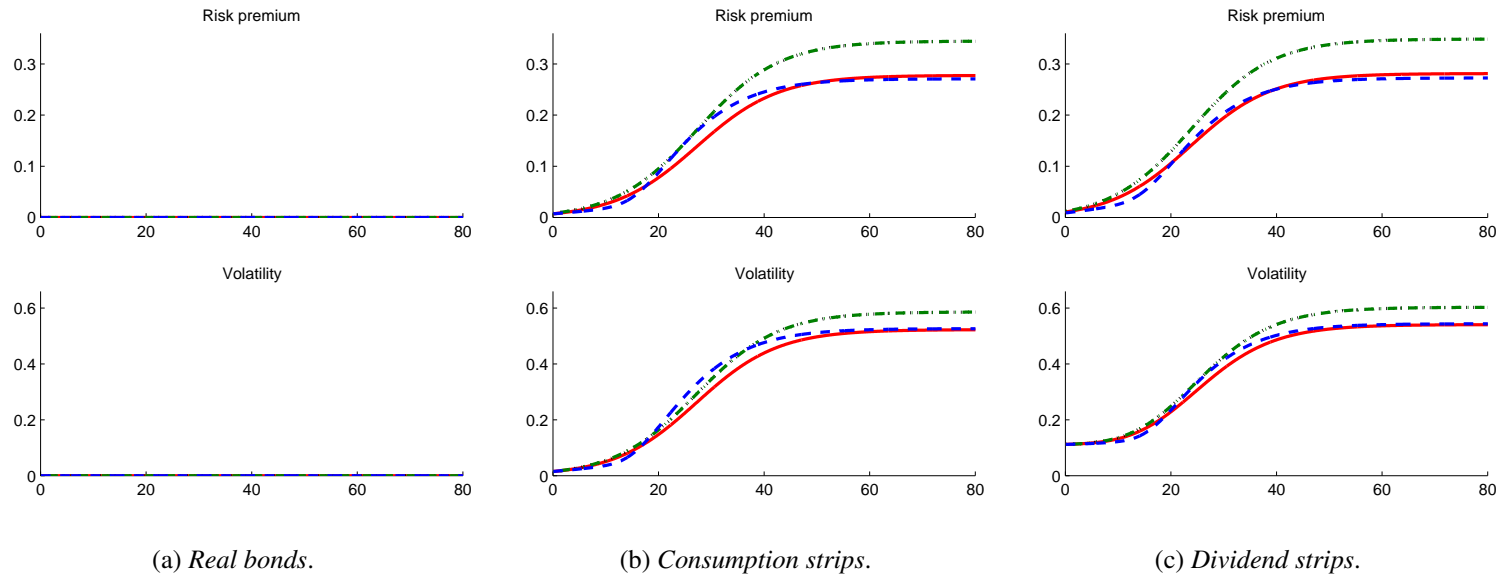


Figure 1: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia  $\{\ln E_t R_{t+1}^{e,(n)}\}$  and volatilities  $\{std_t(r_{t+1}^{(n)})\}$  in the Campbell and Cochrane (1999) model. Generalized affine (solid red), perturbation around the risky steady state (dot-dashed green), standard affine (dotted black), and projected solution using cubic splines collocated over 100 Chebyshev nodes and 6-point Gauss-Hermite quadrature (dashed blue).



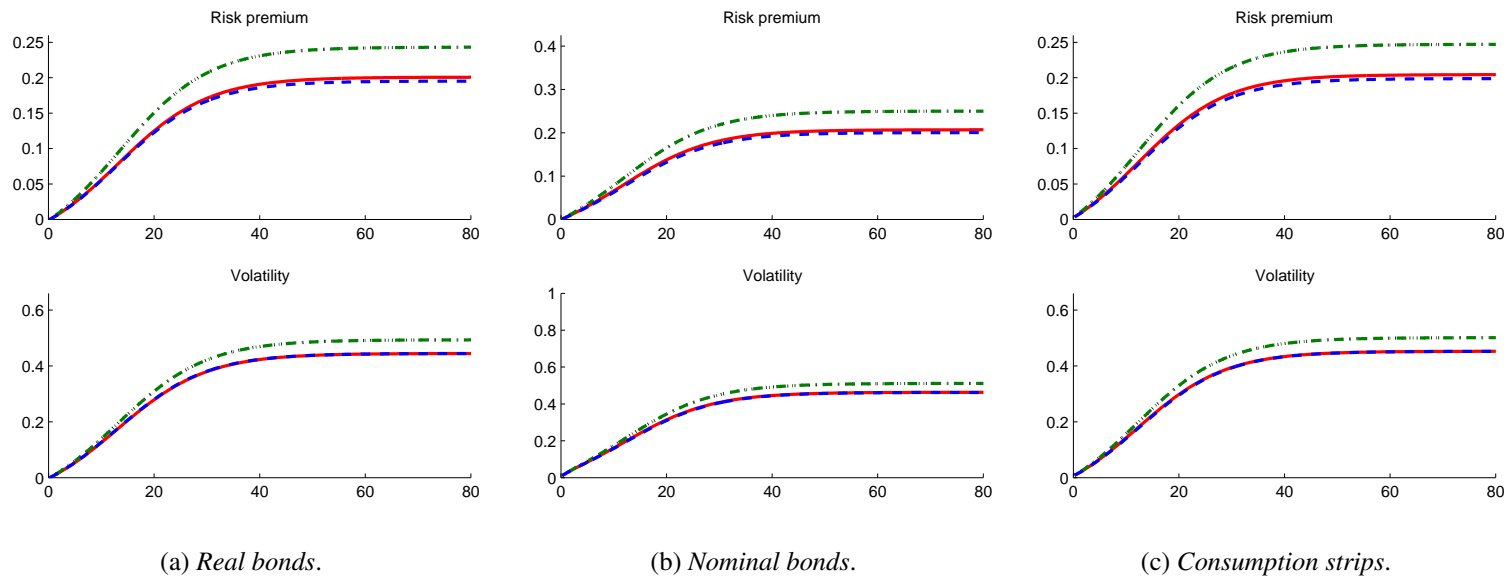


Figure 2: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia  $\{\ln E_t R_{t+1}^{e,(n)}\}$  and volatilities  $\{std_t(r_{t+1}^{(n)})\}$  in the Wachter (2006) model. Generalized affine (solid red), perturbation around the risky steady state (dot-dashed green), standard affine (dotted black), and projected solution using cubic splines collocated over 100 Chebyshev nodes and 6-point Gauss-Hermite quadrature (dashed blue).

