Global Dynamics in a Search and Matching Model of the Labor Market*

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Abstract

We study global and local dynamics of a simple search and matching model of the labor market. We show that the model can be locally indeterminate or have no equilibrium at all, but only for parameterizations that are empirically implausible. In contrast to the local results, we show that the model admits chaotic and periodic behavior for reasonable parameter values. In contrast to earlier literature we establish these results analytically.

JEL Classification:

Key Words: Indeterminacy, Bifurcation, Chaos, Backward Map, Forward Map

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1 Introduction

We study global and local dynamics of a simple search and matching model of the labor market. This modeling framework has proven to be a convenient laboratory for analyzing the joint behavior of unemployment and job vacancies. A particularly convenient facet of the search-and-matching framework is that its dynamic behavior can be reduced to a two-variable system, while still offering rich dynamics. The model is thereby amenable to analytical characterization of its local and global properties. Krause and Lubik (2010) have shown that the simple search-and-matching model can be locally indeterminate or that an equilibrium may not exist. However, these outcomes occur only on the edges of the parameter space that are generally considered to be plausible.

The global dynamics of this framework have been studied by Bhattacharya and Bunzel (2003a,b).[1] They lay out a simple textbook search and matching model, which they solve for the social planner problem. The social planner allocation is a convenient benchmark in the literature as it leads to socially efficient outcomes. They focus on the backward dynamics and show that the simple model has the potential for admitting n-period cycles. Their analysis has been extended by Mendes and Mendes (2008), who use numerical techniques and recent results from the global analysis of forward maps to uncover additional global equilibria. Specifically, they show that the model does allow for chaotic dynamics both in the backward and in the forward dynamics. However, their results are largely numerical and difficult to interpret from an economic perspective.

Our work builds on the papers by Medio and Raines (2007) and Mendes and Mendes (2008).[2] The former authors study backward dynamics in some general economic models while the latter focus on labor market framework that is similar to ours. Mendes and Mendes (2008) show that the backward dynamics in the search and matching model can undergo a period-doubling bifurcation that leads to chaos. However, this result is established under rather strict restrictions on parameter values. Period doubling and existence of periodic

[1] There are substantial differences between Bhattacharya and Bunzel (2003a) and Bhattacharya and Bunzel (2003b). The latter imposes the restriction that job-finding and job-matching rates have to be less than one. Based on this restriction, they then give a result that explosive backward dynamics do not exist. However, this does not rule out stable and chaotic forward dynamics as was later demonstrated by Mendes and Mendes (2008). In their earlier paper, Bhattacharya and Bunzel (2003a) do not impose the said parameteric restriction, which was pointed out by Shimer (2004). In this case they do find various cycles. Our paper can be seen as contribution that unifies and clarifies these previous results.

[2] Mendes and Mendes (2008) restrict their analysis by fixing a large set of parameter values in their model. Their results are thus conditional on these values, whereby they only allow the remaining parameters to vary within a restricted range. Our result is purely analytical; the only restrictions on parameters are for economic reasons. In that sense, our results are significantly stronger, both analytically and in terms of parameter restrictions.

points of period 3 and 5 are demonstrated numerically, rather than analytically. In contrast, we establish existence of periodic and chaotic solutions in the model analytically, without posing any numerical restrictions on parameter values. This significantly extends the range of acceptable parameter values under which cycles and chaos can occur. From a technical perspective, Mendes and Mendes (2008) use symbolic dynamics and inverse limit theory to establish cycles and chaos going forward in time, when backward dynamics exhibits similar behavior. This is done under the same restrictions on parameter values. Using the result established by Kennedy and Stockman (2008), we establish chaotic and periodic solutions in forward time more generally, without imposing numerical restrictions on parameter values.

Our paper differs from these contribution in the following dimensions. First, we consider a wider range of model specifications beyond the social planning solution and risk-neutrality. Specifically, we use the decentralized solution as our benchmark, which introduces an additional degree of parameterization. This turns out to be crucial for generating additional dynamic equilibria both locally and globally. Furthermore, we regard the decentralized allocation as the more plausible since it is validated by empirical studies. While our baseline model assumes risk-neutrality for analytical ease, we also provide some numerical results under risk aversion. Our second contribution is that we provide more extensive analytical results for the local and global dynamics than these earlier contributions, and link them both to the economics and the empirics of the search and matching model.

Two recent papers that are close to ours in spirit are Coury and Wen (2009) and Ernst and Semmler (2010). The former analyze global dynamics in the standard Real Business Cycle (RBC) model. This model can be reduced to a two-equation system, to which the local solution can be found (almost) analytically. It is well known that the basic RBC model has a unique steady state, and that the local dynamics around this steady state are saddle-path stable for the entire admissible parameter region. Using global analysis, Coury and Wen (2009) show that the steady state is surrounded by stable deterministic cycles, which implies global indeterminacy not apparent from a local analysis. Their paper is similar to ours in that we also work in a two-equation environment that is amenable to straightforward, and intuitive, analytical and numerical analysis. Moreover, just as the standard RBC model forms the building block for larger DSGE models that are used for policy analysis, the standard search a model captures the key employment dynamics and optimality condition that are found in more expansive labor market models.

The paper by Ernst and Semmler (2010) is an example of the latter. More specifically it builds a rich environment with capital-based production and investment, where both the labor market and the market for capital is characterized by matching frictions. The authors show that the model has multiple steady states, one of which is a local attractor while another is saddle-path stable. Their analysis is fully numerical based on value-function iteration, whereas we solve the non-linear equilibrium conditions that emerge from the first-order conditions. Moreover, the Ernst and Semmler (2010) model is sufficiently complex to evade simple analytics and intuition. We attempt in our framework to keep the discussion of global analysis as transparent as possible.

The paper is structured as follows. In the next section we describe a simple search and matching model of the labor market and derive the equilibrium conditions that will go into the local and global analysis. Section 3 shows that the model has a unique steady state. We also discuss our calibration approach as it pertains to the interpretation of our results. The following section discusses the local determinacy properties of the model. In section 5 we turn to global dynamics. This section is the central part of the paper, where we study the various global equilibria that arise in different regions of the parameter space. We provide results for both backward dynamics and forward dynamics of the model. We also contrast our findings with those from the local analysis. In section 6 we amend our discussion in three directions. First, we study the corresponding continuous time version of our model; second, we introduce an alternative matching function. The third experiment reintroduces risk-aversion and provides an overview of the global equilibria using numerical methods. The following section conducts an experiment, where we generate data from the global solution of the model under various equilibria, then estimate the locally linear version of the model using likelihood-based methods. The final section concludes.

2 A Simple Search and Matching Model of the Labor Market

We develop a simple version of the search and matching model of the labor market. Our exposition follows Krause and Lubik (2010). Time is discrete. The model period is one quarter. A continuum of identical firms employs workers which inelastically supply one unit of labor.[3] Output y of a typical firm is linear in employment n:

$$y_t = A_t n_t, \tag{1}$$

where A_t is an exogenous aggregate productivity process to be specified later.

