Long-Run Consumption Risk and the Real Yield Curve

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Abstract
This paper estimates a consumption-based, no-arbitrage model of the term structure of real interest rates. The model nests the standard long-run risk model which assumes constant market prices of risk. We find that the long-run consumption risk dominates the short-run and volatility risks and drives most of the movements of bond risk premiums. The risk premium for consumption volatility is negative, suggesting that long-term real bonds provide an effective hedge against the volatility risk in consumption growth. In contrast to the standard long-run risk model, however, we find strong evidence that the market price of long-run consumption risk is time-varying and that stochastic volatilities alone are not sufficient to account for the variations in bond risk premiums.

JEL Classification: G12, E43
Key Words: consumption, long-run risk, the term structure, index-linked bonds

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1 Introduction

Economic theories suggest a close relation between real interest rates and consumption through the elasticity of intertemporal substitution. The expected consumption growth is a key state variable determining real interest rates.\textsuperscript{1} Even though the standard consumption-based asset pricing model is often rejected by financial market data, recent studies that feature more general specifications of investor’s preferences such as those with recursive utility or habit persistence, however, have documented more supporting evidence for consumption risks to explain major financial market phenomena. Among studies that focus on the term structure of interest rates, both Wachter (2006) and Buraschi and Jiltsov (2007) find that a consumption-based model with the habit utility of Campbell and Cochrance (1999) can account for many features of the nominal term structure of interest rates. Piazzesi and Schneider (2006) uses a representative agent asset pricing model with recursive utility preferences of Epstein and Zin (1989) and Weil (1989) to examine the roles of inflation and long-run consumption growth in determining risk premiums on the U.S. Treasury bonds. Using the same recursive utility preferences, Gallmeyer et al. (2007) derives the equilibrium yield curve that conforms with the standard affine term structure model and is able to relate factor loadings and market prices of risk to deep structural parameters.

The state variables in these models include not only consumption growth, but also a latent variable that represents either an exogenous consumption habit or the expected long-run consumption growth. In particular, in the long-run risk model of Bansal and Yaron (2004) and Bansal (2007), consumption growth includes a small but persistent long-run component. Fluctuations of this long-run component, together with stochastic volatilities of consumption growth, drive financial markets as recursive utility preferences

\textsuperscript{1}Some early empirical studies that investigate the relation between real interest rates and consumption growth include Hall (1988), Harvey (1988) and Chapman (1997) among many others. Boudoukh (1993) and Labadie (1994) both estimate equilibrium models of the nominal yield curve in a monetary economy.
generate heightened concerns about long-run growth prospects of the economy and the time-varying levels of economic uncertainty. The model implies that there are three distinct sources of macroeconomic risk that affect asset prices: the short-run, long-run and consumption volatility risks. Bansal and Shaliastovich (2007) shows that the long-run risk model can simultaneously account for many stylized facts in equity, bond and foreign exchange markets.

Motivated by these results, the current paper estimates a consumption-based, no-arbitrage model of the term structure of real interest rates that nests the standard long-run risk model. We seek to understand more precisely the empirical properties of various consumption risks and to examine their roles in determining the long-end of the real yield curve. Long-term interest rates encode information about investor’s intertemporal marginal rate of substitution. And focusing on real interest rates allows us to concentrate on consumption growth while abstracting from the effect of inflation.\(^2\) Unlike the tightly specified equilibrium asset pricing models, this paper doesn’t attempt to identify and estimate the structural parameters that characterize investor’s preferences. Using the no-arbitrage condition, the model adopts a more flexible specification of market prices of risk. It lies within the broad class of dynamic affine models of the term structure of interest rates.\(^3\) Retaining such econometric flexibilities is important if we are to decode information from asset prices about long-run economic growth prospects as well as the required risk compensations.

The paper contributes to a growing literature on the estimation of long-run risk models, including Bansal, Gallant and Tauchen (2007), Bansal, Kiku and Yaron (2007) and Chen, Favilukis and Ludvigson (2007) among others. These studies focus on stock market returns instead of the term

\(^2\)As shown in Gallmeyer et al. (2007), the empirical properties of equilibrium models of the nominal term structure of interest rates depends critically on the econometric assumptions about inflation. In the case of stock marker returns, additional assumptions about the dividend process are needed.

structure of interest rates. Doh (2008) uses Bayesian methods to estimate a Gaussian model of the nominal term structure under the long-run risk assuming a constant growth volatility. In all these models, the stochastic discount factor for asset pricing is derived from Epstein-Zin recursive utility preferences. The log-linearized Euler equation leads to constant market prices of risk. Risk premiums are time-varying solely because of stochastic volatilities of consumption growth. In contrast, the current paper allows for time-varying market prices of risk under the weaker assumption of no-arbitrage. Our empirical results show strong support for this more general specification. We find that the stochastic volatilities alone are not sufficient to account for the time variations in bond risk premiums.

