Durations at the Zero Lower Bound*

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Abstract

Many central banks in developed countries have had very low policy rates for quite some time. A growing number are experimenting with official rates that are negative. We develop a New Keynesian model in which the zero lower bound (ZLB) on nominal interest rates is imposed as an occasionally binding constraint and use this model to examine the duration of ZLB episodes. In addition, we show that capital accumulation and capital adjustment costs can raise significantly the length of time an economy spends at the ZLB, as does the conduct of monetary policy. We identify anticipation effects that make the ZLB more likely to bind and we show that allowing negative nominal interest rates shortens average durations, but only by about one quarter.
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1 Introduction

Japan has been wrestling with a weak economy and with short-term nominal interest rates near zero for more than two decades. More recently, following the global financial crisis, a number of developed countries have joined Japan in having very low policy rates. The U.S. Federal Reserve had its policy target rate below 0.25 percent from December 2008 and only raised the target to 0.25—0.50 percent in December 2015. The Bank of England has kept its policy rate at 0.5 percent since March 2009. Other central banks, including the Riksbank, the Swiss National Bank, Danmarks Nationalbank, the European Central Bank, and the Bank of Japan, following periods of very low policy rates, have recently engineered official rates that are negative, some of the order of -0.50 percent per annum. Although these negative official rates have yet to extend to negative retail rates, these central banks hope that having negative interest rates will stimulate economic activity and lift the economy out of the liquidity trap.

The problems that can arise from very low nominal interest rates, problems relating to the zero lower bound (ZLB), are well-known. Summers (1991) argued that the ZLB is an important reason why central banks should target a positive rate of inflation, providing greater scope to lower nominal rates should the need arise. This theme was picked up by studies such as Reifschneider and Williams (2000) and Coibion, Gorodnichenko, and Wieland (2012) that show that although the ZLB materially hampers monetary policy the ZLB's adverse effects can be mitigated through a higher inflation target. Krugman (1998) identified a lack of commitment as a key impediment to extracting an economy from a liquidity trap, leading to the idea that central banks in a liquidity trap should commit to being "irresponsible"—commit to allowing higher future inflation—an idea explored in Eggertsson (2006).

Our analysis of the ZLB focuses on durations: once the policy rate reaches zero, for how long does it stay at zero? We focus on this question because the examples we have of countries experiencing very low policy rates are examples where the policy rate has remained low for sustained periods. Of the countries listed above, the shortest duration before the policy rate was raised is about one year in length. That example being Canada, which kept its policy rate low from April 21, 2009 through June 1, 2010, when it was raised to 0.5 percent. Contrasting with this experience, studies examining the ZLB have tended to find that durations during which the ZLB binds are typically short, usually well less than one year. For example, Adam and Billi (2006) in their study of optimal commitment policy with the ZLB found that the average duration for a ZLB event was just 1.4 quarters, with less than 2 percent of all ZLB events lasting more than 4 quarters. The low mean-duration that Adam and Billi (2006) find is notable because commitment policies are known for their persistence, which may suggest that alternative policies that do not involve commitment to the fully optimal policy could imply even shorter ZLB durations. Reifschneider and Williams (2000), using the large-scale FRB/US model, which has considerable internal persistence, found a mean duration at the ZLB of 6 quarters when policy targeted zero inflation. Examining a nonlinear New Keynesian model, Fernández-Villaverde, Gordon, Querrón-Quintana, and Rubio-Ramírez, (2015) obtained a mean duration at the ZLB of 2.1 quarters; the inflation target in their Taylor rule was also zero percent.

The model we use is in many ways a standard New Keynesian model in which firms are monopolistically competitive and subject to sticky prices. Unlike much New Keynesian literature, however, we use Rotemberg-pricing (Rotemberg, 1982) rather than Calvo-pricing (Calvo, 1983) to model the price rigidity and we include capital as a productive input. We include capital in the model because the capital accumulation process provides a propagation mechanism that has the potential to extend the time spent at the ZLB and, through investment, it raises the importance of real interest rate movements. Monetary policy is conducted via a Taylor-type rule (Taylor, 1993). We solve the nonlinear model using global methods and consider specifications for which the ZLB on nominal interest rates is either imposed as an occasionally binding constraint or allowed to be relaxed, the latter case recognizing that some central banks have developed mechanisms that allow negative interest rates. We also consider specifications where a second occasionally binding constraint—one preventing negative investment—is imposed.

Our analysis uncovers several new and interesting results. We identify anticipatory effects of the ZLB that raise the likelihood that the ZLB will bind, effects not found in models that exclude capital. As the ZLB is approached, the probability that the ZLB will bind in the future increases. Because the effect of the ZLB is the keep the real interest rate higher than it would be otherwise, households are induced to defer consumption, which raises investment and the capital stock and lowers the marginal product of capital. Because the probability that the ZLB binds is increasing in the capital stock (decreasing in the marginal product of capital), agents' optimal response to the possibility that the ZLB might bind raises the probability that it will actually bind.

Simulating the model, we find that the ZLB binds around 5 percent of the time for our benchmark parameterization, or about one quarter every five years, a frequency that is very similar to Fernández-Villaverde, et al (2015). Allowing negative nominal interest rates, we find that ZLB events have a mean duration of about 4.8 quarters, with this mean duration rising to 5.5 quarters when the ZLB is imposed and to 5.6 quarters when non-negative investment is also imposed. These mean durations are somewhat longer than those obtained by Adam and Billi (2006) and

Fernández-Villaverde, et al (2015), who study models that exclude investment and capital, but are similar to those obtained by Reifschneider and Williams (2000) who use FRB/US. In addition to showing that investment and capital accumulation are important for ZLB durations, an implication of our results is that efforts by central banks to allow negative nominal interest rates may have only a small impact on the economy's duration at the ZLB.

Focusing on the distribution of ZLB events, although the benchmark parameterization delivers a mean-duration that is close to 6 quarters the median-durations is less than 2 quarters, signaling that the preponderance of ZLB events are relatively short-lived. But this is not to say that the model cannot generate long-lived ZLB events with 10 percent of all ZLB events lasting longer than three years. Exploring mechanisms that extend ZLB durations, we consider model specifications that have capital adjustment costs and we consider a range of different policy rules. We consider capital adjustment costs because they weaken the relationship between the real interest rate and the marginal product of capital and different policy rules because they can have important implications for inflation's persistence. We find that adding an iceberg-cost to adjusting capital raises the mean duration of a ZLB event to 25 quarters with 10 percent of events lasting more than 74 quarters; a weak policy response to inflation movements also leads to markedly longer ZLB durations. As noted above, these durations are somewhat longer than those obtained by previous studies that have used models that exclude capital.

Our research is related to work by Christiano (2004), Kato and Nishiyama (2005), Jung, Teranishi, and Watanabe (2005), Adam and Billi (2006, 2007), and Nakov (2008) who examine the ZLB in log-linearized New Keynesian models. Among these, only Christiano (2004) considers the allows for investment and capital, but he focuses on their influence on the frequency with which the ZLB binds rather than on the duration of ZLB events. Our work also differs from Christiano (2004) in that his model allows only shocks to the natural rate of interest and has monetary policy conducted optimally. Less closely, our study is also related to the work by Eggertsson and Woodford (2003) and Eggertsson (2006), as well as to McCallum (2000) and Svensson (2001), which consider ways that a country can extract itself from a liquidity trap. Also related are Nakata (2012), Schmidt (2013), and Fujiwara and Ueda (2013), who look at fiscal policy in a liquidity trap, and Bodenstein, Hebden, and Nunes (2012), which investigates the interaction between central bank credibility and the ZLB. Perhaps the study closest to ours is the one by Fernández-Villaverde, et al (2015). However, their model is somewhat different to ours and their focus is not on ZLB durations, but rather on the importance of nonlinearities when analyzing New Keynesian models with the ZLB (see also Braun, Boneva, and Waki, 2012).

The remainder of this paper is structured as follows. In the following section we develop the New Keynesian model that forms the laboratory for our analysis. Section 3 summarizes the parameterization used to establish benchmark results. Recognizing that some countries have allowed nominal interest rates to become negative, section 4 presents results on ZLB events where the ZLB is not imposed and negative interest rates are allowed. Section 5 introduces the ZLB as an occasionally binding constraint and restricts investment to be non-negative, illustrating the effects these constraints have on ZLB durations. Section 6 extends the benchmark model along several dimensions, introducing factors such as capital adjustment costs and considering alternative policy rule specifications. Some of these extensions are shown to lengthen the mean ZLB duration considerably. Section 7 concludes. Appendices present detailed model derivations and contain technical material relating to model-solution and the imposition of the occasionally binding constraints.

2 The Model

We consider an economy populated by a unit-measure of identical atomistic households, a unit-measure of atomistic monopolistically competitive firms, a fiscal authority, and a central bank. Households own the capital stock and rent their capital and supply their labor to firms in perfectly competitively markets. Capital accumulation is subject to an adjustment cost. Firms hire capital and labor, produce according to a neoclassical production technology and set their price each period subject to a Rotemberg (1982) price-adjustment cost. The central bank conducts monetary policy according to a Taylor-type rule. In formulating the model below, we set the ZLB on the nominal interest rate aside. Our method for computing equilibrium with the ZLB imposed is described in Appendix C.