#### 4.3.1. Risk-adjusted loglinearization

To apply the results of section 2, we set the necessary conditions that describe the competitive equilibrium values of the consumption gap  $\widetilde{c}_t \doteq c_t - a_t$ , with  $a_t$  the level of technology, the inflation rate  $\pi_t$  and the auxiliary control variable  $\ell_t$  in in form (1) as:

$$\begin{aligned}
0 &= \ln E_t e^{\ln(\beta) - \gamma\mu + i^* - \widetilde{\pi} + \phi_\pi \widetilde{\pi}_t + \phi_y \widetilde{c}_t - \gamma(\Delta \widehat{c}_{t+1} + u_t + \varepsilon_{t+1}^a) - \gamma \Delta \widehat{s}_{t+1} - \widehat{\pi}_{t+1}} \\
0 &= \ln E_t e^{\ln(\widetilde{\beta}\eta\widetilde{\Pi}^{\frac{\varepsilon}{1-\alpha}}) + \frac{\varepsilon}{1-\alpha}\widehat{\pi}_{t+1} + (1-\gamma)(u_t + \varepsilon_{t+1}^a) + \Delta \widehat{\ell}_{t+1} - w_{1,t}} \\
0 &= \ln E_t e^{\ln(\widetilde{\beta}\eta\widetilde{\Pi}^{\varepsilon-1}) + (\varepsilon-1)\widehat{\pi}_{t+1} + (1-\gamma)(u_t + \varepsilon_{t+1}^a) + \Delta \widehat{\ell}_{t+1} + \Delta \widehat{w}_{3,t+1} - w_{2,t}} \\
0 &= \ln e^{w_{1,t} - \ln\left(1 - \frac{\varepsilon(1-\tau)}{\varepsilon-1} \frac{\lambda_0}{1-\alpha} \frac{\widetilde{N}}{L} \exp\left[\frac{1}{1-\alpha}\widehat{c}_t - \gamma(1+\xi_2)\widehat{s}_t - \widehat{\ell}_t\right] \left[1 - \widetilde{N}e^{\Delta} \exp\left(\frac{1}{1-\alpha}\widehat{c}_t + \Delta_t\right)\right]^{-\gamma}\right)} \\
0 &= \ln e^{w_{2,t} - \ln\left(1 - \frac{\widetilde{N}(1-\gamma)(1-\alpha)}{w_3 L} \exp\left[(1-\gamma)\widehat{c}_t - \gamma\widehat{s}_t - \widehat{\ell}_t - \widehat{w}_{3,t}\right]\right)} \\
0 &= \ln e^{w_{3,t} - \frac{1-\alpha+\alpha\varepsilon}{(1-\alpha)(\varepsilon-1)} \{\ln(1-\eta)\widetilde{\Pi}^{\varepsilon-1} \exp[(\varepsilon-1)\widehat{\pi}_t] - \ln(1-\eta)\}}
\end{aligned}$$

where the states take form:

$$\begin{aligned}
a_{t+1} &= \mu + a_t + u_t + \begin{bmatrix} \sigma & 0 \end{bmatrix} \varepsilon_{t+1} \\
u_{t+1} &= \rho_u u_t + \begin{bmatrix} \rho\phi\sigma & \sqrt{1-\rho^2}\phi\sigma \end{bmatrix} \varepsilon_{t+1} \\
\widehat{s}_{t+1} &= \rho_s \widehat{s}_t + \Lambda(\widehat{s}_t) \varepsilon_{t+1}^c \\
\Delta_t &= \ln \left[ \eta \widetilde{\Pi}^{\frac{\varepsilon}{1-\alpha}} e^{\Delta} e^{\frac{\varepsilon}{1-\alpha}\pi_t + \widehat{\Delta}_{t-1}} + (1-\eta) \left( \frac{1 - \eta \widetilde{\Pi}^{\varepsilon-1} e^{(\varepsilon-1)\widehat{\pi}_t}}{1-\eta} \right)^{\frac{\varepsilon}{(1-\alpha)(\varepsilon-1)}} \right]
\end{aligned}$$

where  $\varepsilon_t \sim Niid(0, I_2)$ . The level of the interest rate in the risky steady state  $i^*$  and the employment subsidy  $\tau$  are chosen by the government to correct the risky steady state to a model without monopolistic competition and nominal rigidities. ‘Hat’ variables denote deviations from the respective constant terms (the risky steady-state values by proposition 1) and  $\varepsilon_{t+1}^x \doteq x_{t+1} - E_t x_{t+1}$ .

We assume the parametrization:

$$\begin{aligned}
i^* &= -\ln\beta + \gamma\mu - \frac{\gamma(1-\rho_s - \xi_1/\gamma)}{2} - \frac{1}{2}var(\varepsilon^\pi) - \frac{\gamma}{S}cov(\varepsilon^c, \varepsilon^\pi) \\
1-\tau &= \widetilde{MC} = \frac{\varepsilon-1}{\varepsilon} \frac{1-\delta\eta\Upsilon}{1-\delta\eta\Upsilon\Theta} \\
\xi_1 &= \frac{\gamma}{S}cov(\varepsilon^c, \varepsilon^\pi) \\
\xi_2 &= \frac{\delta\eta\Upsilon(1-\Theta)}{1-\delta\eta\Upsilon} - \frac{\delta\psi_{\ell_s}}{\gamma\lambda S}cov(\varepsilon^c, \varepsilon^\pi)
\end{aligned}$$

that implies  $\widetilde{n} = n$ , where  $n$  is the undistorted deterministic steady-state value of hours, and  $\widetilde{\pi} = 0$  and hence  $\Delta_t = 0$  for all  $t$ ; fiscal policy (through  $\tau$ ) and monetary policy (through  $i^*$ ) correct jointly the risky steady state. Under this condition, it turns out that the implicit solution has the approximate form  $\pi_t = \widetilde{\pi} + \psi_\pi u_t$ ,  $\widetilde{c}_t = \widetilde{c} + \psi_c u_t$  and  $\ell_t = \widetilde{\ell} + \psi_{\ell_u} u_t + \psi_{\ell_s} \widehat{s}_t$ . Accordingly, the homoskedasticity of technology implies the homoskedasticity of inflation and consumption. Moreover, we define

	Parameter		Value
New Keynesian block	$\gamma$	Utility curvature in market and home consumption	2
	$1/\varphi$	Quasi-Frisch's labor supply elasticity	1
	$\beta$	Subjective discount factor	.9991
	$1 - \alpha$	Labor share in value added	2/3
	$\varepsilon$	Elasticity of substitution in Dixit-Stiglitz aggregator	6
	$1/(1 - \eta)$	Average price duration (in months)	9
	$\phi_\pi$	Policy response coefficient to inflation movements	1.5
	$\phi_y$	Policy response coefficient to output movements	.5/12
Habit block	$\xi_1$	Financial spillover onto the intertemporal rate of substitution	-.0001
	$\xi_2$	Financial spillover onto the intratemporal rate of substitution	-.0137
	$\rho_s$	Habit persistence	.9940
Exogenous block	$\mu$	Mean technology growth	.0030
	$\rho_u$	Persistence of the conditional mean of technology growth	.8470
	$\sigma$	Conditional volatility of technology	.0222
	$\phi$	Relative volatility of the conditional mean of technology	.1473
	$\rho$	Correlation between short-run and long-run shocks	-.9623