Matching between workers and firms is captured by the function $m(u_t, v_t) = m u_t^{\xi} v_t^{1-\xi}$, with unemployment u and vacancies v, and parameters $m > 0$ and $0 < \xi < 1$. It describes

[3]For expositional convenience, we present the problem of a representative firm only. We abstract from indexing the individual variables.

the number of newly formed employment relationships that arise out of the contacts between unemployed workers and firms seeking to fill open positions. Unemployment is defined as:

$$u_t = 1 - n_t, \tag{2}$$

which is the measure of all potential workers in the economy who are not employed at the beginning of the period and are thus available for job search activities.

We can write the law of motion for employment as follows:

$$n_t = (1-\rho)[n_{t-1} + m(u_{t-1}, v_{t-1})], \tag{3}$$

where new hires add to the existing stock of workers. The end-of-the-period workforce is subject to separation at the rate $0 < \rho < 1$.[4] We define $q(\theta_t)$ as the probability of filling a vacancy, or the firm-matching rate, where $\theta_t = v_t/u_t$ is labor market tightness. In terms of the matching function, we can write this as $q(\theta_t) = m(u_t, v_t)/v_t = m\theta_t^{-\xi}$. Similarly, the job-finding rate is $p(\theta_t) = m(u_t, v_t)/u_t = m\theta_t^{1-\xi}$. An individual firm is atomistic in the sense that it takes the aggregate matching rate $q(\theta_t)$ as given. The employment constraint on the firm's decision problem is therefore linear in vacancy postings:

$$n_t = (1-\rho)[n_{t-1} + v_{t-1}q(\theta_{t-1})]. \tag{4}$$

Firms maximize profits, using the discount factor $\beta^t \frac{\lambda_t}{\lambda_0}$ (to be determined below):

$$\begin{aligned} \max_{\{v_t, n_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t \frac{\lambda_t}{\lambda_0} \left[A_t n_t - w_t n_t - \kappa v_t\right] + \\ & + \sum_{t=0}^{\infty} \beta^t \frac{\lambda_t}{\lambda_0} \mu_t \left[(1-\rho)[n_{t-1} + v_{t-1}q(\theta_{t-1})] - n_t\right]. \end{aligned} \tag{5}$$

Wages paid to the workers are w, while $\kappa > 0$ is a firm's cost of posting a vacancy. μ is the Lagrange-multiplier on the firm's employment constraint. It can be interpreted as the marginal value of a filled position. Firms decide on how many vacancies to post and how many workers to hire. The first order conditions are:

$$n_t \quad : \quad \mu_t = A_t - w_t + \beta(1-\rho)\frac{\lambda_{t+1}}{\lambda_t}\mu_{t+1}, \tag{6}$$

$$v_t \quad : \quad \kappa = \beta(1-\rho)\frac{\lambda_{t+1}}{\lambda_t}\mu_{t+1}q(\theta_t), \tag{7}$$

[4]Note that newly matched workers that are separated from their job within the period reenter the matching pool immediately.

which imply the job creation condition:

$$\frac{\kappa}{q(\theta_t)} = (1-\rho)\beta\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\left[A_t - w_{t+1} + \frac{\kappa}{q(\theta_{t+1})}\right]. \tag{8}$$

This optimality condition trades off expected hiring cost $\kappa/q(\theta_t)$ against the benefits of a productive match. This consists of the output accruing to the firm net of wage payments and the future savings on hiring costs when the current match is successful.

As is common in the literature, we assume that the economy is populated by a representative household. The household is composed of workers, who are either unemployed or employed. If they are unemployed they are compelled to search for a job, but they can draw unemployment benefits b. Employed members of the household receive pay w, but share this with the unemployed. They do not suffer disutility from working and supply a fixed number of hours.[5] Since the household's only choice variable is consumption, and there is no mechanism to transfer resources intertemporally, the utility maximization problem is trivial. Assuming constant relative risk aversion this determines the marginal utility of wealth, $\lambda_t = C_t^{-\sigma}$, where C is consumption and σ^{-1} is the intertemporal elasticity of substitution. In equilibrium, total income accruing to the household equals net output in the economy, which is composed of production less real resources lost in the search process, $Y_t = y_t - \kappa v_t$. Since $C_t = Y_t$, we can now derive the stochastic discount factor $\beta^t \frac{\lambda_t}{\lambda_0} = \beta^t \frac{Y_t^{-\sigma}}{Y_0^{-\sigma}}$.

Finally, we need to derive how wages are determined. We assume that wages are set according to the Nash bargaining solution.[6] As this is a lengthy, but standard, derivation, we relegate discussion to an Appendix. The Nash-bargained wage is thus:

$$w_t = \eta\left(A_t + \kappa\theta_t\right) + (1-\eta)b. \tag{9}$$

This equation can be substituted into the job-creation condition (JCC) to derive:

$$\frac{\kappa}{m}\theta_t^{\xi} = \beta(1-\rho)\frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}}\left[(1-\eta)\left(A_t - b\right) - \eta\kappa\theta_{t+1} + \frac{\kappa}{m}\theta_{t+1}^{\xi}\right]. \tag{10}$$

This completes the description of the model.

[5] We thus assume income pooling between employed and unemployed households, and abstract from potential incentive problems concerning labor market search. This allows us to treat the labor market separate from the consumption choice. See Merz (1995) and Andolfatto (1996) for discussion of these issues.

[6] This is a standard assumption in the literature. Shimer (2005) provides further discussion.

3 Preliminaries: Steady State and Calibration

3.1 Steady State

We first establish that the model has a unique steady state. Steady state θ_{SS} solves the following non-linear equation:

$$\theta_{SS}^{\xi} - \beta(1-\rho)\theta_{SS}^{\xi} = \beta(1-\rho)(1-\eta)m\frac{A-b}{\kappa} - \beta(1-\rho)\eta m\theta_{SS}, \tag{11}$$

which is derived from the JCC (10) after rearranging terms. We now prove the following Lemma.

Lemma 1 *The job creation condition has a unique steady state θ_{SS}.*

Proof. Consider the left-hand side and the right-hand side of the above equation separately. The left-hand side $f_1 = [1-\beta(1-\rho)]\,\theta_{SS}^{\xi}$ has an intercept at the origin and is strictly increasing in θ_{SS} since $1-\beta(1-\rho) > 0$. The right-hand side $f_2 = \beta(1-\rho)(1-\eta)m\frac{A-b}{\kappa} - \beta(1-\rho)\eta m\theta_{SS}$ is linear in θ_{SS} and strictly decreasing with a positive intercept. It therefore follows that the two functions intersect once and that there is a unique steady state θ_{SS}. ■

We thus find that the simple search and matching model does not suffer from the multiple steady state problem identified by Benhabib et al. (2001) in a monetary model with an interest rate feedback rule for monetary policy. They show that the interaction of the Fisher-equation, that is, the relationship between nominal and real interest rates and expected inflation, with an ad-hoc policy rule results in the existence of two steady states, one stable and one unstable globally. The authors show that the globally unstable steady state is locally saddle path stable and is actually the one that is imposed in linearized analyses. They therefore argue that policy recommendations based on local analysis can be perilous in the global context. This is not an issue here. Our object of investigation is whether the unique steady state is locally and globally stable or unstable and whether there are chaotic endogenous dynamics.