2.1 Households

Households supply labor hours, h_t, and rent their capital, k_t, to firms at real factor prices w_t and r_t^k, respectively. In addition to receiving income from supplying capital and labor, households own the equity in firms and receive a dividend payment, D_t, and they receive interest income on their holdings of one-period non-state-contingent nominal bonds, b_t. Households allocate their income among consumption, c_t, investment, i_t, and the purchase of nominal bonds, b_{t+1}, to take into next period. With P_t denoting the aggregate price level and R_t denoting the net nominal interest rate, households take the processes for prices (P_t, w_t, r_t^k, R_t), dividends, D_t, and lump-sum taxes, T_t, as given, and solve the problem

$$u_t = \max_{\{c_t, h_t, i_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}} \left[\frac{\left(c_t^{\theta} \left(1-h_t\right)^{1-\theta} \right)^{1-\sigma} - 1}{1-\sigma} + \mathrm{E}_t \left[\beta_{t+1} u_{t+1} \right] \right],$$

$\theta \in (0,1)$ and $\sigma \in (0,\infty)$, subject to the real flow-budget-constraint

$$c_t + \left(1 + \frac{\gamma}{2} \frac{i_t}{k_t}\right) i_t + T_t + \frac{b_{t+1}}{P_t} = w_t h_t + r_t^k k_t + \left(\frac{1+R_{t-1}}{1+\pi_t}\right) \frac{b_t}{P_{t-1}} + D_t, \tag{1}$$

in which $\gamma \in [0,\infty)$ governs the capital-adjustment cost, and the capital accumulation equation

$$k_{t+1} = (1-\delta) k_t + i_t, \tag{2}$$

in which $\delta \in (0,1)$ represents the depreciation rate.

Departing from the typical representative agent framework, here the discount factor is stochastic and obeys the process

$$\ln(\beta_{t+1}) = (1-\rho_\beta) \ln(\beta) + \rho_\beta \ln(\beta_t) + \zeta_{t+1},$$

where $\beta \in (0,1)$, $\rho_\beta \in (0,1)$ and $\zeta_t \sim i.i.d. \left[0, \sigma_\zeta^2\right]$.

As shown in Appendix A, the first-order conditions from the household's problem are equations (1) and (2) together with

$$c_t^{(\theta-1-\sigma\theta)} (1-h_t)^{(1-\theta)(1-\sigma)} \left(1+\gamma \frac{i_t}{k_t}\right) = \mathrm{E}_t \left[\begin{array}{c} \beta_{t+1} c_{t+1}^{(\theta-1-\sigma\theta)} (1-h_{t+1})^{(1-\theta)(1-\sigma)} \\ \times \left((1-\delta)\left(1+\gamma \frac{i_{t+1}}{k_{t+1}}\right) + r_{t+1}^k + \frac{\gamma}{2} \left(\frac{i_{t+1}}{k_{t+1}}\right)^2 \right) \end{array} \right], \tag{3}$$

$$c_t^{(\theta-1-\sigma\theta)} (1-h_t)^{(1-\sigma)(1-\theta)} = \mathrm{E}_t \left[\beta_{t+1} c_{t+1}^{(\theta-1-\sigma\theta)} (1-h_{t+1})^{(1-\sigma)(1-\theta)} \left(\frac{1+R_t}{1+\pi_{t+1}}\right) \right], \tag{4}$$

$$\frac{c_t}{1-h_t} = \frac{\theta}{1-\theta} w_t. \tag{5}$$

Equations (3) and (4) are Euler equations associated with the purchase of capital and bonds, respectively, while equation (5) characterizes the representative household's labor supply curve.

2.2 Firms

Firms employ capital and labor to produce their output, y_t, according to the neoclassical production technology

$$y_t = k_t^{\alpha} h_t^{1-\alpha}, \tag{6}$$

$\alpha \in (0,1)$. We assume that firms are monopolistically competitive, setting their price subject to the demand curve

$$y_t = \left(\frac{p_t}{P_t}\right)^{-\epsilon_t} Y_t, \tag{7}$$

where p_t denotes the firm's price, Y_t denotes aggregate output, and ϵ_t denotes the elasticity of substitution between goods, which is stochastic and follows the process

$$\epsilon_{t+1} = \epsilon + \rho_\epsilon (\epsilon_t - \epsilon) + \eta_{t+1},$$

where $\epsilon \in (1, \infty)$, $\rho_\epsilon \in (-1, 1)$ and $\eta_t \sim i.i.d.\left[0, \sigma_\eta^2\right]$.

Drawing on Rotemberg (1982), firms face quadratic costs to changing their price. Using $\widetilde{p}_t = \frac{p_t}{P_t}$ to represent the firm's relative price and λ_t to represent the marginal utility of consumption for the representative household, the representative firm chooses $\{k_t, h_t, \widetilde{p}_t\}_{t=0}^{\infty}$ to maximize

$$\mathrm{E}_0 \left[\sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \beta_s \right) \frac{\lambda_t}{\lambda_0} \left((1 - \tau_t) \widetilde{p}_t y_t - w_t h_t - r_t^k k_t - \frac{\psi}{2} \left(\frac{\widetilde{p}_t (1 + \pi_t)}{\widetilde{p}_{t-1}} - 1 \right)^2 \widetilde{p}_t y_t \right) \right],$$

subject to their production technology, equation (6), and their demand curve, equation (7), taking the processes for w_t, r_t^k, P_t, λ_t, and Y_t as given. The parameter $\psi \in (0, \infty)$ reflects the price-adjustment cost while $\tau_t \in (-1, 1)$ represents a production tax, or subsidy when negative.

Appendix A shows that the first order conditions from the firm's problem yield

$$\pi_t (1 + \pi_t) + \frac{(1 - \epsilon_t)}{2} \pi_t^2 = \frac{(1 - \tau_t)(1 - \epsilon_t)}{\psi} + \mathrm{E}_t \left[\beta_{t+1} \frac{\lambda_{t+1} Y_{t+1} \pi_{t+1} (1 + \pi_{t+1})}{\lambda_t Y_t} \right] + \frac{\epsilon_t}{\psi} \omega_t, \quad (8)$$

where real marginal costs, ω_t, are given by

$$\omega_t = \left(\frac{r_t^k}{\alpha} \right)^{\alpha} \left(\frac{w_t}{1 - \alpha} \right)^{1 - \alpha},$$

and where the optimal factor-price ratio satisfies

$$\frac{r_t^k}{w_t} = \frac{\alpha}{1 - \alpha} \frac{h_t}{k_t}.$$

Equation (8) is a form of New Keynesian Phillips curve, relating inflation outcomes to expected future inflation and real marginal costs.

2.3 Fiscal authority

We assume that the non-state-contingent bond is in zero-net supply, which leads to the following government budget constraint

$$T_t + \tau_t Y_t = 0, \quad (9)$$

holding period-by-period. From equation (9), when production is taxed ($\tau_t > 0$), the revenues are remitted lump-sum to households while when production is subsidized ($\tau_t < 0$), the subsidy is financed by a lump-sum tax. As is well-known, the (nonstochastic) steady-state tax rate that offsets the production lost due to monopolistic competition is given by

$$\tau = -\frac{1}{\epsilon - 1},$$

which is negative, implying that in the (nonstochastic) steady state production should be subsidized. In our analysis below, we assume that $\tau_t = \tau \in (-1, 1)$ for all t.

2.4 Central bank

The central bank's policy instrument is the gross nominal interest rate, R_t, on the one-period non-state-contingent nominal bond. We assume that monetary policy is conducted according to the simple nonlinear Taylor-type rule

$$1 + R_t = \overline{R}\,(1 + \pi_t)^{\phi_\pi} \left(\frac{Y_t}{\overline{Y}}\right)^{\phi_y}, \tag{10}$$

where $\overline{Y}$ is the level of aggregate output and $\overline{R}$ is the gross interest rate[1] in the nonstochastic steady state.

The two key policy parameters are ϕ_π and ϕ_y. Consistent with linear Taylor-type rules, we assume that $\phi_\pi > 1$ and that $\phi_y > 0$ so that for a log-linearized version of equation (10) the Taylor principle holds.

3 Parameterization

Table 1 reports the model's benchmark parameterization. The mean value for the discount factor, β, is set to 0.994, which is the same as the value used in Fernández-Villaverde et al (2015). We parameterize the model to a quarterly frequency, so this value for β generates a nonstochastic steady state real interest rate of about 2.4 percent per annum. The form of the momentary utility function implies that the elasticity of substitution between consumption and leisure equals one. We set the inverse elasticity of intertemporal substitution, σ, equal to 1.00, which gives a momentary utility function that is linear in the natural logarithms of consumption and leisure. The parameter θ, which governs the share on consumption in momentary utility, is set to 0.47, which delivers a nonstochastic steady state in which households spend about 40 percent of their time working. Consistent with much business cycle literature the parameter α is set to 0.36. After accounting

[1] As is well-known, the stochastic steady state differs from the nonstochastic steady state because the model is not linear, with inflation on average below target and output on average below potential. To correct for this effect we follow Reifschneider and Williams (2000) and adjust $\overline{R}$ across simulations to ensure that the inflation target is met on average.

for the goods lost due to price-adjustment costs, this value for α implies that labor's share of the remaining output equals 0.64.

Table 1: Benchmark Parameterization

Interpretation	Parameter	Value
Mean discount factor	β	0.994
Inverse elasticity of intertemporal substitution	σ	1.00
Consumption-share in momentary utility	θ	0.47
Cobb-Douglas coefficient on capital	α	0.36
Capital adjustment costs	γ	0.00
Price adjustment cost	ψ	80.0
Mean elasticity of substitution between goods	ϵ	11.0
Depreciation rate	δ	0.015
Steady state production tax/subsidy	τ	0.00
Policy coefficient on inflation	ϕ_π	1.50
Policy coefficient on output	ϕ_y	0.00
Discount-factor shock persistence	ρ_β	0.85
Elasticity shock persistence	ρ_ϵ	0.80
Standard deviation of discount-factor innovation	σ_ζ	0.008
Standard deviation of elasticity innovation	σ_η	0.40

For the benchmark model, we assume that there are no capital adjustment costs, $\gamma = 0.00$, and no production subsidy, $\tau = 0.00$; we vary γ and τ from these benchmark values in subsequent sections. We further assume that the Rotemberg price-adjustment parameter, ψ, equals 80.0, that the mean elasticity of substitution between goods, ϵ, equals 11.0, and that the (quarterly) depreciation rate, δ, equals 0.015. This value for δ leads to an annual depreciation rate on capital of about 6 percent. The value for ϵ implies a non-stochastic steady state markup of 10 percent. With $\epsilon = 11.0$, setting ψ equal to 80.0 generates a coefficient on real marginal costs in the Phillips curve of 0.134. Turning to the coefficients in the monetary policy rule, the weight on gross inflation, ϕ_π, is set to 1.50 while the weight on output, ϕ_y, is set to 0.00; both ϕ_π and ϕ_y are varied from these benchmark values in section 7. The persistence and standard deviations for the two shocks imply that the discount factor spends 90 percent of its time between 0.966 and 1.015 and that the elasticity of substitution spends 90 percent of its time between 9.9 and 12.1.