Table 2: Deep parameters and their calibration (monthly frequency) in the model by Lopez et al. (2015).

$$\Upsilon \doteq e^{\frac{1}{2} \text{var}(\frac{\varepsilon}{1-\alpha} \varepsilon_{t+1}^\pi + (1-\gamma) \varepsilon_{t+1}^a + \psi_{\ell u} \varepsilon_{t+1}^u + \psi_{\ell s} \lambda(0) \varepsilon_{t+1}^c)} \quad \text{and} \quad \Theta \doteq e^{\frac{1}{2} \text{var}(\frac{\varepsilon}{1-\alpha} \varepsilon_{t+1}^\pi + (1-\gamma) \varepsilon_{t+1}^a + \psi_{\ell u} \varepsilon_{t+1}^u + \psi_{\ell s} \lambda(0) \varepsilon_{t+1}^c - \frac{1-\alpha+\alpha\varepsilon}{(1-\alpha)(1-\eta)} \varepsilon_{t+1}^\pi)} / \Upsilon.$$

We identify the conjecture by the method of undetermined coefficients.

#### 4.3.2. Generalized affine pricing

Accordingly, cashflows can be written in form (4) as:

$$\begin{aligned} \Delta c_{t+1} &= \mu + [1 - (1 - \rho_u) \psi_c] u_t + \left[ (1 + \psi_c \rho \phi) \sigma \quad \sqrt{1 - \rho^2} \psi_c \phi \sigma \right] \varepsilon_{t+1} \\ -\pi_{t+1} &= -\rho_u \psi_\pi u_t + \left[ -\psi_\pi \rho \phi \sigma \quad -\sqrt{1 - \rho^2} \psi_\pi \phi \sigma \right] \varepsilon_{t+1} \end{aligned}$$

with market dividends in the approximate relationship with consumption  $d_t = a_t - \frac{\gamma(1-\alpha)+\widehat{\varphi}}{\alpha} \widehat{c}_t + \frac{\gamma(1-\alpha)}{\alpha} \xi_2 \widehat{s}_t$ .

We calibrate the parameters with the values in table 2. Figure 3 compares the global solution with our generalized affine approximation method as well as with the benchmark approximations. The plots show an overall good performance of our generalized affine approximation which captures well the level, amplitude and shape of the term structures of risk premia and volatilities, including the initially downward-sloping term structure of market equity. The linear perturbation approximation around the risky steady state displays the usual bias in representing long-run pricing properties due to the small radius of convergence of map  $\Lambda$ . The alternative affine approximation is severely inaccurate by its inability to capture a dynamic risk adjustment due to the interaction between the time-varying function  $\Lambda$  and endogenous innovations to consumption. The standard affine method has therefore a hard time in accounting for a precautionary savings effect. This distortion is particularly severe for the term structure of nominal yields, as the spillover of surplus consumption

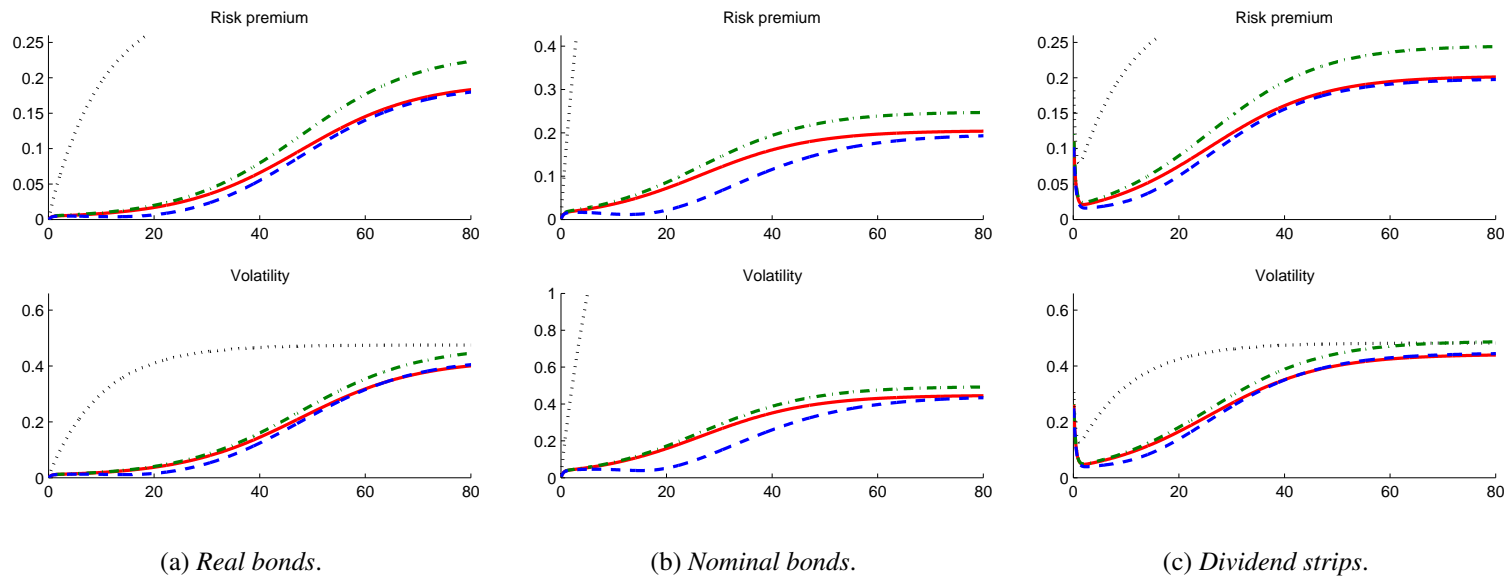


Figure 3: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia  $\{\ln E_t R_{t+1}^{e,(n)}\}$  and volatilities  $\{std_t(r_{t+1}^{(n)})\}$  in the Lopez et al. (2015) model. Generalized affine (solid red), perturbation around the risky steady state (dot-dashed green), standard affine (dotted black), and projected solution using Chebyshev polynomials of up to degree eight collocated over a Smolyak grid and 6-point Gauss-Hermite quadrature (dashed blue).

on inflation has an incorrect effect on its volatility.