The current paper is also closely related to the literature of no-arbitrage term structure models with macroeconomic fundamentals, including Evans and Marshall (2001), Ang and Piazzesi (2003), Hördahl, Tristani and Vestin (2006), Bekaert, Cho and Moreno (2006), Bikov and Chernov (2006), Ang, Dong and Piazzesi (2007), Rudebusch and Wu (2007), Ang, Bekaert and Wei (2008) among others. These studies extend the latent-factor term structure models by including observable macroeconomic variables as additional state variables. This extension enables the researchers to examine the effects of various macroeconomic shocks, especially monetary shocks, on the nominal term structure of interest rates. Our focus, however, is on the relation between consumption growth and the real yield curve. The underlying state variables are based on a long-run risk model and have clear economic interpretations. Another difference is that the existing no-arbitrage, discrete-time term structure models with macroeconomic fundamentals are Gaussian with constant volatility. In this paper, consumption growth exhibits stochastic volatilities as in the standard long-run risk model. We find that a similar term structure model with a constant growth volatility has a much poorer in-sample fit.

The rest of the paper is organized as follows. Section 2 provides summary statistics about consumption growth and long-term real interest rates. Section 3 presents the arbitrage-free dynamic model of the real yield curve
under the long-run consumption risk. The estimation and empirical results are discussed in section 4. Section 5 contains some concluding remarks.

2 Data and Summary Statistics

We study the joint dynamics of aggregate consumption growth and long-term real interest rates. Ex-ante real interest rates are not observable. The best approximations are yields on inflation index-linked government bonds. The U.K. market for inflation index-linked government debts was started in 1981 and has the longest time series on such yields. Yields on Treasury Inflation Protected Securities (TIPS) in the U.S. also provide close approximations to real interest rates. But the U.S. market was started in the 1990s and has much shorter time series. Since the data on aggregate consumption are available on a quarterly basis, we use the U.K. data on real interest rates and consumption in this paper.\(^4\) Consumption data are obtained from International Financial Statistics (IFS). Consumption growth rates are calculated as quarterly percentage changes of seasonally adjusted per-capita real consumption on non-durable goods and services. Data on real interest rates are obtained from Bank of England which also provides detailed explanations of the estimation of zero-coupon yields from bond prices. We only use the long end of the real yield curve (5-year to 15-year) in this study. One reason is that the U.K. index-linked bonds have their coupon and principal payments effectively linked to the Retail Price Index published approximately eight months prior to the payment date. While this “indexation lag problem” may create serious errors in the estimates of short-term real interest rates, its effect on the long end of the yield curve should be negligible. We use data from the second quarter of 1985 to the first quarter of 2007. Interest rates are collected at the beginning of each quarter. We exclude the data from 1981 to 1984 mainly because the market for index-linked bonds was known to be not very liquid in the early years of its development, and the

\(^4\)Other studies of the real term structure that also utilize the U.K. index-linked bond yields include Evans (1998, 2003) and Seppälä (2005) among others.
bond yields might include a significant liquidity premium.

In Table 1 we report the summary statistics on consumption growth and the long-term real interest rates. During 1985 and 2007, aggregate consumption grows at an annual rate of about 2.4%. Compared to long-term real interest rates, consumption growth is much more volatile with a standard deviation of 3.4%. The standard deviations of long-term interest rates are all smaller than 1%. During the sample period, consumption growth is negatively correlated with an autocorrelation coefficient of -0.22. In contrast, interest rates are very persistent. The autocorrelation coefficients for the long-term real interest rates range from 0.91 to 0.95. The real yield curve has a positive slope on average during 1985-2007. The mean of the 5-year rate is 2.96%, while the mean of the 15-year rate is 3.09%. But if we break the sample into two, we find that the yield curve is downward sloping since 1998 (see Figure 5). One often observed feature of the term structure of interest rates is that interest rate volatility doesn’t seem to attenuate as maturity increases. This is also true about the long-term real interest rates. As we can see from Table 1, as interest rate maturity increases from 5 years to 15 years, the standard deviation instead of declining, it actually increases slightly from 0.81% to about 0.99%. Our term structure model is able to capture these empirical properties of the real yield curve.