4 Results

In this section we present some results for the benchmark parameterization. We solve the model and use stochastic simulation to construct the unconditional distributions of the model's main variables and to compute the unconditional probability that the ZLB on the nominal interest rate will be violated. Focusing on events where the ZLB is violated, we compute the durations of these events and report their distribution. While the results obtained without imposing the ZLB provide a useful benchmark and are interesting in their own right, by not imposing the ZLB the results in this section can be thought of as corresponding to an economy for which the ZLB does not bind, perhaps due to costs or taxes associated with holding and transporting paper money (Goodfriend, 2000), due to the absence of paper money or non-convertibility of bonds for money (Buiter, 2009), or due to paper money and electronic money not exchanging at parity (Buiter, 2009; Agarwal and Kimball, 2015).

4.1 Decision rules

Figure 1 displays the decision rules for the key model variables as a function of the capital stock with the values for the two shocks ($\ln(\beta_t)$ and ϵ_t) set to their mean values of $\ln(\beta)$ and ϵ, respectively. Panels A and B show that consumption and leisure are both increasing functions of the capital stock, implying that households consume more and work less as their wealth rises. Panel C shows that inflation is a decreasing function of capital. To understand why inflation falls as capital rises it is useful to recognize that when capital is low households and firms expect output to be rising over time as the economy transitions toward steady state. With demand for their good expected to be higher tomorrow than it is today firms optimally raise their price. As capital continues to rise the demand-differential between tomorrow and today declines so prices are raised by a smaller amount. When capital is high the transition toward steady state involves demand declining over time so firms optimally lower their price, which leads to negative inflation.

We see from panel D that investment is a decreasing function of capital, a consequence of intertemporal consumption smoothing. When capital is relatively low the marginal product of capital is high, leading to a higher real interest rate and a greater incentive for households to defer consumption and invest the savings. Panel E shows that the nominal interest is also a decreasing function of the capital stock. The rise in capital leads to an increase in the capital-labor ratio which lowers the marginal product of capital, the rental rate of capital, and the real interest rate on bonds. With the real interest rate and the inflation rate both declining as capital increases, the nominal interest rate is also a decreasing function of capital.

4.2 Unconditional distributions

Table 2 reports the unconditional means for the model's main variables, where the means are calculated by taking the average over two million quarters of simulated data. Also shown are the values associated with the model's nonstochastic steady state.

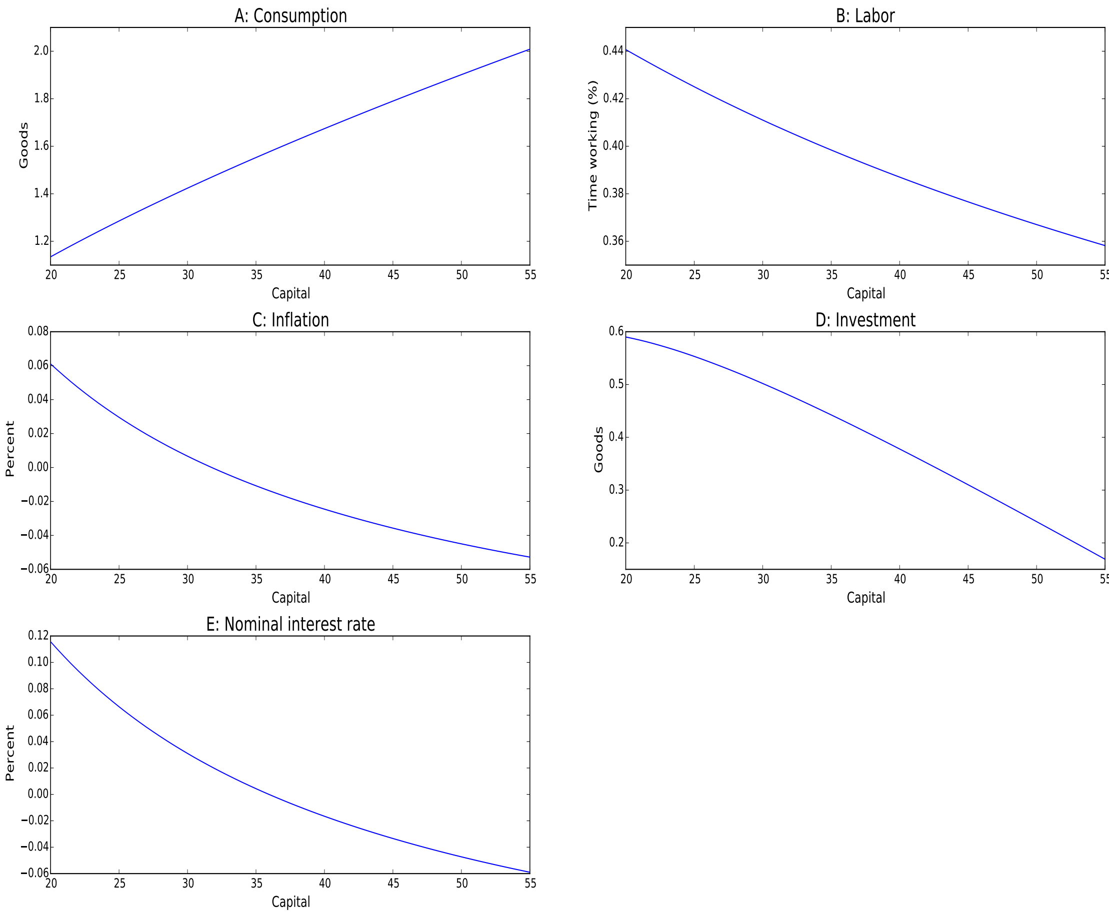

Figure 1: Decision rules for benchmark parameterization

Table 2: Mean Outcomes for Benchmark Parameterization

Variable	Nonstochastic	Stochastic
Output	1.884	1.952
Consumption	1.444	1.473
Investment	0.440	0.478
Dividend	0.348	0.333
Labor	0.402	0.406
Capital	29.304	31.885
Inflation	0.000	0.000
Nominal Interest Rate	2.414	2.100
$Prob\,(R_t < 0)$	$---$	0.052

Comparing mean outcomes across the stochastic and nonstochastic economies, the presence of shocks in the model leads to an increase in the capital stock from about 29.3 to around 31.9, a 10 percent increase. This increase is driven by a precautionary saving motive and is absent from models that exclude assets such as capital. Higher capital in the stochastic economy leads—on average—to higher output, consumption, and investment. Hours worked also increase slightly while dividends fall. Our parameterization of the model keeps the mean value for inflation at zero in the stochastic economy, however the higher capital stock lowers the marginal product of capital and the shadow real return on the bond, leading to a decline in the average nominal interest rate. The bottom row of Table 2 reports that the probability that the simulated nominal interest rate is negative is about 0.05 in the stochastic economy, or about one quarter in any given five-year period.

Where Table 2 reports the unconditional means for key variables, Figure 2 displays their unconditional distributions generated using the simulated data. Several things about Figure 2 are worth noting. First, the unconditional distributions are all largely symmetric, but with the greatest asymmetry residing in the nominal variables: inflation (panel I) and the nominal interest rate (panel G). Second, as reported in Table 2, about 5 percent of the distribution for the nominal interest rate is negative. Third, some of the simulated outcomes for investment are negative, implying that in equilibrium capital goods are occasionally converted back into output goods and consumed. Finally, panel G, H and I suggests that when the nominal interest rate becomes negative it does so not because the real interest rate is low, but because inflation is negative.

4.3 ZLB durations

Focusing on events where the nominal interest rate is non-positive, Figure 3 plots a range of statistics associated with these ZLB events. Panel A reports the proportion of ZLB events according to their duration. For a duration of one quarter the proportion is just below 0.4, which means that almost 40 percent of all ZLB events last a single quarter. Moreover, panel A shows that the proportions decline rapidly as the duration increases, suggesting that most ZLB events are of relatively short duration.