The remaining steady state values can be computed in a straightforward manner. The steady-state unemployment rate u_{SS} can be computed from the law of motion for employment; that is, $\frac{\rho}{1-\rho}\frac{1-u_{SS}}{u_{SS}} = m\theta_{SS}^{1-\xi}$. The rest of the variables then follow immediately. Given the specific type of non-linearity of the steady-state JCC there is no closed-form solution for θ_{SS}. Instead, we have to compute the value of θ_{SS} numerically for a given parameterization. This makes it more burdensome to study the dynamic properties of the model since they have to be evaluated for each new set of parameter values. An alternative to computing the steady state values directly is to target specific steady state values and thereby impute the implied values of fixed parameters. If done judiciously, this would help avoid issues with non-linearity when solving for the steady state. We describe such an approach in the next section when we discuss calibration.

3.2 Calibration

We now describe our benchmark choice for parameter values which we use in the numerical analysis of the local and global properties. In our baseline specification we assume, as in Shimer (2005), that households are risk-neutral, that is, $\sigma = 0$. This simplifies derivation of analytical results considerably. We also conduct a robustness check in which we impose risk-aversion with $\sigma = 1$. In this case, almost all of our results are of a numerical nature.

We pursue two different strategies to assign numerical values to the structural model parameters. One strategy sets the parameters values directly. The advantage is that we can directly determine the impact of any parameter changes on the behavior of the model. A drawback is that there is not much independent information available for some of the parameters, and certain parameter choices can lead to a priori implausible steady-state values. We can address this issue by imposing plausible bounds on parameter values. Nevertheless, the steady state of the model is given by a unique mapping from the structural parameters to the endogenous variables, as we saw above.

However, the mapping from the parameters to objects of interests may not admit an analytical solution, of which one example is the computation of the steady state. Our alternative strategy treats endogenous variables as parameters to be calibrated. In order to obtain a specific target value a parameter thus needs to be adjusted endogenously. The advantage of this approach is that a judicious choice of fixing steady state values can allow analytical solutions.

We set the discount factor $\beta = 0.99$ and normalize the productivity level $A = 1$. The separation rate $\rho \in (0, 1)$. A typical value for quarterly data is $\rho = 0.1$, which is consistent with the evidence reported in Shimer (2005). The bargaining parameter $\eta \in (0, 1)$. The vast majority of the literature assumes $\eta = 0.5$, as independent observations on its value are not obvious to obtain. The match elasticity $\xi \in (0, 1)$. In a well-known study, Petrongolo and Pissarides (2001) advocate values between 0.5 and 0.7. The plausibility of this range is supported by the evidence in Lubik (2013). However, values outside this range can be considered as well. The level parameter in the matching function $m > 0$ can be used to scale, for instance, the unemployment rate, but it is otherwise left unrestricted in the literature.

However, we restrict this parameter to obey $m \in (0,1)$ based on the following argument.

The job matching and job finding rates are defined as, respectively, $q(\theta) = m\theta^{-\xi}$ and $p(\theta) = m\theta^{1-\xi}$. These should properly be interpreted as the probabilities of a firm filling a vacancy and a worker finding a job. It is a quirk of the discrete-time matching model that mathematically these variables can take on values above one. Intuitively, at a low enough frequency, everyone in the pool of searchers will transition out of unemployment at least once, which translates into a job finding rate of above one. While this is conceptually valid - the rate counts the number of new matches per searchers over a long enough period -, it violates the spirit of the search and matching model in that successful matching is probabilistic. We note that this is not a problem for the continuous time version of the search and matching model since $q(\theta)$ and $p(\theta)$ are instantaneous transition rates and thus are true probabilities. In what follows we therefore restrict these rates to lie on the unit interval (see also Bhattacharya and Bunzel, 2003b, and Shimer, 2004). The following Lemma establishes the necessary parameteric restriction.

Lemma 2 *The transition rates $q(\theta)$ and $p(\theta)$ are less than one if $m < 1$.*

Proof. Define $q(\theta) = m\theta^{-\xi}$ and $p(\theta) = m\theta^{1-\xi}$. $q(\theta) < 1$ implies $\theta > \left(\frac{1}{m}\right)^{-1/\zeta}$; $p(\theta) < 1$ implies $\theta < \left(\frac{1}{m}\right)^{1/(1-\xi)}$. For both transition rates to be less than one, this requires: $\left(\frac{1}{m}\right)^{-1/\zeta} < \theta < \left(\frac{1}{m}\right)^{1/(1-\xi)}$. This is a non-empty interval for θ if $m < 1$. ■

As for the remaining parameters, benefits $b \in (0, A)$ since they cannot exceed the marginal product of the firm, in which case a firm could not offer any wage that would induce an unemployed person to work. Given our normalization $A = 1$ this restricts b to the unit interval. Typical values in the literature range from $b = 0.4$ (Shimer, 2005) to $b = 0.9$ (Hagedorn and Manovskii, 2008). Vacancy posting cost $\kappa > 0$. It is a scale variable that can be measured in terms of resource loss as a percentage of GDP. Typical values are in the low percentage points.

The alternative strategy would fix the steady-state value of an easily observable variable, which is then used to back out the value of a parameter. For instance, it is often convenient to calibrate the steady state unemployment rate u_{SS}. Using the law of motion for employment (3), we can then compute $\theta_{SS} = \left(\frac{1}{m}\frac{\rho}{1-\rho}\frac{1-u_{SS}}{u_{SS}}\right)^{1/(1-\xi)}$ without having to solve the non-linear equation (11). Similarly, we can fix the job finding rate $p_{SS} = p(\theta_{SS})$, which implies $\theta_{SS} = (p_{SS}/m)^{1/(1-\xi)}$. The JCC then delivers the following restriction on the imputed parameter: $\frac{A-b}{\kappa} = \frac{\eta}{1-\eta}\theta_{SS} + \frac{1}{1-\eta}\frac{1-\beta(1-\rho)}{\beta(1-\rho)}\frac{\theta_{SS}^{\xi}}{m}$, from which we can obtain either b, κ, or even η. We note that in this expression b and κ are not separately identifiable. However, the term $\frac{A-b}{\kappa}$ scales various expressions, and we will discuss its importance further below.