### 4.3.3. Euler Equation Errors

To show the accuracy of our approximations we compute the errors in the  $n$ -period Euler equation for the three solutions of the model. In the model, one-period Euler equation errors are defined as in Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015) from a solution for consumption  $c^{(0)}(z_t)$  as

$$\begin{aligned} EEE^{(1)}(z_t) &\doteq \log_{10} \left| 1 - e^{c^{(1)}(z_t) - c^{(0)}(z_t)} \right|, \\ c_t^{(1)} &\doteq -\frac{1}{\gamma} \ln E_t \left( \beta e^{-\gamma c_{t+1}^{(0)} - \gamma \Delta s_{t+1}^{(0)} + r_t} \right) \end{aligned}$$

for points  $z_t$  that cover a high-probability region of the state space, and where  $s^{(0)}$  denotes its dependence on the functional  $c^{(0)}$ . An EEE of  $-\varepsilon$  implies that the consumer makes a one dollar mistake in how much she decides to save for every  $10^\varepsilon$  dollars spent.

We propose a multiperiod version of the Euler equation errors as a metric to test the accuracy of our approximation for the  $n$ -period equilibrium term structures

$$\begin{aligned} EEE^{(n)}(z_t) &\doteq \log_{10} \left| 1 - e^{c^{(n)}(z_t) - c^{(0)}(z_t)} \right| \\ c_t^{(n)} &\doteq -\frac{1}{\gamma} \ln E_t \left( \beta e^{-\gamma c_{t+1}^{(n-1)} - \gamma \Delta \widehat{s}_{t+1}^{(n-1)} + r_t} \right) = -\frac{1}{\gamma} \ln E_t \left( \beta^n e^{-\gamma c_{t+n}^{(0)} - \gamma \sum_{j=1}^n \Delta \widehat{s}_{t+j}^{(n-j)} + \sum_{j=0}^{n-1} r_{t+j}} \right) \end{aligned}$$

with  $\widehat{s}_{t+1}^{(j)} \doteq \rho_s \widehat{s}_t + \Lambda(\widehat{s}_t)(E_{t+1} - E_t)c_{t+1}^{(j)}$ . In this context, a  $n$ -period Euler equation error of  $-\varepsilon$  implies that the consumer is making a one dollar mistake in how much she decides to save over a  $n$ -period horizon for every  $10^\varepsilon$  dollars spent. Since the errors accumulate as the horizon increase, multiperiod Euler equation errors provide an indication of how good the approximation is for long-term valuations.

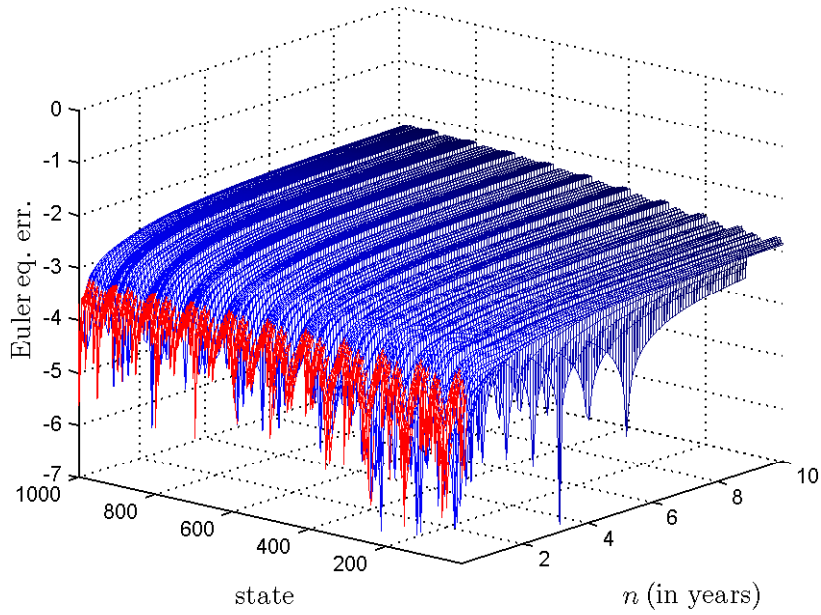
Figure 4 shows multiperiod Euler equation errors. The accuracy of our global solution in terms of conventional 1-step ahead Euler equation errors is consistently lower than  $-3$  and is comparable to values typically retained in the extant literature (e.g., Fernández-Villaverde et al., 2015), and remains below  $-2$  over arbitrarily long horizons. The risk-adjusted loglinearized solution for quantities that forms the basis of the generalized affine approximation also shows relatively small errors; notably, the lower accuracy of the loglinearized solution does not translate in substantially larger Euler equation errors over long horizons when contrasted with the projected solution.

## 5. Conclusion

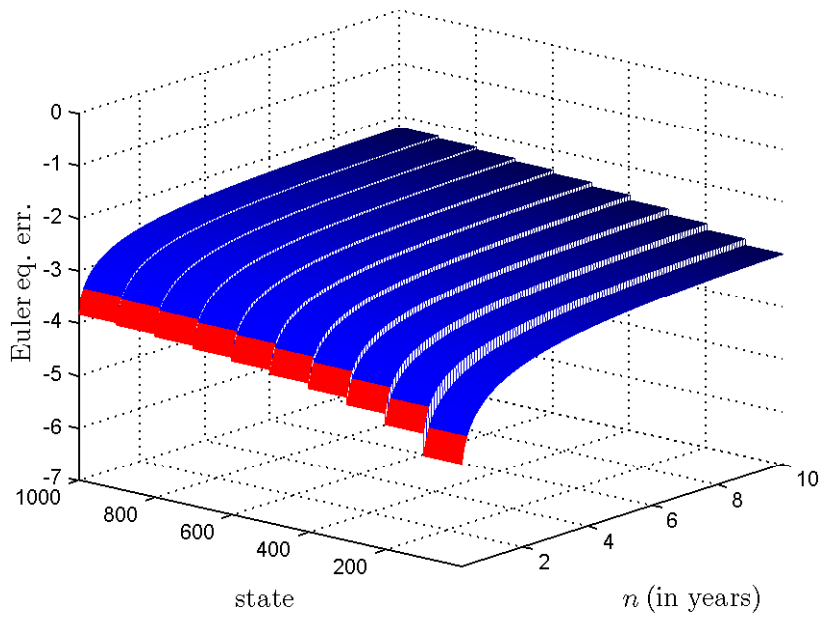
We propose an exponential-affine approximation technique to solve analytically for approximate equilibrium asset prices and quantities of dynamic macro-finance models with state-dependent risk. The proposed generalized affine approximation performs similarly to global projection methods and significantly outperforms alternative affine approximations.

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(a) *Projected solution for quantities.*



(b) *Loglinearized solution for quantities.*

Figure 4: Multiperiod Euler equation errors in the Lopez et al. (2015) model. Errors are expressed in  $\log_{10}$ . Values in the state dimension index different triplets  $[u_t, \zeta_t, \Delta_{t-1}]$  built as the Cartesian product of 10 equidistant points along each dimension. The red region associates with conventional 1-step ahead Euler equation errors.