The close relation between real interest rates and consumption growth can be seen from Table 2, where we report the cross correlations between the long-term real interest rates and the consumption growth rate. The table clearly shows that, consistent with economic theories, real interest rates are strongly and positively correlated with consumption growth. In particular, the real interest rates in our sample are most correlated with 1-year or 2-year ahead consumption growth (Δc_{t+4} or Δc_{t+8}). For example, the correlation between the 5-year real interest rate, R_{5,t}, with Δc_{t+4} is nearly 0.28. The correlation between the real interest rates and lagged consumption growth

\footnote{For nominal interest rates in the U.K. over the same sample period, the standard deviation decreases slightly with maturity. The difference between the term structure of nominal rate volatilities and that of real rate volatilities is a topic for future research.}
is much weaker. The highest correlation is between $R_{5,t}$ and $\Delta c_{t-8}$ at about 0.10. These results suggest that, if consumption growth is predictable, the state variables that predict future consumption growth are likely to play an important role in determining real interest rates as well.

To see that the aggregate consumption growth includes a predictable component, we regress future consumption growth on lagged real long-term interest rates and consumption growth. The results are reported in Table 3. The first panel shows the regression result for forecasting the short-run (one quarter ahead) consumption growth, $\Delta c_{t+1} = \log C_{t+1} - \log C_t$. Variables with statistically insignificant and economically unimportant coefficients are excluded from the forecasting regression reported in Table 3. Both the 5-year interest rate, $R_{5,t}$ and the yield spread, $SP_{15,t} = R_{15,t} - R_{5,t}$, have significant forecasting powers for the short-run consumption growth. Together with lagged consumption growth rates, they account for 38% of the variations in $\Delta c_{t+1}$. Panel 2 and Panel 3 show the regression results for forecasting long-run consumption growth. A 4-quarter or 8-quarter moving average of the quarterly growth rate is used to measure long-run consumption growth. The constructed long-run growth rate is highly autocorrelated because of overlaps. We include the lagged long-run consumption growth in our forecasting regression. We find that both the 5-year rate and the yield spread still have significant forecasting powers for future long-run consumption growth even when the lagged long-run growth rate is included in the regression. The adjusted R-square from the forecasting regression is above 80% and there is little autocorrelation in the regression residuals according to the Dubin-Watson statistics. These forecasting regressions show a very robust result; a higher level of real interest rates predicts higher future consumption growth while a bigger yield spread predicts lower future consumption growth. Early empirical studies on the predictive power of the yield curve for future economic activities have focused almost exclusively on the slope of the yield curve. Our result is consistent with that of Ang and Piazzesi (2006) which shows that the level of interest rates may have more predicative power for GDP growth than any yield spread does.
3 A Consumption-Based Term Structure Model

Motivated by the empirical results above, in this section, we construct a consumption-based asset pricing model of the terms structure of interest rates. The model allows us to estimate the joint dynamics of consumption growth and the long-end of the real yield curve while imposing the no-arbitrage condition on the cross-section of real interest rates.

3.1 State variables

Following the long-run risk models, we assume that consumption growth $\Delta c_t$ contains a small but persistent long-run component so that $\Delta c_t$ can be described by a mean-reverting process as follows

$$\Delta c_{t+1} - \Delta c_t = k(\mu_t - \Delta c_t) + \xi_{t+1}$$

where $|k| < 1$ and $\mu_t$ can be thought of as the expected long-run growth path of consumption. It turns out that this long-run component of consumption growth plays an important role in determining the real yield curve and bond risk premiums. We assume that $\mu_t$ follows a stationary AR(1) process

$$\mu_{t+1} = \psi_0 + \psi_1 \mu_t + \nu_{t+1}$$

where $|\psi_1| < 1$. $\xi_{t+1}$ and $\nu_{t+1}$ are innovations to $\Delta c_{t+1}$ and $\mu_{t+1}$ respectively. We assume that they are correlated and have stochastic volatilities as follows