In terms of numerical values assigned to the steady-state values, u_{SS} can be chosen to correspond to observed sample means, which typically is around 5%. An alternative approach is target the observed employment rates, which would imply an unemployment rate that is much higher, for instance, 25%. Both approaches have been used in the literature, with different implications for the dynamic behavior of the calibrated model.[7] In choosing the steady state job matching rate we follow den Haan et al. (2000) who set $\overline{q} = 0.7$.

4 Local Dynamics

The local dynamics of the simple search and matching model have been studied by Krause and Lubik (2010). We refer to their paper for more detail. We proceed in two steps. First, we study the local properties of the benchmark specification with risk-neutral households. This can be done analytically since the model has a recursive structure, which allows for (almost) closed-form solutions for the eigenvalues. In the second step, we compute the local properties under risk aversion using numerical techniques.

In our benchmark specification , we consider the case $\sigma = 0$. The job creation condition then becomes:

$$\frac{\kappa}{m}\theta_t^{\xi} = \beta(1-\rho)\left[(1-\eta)(A-b) - \eta\kappa\theta_{t+1} + \frac{\kappa}{m}\theta_{t+1}^{\xi}\right]. \tag{12}$$

We now linearize this equation around the steady state, which results in:

$$\widehat{\theta}_t = \beta(1-\rho)\left(1 - \frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right)\widehat{\theta}_{t+1}, \tag{13}$$

where $\widehat{\theta}_t = \theta_t - \theta_{SS}$ is the deviation from the steady state θ_{SS}.[8] This is an autonomous first-order linear difference equation in θ, the dynamic properties of which depend on the coefficient $\beta(1-\rho)\left[1 - \frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right]$. Since this is a forward-looking equation, a unique and determinate equilibrium requires that the eigenvalue lies within the unit circle (see Blanchard and Kahn, 1980). More formally, we establish the following Theorem.

Theorem 3 *The equilibrium dynamics of the job-creation condition is locally unique if*

$$0 < p(\theta_{SS}) < \frac{1+\beta(1-\rho)}{\beta(1-\rho)}\frac{\xi}{\eta}.$$

[7] The idea is to capture both measured unemployment in terms of recipients of unemployment benefits and potential job searchers that are only marginally attached to the labor force, but are open to job search. Since we do not model labor force participation decisions, this is a shortcut to capturing effective labor market search. This approach has been taken by Cooley and Quadrini (1999) and Trigari (2009).

[8] *Log*-linearizing this equation around the steady state would result in the same dynamic properties.

The equilibrium dynamics are locally indeterminate if

$$1 > p(\theta_{SS}) > \frac{1+\beta(1-\rho)}{\beta(1-\rho)} \frac{\xi}{\eta}$$

Proof. The equilibrium is locally unique if $\left|\beta(1-\rho)\left(1-\frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right)\right| < 1$. Consider the boundaries in turn. Denote $p(\theta_{SS}) = m\theta_{SS}^{1-\xi}$. $\beta(1-\rho)\left(1-\frac{\eta}{\xi}p(\theta_{SS})\right) < 1$ implies $p(\theta_{SS}) > 0 > -\frac{\xi}{\eta}\frac{1-\beta(1-\rho)}{\beta(1-\rho)}$, which is always true. Second, $-1 < \beta(1-\rho)\left(1-\frac{\eta}{\xi}p(\theta_{SS})\right)$ implies $p(\theta_{SS}) < \frac{\xi}{\eta}\frac{1+\beta(1-\rho)}{\beta(1-\rho)}$. Since $0 < p(\theta_{SS})$ this proves the first part of the Theorem. The equilibrium is indeterminate if $\left|\beta(1-\rho)\left(1-\frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right)\right| > 1$ ■

The Theorem is stated in terms of boundary conditions for the job finding rate $p(\theta_{SS})$. We find this useful for developing intuition as to the determinants of local and global dynamics in the model. Moreover, as we pointed out above, the search and matching model is in practice often calibrated in terms of target values for job matching and job finding rates. The Theorem establishes that for a wide range of parameter values the dynamic equilibrium is locally unique. Since the forward dynamics, obtained by iterating the linearized JCC forward, are stable, the unique equilibrium is $\widehat{\theta}_t = 0$, which puts labor market tightness at its steady-state value as the only possible equilibrium. The steady state θ_{SS} is locally unstable in this case. As time goes forward, the path for tightness will become unbounded unless the economy is placed on the initial condition $\theta_0 = \theta_{SS}$. Starting values in a small neighborhood of the steady state will lead to explosive paths.

An alternative way to see this is by inverting the linearized JCC. This implies the backward-looking representation $\widehat{\theta}_{t+1} = \left[\beta(1-\rho)\left(1-\frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right)\right]^{-1}\widehat{\theta}_t$. The root of this representation is the inverse of the root of the forward equation. Forward stability therefore implies backward instability, and vice versa. Given the parameteric restrictions established in the Theorem, the JCC would have explosive dynamics if expressed backward. Consequently, the only solution to be consistent with local stability is $\widehat{\theta}_t = 0$. One important insight is that in the linear case the roots of the forward- and the backward-representation of the difference equation in question are the inverse of each other. Local analysis can therefore rely on either representation. This is in general not the case for global dynamics, as we will see below.

The equilibrium is locally indeterminate if the job-finding rate is high enough. In this case, the JCC is a stable difference equation, one solution of which is:

$$\widehat{\theta}_{t+1} = \left[\beta(1-\rho)\left(1-\frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right)\right]^{-1}\widehat{\theta}_t. \tag{14}$$

The steady state is therefore an attractor. All paths with starting values in a small neighborhood around θ_{SS} converge to it. The different adjustment paths are indexed by their starting values θ_0^i, which correspond to the various dynamic equilibria.

The threshold for switching from a unique to an indeterminate local equilibrium is given by the term $\frac{1+\beta(1-\rho)}{\beta(1-\rho)}\frac{\xi}{\eta}$. High enough job-finding rates result in indeterminacy. A special case is when $\xi = \eta$. This parameterization implements the so-called Hosios-condition, under which the decentralized allocation is identical to the social planner solution. In this case, $p(\theta_{SS}) < 1 < \frac{1+\beta(1-\rho)}{\beta(1-\rho)}$, and indeterminacy can never arise, which is the main finding by Bhattacharya and Bunzel (2003a,b). However, the Hosios-condition is empirically violated as the literature has amply demonstrated, and we do not regard this as a likely parameterization.[9] The threshold is tightened as the term $\frac{\xi}{\eta}$ becomes smaller. Low values of the match elasticity and high values for the bargaining share are therefore more likely to imply indeterminacy. For instance, for $\beta = 0.99$, $\rho = 0.1$, $\xi = 0.4$, and $\eta = 0.9$, the threshold coefficient is 0.94. den Haan et al. (2000) report an estimate for $p(\theta)$ of 0.45. Although this is far away from the threshold, we nevertheless regard the possibility of local indeterminacy as more than a curiosity.