Figure 3, panel B, reports the proportion of ZLB events that last longer than a given time, or duration. Thus, consistent with panel A, just over 60 percent of all events last more than one quarter. Looking at longer-duration events, it is clear from Panel B that while most ZLB events tend to be relatively short, long durations at the ZLB are certainly possible—about 10 percent of events last more than 10 quarters. Panel C reports the probability that a ZLB event will continue for one or more quarters, given the length of time it has spent at the ZLB. Thus about 73 percent of

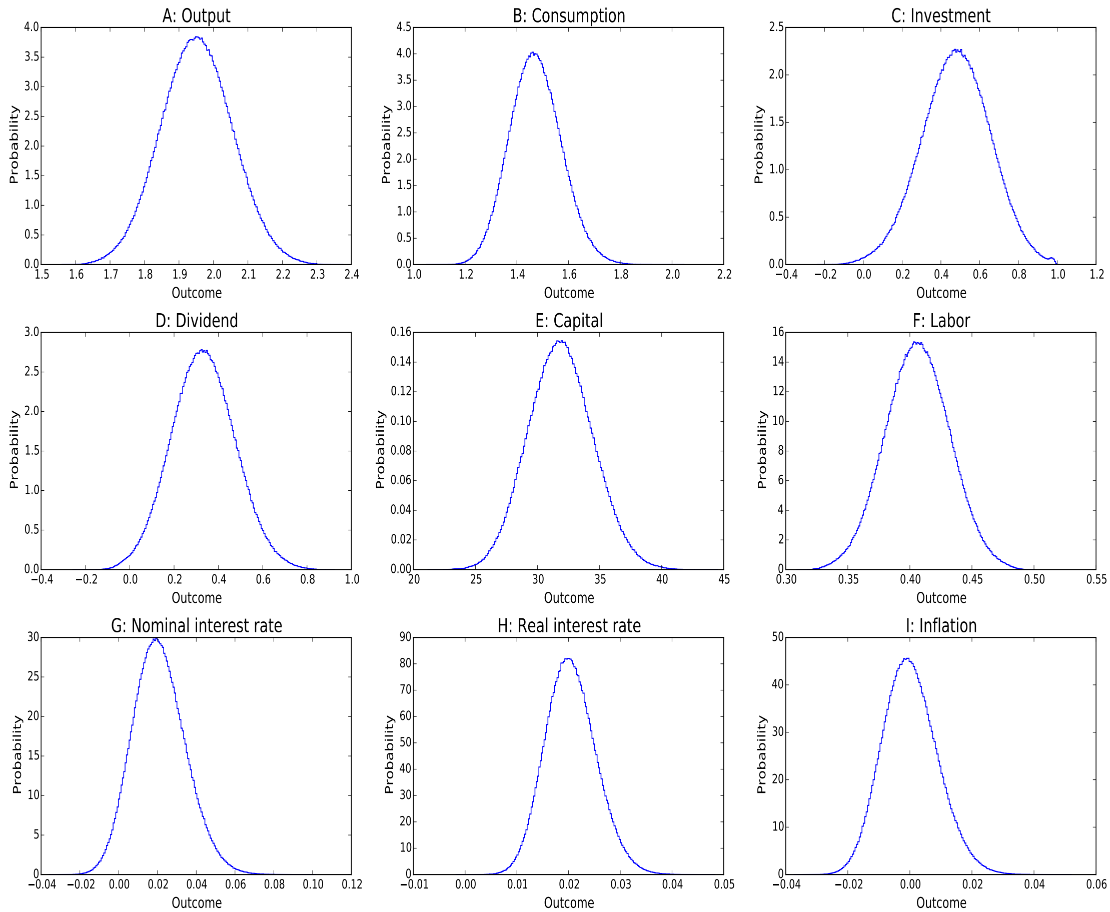

Figure 2: Unconditional distributions for benchmark parameterization

ZLB events that have lasted for one quarter will continue to last for two or more quarters. Notably, as the length of time the economy has already spent at the ZLB rises, the probability that it will continue to be at the ZLB for one or more quarters rises, leading to some extremely long-duration ZLB events.[2]

Table 3 presents some summary statistics associated with the simulated ZLB events, complementing Figure 3. The mean duration at the ZLB is 4.8 quarters, somewhat longer than the mean duration found by Adam and Billi (2006) and Fernández-Villaverde, et al (2015). However, really long durations at the ZLB remain relatively rare with 90 percent of all ZLB events lasting less than

[2]The increased volatility in the statistics reported in Figure 3, panel C, are simply a consequence of there being increasingly fewer events in the sample as the duration length increases.

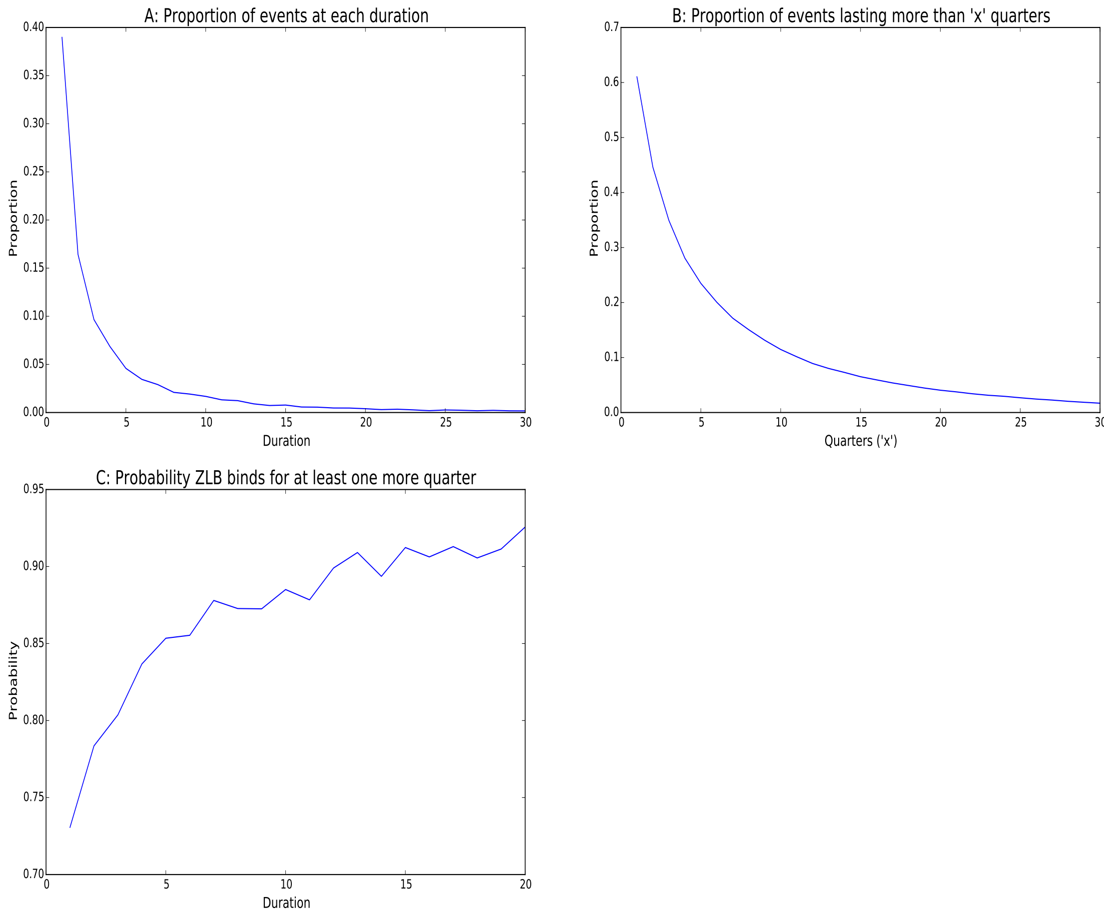

Figure 3: Statistics on ZLB events

11 quarters.

Table 3: Some Statistics on Durations at the ZLB

Mean	Median	75'th percentile	90'th percentile	Longest	Pr(dur> 12)	Pr(dur> 24)
4.84	1.67	4.67	11.12	137	0.089	0.029

As suggested by Figure 2, one reason that ZLB durations are not longer might be that model-generated ZLB events occur due to negative inflation rather than due to a low real interest rate. Given the persistence arising through capital accumulation, longer durations at the ZLB might arise were ZLB events a consequence of low real interest rates. However, because the marginal product of capital cannot be negative, generating low real interest rates requires weakening the real interest rate's connection to the marginal product of capital.

5 Introducing the ZLB and non-negative investment

The previous section treated ZLB events as episodes during which the nominal interest rate was negative. In this section we explicitly impose the ZLB as a constraint and treat ZLB events as episodes during which the constraint binds. In addition to the ZLB constraint on the nominal interest rate, we also consider specifications for which investment is constrained to be non-negative, preventing capital from being consumed. Our procedure for imposing these occasionally binding constraints is described in Appendix C. Denoting the investment-lower-bound constraint by ILB, the effect of the ZLB and the ILB, imposed individually and jointly, on the decision rules are shown in Figure 4.

Although the ILB constraint has relatively little effect on the model, this is not the case for the ZLB constraint. Figure 4 shows that imposing the ZLB constraint has big effects on the behavior of all variables except inflation, with the effects becoming more pronounced when capital is large. To the extent that the ILB constraint impacts the decision rules it does so when capital is large, which is when transitions toward the economy's steady state are more likely to involve efforts to consume the capital stock. Looking again at the ZLB constraint, in states where the ZLB binds Figure 4 shows that its effect is to keep the real interest rate higher than it would otherwise be (because inflation continues to fall while the nominal interest rate is constrained), leading to a decline in consumption (Panel A),as households choose to defer consumption, and to a rise in investment (Panel D), and to an increase in hours worked (Panel B).

Also shown in Figure 4 are vertical lines indicating the value for capital at which the ZLB binds (with $\ln(\beta_t)$ and ϵ_t set equal to their mean values). Clearly, to the right of the vertical line the decision rules are affected by the fact that the ZLB binds. To the left of the vertical line, however, it is apparent that the decision rules are affected by the ZLB even though the ZLB does not actually bind. The effects seen here arise in states where the ZLB might bind tomorrow, even though it does not bind today, and represent anticipation effects. Panels C and E suggest that the anticipation effect is small for inflation and for the nominal interest rate itself, i.e., the fact that the ZLB may bind imminently does not cause the central bank to lower the nominal interest rate either more quickly or more slowly. But the anticipation effects are somewhat more pronounced for consumption (panel A), labor (panel B), and investment (panel D) where we see that the anticipation of the ZLB binding induces households to lower their consumption and raise their investment and their supply of labor. Importantly, to the extent that the ZLB is more likely