$$\left( \begin{array}{c} \nu_{t+1} \\ \xi_{t+1} \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ h & 1 \end{array} \right) \left( \begin{array}{c} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{array} \right)$$

where $\varepsilon_{1,t+1}$, an exogenous shock to $\mu_{t+1}$, represents the long-run consumption risk. When $h \neq 0$, $\varepsilon_{1,t+1}$ also affects consumption growth rate, $\Delta c_{t+1}$. $\varepsilon_{2,t+1}$ is another shock to $\Delta c_{t+1}$ that is orthogonal to $\varepsilon_{1,t+1}$. $\varepsilon_{2,t+1}$ drives

\[\text{In the existing long-run risk models } \xi_{t+1} \text{ and } \nu_{t+1} \text{ are usually assumed to be mutually independent. See Bansal and Yaron (2004), Gallmeyer et al. (2007).}\]
Δc_{t+1} away from its long-run path, therefore representing the short-run consumption risk. We assume \( \varepsilon_{1,t+1} \) and \( \varepsilon_{2,t+1} \) are conditional Gaussian as follows

\[
\begin{pmatrix}
\varepsilon_{1,t+1} \\
\varepsilon_{2,t+1}
\end{pmatrix} \sim N(0, \Sigma_t)
\]

where

\[
\Sigma_t = \begin{pmatrix}
\sqrt{\alpha_1 + \beta_1 z_t} & 0 \\
0 & \sqrt{\alpha_2 + \beta_2 z_t}
\end{pmatrix}
\]

\( z_t \) is a positive latent factor that drives the volatility of consumption growth \((\alpha_i \geq 0, \beta_i \geq 0 \text{ for } i = 1, 2)\). Most discrete-time arbitrage-free term structure models with macroeconomic fundamentals are Gaussian with constant volatility (see Ang and Piazzesi (2003) for example). Even in models with stochastic volatilities (see Bansal and Yaron (2004) and Gallmeyer et al. (2007) for example), \( z_t \) is assumed to be Gaussian for convenience. But \( z_t \) needs to be non-negative for \( \Sigma_t \) to be well defined. In this paper, we assume that \( z_t \) follows an autoregressive gamma process as in Gourieroux and Jasiak (2006) and Dai, Le and Singleton (2007). That is, conditional on \( z_t \),

\[
\frac{z_{t+1}}{c} \sim \text{gamma}(\delta + P)
\]

where

\[
P \sim \text{Poisson} \left( \frac{\rho z_t}{c} \right)
\]

for some positive parameters \( \delta > 0, 0 < \rho < 1 \) and \( c > 0 \). We also assume that \( z_{t+1} \) is conditionally independent of \((\mu_{t+1}, \Delta c_{t+1})'\). To prevent \( z_t \) from attaining the zero lower bound, we further restrict \( \delta > 1 \) in the empirical estimation below.\(^7\)

\(\makebox[1\textwidth]{7}\)The conditional density of \( z_{t+1} \) given \( z_t \) takes the following form \( f(z_{t+1}|z_t) = \exp(-\frac{z_{t+1}}{c}) \sum_{k=0}^{\infty} \left( \frac{\delta}{c} \right)^{k+1} \frac{1}{k!} \exp(-\rho z_t/c) \left( \frac{\rho z_t}{c} \right)^k \). In continuous-time limit, \( z_t \) converges to the square-root process \( dz(t) = k(\theta - z(t))dt + \sigma \sqrt{z(t)} dB(t) \) where \( k \Delta t = 1 - \rho, \frac{1}{2} \sigma^2 \Delta t = c \) and \( \frac{2k\theta}{c} = \delta \). See Gourieroux and Jasiak (2006) for more detailed discussions of the properties of \( z_t \). Dai, Le and Singleton (2007) provides a multivariate extension of the autoregressive gamma process.