5 Global Dynamics

We now turn to an analysis of the global dynamics. We proceed in three steps. We first provide some general insights into the properties of the non-linear search and matching model and set up the map that we use to study global equilibria. The second step analyzes the backward dynamics of the model, while the third step considers the dynamics of the forward representation.

We analyze the benchmark case of an economy with risk-neutral agents. This implies the job-creation condition which we replicate here for convenience:

$$\frac{\kappa}{m}\theta_t^{\xi} = \beta(1-\rho)\left[(1-\eta)(A-b) - \eta\kappa\theta_{t+1} + \frac{\kappa}{m}\theta_{t+1}^{\xi}\right]. \tag{15}$$

This is an autonomous first-order non-linear difference equation in θ. It describes the evolution of labor market tightness θ_t, which can be solved independently from the rest of the model. As in the discussion of the local dynamics, this allows to study the properties of θ_t in isolation. With risk aversion, the dynamics would depend on the time path of output y_t. This, in turn, is a function of employment n_t, which evolves based on the law of motion

[9] See Lubik (2013) and the references cited therein.

(3). Since this feeds back onto the job creation condition via the definition of $\theta_t = v_t/u_t$, it results in an interconnected two-equation system that cannot be solved analytically.

We rewrite the JCC by isolating terms in θ_{t+1} on the left-hand side:

$$\beta(1-\rho)\theta_{t+1}^{\xi} - \beta(1-\rho)m\eta\theta_{t+1} = \theta_t^{\xi} - \beta(1-\rho)(1-\eta)m\frac{A-b}{\kappa}, \tag{16}$$

or in coefficient form:

$$a\theta_{t+1}^{\xi} - c\theta_{t+1} = \theta_t^{\xi} - d, \tag{17}$$

where:

$$\begin{aligned} 0 &< a = \beta(1-\rho) < 1, \\ 0 &< c = \beta(1-\rho)m\eta < 1, \\ 0 &< d = \beta(1-\rho)(1-\eta)m\frac{A-b}{\kappa}. \end{aligned}$$

Equation (17) defines the forward map of the model. It maps θ_t into θ_{t+1}. Similarly, we can define the backward map $g(\theta)$ by rearranging (17) to isolate θ_t:

$$\theta_t = \left(a\theta_{t+1}^{\xi} - c\theta_{t+1} + d\right)^{1/\xi} = g\left(\theta_{t+1}\right). \tag{18}$$

The forward map is not invertible; that is, we cannot express this relationship as one of θ_{t+1} into θ_t. In other words, the forward dynamics are captured by a correspondence, whereas the backward dynamics are well defined. It is therefore much more convenient in this case to study the global properties using the backward dynamics. We will turn to this in the next section.[10] It is straightforward to see that the steady state of the forward and the backward map is the same as established in Theorem 1; that is, $\theta_{SS} = g\left(\theta_{SS}\right)$. We also note that the coefficients are independent of the match elasticity ξ, which therefore only determines the shape of the mapping but not its location in (θ_{t+1}, θ_t)-space. Furthermore, the term $\frac{A-b}{\kappa}$scales the intercept d, but does not affect other coefficients.

We also find it convenient to introduce another representation of the backward map. This involves a change of variables such that for $\theta_t \geq 0$:

$$z_t = \theta_t^{\xi}. \tag{19}$$

[10]The relationship between the backward and forward dynamics of non-linear system is an area of active research (see, for example, Kennedy and Stockman (2008) and references thereof). This distinction is immaterial for the study of linear systems since they are always invertible. That is, the properties of the forward dynamics are the 'inverse' of the properties of the backward map. If on the other hand, one of the dynamic maps is a correspondence this equivalence fails.

This allows us to rewrite the equation in (17) as the following backward recurrence relation:

$$z_t = az_{t+1} - cz_t^{1/\xi} + d, \tag{20}$$

whereby the coefficients are as defined above. Backward solutions of (17) and (20) are well defined for any $\xi \in (0,1)$, if z_t and θ_t are non-negative. Note that this condition is also necessary for the solutions to be economically meaningful. As long as the solutions of (20) are nonnegative, the solutions of (17) can be obtained by using $\theta_t = z_t^{1/\xi}$. The model in the form of (20) provides us with a more convenient means of studying periodic and chaotic solutions.

5.1 Backward Dynamics

We now study the dynamics of the backward map $\theta_t = g(\theta_{t+1})$. We first establish the properties of the function g. We then study the stability properties of the steady state, where we distinguish between two broad areas of dynamics in the backward map, namely stable and unstable. We focus on the bifurcation that occurs when the dynamics switch between the two regions. Finally, we establish the presence of chaotic dynamics in the backward map.

5.1.1 Preliminary Results

The backward dynamics are governed by the properties of the function $g(\theta) = \left(a\theta^{\xi} - c\theta + d\right)^{1/\xi}$. We first establish some analytical results. $g(0) = d^{1/\xi} > 0$ defines the intercept. $g(\theta_0) = 0$ implicitly defines the unique intersection point θ_0 where $a\theta_0^{\xi} = c\theta_0 - d$.[11] The first derivative of g is given by:

$$g'(\theta) = \left(a\theta^{\xi} - c\theta + d\right)^{\frac{1-\xi}{\xi}} \left(a\theta^{\xi-1} - \frac{c}{\xi}\right),$$

while the second derivative is:

$$g''(\theta) = (1-\xi)\left(a\theta^{\xi} - c\theta + d\right)^{\frac{1-\xi}{\xi}} \left[\frac{\left(a\theta^{\xi-1} - \frac{c}{\xi}\right)^2}{a\theta^{\xi} - c\theta + d} - a\theta^{\xi-2}\right].$$

g has a global maximum at $g'(\theta_{\max}) = 0$, where $\theta_{\max} = \left(\frac{a\xi}{c}\right)^{\frac{1}{1-\xi}} > 0$, since $g''(\theta_{\max}) < 0$, as is easily verified. The function g has a saddle point at θ_0 since $g'(\theta_0) = g''(\theta_0) = 0$. Since $g''(\theta_{\max}) < 0$, $g''(\theta_0) = 0$, and $\theta_{\max} < \theta_0$, there is a second inflection point $\theta_{\inf}$ at which $g''(\theta_{\inf}) = 0$, and $\theta_{\max} < \theta_{\inf} < \theta_0$.

[11] It is straightforward to show that θ_0 is unique. Both $f_1 = a\theta_0^{\xi}$ and $f_2 = c\theta_0 - d$ are strictly increasing and concave over $[0,\infty)$. Since $f_1(0) = 0 > f_2(0) = -d$, the two functions intersect only once at θ_0.