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## Appendix

### A. Proof of proposition 1

*Proof of proposition 1.a.* Since the risky steady state is saddle-point stable, we can invoke the implicit function theorem to guarantee a solution to system (3) with form

$$v_t^q = v(z_t, q), \quad y_t^q = y(z_t, q), \quad z_{t+1}^q = z(z_t, q, q\varepsilon_{t+1}), \quad x_{t+1}^q = x(z_t, q, q\varepsilon_{t+1})$$

that is locally unique and differentiable at  $q = 1$ .

We approximate around the point  $(y_t, z_t, q) = (\bar{y}, \bar{z}, 1)$ . We keep the constant values  $\bar{y}$  and  $\bar{z}$  arbitrary for the moment, although we will use the risky steady-state values of  $y_t$  and  $z_t$ , respectively.

We are looking to identify the approximate functions:

$$\begin{aligned} y_t^q &= y(\bar{z}, q) + y_1(\bar{z}, q)(z_t - \bar{z}) \\ z_{t+1}^q &= z(\bar{z}, q, q\varepsilon_{t+1}) + z_1(\bar{z}, q, q\varepsilon_{t+1})(z_t - \bar{z}) \\ x_{t+1}^q &= x(\bar{z}, q, q\varepsilon_{t+1}) + x_1(\bar{z}, q, q\varepsilon_{t+1})(z_t - \bar{z}) \\ v_t^q &= v(\bar{z}, q) + v_1(\bar{z}, q)(z_t - \bar{z}) \end{aligned}$$

which will be evaluated *ex post* at  $q = 1$  and  $\varepsilon_{t+1} = 0$ , consistent with the definition of risky steady state.

A first-order Taylor expansion around the point  $z_t = \bar{z}$  yields:

$$\begin{aligned} z(\bar{z}, q, q\varepsilon_{t+1}) &= g[y(\bar{z}, q), \bar{z}] + \lambda(\bar{z})(E_{t+1} - E_t)y[z(\bar{z}, q, q\varepsilon_{t+1}), q] + \sigma(\bar{z})q\varepsilon_{t+1} \\ &= g[y(\bar{z}, q), \bar{z}] + \sigma_z(\bar{z}, q)\varepsilon_{t+1}, \quad \sigma_z(z_t, q) \doteq q[I - \lambda(z_t)y_1(\bar{z}, q)]^{-1}\sigma(z_t) \\ z_1(\bar{z}, q, q\varepsilon_{t+1})(z_t - \bar{z}) &= [g_1^q y_1(\bar{z}, q) + g_2^q](z_t - \bar{z}) + \lambda_1(\bar{z}) \{ (z_t - \bar{z}) \otimes (E_{t+1} - E_t)y[z(\bar{z}, q, q\varepsilon_{t+1}), q] \} \\ &\quad + \lambda(\bar{z})(E_{t+1} - E_t)y_1[z(\bar{z}, q, q\varepsilon_{t+1}), q]z_1(\bar{z}, q, q\varepsilon_{t+1})(z_t - \bar{z}) + \sigma_1(\bar{z})[(z_t - \bar{z}) \otimes q\varepsilon_{t+1}] \\ &= [g_1^q y_1(\bar{z}, q) + g_2^q](z_t - \bar{z}) + \sigma_{1,z}(\bar{z}, q)[(z_t - \bar{z}) \otimes \varepsilon_{t+1}] \end{aligned}$$



with the auxiliary variable:

$$\begin{aligned}
x(\bar{z}, q, q\varepsilon_{t+1}) &= f[y(\bar{z}, q), \bar{z}] + f_3 y[z(\bar{z}, q, q\varepsilon_{t+1}), q] + f_4 z(\bar{z}, q, q\varepsilon_{t+1}) \\
&= f[y(\bar{z}, q), \bar{z}] + f_3 y(\bar{z}, q) + f_4 \bar{z} + [f_3 y_1(\bar{z}, q) + f_4] \sigma_z(\bar{z}, q) \varepsilon_{t+1} \\
x_1(\bar{z}, q, q\varepsilon_{t+1})(z_t - \bar{z}) &= \left[ f_1^q y_1(\bar{z}, q) + f_2^q + f_3 y_1[z(\bar{z}, q, q\varepsilon_{t+1}), q] z_1(\bar{z}, q, q\varepsilon_{t+1}) + f_4 z_1(\bar{z}, q, q\varepsilon_{t+1}) \right] (z_t - \bar{z}) \\
&= \left[ f_1^q y_1(\bar{z}, q) + f_2^q + [f_3 y_1(\bar{z}, q) + f_4] \left( g_1^q y_1(\bar{z}, q) + g_2^q \right) \right] (z_t - \bar{z}) \\
&\quad + [f_3 y_1(\bar{z}, q) + f_4] \sigma_{1,z}(\bar{z}, q) [(z_t - \bar{z}) \otimes \varepsilon_{t+1}]
\end{aligned}$$

and approximate entropy:

$$\begin{aligned}
v(\bar{z}, q) &= \mathcal{V}_t [\exp(x(\bar{z}, q, q\varepsilon_{t+1}) + x_1(\bar{z}, q, q\varepsilon_{t+1})(z_t - \bar{z}))] \Big|_{z_t = \bar{z}} \\
v_1(\bar{z}, q) &= \frac{\partial \mathcal{V}_t [\exp(x(\bar{z}, q, q\varepsilon_{t+1}) + x_1(\bar{z}, q, q\varepsilon_{t+1})(z_t - \bar{z}))]}{\partial z_t^\top} \Big|_{z_t = \bar{z}}
\end{aligned}$$

with  $f_1^q \doteq f_1[y(\bar{z}, q), \bar{z}]$ ,  $f_2^q \doteq f_2[y(\bar{z}, q), \bar{z}]$ ,  $g_1^q \doteq g_1[y(\bar{z}, q), \bar{z}]$  and  $g_2^q \doteq g_2[y(\bar{z}, q), \bar{z}]$ . In the derivation we used:

$$\begin{aligned}
y[z(\bar{z}, q, q\varepsilon_{t+1}), q] &= y(\bar{z}, q) + y_1(\bar{z}, q) \sigma_z(\bar{z}, q) \varepsilon_{t+1} \\
y_1[z(\bar{z}, q, q\varepsilon_{t+1}), q] z_1(\bar{z}, q, q\varepsilon_{t+1}) &= y_1(\bar{z}, q) z_1(\bar{z}, q, q\varepsilon_{t+1})
\end{aligned}$$

that follows from  $y_{t+1} = y(\bar{z}, q) + y_1(\bar{z}, q)(z_{t+1} - \bar{z})$ , or  $y[z(z_t, q, q\varepsilon_{t+1}), q] = y(\bar{z}, 1) + y_1(\bar{z}, 1)[z(z_t, q, q\varepsilon_{t+1}) - \bar{z}]$ .