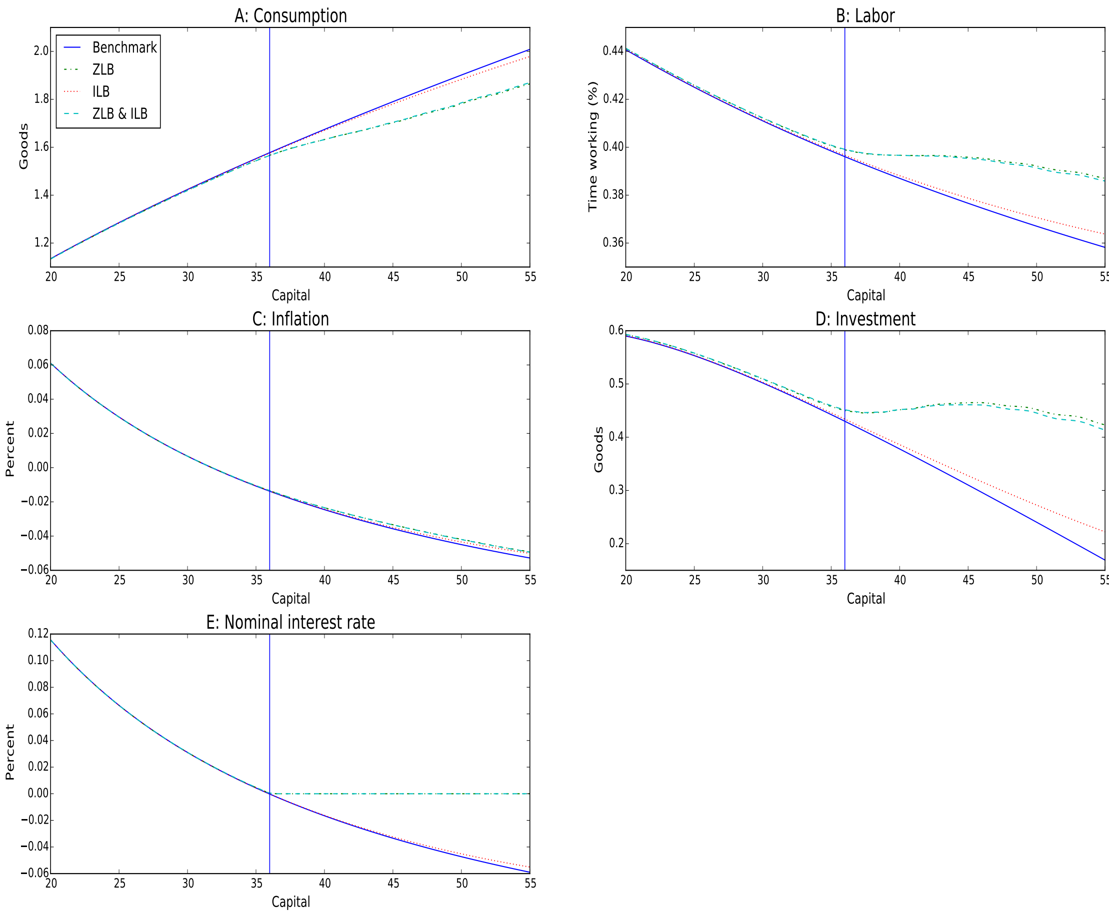

Figure 4: The effects of ZLB and ILB constraints on decision rules

to bind at higher capital stocks, these anticipation effects serve to increase the likelihood that the ZLB will actually bind. In other words, the anticipation that the ZLB *may* bind raises the likelihood that it *will* bind.

Figure 5 shows the effects of the ZLB and ILB constraints on the unconditional distributions for the model's main variables. Although imposing ILB constraint leads to a spike in investment at zero (Panel C) the main effects of the model come from imposing the ZLB constraint.

The most obvious effect of imposing the ZLB constraint is to create a spike in the distribution of the nominal interest rate at zero (Panel G), which largely eliminates the left-tail of the distribution of the real interest rate (Panel H). However, it also leads to a lower mean and a larger left-tail

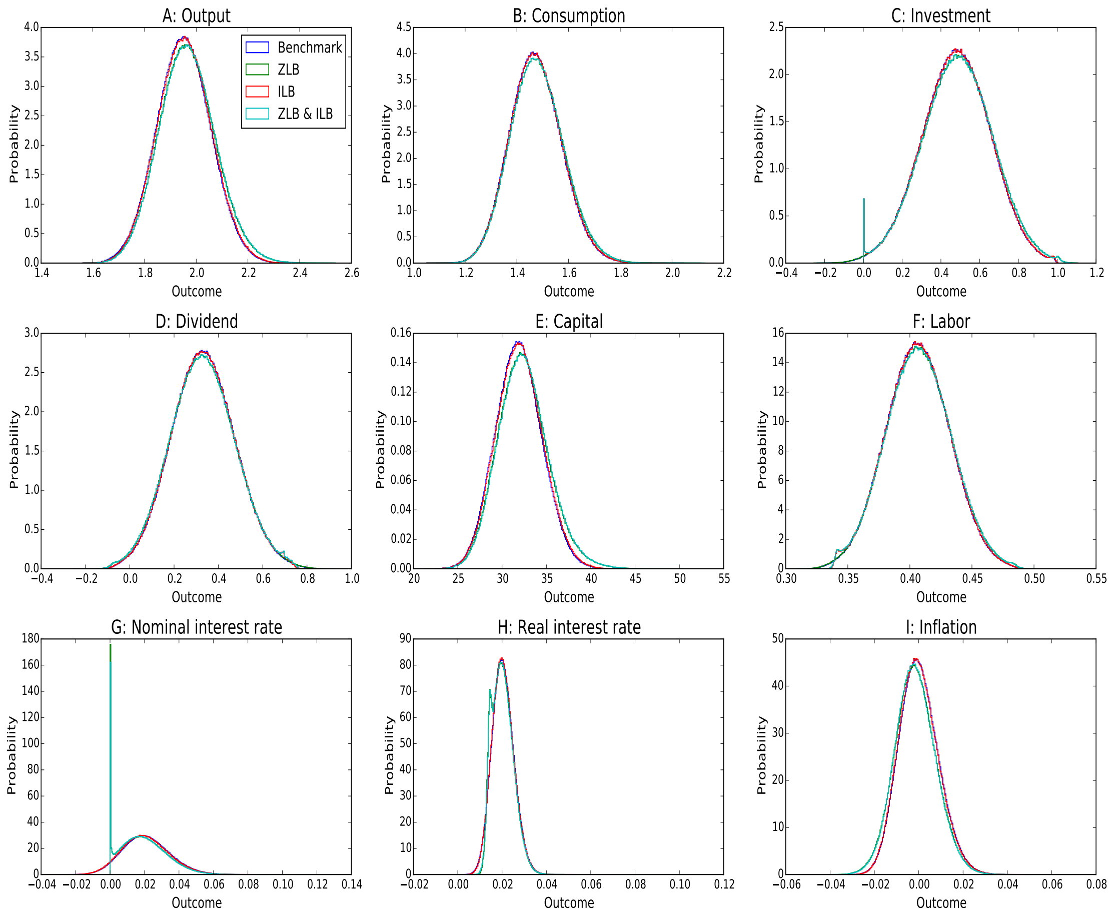

Figure 5: The effects of the ZLB and ILB constraints on unconditional distributions

in the distribution of inflation (Panel I) as the ZLB makes it difficult for monetary policy to raise inflation when inflation is low. Lacking the ability to stimulate the economy through lower interest rates, low inflation outcomes—particularly negative outcomes—become more persistent. Figure 5 also shows that the ZLB constraint creates an asymmetry in the distribution of capital, with high outcomes for capital enduring for longer as the ZLB's effect of the real return to saving leads to higher investment and sustains higher capital. The asymmetry in the distribution of capital leads to corresponding asymmetries in the distributions of output, consumption, and investment.

Table 4 presents the effects that the ZLB and ILB constraints have on ZLB durations. Consistent with what we have seen in Figures 4 and 5, it is the ZLB constraint rather than the ILB constraint that has greater impact on the duration the economy spends at the ZLB, raising the mean duration from around 4.8 quarters in the benchmark specification to almost 5.5 quarters. This increase in the mean duration when the ZLB is imposed can be interpreted in a couple of different ways. One interpretation is that it demonstrates that computing ZLB durations without imposing the constraint explicitly leads to durations being slightly understated, which is intuitive as the not imposing the ZLB constraint allows monetary policy to stimulate the economy through negative nominal interest rates. Another interpretation is that it illustrates that mechanisms that allow central banks to achieve negative interest rates can shorten ZLB durations, although perhaps by less than might be hoped.

Table 4: Summary Statistics on ZLB Durations

Specification	Mean	Median	75 percent	90 percent	Longest	Pr(dur> 12)	Pr(dur> 24)
Benchmark	4.840	1.671	4.674	11.132	137	0.089	0.029
ILB binds	5.105	1.758	4.977	11.906	139	0.099	0.032
ZLB binds	5.472	1.591	4.747	12.316	203	0.103	0.041
ILB/ZLB bind	5.601	1.639	4.919	12.651	195	0.106	0.043

Imposing both the ZLB constraint and the ILB constraint raises the mean duration to 5.6 quarters—about a year and a half—, with the longest duration found in the simulated sample of $2,000,000$ observations equaling 203 quarters, or a little over 50 years.

The statistics on durations shown in Table 4 differs somewhat from Adam and Billi (2006) who, for a simple two-equation New Keynesian model with policy conducted optimally under commitment, found an average duration of just 1.4 quarters and a probability that the duration will exceed 4 quarters of only 1.8 percent. The longer durations that we find are likely a consequence of several factors. One important factor is the presence of capital and investment in our model, which provides an endogenous propagation mechanism not present in Adam and Billi (2006). A second important factor is the form of the monetary policy rule, which is assumed to be optimal (under commitment) in Adam and Billi (2006) and to follow a Taylor-type rule here. With the ability to commit and with policy conducted optimally the central bank is able to both respond efficiently to shocks, making the ZLB constraint less likely to bind—and more efficiently extract the economy from a liquidity trap should it arise. Our average duration also differs from the study by Fernández-Villaverde et al (2015) that using a more sophisticated New Keynesian model than Adam and Billi (2006) found a mean duration at the ZLB of 2.06 quarters. Like ourselves, Fernández-Villaverde et al (2015) assume that monetary policy is conducted using a Taylor-type rule, however they exclude the potential for propagation arising through investment and capital accumulation.