We can express the maximum of g in terms of the structural parameters of the model: $\theta_{\max} = \left(\frac{a\xi}{c}\right)^{\frac{1}{1-\xi}} = \left(\frac{1}{m}\frac{\xi}{\eta}\right)^{\frac{1}{1-\xi}}$. $\theta_{\max}$ only depends on three parameters, which reduce to two in terms of the match efficiency m under the Hosios-condition. In this special case, $\theta_{\max} > 1$ since $m < 1$. In the general case, $\theta_{\max}$ can be less than one if $m > \frac{\xi}{\eta}$. Notably, the location of the maximal point does not depend on other parameters, chiefly the scale term $\frac{A-b}{\kappa}$.

In what follows, we discuss two separate regions for the support of the backward map: (i) for $0 < \theta < \theta_{\max}$, $g'(\theta) > 0$; (ii) for $\theta_{\max} < \theta < \theta_0$, $g'(\theta) < 0$. The monotonicity of the map within the two regions follows immediately from the previously established results. We now analyze the stability properties of the steady state for each region in turn. These properties are determined around the thresholds ± 1 and are given by the first derivative $g'(\theta_{SS})$. If $|g'(\theta_{SS})| < 1$, the map is stable in its backward dynamics, and unstable otherwise. At $|g'(\theta_{SS})| = 1$ a bifurcation occurs, and the dynamics switch from stable to unstable. We note for analytical tractability that substituting the implicit definition of the steady state $\theta_{SS}^{\xi} = a\theta_{SS}^{\xi} - c\theta_{SS} + d$ into the first derivative $g'(\theta)$ and collecting terms results in $g'(\theta_{SS}) = a - \frac{c}{\xi}\theta_{SS}^{1-\xi}$.

For $0 < \theta_{SS} < \theta_{\max}$, $g'(\theta_{SS}) > 0$. $g'(\theta_{SS}) < 1$ implies $\theta_{SS}^{1-\xi} > 0 > -(1-a)\frac{\xi}{c}$ which always holds. This rules out $g'(\theta_{SS}) \geq 1$. If the steady state is to the left of the maximal point in the map, then the backward dynamics will be always stable, and the steady state is an attractor in the backward map. This implies that the forward dynamics are unstable under this parameterization and the steady state is globally unstable. Hence, the only global solution is $\theta_t = \theta_{SS}$. We also note that the adjustment dynamics of the backward map are monotonic since $0 < g'(\theta_{SS}) < 1$. Since a bifurcation cannot occur in this region of the map, we will focus our attention on the other case.

For $\theta_{\max} < \theta_{SS} < \theta_0$, $g'(\theta_{SS}) < 0$, so that the steady state is to the right of the maximum. If $-1 < g'(\theta_{SS}) < 0$ the steady state is again stable in the backward dynamics and is a global attractor, which renders the forward dynamics unstable. Consequently, for this parameterization the steady state is again globally unstable and the only solution that obeys transversality. Adjustment dynamics in the backward map are oscillatory, however, since the first derivative is negative.

If $g'(\theta_{SS}) < -1$ the backward dynamics are unstable. This can be expressed as

$$g'(\theta_{SS}) = \left(a\theta_{SS}^{\xi} - c\theta_{SS} + d\right)^{\frac{1-\xi}{\xi}} \left(a\theta_{SS}^{\xi-1} - \frac{c}{\xi}\right) < -1$$

which also implies $\theta_{SS} > \left(\frac{1+a}{c}\xi\right)^{1/(1-\xi)}$. This translates into stable adjustment dynamics for the forward map. The solution to the model is such that starting from an initial value, the economy will converge non-monotonically (since $g'(\theta_{SS}) < 0$) to the steady state, which is a global attractor in the forward dynamics. The law of motion is given by the correspondence (17). This is the counterpart to the case identified for local dynamics under indeterminacy, which required a high enough job-finding rate or a high steady-state value θ_{SS}, Similarly, in the global case θ_{SS} needs to be high enough.

5.1.2 Bifurcation Analysis

We can now provide further insight into the determinants of the various equilibria by substituting back in the structural parameters. We showed above that the location of $\theta_{\max} = \left(\frac{1}{m}\frac{\xi}{\eta}\right)^{\frac{1}{1-\xi}}$ is determined by only three parameters. We therefore find it convenient to analyze the equilibria in terms of the other model parameters, which affect the type of equilibria and the shape of the map. Furthermore, we keep the separation rate ρ and the discount factor β fixed for the purposes of this analysis. This leaves the scale term $\frac{A-b}{\kappa}$ as the crucial coefficient to analyze, whereby we normalize $A = 1$. It can be ascertained immediately that $\frac{A-b}{\kappa}$ shifts the map $g(\theta)$ vertically, thereby changing the location of the steady state θ_{SS} and the intercept θ_0 with the zero line. The shape of the map, however, is unaffected.

Furthermore, we also find it convenient to describe the analytical properties of the map in terms of the steady state values of endogenous parameters. Specifically, we implement the alternative calibration strategy discussed above. We thus consider the steady state unemployment rate u_{SS} a parameter to be calibrated. We can then back out the implied $\theta_{SS} = \left(\frac{1}{m}\frac{\rho}{1-\rho}\frac{1-u_{SS}}{u_{SS}}\right)^{1/(1-\xi)}$ from the law of motion for employment. This restricts the parameterization of either b or κ based on the steady-state JCC, $\frac{A-b}{\kappa} = \frac{\eta}{1-\eta}\theta_{SS} + \frac{1}{1-\eta}\frac{1-\beta(1-\rho)}{\beta(1-\rho)}\frac{\theta_{SS}^{\xi}}{m}$ and leaves one remaining parameter for which we can do comparative static analysis. Intuitively, this approach targets a specific unemployment rate by setting, for instance, benefits b at a specific level. Instead of discussing the effects of changes in this parameter on the implied u_{SS}, this reparameterization allows us a more direct and economically intuitive consideration.

We now establish the following Lemma, which describes the regions of the parameter space for which the first derivative of the map g is negative. We concentrate on this case since it admits a bifurcation.

Lemma 4 *For $\theta_{\max} < \theta_{SS} < \theta_0$, $g'(\theta_{SS}) < 0$. We distinguish three different regions of the parameter space:*

1. $0 > g'(\theta_{SS}) > -1$:

$$\frac{1}{1+\frac{\beta^{-1}+(1-\rho)}{\rho}\frac{\xi}{\eta}} < u_{SS} < \frac{1}{1+\frac{1-\rho}{\rho}\frac{\xi}{\eta}}$$

2. $g'(\theta_{SS}) = -1$:

$$u_{SS} = \frac{1}{1+\frac{\beta^{-1}+(1-\rho)}{\rho}\frac{\xi}{\eta}}$$

3. $-1 > g'(\theta_{SS})$:

$$u_{SS} < \frac{1}{1+\frac{\beta^{-1}+(1-\rho)}{\rho}\frac{\xi}{\eta}}$$

5.1.3 Periodic and Chaotic Solutions

We establish existence of periodic and chaotic solutions with the help of the backward recurrence in (20), i.e.

$$z_t = az_{t+1} - cz_t^{1/\xi} + d \tag{21}$$

From (21), we define the map

$$f(z) = az - cz^{1/\xi} + d \tag{22}$$

Since $f(0) = d > 0$, and $f'(z) = a - \frac{c}{\xi}z^{\frac{1-\xi}{\xi}}$, then f attains a maximum at

$$\left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}} := z_{max} \tag{23}$$

Given that f is increasing on $[0, z_{max})$ and decreasing on (z_{max}, ∞), the map f is a Type-B map as defined by Medio and Raines (2007). In the Appendix, sufficient conditions are established for existence of periodic and chaotic solutions of systems defined by these class of maps. These results are in Theorems 15, 16 and Corollary 17 and can be readily applied to the map in (22). In particular, Theorems 15 and 16 require that $z_{max} > d$, and $f(z_{max}) \leq q$ where $q > z_{max}$ is the preimage of 0, i.e. $f(q) = 0$.