\(\makebox[1\textwidth]{8}\)See Cheridito, Filipovic and Kimmel (2007) and Dai, Le and Singleton (2007) for more discussions on this issue.
Note that the conditional mean and variance of $z_{t+1}$ are given by

$$E_t(z_{t+1}) = c\delta + \rho z_t$$

$$V_t(z_{t+1}) = c^2\delta + 2\epsilon\rho z_t$$ 

We collect all the state variables in a $3 \times 1$ vector $X_t = (z_t, \mu_t, \Delta c_t)'$. We also denote $Y_t = (\mu_t, \Delta c_t)'$. We notice that $X_t$ is a Markov process and $Y_t$ is AR(1) that is conditionally independent of $z_t$ with conditional mean

$$E_t(Y_{t+1}) \equiv \bar{Y}_{t+1}^P = \Phi_0 + \Phi_1 Y_t$$

where

$$\Phi_0 = \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} \psi_1 & 0 \\ k & 1 - k \end{pmatrix}$$

and conditional variance

$$\text{Var}_t(Y_{t+1}) \equiv \Omega_t = S \Sigma_t^2 S' = \Omega_0 + \Omega_1 z_t$$

where

$$S = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}, \quad \Omega_0 = S \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} S', \quad \Omega_1 = S \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} S'$$

### 3.2 The stochastic discount factor

Instead of making explicit assumptions about investor’s intertemporal utility function as in equilibrium asset pricing models, we use the no-arbitrage approach in the current paper to obtain the term structure of interest rates. In particular, the weaker condition of no-arbitrage implies that there exists a positive stochastic discount factor, $M_{t,t+1}$, such that for an asset with a payoff $D_{t+1}$ at $t + 1$, its time-$t$ price is given by

$$P_t = E_t(M_{t,t+1} D_{t+1})$$
Dai, Le and Singleton (2007) suggests a general specification for the stochastic discount factor as follows

\[ M_{t,t+1} = e^{-r_t} \frac{e^{-\lambda'_t X_{t+1}}}{L_t^P(\lambda_t)} \] (12)

where \( r_t \) is the 1-period risk-free rate, \( \lambda_t = (\lambda_{z,t}, \lambda_{\mu,t}, \lambda_{c,t})' \), a 3 \times 1 vector of “market prices of risk” (see more discussions below), and \( L_t^P(u) \) is the conditional (two-sided) Laplace transforms of \( X_{t+1} \), which under conditional independence between \( z_{t+1} \) and \( Y_{t+1} \) is given by,

\[ L_t^P(u) = E_t(e^{-u'X_{t+1}}) = e^{-a_z(u_z)-b_z(u_z)z_t} \times e^{-u'_z\gamma_t^P + \frac{1}{2}u'_z\Omega_t u_z} \] (13)

where \( u = (u_z, u'_y)' = (u_z, u_\mu, u_c)' \) is a 3 \times 1 vector and,

\[ a_z(u_z) = \delta \log(1 + u_z c), \quad b_z(u_z) = \frac{u_z \rho}{1 + u_z c} \]

The specification of the stochastic discount factor above closely mimics that of the intertemporal marginal rate of substitution (IMRS) in the long-run risk models. For example, Bansal and Yaron (2004) shows that, under recursive utility preferences, the log of IMRS can be obtained as

\[ m_{t,t+1} - E_t(m_{t,t+1}) = -\lambda_{m,\eta}\sigma_t\eta_{t+1} - \lambda_{m,e}\sigma_t e_{t+1} - \lambda_{m,w}\sigma_w w_{t+1} \] (14)

where \( \eta_{t+1} \), \( e_{t+1} \) and \( w_{t+1} \) are exogenous shocks that represent, respectively, short-run, long-run and consumption volatility risks, and \( \lambda_{m,\eta} \), \( \lambda_{m,e} \) and \( \lambda_{m,w} \) are the corresponding market prices of risk. \( \sigma_t \) is the stochastic volatility of consumption growth. While \( \lambda_{m,\eta} \), \( \lambda_{m,e} \) and \( \lambda_{m,w} \) depend on deep structural parameters, movements in the risk premiums are driven by the time-varying volatilities of consumption growth.

To compare the model presented in this paper with the long-run risk models, we can rewrite (12) in a similar form as (14)

\[ m_{t,t+1} - E_t(m_{t,t+1}) = -\lambda_{z,t}(z_{t+1} - E_t z_{t+1}) - (\lambda_{\mu,t} + h\lambda_{c,t})\varepsilon_{1,t+1} - \lambda_{c,t}\varepsilon_{2,t+1} \] (15)
where \( z_{t+1}, \varepsilon_{i,t+1} (i = 1,2) \) are defined in Section 3.1. They represent, respectively, growth volatility, long-run and short-run consumption risks just as in Bansal and Yaron (2004). Notice that the stochastic volatility of \( \varepsilon_{i,t+1} \) is given by \( \sqrt{\alpha_i + \beta_i z_t} \), and they correspond to \( \sigma_t \) in Bansal and Yaron (2004). In this paper, we don’t restrict the volatilities of long-run and short-run consumption growth to be the same. Also notice that the variance of the uncertainty shock \( z_{t+1} - E_t z_{t+1} \) is given by \( c^2 \delta + 2c\rho z_t \) in this paper. In Bansal and Yaron (2004), it is assumed to be a constant \( \sigma_w^2 \).