We can identify the solution  $[y(\bar{z}, q), y_1(\bar{z}, q)]$  using the equilibrium condition  $E_t x_{t+1}^q + v_t^q = 0$ , which results in the nonlinear matrix equations:

$$\begin{aligned}
0 &= f[y(\bar{z}, q), \bar{z}] + f_3 y(\bar{z}, q) + f_4 \bar{z} + \kappa [(f_3 y_1(\bar{z}, q) + f_4) \sigma_z(\bar{z}, q); \bar{z}] \tag{A.1} \\
0 &= f_1^q y_1(\bar{z}, q) + f_2^q + [f_3 y_1(\bar{z}, q) + f_4] [g_1^q y_1(\bar{z}, q) + g_2^q] + \frac{\partial \mathcal{V}_t \left( e^{[f_3 y_1(\bar{z}, q) + f_4] \{ \sigma_z(\bar{z}, q) \varepsilon_{t+1} + \sigma_{1,z}(\bar{z}, q) [(z_t - \bar{z}) \otimes \varepsilon_{t+1}] \}} \right)}{\partial z_t^\top} \Big|_{z_t = \bar{z}}
\end{aligned}$$

We can then evaluate the solution at  $q = 1$ :

$$\begin{aligned}
y_t &= y(\bar{z}, 1) + y_1(\bar{z}, 1)(z_t - \bar{z}) \\
z_{t+1} &= g[y(\bar{z}, q), \bar{z}] + [g_1^q y_1(\bar{z}, q) + g_2^q] (z_t - \bar{z}) + \sigma_z(\bar{z}, q) \varepsilon_{t+1} + \sigma_{1,z}(\bar{z}, q) [(z_t - \bar{z}) \otimes \varepsilon_{t+1}]
\end{aligned}$$

We define the risky steady state  $(y_t, z_t, q) = (\bar{y}, \bar{z}, 1)$  as the solution evaluated at  $z_t = \bar{z}$  and  $\varepsilon_{t+1} = 0$ :

$$\begin{aligned}
0 &= f(\bar{y}, \bar{z}) + f_3 \bar{y} + f_4 \bar{z} + \kappa [(f_3 y_1(\bar{z}, 1) + f_4) \sigma_z(\bar{z}, 1); \bar{z}] \\
\bar{z} &= g(\bar{y}, \bar{z})
\end{aligned}$$

i.e., as the point where agents decide to stay while expecting shocks in the future under rational expectations (i.e., beliefs consistent with the solution of the model) and when ex-post realized

shocks are zero. Since people's expectations require knowledge of the probability distribution of endogenous variables, the risky steady state depends on the approximate solution.

Finally, matrix equations (A.1) at  $q = 1$  coincide with matrix equations (2), so the affine coefficients  $\widetilde{y} = y(\widetilde{z}, 1)$  and  $\widetilde{\Psi} = y_1(\widetilde{z}, 1)$  can be interpreted as the coefficients from a first-order perturbation around the risky steady state  $(y_t, z_t) = (\widetilde{y}, \widetilde{z})$  evaluated at  $q = 1$  and  $\varepsilon_{t+1} = 0$ .  $\square$

*Proof of proposition 1.b.* We follow Klein (2000) and consider the generalized Schur factorization of  $\Gamma$  and  $\Xi$ , with unitary  $Q, Z \in \mathbb{C}^{n_y+n_z \times n_y+n_z}$  and upper triangular matrices  $S, T \in \mathbb{C}^{n_y+n_z \times n_y+n_z}$  such that:

$$Q\Gamma Z = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \quad Q\Xi Z = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \quad Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \quad Z^* = \begin{bmatrix} Z_{11}^* & Z_{21}^* \\ Z_{12}^* & Z_{22}^* \end{bmatrix}$$

with  $Z^*$  the conjugate transpose of  $Z$ , where  $S_{11}, T_{11} \in \mathbb{C}^{n_z \times n_z}$ ,  $S_{22}, T_{22} \in \mathbb{C}^{n_y \times n_y}$ ,  $Z_{11} \in \mathbb{C}^{n_z \times n_z}$ ,  $Z_{12} \in \mathbb{C}^{n_z \times n_y}$ , and matrices  $S, T$  are sorted with generalized eigenvalues  $\alpha(\Gamma, \Xi) = \{t_{ii}/s_{ii}, i = 1, \dots, n_y + n_z\}$  in increasing order as  $|t_{ii}/s_{ii}| < 1, i = 1, \dots, n_z$  and  $|t_{ii}/s_{ii}| > 1, i = n_z + 1, \dots, n_z + n_y$ .

We rewrite the matrix equation that describes the affine solution (2) as:

$$\begin{aligned} & \Gamma \begin{bmatrix} I_{n_z} \\ \widetilde{\Psi} \end{bmatrix} [g(\widetilde{y}, 0)\widetilde{\Psi} + g_2(\widetilde{y}, 0)](z_t - \widetilde{z}) = \Xi \begin{bmatrix} I_{n_z} \\ \widetilde{\Psi} \end{bmatrix} (z_t - \widetilde{z}) \\ \text{or: } & Q\Gamma ZZ^* \begin{bmatrix} I_{n_z} \\ \widetilde{\Psi} \end{bmatrix} E_t(z_{t+1} - \widetilde{z}) = Q\Xi ZZ^* \begin{bmatrix} I_{n_z} \\ \widetilde{\Psi} \end{bmatrix} (z_t - \widetilde{z}) \Leftrightarrow S E_t \begin{bmatrix} x_{z,t+1} \\ x_{y,t+1} \end{bmatrix} = T \begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix} \end{aligned} \quad (\text{A.2})$$

with

$$\begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix} \doteq Z^* \begin{bmatrix} I_{n_z} \\ \widetilde{\Psi} \end{bmatrix} (z_t - \widetilde{z}), \quad x_{z,t} \in \mathbb{R}_t^{n_z}, \quad x_{y,t} \in \mathbb{R}_t^{n_y} \quad (\text{A.3})$$

Note that the upper triangular matrices  $S_{11}$  and  $T_{22}$  are invertible, as their respective eigenvalues  $\{s_{ii}, i = 1, \dots, n_z\}$  and  $\{t_{ii}, i = n_z + 1, \dots, n_z + n_y\}$  are nonzero by the conditions for saddle-path stability.