For the condition $f(z_{max}) = q$, it is necessary that the second iterate of f to map to zero, i.e. $f^2(z_{max}) = 0$. Similarly $f(z_{max}) \leq q$ implies that $f^2(z_{max}) \geq 0$. Now

$$f(z_{max}) = a\left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}} - c\left(\frac{a\xi}{c}\right)^{\frac{1}{1-\xi}} + d$$

and

$$f^2(z_{max}) = a^2\left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}} - ac\left(\frac{a\xi}{c}\right)^{\frac{1}{1-\xi}} + ad - c\left[a\left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}} - c\left(\frac{a\xi}{c}\right)^{\frac{1}{1-\xi}} + d\right]^{\frac{1}{\xi}} + d$$

This gives us the following inequalities

$$d < (\frac{a\xi}{c})^{\frac{\xi}{1-\xi}} \tag{24}$$

and

$$a^2\left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}} - ac\left(\frac{a\xi}{c}\right)^{\frac{1}{1-\xi}} + (1+a)d - c\left[a\left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}} - c\left(\frac{a\xi}{c}\right)^{\frac{1}{1-\xi}} + d\right]^{\frac{1}{\xi}} \geq 0 \tag{25}$$

The following result is a corollary to Theorem 15 that establishes invariance and existence of nonnegative solutions globally:

Theorem 5 *If (25) holds, then the interval* $[0, q]$ *is invariant under* f*, where* $f : \mathbb{R} \to \mathbb{R}$ *is the map defined in (22) with a maximum critical point* z_{max} *given in (23) and* $q > z_{max}$ *is the point such that* $f(q) = 0$*.*

Note that Theorem 5 establishes sufficient conditions for existence of solutions of (21) and therefore, for (17), since solutions obtained from initial values in the interval $[0, q]$ stay bounded above by zero.

The main result of this section is on existence of periodic and chaotic solutions.

Theorem 6 *Let* a, c, ξ, d *satisfy (24) and further assume that the inequality in (25) is binding. Then the equations in in 20) and (17) have periodic solutions of all possible periods, as well as chaotic solutions in the sense of Li and Yorke, Block and Coppell, and Devaney.*

When the inequality in (25) becomes binding, it defines an implicit surface in four dimensional parameter space for a, d, c, ξ. For any fixed values of two of these parameters, one can plot the implicit curve in two dimensions to demonstrate ranges in parameters that give rise to periodic and chaotic solutions for infinitely many values of the remaining two parameters. This is an important improvement over the results obtained in Mendes and Mendes (2008) that demonstrate existence of chaotic solutions for (17) under very restrictive parameter values.

5.2 Forward Dynamics

In this section, we connect periodic and chaotic solutions found in the backward dynamics of the model to its forward dynamics. From the equation in (17), it is apparent that the dynamics of the system are multi-valued going forward - i.e. the backward map defining

(17) is not invertible. For establishment of periodic solutions in the forward map, existence of periodic solutions in the backward map is sufficient. To establish chaotic solutions that lead into the future, we use the following result from Kennedy and Stockman (2008). Using the same notation as in Kennedy and Stockman (2008), the map f^{-1} is defined for the map f on a metric space X with $f : X \to X$, regardless whether f is multi-valued or not. Their main result states:

Theorem 7 Let $f : X \to X$ be continuous on a metric space X. Then f is chaotic on X in the sense of Devaney if and only if f^{-1} is chaotic on X.

The above theorem is an important result showing that models with backward dynamics are chaotic going forward in time if and only if they are chaotic going backward in time. Hence, establishment of chaotic solutions in the previous section in backward dynamics is sufficient for existence of chaotic forward dynamics.

6 Extensions [To be written]

6.1 A Continuous Time Version of the Search and Matching Model

6.2 An Alternative Matching Function

6.3 Introducing Risk Aversion

7 Application: Local Estimation of Global Data

In this section, we discuss some implications of the differences between local and global equilibrium analysis for the empirics of the search and matching model. The main issue we want to address is to what extent it is possible, if at all, to detect different equilibria in actual labor market data. We assume that the true data-generating process (DGP) is given by the non-linear equation system derived from the optimizing behavior of households and firms. The analysis above demonstrated that the basic search and matching model admits a wide range of global dynamic behavior for different parameterization. The specific question is whether standard empirical methods that are typically used for linear models can identify global dynamics. A corollary to this question is whether global dynamics can potentially be distinguished from local dynamics. We approach these questions from a simulation analysis in that we generate data from the local and global model for different equilibria. We then analyze the simulated time series using methods of moments and likelihood-based methods.

[To be written]

8 Conclusion

[To be written]

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A Mathematical Appendix

In this appendix we list definitions and results necessary for establishment of periodic and chaotic solutions in the search and matching model, as discussed in Section 5.

A.1 Preliminaries

Let $f : \mathbb{R} \to \mathbb{R}$ be a map and consider the first order difference equation given by

$$x_{t+1} = f(x_t) \tag{26}$$

Definition 8 *(Invariance)* The interval $I \subset \mathbb{R}$ is *invariant* under f if $f(I) \subseteq I$. For the first order equation in (26), the above definition implies that if the initial value $x_0 \in I$ then $x_t \in I$ for $t > 0$.

Definition 9 *(Periodic points)* Let p be a nonnegative integer and let f^p be the composition of f with itself p times. The point $s \in \mathbb{R}$ is a p-periodic point of the map f if $f^p(s) = s$. The first order equation in (26) has a periodic solution of period p if the map f has a p-periodic point. In this case we say that the equation in (26) has a periodic solution of period p, i.e. $x_{t+p} = x_t$ for all $t \geq 0$.

The following result in Block and Coppel (1986) establishes sufficient conditions for existence of periodic points of odd periods.

Lemma 10 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. If for some odd integer $p > 1$ there exists a point x such that*

$$f^p(x) \leq x < f(x) \quad or \quad f(x) < x \leq f^p(x)$$

then f has a periodic point of period p.