6 Further results

In this section we explore a range of alternative specifications, changing the model from its benchmark parameterization and investigating the effect that each change has on the distribution of ZLB durations. In particular, for each of the following five alternative model specifications we solve the model imposing the ZLB constraint and the ILB constraint and simulate the model generating $2,000,000$ quarters of data from which we construct the distributions of ZLB durations.

- *Capital adjustment costs*: Where the benchmark specification set $\gamma = 0$, for the analysis on capital adjustment costs we set $\gamma = 0.5$. Capital adjustment costs weaken the relationship between the real interest rate and the marginal productivity of capital, potentially allowing more ZLB events to be associated with low real interest rates. The value for γ implies that capital adjustment costs amount on average to slightly less than one-tenth of a percent of output each quarter.

- *Strong monetary policy*: The benchmark monetary policy has $\phi_\pi = 1.5$, consistent with Taylor (1993). For the strong monetary policy alternative we raise this parameter and set $\phi_\pi = 1.7$, implying a slightly stronger and more aggressive response by interest rates to an increase in actual inflation.

- *Weak monetary policy*: For the weak monetary policy alternative we lower the coefficient on inflation in the monetary policy rule, ϕ_π, from its benchmark value of 1.5 to 1.3, allowing monetary policy to respond less strongly to an increase in inflation.

- *Stickier prices*: The stickier-price alternative doubles the cost associated with changing prices from $\phi = 80$ in the benchmark model to $\phi = 160$, halving the slope of the Phillips curve and increasing the persistence of aggregate prices.

- *Output policy response*: For this alternative specification we introduce a small response coefficient on output in the monetary policy rule. Specifically, we set $\phi_y = 0.025$ where it equaled 0 in the benchmark specification.[3]

6.1 Discussion

The decision rules associated with the five alternative specifications are presented in Figure 6 while summary statistics on ZLB durations are reported in Table 4. The effect that stronger policy

[3]Because the policy rule is specified in terms of quarterly rates, $\phi_y = 0.025$ would correspond to a coefficient of 0.1 for a rule specified in terms of annualized rates.

and weaker policy have on the economy are clearly shown in Figure 6, panel C, which depicts the equilibrium relationship between inflation and the capital stock (keeping the values for the two shocks at their mean values). Relative to the benchmark parameterization but with the ZLB constraint and the ILB constraint imposed, the alternative with stronger monetary policy has inflation less response to capital while the alternative with weaker policy has inflation more responsive to capital. For any given variance of capital, what this translates into is less inflation volatility under the stronger monetary policy and more inflation volatility under the weaker monetary policy, which leads to the ZLB constraint binding less frequently under the stronger monetary policy and more frequently under the weaker monetary policy.

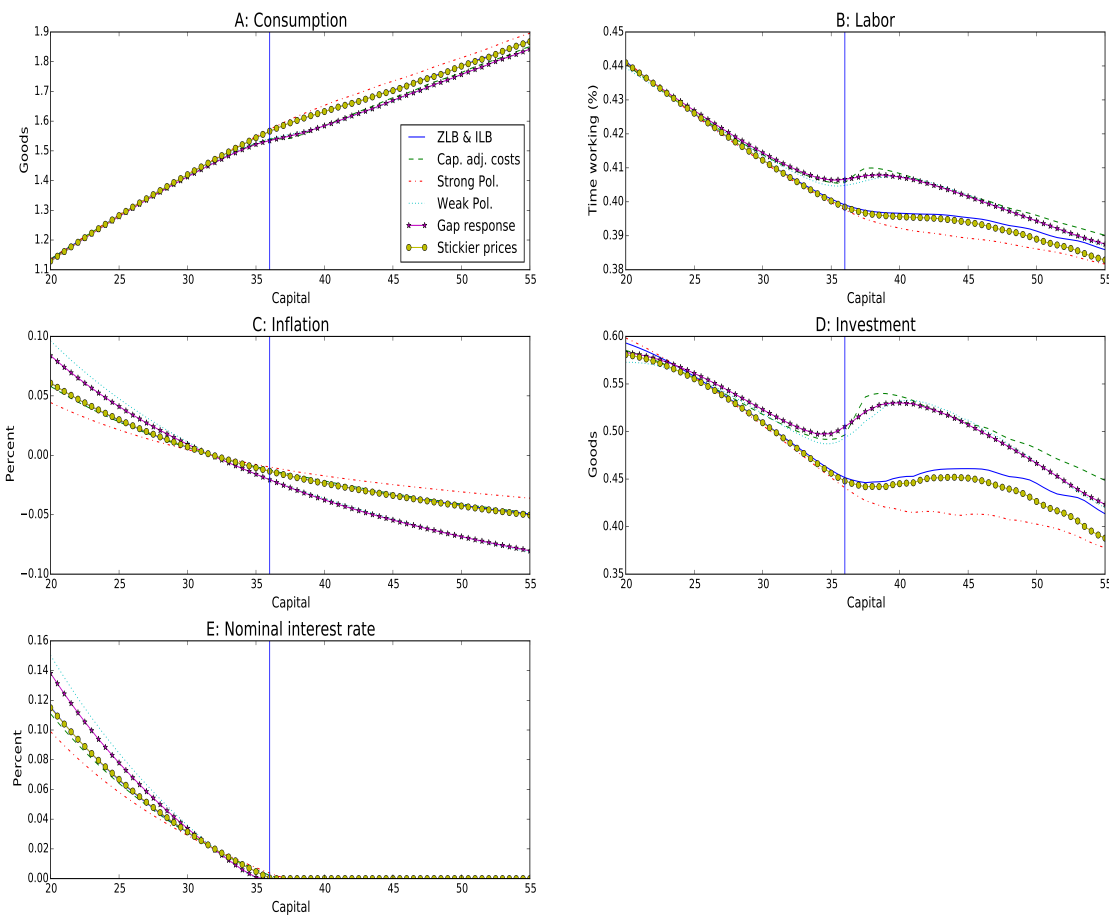

Figure 6: Decision rules under alternative parameterizations

Capital adjustment costs and a policy response to output each have important implications for equilibrium behavior. The effects of capital adjustment costs are clearly evident in panel D which displays the decision rules for investment. Once the ZLB constraint binds further increases in the capital stock lead to a rise in investment because the ZLB causes the real interest rate to rise inducing households to defer consumption. However, the rise in investment relative to capital comes at an increasingly larger installation cost, to the point where investment eventually begins falling relative to capital despite the high real interest rate. The alternative that has policy respond to output also has a large impact on the decision rule for investment. With inflation declining as capital rises (panel C) and output rising as capital increases, the policy response to output leads to the real interest rate falling less quickly than the benchmark policy as capital rises, leading to a larger decline in consumption (panel A) and to more investment (panel D), as capital rises.

Looking now at panel E, which reports the decision rules for the nominal interest rate, it is clear that the ZLB constraints tends to bind at a lower level of capital—and hence binds more frequently—for the alternative specifications for which monetary policy either responds less strongly to inflation or responds to output. Less clearly, the same is true, but to a lesser extent, for the specification with capital adjustment costs.

Table 4: Summary Statistics on ZLB Durations: Alternative Specifications

Specification	Mean	Median	75 percent	90 percent	Longest	Pr(dur> 12)	Pr(dur> 24)
Benchmark	4.840	1.671	4.674	11.132	137	0.089	0.029
CAC	13.889	6.966	18.347	35.245	158	0.362	0.182
ZLB & ILB bind	5.601	1.639	4.919	12.651	195	0.106	0.043
CAC	24.914	5.662	27.653	73.994	581	0.383	0.272
Strong policy	3.845	1.320	3.572	8.363	88	0.059	0.015
Weak policy	14.014	2.341	9.539	35.846	594	0.218	0.138
Gap response	24.490	5.846	23.872	71.563	756	0.361	0.249
Stickier prices	5.231	1.587	4.647	11.764	165	0.097	0.037

Table 4 reports summary statistics on the distribution of ZLB events obtained for each of the five alternative model parameterizations, alongside those for the benchmark specifications, a specification with capital adjustment costs (but without the ZLB and ILB constraints), and the specification with the ZLB and ILB constraints imposed. Relative to the benchmark specification, which has a mean duration for ZLB events of 4.8 quarters, introducing capital adjustment costs in isolation lengthens the average duration to nearly 14 quarters. Furthermore, imposing the ZLB and ILB constraints and then introducing capital adjustment costs raises the average duration of ZLB events to around 25 quarters, and pushes the median duration up to 5.7 quarters. For this specification nearly 40 percent of all ZLB events last more than 12 quarters and about 27 percent last for more than 24 quarters. Clearly capital adjustment costs are an important determinant of how long the economy stays at the ZLB once it is encountered. As observed earlier, capital adjustment costs weaken the connection between the real interest rate and the marginal product of capital, allowing low real interest rates to be a greater cause of ZLB events. In addition, capital adjustment costs delay economic adjustment, making it more difficult for the economy to escape the ZLB and prolonging ZLB events.

However, capital adjustment costs are not the only mechanism capable of generating longer ZLB events. The behavior of monetary policy is also a key determinant, with longer durations associated with both a weaker policy response to inflation and a response by policy to output. Weaker monetary policy raises the average duration to about 14 quarters and leads to around 14 percent of all ZLB events lasting more than 24 quarters. One explanation for this effect is that a weaker policy repose to inflation allows greater inflation persistence that when built into inflation expectations leads to longer ZLB durations. The policy that responds to output operates similarly, prolonging ZLB events by allowing greater inflation persistence and by inheriting persistence coming from capital accumulation that is manifest in output through output's dependence on capital.

Interestingly, the specification with stickier prices generates slightly longer ZLB events, but the effect is small with the average duration rising to 5.2 quarters and the median duration largely unchanged. This result suggests that it is inflation persistence rather than price-level persistence that is central for generating extended ZLB durations.