By the stability requirement,  $\lim |E_t z_{t+N}| < \infty$ , equation (A.2) implies:

$$x_{y,t} = (T_{22})^{-1} S_{22} E_t x_{y,t+1} = [(T_{22})^{-1} S_{22}]^N E_t x_{y,t+N} \xrightarrow{N \rightarrow \infty} 0$$

as the eigenvalues of the upper triangular matrix  $(T_{22})^{-1} S_{22}$  coincide with  $\{s_{ii}/t_{ii}, i = n_z + 1, \dots, n_z + n_y\}$ , hence lie within the unit circle. Using definition (A.3), it follows that  $\widetilde{\Psi} = -(Z_{22}^*)^{-1} Z_{12}^* = Z_{21} (Z_{11})^{-1}$ , where the last equality and invertibility owe to the orthonormality of matrix  $Z$ . Orthonormality of  $Z$  also implies  $Z_{11}^* - Z_{21}^* (Z_{22}^*)^{-1} Z_{12}^* = (Z_{11})^{-1}$ . Therefore, equation (A.2) implies:

$$E_t x_{z,t+1} = (S_{11})^{-1} T_{11} x_{z,t}, \quad x_{z,t} = (Z_{11}^* + Z_{21}^* \widetilde{\Psi})(z_t - \widetilde{z}) = (Z_{11})^{-1} (z_t - \widetilde{z})$$

hence  $E_t(z_{t+1} - \bar{z}) = Z_{11}(S_{11})^{-1}T_{11}(Z_{11})^{-1}(z_t - \bar{z})$ , so the spectrum of matrix  $g_1(\bar{y}, 0)\tilde{\Psi} + g_2(\bar{y}, 0)$  is:

$$\{\lambda \in \mathbb{C} : \det[Z_{11}(S_{11})^{-1}T_{11}(Z_{11})^{-1} - \lambda I_{n_z}] = 0\} = \left\{ \frac{t_{ii}}{s_{ii}}, i = 1, \dots, n_z \right\}$$

Since the spectral radius of the matrix is less than unity, the state vector has stable dynamics.  $\square$

### B. Proof of proposition 2

A function with argument  $q \in [0, 1]$  and a convergent Taylor series in  $q = 0$  with radius of convergence larger than unity can be reconstructed by a perturbed function around  $q = 0$  with order of approximation  $N \rightarrow \infty$  (e.g., Judd, 1998). This implies that there is always a way to construct uniquely  $y(0, q)$  and  $y_1(0, q)$ , and hence  $\bar{y}$  and  $\tilde{\Psi}$  when  $\bar{z} = 0$ , despite the nonlinearity of the matrix equations they solve and to check whether the solution is a saddle path. In particular:

$$y(0, q) = y_2(0, 0)q + \frac{1}{2!}y_{22}(0, 0)q^2 + \dots, \quad y_1(0, q) = y_1(0, 0) + y_{12}(0, 0)q + \frac{1}{2!}y_{122}(0, 0)q^2 + \dots$$

where  $y_i(0, 0)$ ,  $y_{ij}(0, 0)$ ,  $y_{ijk}(0, 0)$ , ... are the output of regular perturbations around the deterministic steady state  $(z_t, q) = (0, 0)$  (e.g., Schmitt-Grohé and Uribe, 2004).  $\square$

### C. Proof of proposition 3

First start by noting that the risk-neutral dynamics of the vector process  $[z; \Delta d]$  are

$$\begin{aligned} \ln E_t^{\mathbb{Q}}[e^{u'_z \widehat{z}_{t+1} + u'_d \Delta d_{t+1}}] &= \ln E_t^{\mathbb{P}}[e^{-\frac{1}{2}\gamma(\widehat{z}_t)' \gamma(\widehat{z}_t) - \gamma(\widehat{z}_t)' \varepsilon_{t+1} + u'_z \widehat{z}_{t+1} + u'_d \Delta d_{t+1}}] \\ &= -\frac{1}{2}\gamma(\widehat{z}_t)' \gamma(\widehat{z}_t) + u'_z E_t^{\mathbb{P}} \widehat{z}_{t+1} + u'_d E_t^{\mathbb{P}} \Delta d_{t+1} + \mathcal{V}_t^{\mathbb{P}}(e^{-\gamma(\widehat{z}_t)' \varepsilon_{t+1} + u'_z \widehat{z}_{t+1} + u'_d \Delta d_{t+1}}) \\ &= e^{u' \mu + \frac{1}{2} u' \Sigma(\widehat{z}_t) \Sigma(\widehat{z}_t)' u + u' [\Phi \widehat{z}_t - \Sigma(\widehat{z}_t) \gamma(\widehat{z}_t)]} \\ &= e^{u' [\mu - \pi(0)] + \frac{1}{2} u' \Sigma(\widehat{z}_t) \Sigma(\widehat{z}_t)' u + u' [\Phi - \pi_1(0)] \widehat{z}_t} \end{aligned} \quad (\text{C.4})$$

where we use a linear approximation of the risk premium  $\pi(\widehat{z}_t)$ , which is necessary for expression (C.4) to reduce to an affine form in the state vector up to a term of second order.

This expression implies that the vector process  $[z; \Delta d]$  is affine under  $\mathbb{Q}$  up to a second-order term, and its risk-neutral dynamics can be written as

$$\begin{bmatrix} \widehat{z}_{t+1} \\ \Delta d_{t+1} \end{bmatrix} = [\mu - \pi(0)] + [\Phi - \pi_1(0)] \widehat{z}_t + \Sigma(\widehat{z}_t) \varepsilon_{t+1}^*$$

where the  $q$ -dimensional shock process  $\varepsilon^*$  is distributed as  $\varepsilon_t^* \sim Niid(0, I_q)$  under the risk-neutral measure  $\mathbb{Q}$ . The first-order effect of the risk-neutral distribution is to increase the drift and the persistence of the states of the world that associate with high marginal utility; symmetrically, it underweights more pleasant states of the world.

We then guess that the price-dividend ratio of the  $n$ -period ahead cashflow strip has the

exponential-affine shape  $P_{d,t}^{(n)}/D_t = e^{A^{(n)}+B_z^{(n)}\widehat{z}_t}$ , and use the no-arbitrage relation

$$P_{d,t}^{(n)} = e^{-r(\widehat{z}_t)} E_t^{\mathbb{Q}} P_{d,t+1}^{(n-1)}, \quad P_{d,t}^{(0)} = D_t$$

to verify the conjecture as

$$\begin{aligned} e^{A^{(n)}+B_z^{(n)}\widehat{z}_t} &= e^{-r(\widehat{z}_t)+A^{(n-1)}} E_t^{\mathbb{Q}} [e^{B_z^{(n-1)}\widehat{z}_{t+1}+\Delta d_{t+1}}] \\ &= e^{-r(\widehat{z}_t)+A^{(n-1)}+u'_{(n-1)}[\mu-\pi(0)]+\frac{1}{2}u'_{(n-1)}\Sigma(\widehat{z}_t)\Sigma(\widehat{z}_t)'u_{(n-1)}+u'_{(n-1)}[\Phi-\pi_1(0)]\widehat{z}_t} \\ &= e^{A^{(n-1)}-r(0)+u'_{(n-1)}[\mu-\Sigma(0)\gamma(0)]+\frac{1}{2}u'_{(n-1)}\Sigma(0)\Sigma(0)'u_{(n-1)}+u'_{(n-1)}[\Phi-\pi_1(0)]\widehat{z}_t+u'_{(n-1)}\Sigma(0)\Sigma_1(0)'(I_{n_z+1}\otimes\widehat{z}_t)u_{(n-1)}-r_1(0)\widehat{z}_t} \end{aligned} \quad (\text{C.5})$$

with  $u_{(n-1)} = [B_z^{(n-1)'}; 1]$ , and where we used the approximate  $\mathbb{Q}$ -dynamics, the loglinearized real risk-free rate, and the linear approximation to  $u'_{(n-1)}\Sigma(\widehat{z}_t)\Sigma(\widehat{z}_t)'u_{(n-1)}$  as the three necessary linearizations in order for expression (C.5) to reduce to an affine form. Thus, we are able to match the corresponding coefficients to verify the initial guess and to identify it as the solution of the quadratic matrix equation (6).