More importantly, \( \lambda_{z,t}, \lambda_{\mu,t} + h\lambda_{c,t} \) and \( \lambda_{c,t} \) correspond to the market prices of risk \( \lambda_{m,w}, \lambda_{m,e} \) and \( \lambda_{m,\eta} \) in the long-run risk model of Bansal and Yaron (2004), respectively. While \( \lambda_{m,w}, \lambda_{m,e} \) and \( \lambda_{m,\eta} \) are assumed to be constant, \( \lambda_{z,t}, \lambda_{\mu,t} + h\lambda_{c,t} \) and \( \lambda_{c,t} \) are time-varying and are nonlinear functions of the underlying state variables in the present paper (see more details below). Here there two separate sources of time-variations in risk premiums: one is the time-varying market prices of risk, \( \lambda_t = (\lambda_{z,t}, \lambda_{\mu,t}, \lambda_{c,t})' \); and the other is the stochastic volatilities of fundamental shocks, \( (z_{t+1} - E_t z_{t+1}), \varepsilon_{t+1} \) and \( \varepsilon_{t+1} \). Retaining such econometric flexibilities is important for our exercise as our goal is to understand more precisely the empirical properties and relative importance of different consumption risks from prices of long-lived assets. The tradeoff is that we are not able to match these market prices of risk with the deep structural parameters that characterize investor’s preferences.

### 3.3 The term structure of interest rates

We seek a tractable solution to the term structure of interest rates that lies within the broad class of affine models. Recently Dai, Le and Singleton (2007) develops a rich class of discrete-time, nonlinear dynamic term structure models. Starting from the distribution of state variables that resides within a family of discrete-time affine process under the risk-neutral probability measure, Dai, Le and Singleton (2007) allows flexible specifications of the market prices of risk so that the distribution of the state variables
under the physical probability measure can have any nonlinear dynamics as long as it is stationary or ergodic. Since the purpose of the current paper is to investigate the empirical nature of consumption risk, we start by assuming the distributions of $\Delta c_{t+1}$, $\mu_{t+1}$ and $z_{t+1}$ under the physical probability measure so that our specification is consistent with the standard long-run risk model. The resulting model is a more restrictive version of that of Dai, Le and Singleton (2007). Nonetheless, it still has the maximum econometric flexibility within the affine framework.

3.3.1 The risk-neutral probability measure and the market prices of risk

The solution to the term structure of interest rates can be obtained by a change of probability measures. In particular, we define the risk-neutral probability measure by the following Radon-Nykodym derivative

$$\xi_{t,t+1} = \left( \frac{dQ}{dP} \right)_{t,t+1} = e^{-\lambda^t X_{t+1}}$$

(16)

We can then show that the conditional Laplace transforms of $X_{t+1} = (z_{t+1}, Y'_{t+1})'$ under the risk-neutral probability measure $Q$ is given by

$$L_Q^t(u) = \frac{L_P^t(u + \lambda_t)}{L_P^t(\lambda_t)} = e^{-a_t^*(u_z) - b_t^*(u_z)Z_t} \times e^{-u^t_y Y_t^Q + \frac{1}{2} u^t_y \Omega_t u_y}$$

(17)

where

$$c_t^* = \frac{c}{1 + \lambda_{z,t} c}, \quad \rho_t^* = \frac{\rho}{(1 + \lambda_{z,t} c)^2}$$

$$a_t^*(u_z) = \delta \log(1 + u_z c_t^*), \quad b_t^*(u_z) = \frac{u_z \rho_t^*}{1 + u_z c_t^*}$$

and

$$Y_t^Q = Y_t^P - \Omega_t \lambda_{y,t}$$

$Y_t^P$ and $\Omega_t$ are given in (9) and (10) respectively. In order to get an analytical
solution of the term structure of interest rates, we assume that $\lambda_{z,t}$ is a constant

$$\lambda_{z,t} = \lambda_z$$  \hspace{1cm} (18)