7 Conclusion

We use a nonlinear New Keynesian model to study the duration of ZLB events. Importantly, our model contains capital accumulation, a propagation mechanism omitted from most previous work looking at the ZLB. Where most previous studies document ZLB durations that often average just two quarters or less, even for our most simple benchmark specification we obtain mean durations of the order of almost six quarters, suggesting that capital accumulation is an important propagating mechanism for ZLB events. We also find several other interesting and important results. Allowing the ZLB to be violated, approximating mechanisms such as taxes on money holdings that facilitate negative nominal interest rates, leads to ZLB durations that are only marginally shorter than those obtained when the ZLB binds. In our simulations, allowing nominal interest rates to be negative shortened the average duration for ZLB events by just 0.7 quarters. This result raises questions about the usefulness of negative nominal interest rates and about the institutional and legal changes required to make negative nominal interest rates possible. Our analysis also identified anticipatory effects that made the ZLB more likely to bind as the ZLB is approached. Underpinning this finding was the fact that the real interest rate would be higher if the ZLB was binding, which gave households greater incentive to defer consumption and accumulate more capital, which increased the prospect that the ZLB would actually bind.

Extending our model, while imposing the ZLB and a constraint to prevent negative investment, uncovered several interesting findings. In particular, the mean- and median-durations for ZLB events were increased markedly by capital adjustment costs, by a relative weak monetary policy response to inflation, and by a monetary policy response to output. The latter two mechanisms lengthened ZLB durations by raising the persistence of inflation; the former mechanism by breaking the tight connection between the real interest rate and the marginal productivity, allowing greater real-interest-rate persistence and variability.

Further extensions of the model would be both desirable and interesting. Consumption habits are a popular feature in New Keynesian models and commonly touted for generating hump-shaped impulse response functions. However, through the bond-pricing equation they may also produce real interest rate persistence over-and-above that generated through capital accumulation alone. In addition, although greater price stickiness may not cause longer-duration ZLB events, to the extent that ZLB events arise through negative or low inflation, inflation persistence or "sticky inflation" could be an important mechanism for extending the duration of ZLB events. It would also be interesting and potentially important to introduce a banking sector and financial frictions, to accommodate variable capacity utilization, and to allow for a wider array of shocks. By way of application, it would be interesting to use this model to investigate the "forward guidance puzzle" (Del Negro, Giannoni, and Patterson, 2012; McKay, Nakamura, and Steinsson, 2005) and the extent to which the puzzle can be resolved by capital adjustment costs. All of these extensions are left for future work.

A Appendix: Decision problems and equilibrium conditions

This appendix presents the decision problems facing households and firms in the benchmark economy and derives the conditions that govern equilibrium outcomes.

A.1 Household's problem

The Lagrangian for the representative household's problem is given by

$$\mathcal{L} = \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \beta_s \right) \left(\begin{array}{c} \frac{\left(c_t^{\theta}(1-h_t)^{1-\theta}\right)^{1-\sigma}-1}{1-\sigma} + \mu_t \left((1-\delta) k_t + i_t - k_{t+1}\right) \\ \lambda_t \left(w_t h_t + r_t^k k_t + \left(\frac{1+R_{t-1}}{1+\pi_t}\right) \frac{b_t}{P_{t-1}} + D_t - c_t - \left(1 + \frac{\gamma}{2} \frac{i_t}{k_t}\right) i_t - T_t - \frac{b_{t+1}}{P_t} \right) \end{array} \right) \right], \tag{11}$$

where μ_t and λ_t are Lagrange multipliers.

The first-order conditions can be written as

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &: \quad \theta c_t^{(\theta-1-\sigma\theta)} (1-h_t)^{(1-\sigma)(1-\theta)} - \lambda_t = 0, && (12)\\
\frac{\partial \mathcal{L}}{\partial h_t} &: \quad (1-\theta) c_t^{\theta(1-\sigma)} (1-h_t)^{-(\sigma+\theta-\sigma\theta)} - w_t \lambda_t = 0, && (13)\\
\frac{\partial \mathcal{L}}{\partial i_t} &: \quad -\lambda_t \left(1+\gamma \frac{i_t}{k_t}\right) + \mu_t = 0, && (14)\\
\frac{\partial \mathcal{L}}{\partial b_{t+1}} &: \quad -\lambda_t + \mathrm{E}_t \left[\beta_{t+1} \left(\frac{1+R_t}{1+\pi_{t+1}}\right) \lambda_{t+1}\right] = 0, && (15)\\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} &: \quad -\mu_t + \mathrm{E}_t \left[\beta_{t+1} (1-\delta) \mu_{t+1}\right] + \mathrm{E}_t \left[\beta_{t+1} \left(r_{t+1}^k \lambda_{t+1} + \frac{\gamma}{2} \left(\frac{i_{t+1}}{k_{t+1}}\right)^2 \lambda_{t+1}\right)\right] = 0, && (16)
\end{aligned}$$

along with the capital accumulation equation and the flow-budget constraint

$$\begin{aligned}
k_{t+1} &= (1-\delta) k_t + i_t, && (17)\\
c_t + \left(1+\frac{\gamma}{2}\frac{i_t}{k_t}\right) i_t + T_t + \frac{b_{t+1}}{P_t} &= w_t h_t + r_t^k k_t + \left(\frac{1+R_{t-1}}{1+\pi_t}\right) \frac{b_t}{P_{t-1}} + D_t. && (18)
\end{aligned}$$

A.2 Firm's problem

The representative firm must first determine how to employ capital and labor to efficiently produce its good given its production technology. This leads to the Lagrangian

$$\mathcal{L} = w_t h_t + r_t^k k_t - \omega_t \left(k_t^{\alpha} h_t^{1-\alpha} - y_t\right), \tag{19}$$

where ω_t is a Lagrange multiplier, whose first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial k_t} &: \quad r_t^k - \omega_t \alpha k_t^{\alpha-1} h_t^{1-\alpha} = 0, && (20)\\
\frac{\partial \mathcal{L}}{\partial h_t} &: \quad w_t - \omega_t (1-\alpha) k_t^{\alpha} h_t^{-\alpha} = 0, && (21)\\
\frac{\partial \mathcal{L}}{\partial \omega_t} &: \quad k_t^{\alpha} h_t^{1-\alpha} - y_t = 0. && (22)
\end{aligned}$$

From the Lagrangian, ω_t represents the real cost of producing the marginal unit of output: real marginal costs.

Having determined how to produce efficiently, the firm must now choose the price at which it sells its good. With $\widetilde{p}_t$ representing the ratio of the firm's price to the aggregate price, i.e., $\widetilde{p}_t = \frac{p_t}{P_t}$, the firm chooses $\{\widetilde{p}_t\}_{t=0}^{\infty}$ to maximize

$$\mathcal{V} = \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \beta_s\right) \frac{\lambda_t}{\lambda_0} \left(\left((1-\tau_t) \widetilde{p}_t - \omega_t\right) \widetilde{p}_t^{-\epsilon_t} Y_t - \frac{\psi}{2} \left(\frac{\widetilde{p}_t (1+\pi_t)}{\widetilde{p}_{t-1}} - 1\right)^2 \widetilde{p}_t^{1-\epsilon_t} Y_t\right)\right], \tag{23}$$

taking the processes for τ_t, π_t, ω_t, Y_t, and λ_t as given.

The first-order condition is

$$\begin{aligned}
\frac{\partial \mathcal{V}}{\partial \widetilde{p}_t} &: \quad (1-\tau_t)(1-\epsilon_t) \widetilde{p}_t^{-\epsilon_t} Y_t + \epsilon_t \omega_t \widetilde{p}_t^{-(1+\epsilon_t)} Y_t \\
&: \quad -\psi \left(\left(\widetilde{p}_t^{1-\epsilon_t} \left(\frac{\widetilde{p}_t (1+\pi_t)}{\widetilde{p}_{t-1}} - 1\right) \left(\frac{1+\pi_t}{\widetilde{p}_{t-1}}\right) + \frac{1}{2} \left(\frac{\widetilde{p}_t (1+\pi_t)}{\widetilde{p}_{t-1}} - 1\right)^2 (1-\epsilon_t) \widetilde{p}_t^{-\epsilon_t}\right) Y_t\right) \\
&: \quad +\psi \mathrm{E}_t \left[\beta_{t+1} \frac{\lambda_{t+1}}{\lambda_t} \widetilde{p}_{t+1}^{1-\epsilon_t} Y_{t+1} \left(\frac{\widetilde{p}_{t+1} (1+\pi_{t+1})}{\widetilde{p}_t} - 1\right) \left(\frac{\widetilde{p}_{t+1} (1+\pi_{t+1})}{\widetilde{p}_t^2}\right)\right] = 0. && (24)
\end{aligned}$$

A.3 Policy

The central bank conducts monetary policy by setting the nominal return on the one-period non-state-contingent nominal bond according to the Taylor-type rule

$$1 + R_t = \frac{1}{\beta} (1 + \pi_t)^{\phi_\pi} \left(\frac{y_t}{\overline{y}} \right)^{\phi_y}, \tag{25}$$

where $\overline{y}$ denotes the nonstochastic steady state level of output.

The optimal production tax is given by

$$\tau_t = -\frac{1}{\epsilon_t - 1}, \tag{26}$$

which is negative implying that output must optimally be subsidized in order to offset the effects of monopolistic competition. For much of our analysis we assume that the steady state is inefficient and set $\tau_t = 0 \ \forall \ t$.