Theorem 11 *(Li and Yorke, 1975)* Let $f : I \to I$ be a continuous map on an interval $I \subseteq \mathbb{R}$. If there is a period point in I of period 3, then for every integer $k \geq 1$, there is periodic point in I having period k.

There are several, not necessarily equivalent definitions of chaos in mathematical literature. The more commonly used ones are those in the sense of Li and Yorke, Devaney, and Block and Coppel (see Aulbach and Kieninger, 2001 for more details). For the purpose of this paper, below we list the definition of chaos in the sense of Block and Coppel, and refer to the result in Aulbach and Kieninger (2001) that establishes equivalence between chaos in the sense of Block and Coppel to that of Devaney.

Definition 12 A map $f : I \to I$ is called *turbulent* if there exist compact subintervals J, K of I with at most one common point such that

$$J \cup K \subseteq f(J) \cap f(K)$$

If J and K are disjoint, then f is said to be *strictly turbulent*.

Theorem 13 *(Chaos in the sense of Block and Coppel)* A continuous map $f : I \to I$ on a nontrivial compact interval I is chaotic in the sense of Block and Coppel if and only if one of the following equivalent conditions is satisfied:

(i) f^m is turbulent for some $m \in \mathbb{N}$.

(ii) f^m is strictly turbulent for some $m \in \mathbb{N}$.

(iii) f has a periodic point whose period is not a power of 2.

Theorem 14 A continuous map $f : I \to I$ on an interval I is chaotic in Devaney sense if and and only if it is chaotic in Block and Coppel sense.

A.2 Chaos in Type-B maps

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function with a critical point $m > 0$ such that f is increasing on $[0, m)$ and decreasing on (m, ∞) and $f(0) = d > 0$. Under appropriate scaling, this type of a map has been characterized by Medio and Raines (2007) as a Type-B map. We establish sufficient conditions for existence of periodic and chaotic solutions for a general class of such maps.

Given that f is decreasing on (m, ∞), there exists a real number $q > m$, such that $f(q) = 0$ (i.e. q is the preimage of 0). This gives us the following result.

Theorem 15 *If $f(m) \leq q$, then the interval $[0, q]$ is invariant under f.*

Proof. Let $x \in [0, q]$. Then $f(x) \leq f(m) \leq q$ for all $x \geq 0$. Further, if $0 \leq x \leq m$, then $f(x) \geq f(0) = d > 0$ since f is increasing on $[0, m)$, and if $m \leq x \leq q$, then $f(x) \geq f(q) = 0$ since f is decreasing on $[0, q]$. ■

Theorem 16 *Let $m > d$. If $f(m) = q$, then f has a periodic point of period 3 in* $[0, q]$.

Proof. By Elaydi and Sacker (2004), for any point $y \in [0, f(m))$, there exist points y_- and y_+ with $y_- < m < y_+$ such that $f(y_-) = f(y_+) = y$. Moreover, if $z < y$, then

$$z_- < y_- < m < y_+ < z_+$$

By the above, there exist points m_- and m_+ such that $f(m_-) = f(m_+) = m$, and points d_- and d_+ such that $f(d_-) = f(d_+) = d$. Now, $d_- = 0$ and $0_+ = q$, and since $m > d$, then we have

$$0 = d_- < m_- < m < q$$

If we set $x = m_-$, then $f(x) = m > m_-$, $f^2(x) = f(m) = q$, and $f^3(x) = f(f(m)) = f(q) = 0$. Hence

$$0 = f^3(x) < x < f(x)$$

and by Lemma 10 f has a periodic point of period 3. ■

As a corollary, we also have the following result.

Corollary 17 Let $m > d$. If $f(m) = q$, then f has a periodic points of every period in $[0, q]$, and is chaotic in the sense of Block and Coppel, Devayney, as well as Li and Yorke.

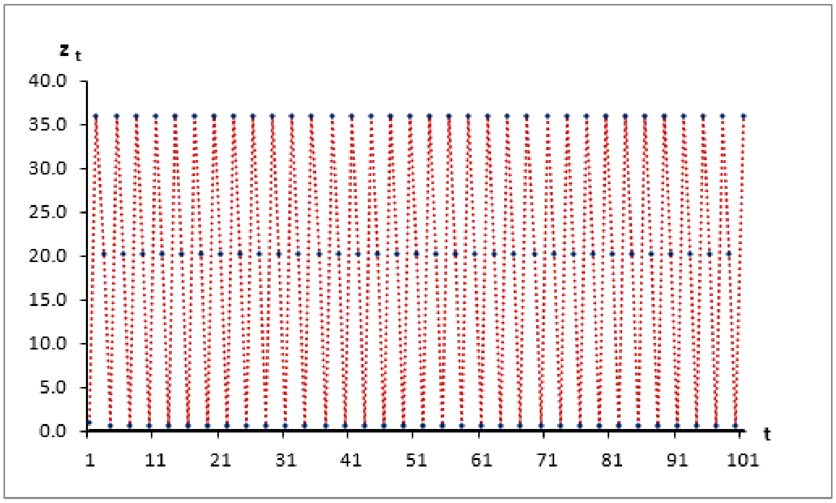

Figure 1: Period 3 solutions of the backward map for $a = 0.7, c = 0.4, d = 5.7, \xi = 0.5$

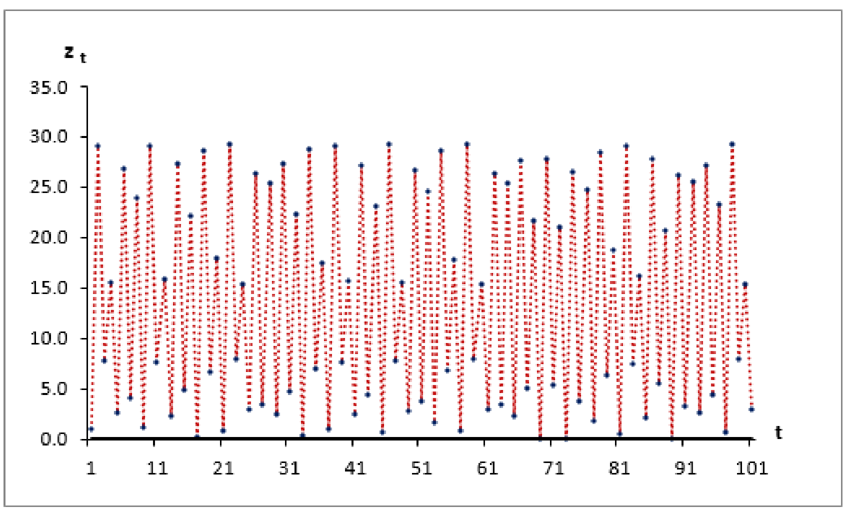

Figure 2: Chaotic solutions of the backward map for $a = 0.7, c = 0.4, d = 5.7, \xi = 0.5$