Hold-to-maturity risk premia can be derived from the equilibrium expression for yields and the term structure of anticipated cashflow growth,  $G_{d,t}^{(n)} \equiv E_t^{\mathbb{P}} [D_{t+n}/D_t]$ , which has the recursive structure

$$G_{d,t}^{(n)} = E_t^{\mathbb{P}} \left( \frac{D_{t+1}}{D_t} G_{d,t+1}^{(n-1)} \right)$$

with boundary condition  $G_{d,t}^{(0)} = 1$ , and hence implies  $G_{d,t}^{(n)} = e^{A_g^{(n)}+B_g^{(n)}\widehat{z}_t}$  up to a second-order term.

#### D. Proof of the corollary to proposition 3

Using  $E_t R_{t+1}^m = \sum_{n=1}^{\infty} \omega_{n,t} E_t R_{d,t+1}^{(n)}$ , straightforward expansions yield

$$\begin{aligned} \ln E_t R_{t+1}^m &= \ln \left( \sum_{n=1}^{\infty} \omega_{n,t} e^{\ln E_t R_{d,t+1}^{(n)}} \right) \\ &\approx \sum_{n=1}^{\infty} \omega_n \ln E_t R_{d,t+1}^{(n)} \\ r_{t+1}^m &= \ln \left( \sum_{n=1}^{\infty} \omega_{n,t} e^{r_{d,t+1}^{(n)}} \right) \\ &\approx \sum_{n=1}^{\infty} \omega_n r_{d,t+1}^{(n)} \end{aligned}$$

and we use the main proposition to derive approximate expressions for the equity premium.

### E. Proof of proposition 4

The exponential-affine solution of the Perron-Frobenius eigenfunction problem can be verified:

$$\begin{aligned}\delta e^{u' \widehat{z}_t} &= e^{-r_t} E_t^{\mathbb{Q}} [e^{u' \widehat{z}_{t+1}}] \\ &= e^{-r(0)-r_1(0)\widehat{z}_t + u'[\mu-\pi(0)] + \frac{1}{2}u'\Sigma(\widehat{z}_t)\Sigma(\widehat{z}_t)'u + u'[\Phi-\pi_1(0)]\widehat{z}_t} \\ &= e^{-r(0)-r_1(0)\widehat{z}_t + u'[\mu-\pi(0)] + \frac{1}{2}u'\Sigma(0)\Sigma(0)'u + u'[\Phi-\pi_1(0)]\widehat{z}_t + u'\Sigma(0)\Sigma_1(0)'(I_{n_z} \otimes \widehat{z}_t)u}\end{aligned}$$

where we used a similar strategy as in the proof of proposition 3.

### F. Borovicka-Hansen elasticities

To derive the approximate expressions for shock-exposure and shock-price elasticities, define  $h_{t+1}(r) \equiv r\alpha_t \varepsilon_{t+1} - \frac{r^2}{2} \|\alpha_t\|^2$  and note that, by the law of iterated expectations,

$$\begin{aligned}E_t^{\mathbb{P}} \left[ e^{h_{t+1}(r)} \frac{D_{t+n}}{D_t} \right] &= E_t^{\mathbb{P}} \left[ e^{h_{t+1}(r) + \Delta d_{t+1}} E_{t+1}^{\mathbb{P}} \left( \frac{D_{t+n}}{D_{t+1}} \right) \right] \\ &= E_t^{\mathbb{P}} \left[ e^{h_{t+1}(r) + \Delta d_{t+1}} F_{g,t+1}^{(n-1)} \right] \\ E_t^{\mathbb{P}} \left[ e^{h_{t+1}(r)} M_{t,t+n} \frac{D_{t+n}}{D_t} \right] &= E_t^{\mathbb{P}} \left[ e^{h_{t+1}(r) + m_{t+1} + \Delta d_{t+1}} F_{v,t+1}^{(n-1)} \right]\end{aligned}$$

with the recursive structures

$$\begin{aligned}F_{g,t}^{(n)} &= E_t^{\mathbb{P}} \left[ e^{\Delta d_{t+1}} F_{g,t+1}^{(n-1)} \right], \quad F_{g,t}^{(0)} = 1 \\ F_{v,t}^{(n)} &= E_t^{\mathbb{P}} \left[ e^{m_{t+1} + \Delta d_{t+1}} F_{v,t+1}^{(n-1)} \right], \quad F_{v,t}^{(0)} = 1\end{aligned}$$

Therefore,

$$\begin{aligned}\varepsilon_{g,t} &= \frac{d}{dr} \ln E_t^{\mathbb{P}} \left[ e^{h_{t+1}(r) + \Delta d_{t+1}} F_{g,t+1}^{(n-1)} \right]_{r=0} \\ &= \frac{d}{dr} \ln E_t^{\mathbb{P}} \left[ e^{h_{t+1}(r) + \Delta d_{t+1} + A_g^{(n-1)} + B_g^{(n-1)} \widehat{z}_{t+1}} \right]_{r=0} \\ &= \alpha_t [D(\widehat{z}_t) + B_g^{(n-1)} B(\widehat{z}_t)]' \\ \varepsilon_{p,t} &= \varepsilon_{g,t} - \frac{d}{dr} \ln E_t^{\mathbb{P}} \left[ e^{h_{t+1}(r) + m_{t+1} + \Delta d_{t+1}} F_{g,t+1}^{(n-1)} \right]_{r=0} \\ &= \varepsilon_{g,t} - \frac{d}{dr} \ln E_t^{\mathbb{P}} \left[ e^{h_{t+1}(r) + m_{t+1} + \Delta d_{t+1} + A_d^{(n-1)} + B_{d,z}^{(n-1)} \widehat{z}_{t+1}} \right]_{r=0} \\ &= \alpha_t [\gamma(\widehat{z})' + B_g^{(n-1)} B(\widehat{z}_t) - B_z^{(n-1)} B(\widehat{z}_t)]'\end{aligned}$$