A.4 Equilibrium conditions

In a symmetric equilibrium, after aggregating across agents, we obtain the following equilibrium conditions. From equations (12) and (13) we obtain the labor supply equation

$$\frac{C_t}{1 - H_t} = \frac{\theta}{1 - \theta} w_t. \tag{27}$$

Equations (12), (14), (15) and (16) deliver the two Euler equations

$$C_t^{(\theta-1-\sigma\theta)} (1 - H_t)^{(1-\theta)(1-\sigma)} = \mathrm{E}_t \left[\begin{array}{c} \beta_{t+1} C_{t+1}^{(\theta-1-\sigma\theta)} (1 - H_{t+1})^{(1-\theta)(1-\sigma)} \\ \times \left(\frac{1+R_t}{1+\pi_{t+1}} \right) \end{array} \right], \tag{28}$$

$$C_t^{(\theta-1-\sigma\theta)} (1 - H_t)^{(1-\theta)(1-\sigma)} \left(1 + \gamma \frac{I_t}{K_t} \right) = \mathrm{E}_t \left[\begin{array}{c} \beta_{t+1} C_{t+1}^{(\theta-1-\sigma\theta)} (1 - H_{t+1})^{(1-\theta)(1-\sigma)} \\ \times \left(\begin{array}{c} (1-\delta) \left(1 + \gamma \frac{I_{t+1}}{K_{t+1}} \right) \\ + r_{t+1}^k + \frac{\gamma}{2} \left(\frac{I_{t+1}}{K_{t+1}} \right)^2 \end{array} \right) \end{array} \right]. \tag{29}$$

Equation (17) yields

$$K_{t+1} = (1 - \delta) K_t + I_t. \tag{30}$$

From the firm's problem, equations (20)—(22) give

$$\frac{r_t^k}{w_t} = \frac{\alpha}{1-\alpha} \frac{H_t}{K_t}, \tag{31}$$

$$\omega_t = \left(\frac{r_t^k}{\alpha} \right)^{\alpha} \left(\frac{w_t}{1-\alpha} \right)^{(1-\alpha)}, \tag{32}$$

while equation (24) yields the Phillips curve

$$\begin{aligned} \pi_t (1 + \pi_t) = {} & \frac{(1 - \tau_t)(1 - \epsilon_t)}{\psi} + \frac{(\epsilon_t - 1)}{2} \pi_t^2 + \frac{\epsilon_t}{\psi} \omega_t \\ & + \mathrm{E}_t \left[\frac{\beta_{t+1} C_{t+1}^{(\theta-1-\sigma\theta)} (1 - H_{t+1})^{(1-\theta)(1-\sigma)} K_{t+1}^{\alpha} H_{t+1}^{1-\alpha} \pi_{t+1} (1 + \pi_{t+1})}{C_t^{(\theta-1-\sigma\theta)} (1 - H_t)^{(1-\theta)(1-\sigma)} K_t^{\alpha} H_t^{1-\alpha}} \right]. \end{aligned} \tag{33}$$

The economy's resource constraint is

$$C_t + \left(1 + \frac{\gamma}{2} \frac{I_t}{K_t} \right) I_t = \left(1 - \frac{\psi}{2} \pi_t^2 \right) K_t^{\alpha} H_t^{1-\alpha}. \tag{34}$$

B Appendix: Computing equilibrium

Our benchmark model, which sets occasionally binding constraints aside, can be solved in a number of ways. Rather than solve it using perturbation methods to obtain a local solution, we solve the model globally using a projection method. Anticipating the introduction of occasionally binding constraints, we solve the model using linear splines to approximate functions. Before solving the model, we use equations (27), (31), and (32) to solve for factor prices and real marginal costs as a function of the real allocation

$$
\begin{aligned}
w_t &= \frac{1-\theta}{\theta}\frac{C_t}{1-H_t}, &(35)\\
r_t^k &= \frac{\alpha}{1-\alpha}\frac{1-\theta}{\theta}\frac{H_t}{K_t}\frac{C_t}{1-H_t}, &(36)\\
\omega_t &= \frac{1}{1-\alpha}\frac{1-\theta}{\theta}\left(\frac{H_t}{K_t}\right)^{\alpha}\frac{C_t}{1-H_t}. &(37)
\end{aligned}
$$

With these equations for factor prices and real marginal costs, the system to be solved can be written as

$$
\begin{aligned}
\lambda_t &= \mathrm{E}_t\left[\beta_{t+1}C\lambda_{t+1}\left(\frac{1+R_t}{1+\pi_{t+1}}\right)\right], &(38)\\
\lambda_t\left(1+\gamma\frac{I_t}{K_t}\right) &= \mathrm{E}_t\left[\beta_{t+1}\lambda_{t+1}\left(\begin{array}{c}(1-\delta)\left(1+\gamma\frac{I_{t+1}}{K_{t+1}}\right)\\ +\frac{\alpha}{1-\alpha}\frac{1-\theta}{\theta}\frac{H_{t+1}}{K_{t+1}}\frac{C_{t+1}}{1-H_{t+1}}+\frac{\gamma}{2}\left(\frac{I_{t+1}}{K_{t+1}}\right)^2\end{array}\right)\right], &(39)\\
\pi_t\left(1+\pi_t\right) &= \left[\begin{array}{c}\frac{(1-\tau_t)(1-\epsilon_t)}{\psi}+\frac{(\epsilon_t-1)}{2}\pi_t^2+\frac{\epsilon_t}{\psi}\frac{1}{1-\alpha}\frac{1-\theta}{\theta}\left(\frac{H_t}{K_t}\right)^{\alpha}\frac{C_t}{1-H_t}\\ +\mathrm{E}_t\left[\frac{\beta_{t+1}\lambda_{t+1}K_{t+1}^{\alpha}H_{t+1}^{1-\alpha}\pi_{t+1}(1+\pi_{t+1})}{\lambda_t K_t^{\alpha}H_t^{1-\alpha}}\right]\end{array}\right], &(40)\\
\lambda_t &= C_t^{(\theta-1-\sigma\theta)}\left(1-H_t\right)^{(1-\theta)(1-\sigma)}, &(41)\\
K_{t+1} &= (1-\delta)K_t+I_t, &(42)\\
C_t+\left(1+\frac{\gamma}{2}\frac{I_t}{K_t}\right)I_t &= \left(1-\frac{\psi}{2}\pi_t^2\right)Y_t, &(43)\\
1+R_t &= \frac{1}{\beta}\left(1+\pi_t\right)^{\phi_\pi}\left(\frac{y_t}{\overline{y}}\right)^{\phi_y}. &(44)
\end{aligned}
$$

The state variables for the system are K_t, ϵ_t, and β_t. To compute the equilibrium we approximate four objects: the decision rules for consumption, labor, investment, and inflation as functions of the three state variables. As mentioned earlier, we approximate these four functions using linear splines. To compute expectations we numerically integrate using Gauss-Hermite quadrature. For the spline approximations, we use 51 nodes for K_t and 31 nodes each for ϵ_t, and β_t, with these nodes spaced uniformly over the intervals $[20, 45]$, $[\epsilon - 3\sigma_\epsilon, \epsilon + 3\sigma_\epsilon]$, and $[\beta - 3\sigma_\beta, \beta + 3\sigma_\beta]$, respectively. For the integration step, 15 nodes for η_t and ζ_t were used. Numerical accuracy was evaluated by computing the errors to the Euler equations, equations (38) and (39).

C Appendix: Computing equilibrium with occasionally binding constraints

At its most general, the model has two occasionally binding constraints. The first requires the nominal interest rate to be non-negative, $R_t \geq 0$ while the second requires investment to be non-negative, $I_t \geq 0$. To allow for both occasionally binding constraints we solve the following system using linear splines to approximate the unknown functions and integrating using Gauss-Hermite quadrature:

$$\lambda_t = \mathrm{E}_t \left[\beta_{t+1} C \lambda_{t+1} \left(\frac{1+R_t}{1+\pi_{t+1}} \right) \right], \tag{45}$$

$$\lambda_t \left(1 + \gamma \frac{I_t}{K_t} \right) = \mathrm{E}_t \left[\nu_t + \beta_{t+1} \lambda_{t+1} \left(\begin{array}{c} (1-\delta) \left(1 + \gamma \frac{I_{t+1}}{K_{t+1}} \right) \\ + \frac{\alpha}{1-\alpha} \frac{1-\theta}{\theta} \frac{H_{t+1}}{K_{t+1}} \frac{C_{t+1}}{1-H_{t+1}} + \frac{\gamma}{2} \left(\frac{I_{t+1}}{K_{t+1}} \right)^2 \end{array} \right) \right], \tag{46}$$

$$\pi_t (1+\pi_t) = \left[\begin{array}{c} \frac{(1-\tau_t)(1-\epsilon_t)}{\psi} + \frac{(\epsilon_t - 1)}{2} \pi_t^2 + \frac{\epsilon_t}{\psi} \frac{1}{1-\alpha} \frac{1-\theta}{\theta} \left(\frac{H_t}{K_t} \right)^{\alpha} \frac{C_t}{1-H_t} \\ + \mathrm{E}_t \left[\frac{\beta_{t+1} \lambda_{t+1} K_{t+1}^{\alpha} H_{t+1}^{1-\alpha} \pi_{t+1} (1+\pi_{t+1})}{\lambda_t K_t^{\alpha} H_t^{1-\alpha}} \right] \end{array} \right], \tag{47}$$

$$\lambda_t = C_t^{(\theta - 1 - \sigma\theta)} (1 - H_t)^{(1-\theta)(1-\sigma)}, \tag{48}$$

$$K_{t+1} = (1-\delta) K_t + I_t, \tag{49}$$

$$C_t + \left(1 + \frac{\gamma}{2} \frac{I_t}{K_t} \right) I_t = \left(1 - \frac{\psi}{2} \pi_t^2 \right) K_t^{\alpha} H_t^{1-\alpha}, \tag{50}$$

$$1 + R_t = \max \left[1, \frac{1}{\beta} (1+\pi_t)^{\phi_\pi} \left(\frac{Y_t}{\overline{Y}} \right)^{\phi_y} \right], \tag{51}$$

$$I_t \geq 0, \nu_t \geq 0, I_t \nu_t = 0, \tag{52}$$

where ν_t in equation (46) is the Kuhn-Tucker multiplier associated with the inequality constraint $I_t \geq 0$.

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