In this case, we have

$$L_t^Q(u) = e^{-a^*(u_z)z_t} \times e^{-u_y^*Y_t^Q + \frac{1}{2}u_y^* \Omega_t u_y}$$  \hspace{1cm} (19)

where

$$a^*(u_z) = \delta \log(1 + u_z c^*) \quad b^*(u_z) = \frac{u_z \rho^*}{1 + u_z c^*}$$

$$c^* = \frac{c}{1 + \lambda_z c}, \quad \rho^* = \frac{\rho}{(1 + \lambda_z c)^2}$$

In other words, $z_t$ still follows an autoregressive gamma process that is conditionally independent of $Y_t$ under $Q$.

Moreover, we assume $Y_{t+1}$ is also AR(1) under $Q$ with a conditional mean of

$$\bar{Y}_{t+1}^Q = \Phi_0^* + \Phi_1^* Y_t$$  \hspace{1cm} (20)

Note that the above assumption on the distribution of $Y_{t+1}$ under $Q$ implicitly defines the market prices of risk associated with the state variables included in $Y_{t+1}$. That is, $\lambda_{y,t} = (\lambda_{\mu,t}, \lambda_{c,t})'$ can be obtained as

$$\lambda_{y,t} = \Omega_t^{-1} \left( \bar{Y}_{t+1}^P - \bar{Y}_{t+1}^Q \right) = \Omega_t^{-1} \left[ (\Phi_0^* - \Phi_0^1) + (\Phi_1^* - \Phi_1^1) Y_t \right]$$  \hspace{1cm} (21)

where $\Omega_t$ is given in (10). We can easily see that, unlike the existing long-run risk models, the market prices of long-run and short-run consumption risks in the current paper are time-varying and are non-linear functions of the underlying state variables, $z_t$, $\mu_t$ and $\Delta c_t$. Moreover, even though the market price of the volatility risk, $\lambda_z$, is assumed to be constant, the stochastic volatility of $z_{t+1} - E_t z_{t+1}$ will still make the volatility risk premiums time-varying.
3.3.2 An affine solution

Now we consider the market for zero-coupon bonds that are free of default risk. Let \( P_{n,t} \) denote the real price at time \( t \) of a \( n \)-period bond that pays one unit of consumption goods when it matures. In the absence of arbitrage opportunities, we must have

\[
P_{n,t} = E_t^P(M_{t,t+1}P_{n-1,t+1}) = e^{-r_t}E_t^Q(P_{n-1,t+1}) \tag{22}
\]

where the first expectation \( E_t^P(\cdot) \) is taken with respect to the physical probability measure \( \mathbb{P} \) and the second expectation \( E_t^Q(\cdot) \) is taken with respect to the risk-neutral probability measure \( \mathbb{Q} \).

The model is completed by assuming that the short-term interest rate, \( r_t \), is given by

\[
r_t = A_1 + B'_1X_t \tag{23}
\]

Let \( B_n = (B_{z,n}, B'_{y,n}) = (B_{z,n}, B_{\mu,n}, B_{c,n}) \), it then follows that \( P_{n,t} \) can be obtained as

\[
P_{n,t} = e^{-A_n - B'_nX_t} = e^{-A_n -B_{z,n}z_{t} -B'_{y,n-1}Y_t} \tag{24}
\]

where the coefficient \( A_n \) and \( B_n \) are determined by the following system of difference equations, starting from \( A_0 = B_0 = 0 \),

\[
A_n = A_1 + A_{n-1} + a^*(B_{z,n-1}) + B'_{y,n-1}\Phi'_0 - \frac{1}{2}B'_{y,n-1}\Omega_0B_{y,n-1} \tag{25}
\]

\[
B_{z,n} = B_{z,1} + b^*(B_{z,n-1}) - \frac{1}{2}B'_{y,n-1}\Omega_1B_{y,n-1} \tag{26}
\]

\[
B_{y,n} = B_{y,1} + \Phi'_1B_{y,n-1} \tag{27}
\]

where

\[
a^*(B_{z,n-1}) = \delta \log(1 + c^*B_{z,n-1})
\]

\[
b^*(B_{z,n-1}) = \frac{\rho^*B_{z,n-1}}{1 + c^*B_{z,n-1}}
\]
and
\[
c^* = \frac{c}{1 + \lambda c}, \quad \rho^* = \frac{\rho}{(1 + \lambda c)^2}
\]

Continuously compounding \(n\)-period interest rate, \(R_{n,t}\), is defined by
\[
P_{n,t} = e^{-nR_{n,t}}
\]
and we have
\[
R_{n,t} = \frac{A_n}{n} + \frac{B'_nX_t}{n}
\] (28)

4 Estimation and Empirical Results

We estimate the joint dynamics of consumption growth and long-term real interest rates based on the term structure model developed above. As discussed in Section 2, the data on consumption are quarterly percentage changes of seasonally adjusted per-capita real consumption of non-durable goods and services from Britain during the second quarter of 1985 to the first quarter of 2007. We use yields on 5-year, 7-year, 10-year, 12-year and 15-year inflation index-linked zero-coupon bonds as ex-ante real interest rates. We assume that 5-year and 15-year bonds are priced without any error. We can, therefore, solve for \((z_t, \mu_t)\)' using the 5-year and 15-year rates, \((R_{5,t}, R_{15,t})\)'. We assume that the pricing errors for 7-year, 10-year and 12-year bonds have normal distributions and are serially and cross-sectionally independent with zero means and standard deviations of \(\sigma_7\), \(\sigma_{10}\) and \(\sigma_{12}\) respectively.

These assumptions together with those on the distribution of state variables \(\Delta c_t, \mu_t\) and \(z_t\) (under probability measure \(P\)) in Section 3.1 enable us to write down the joint likelihood function for \(\{\Delta c_t, R_{5,t}, R_{7,t}, R_{10,t}, R_{12,t}, R_{15,t}\}\) where \(R_{n,t}\) denotes \(n\)-year real interest rate. The parameters to be estimated in this models include: (1) parameters that govern the \(P\)-distribution of the state variables \(\{k, \psi_0, \psi_1, h, \alpha_i, \beta_i, c, \delta, \rho\}\) \((i = 1, 2)\); (2) parameters that determine the market prices of risk or the \(Q\)-distribution of the state variables \(\{\lambda_z, \Phi_0^*, \Phi_1^*\}\), where \(\Phi_0^*\) is a \(2 \times 1\) vector; \(\Phi_1^*\) is a \(2 \times 2\) matrix; (3) parameters
that determine the short-term interest rate \( \{ A_1, B_1 \} \), where \( A_1 \) is a constant and \( B_1 = (B_{z,1}, B_{\mu,1}, B_{\kappa,1})' \) is a 3 \times 1 vector; and (4) standard deviations of the pricing errors for 7-year, 10-year and 12-year bonds \( \{ \sigma_7, \sigma_{10}, \sigma_{12} \} \).

Given the affine structure of the term structure model, it is well known that the model is invariant with respect to certain linear transforms and it is necessary to normalize some parameters to achieve identification. First, we use de-meaned consumption growth rate \( \Delta c_t \) in the estimation. This allows us to fix \( \psi_0 \) at 0. Secondly, following Dai and Singleton (2000), we fix \( \alpha_1 \) and \( c \) at some positive constant and restrict \( B_1 \) to be positive. To prevent \( z_t \) from attaining the zero lower bound, we restrict \( \delta > 1 \). And to further reduce the burden of estimation, we assume that \( \Delta c_t \) still follows a mean-reverting process under \( Q \), that is

\[
\Delta c_{t+1} - \Delta c_t = k^* (\mu_t - \Delta c_t) + \xi_{t+1}^*
\]

(29)

Under this assumption, \( \Phi_0^* \) and \( \Phi_1^* \) are simplified to

\[
\Phi_0^* = \begin{pmatrix} \psi_0^* \\ 0 \end{pmatrix}, \quad \Phi_1^* = \begin{pmatrix} \psi_1^* & 0 \\ k^* & 1 - k^* \end{pmatrix}
\]

Moreover, we assume that the long-run component of consumption growth, \( \mu_t \), has a constant volatility. We fix \( \beta_1 \) at zero. With these normalization and simplifications, the total number of parameters that we need to estimate is reduced to 18. We use quasi-maximum likelihood estimation and the results are reported in Table (4).

4.1 Goodness-of-fit

In the upper panel of Figure 1, we plot the sample mean of the actual index-linked zero yields (maturity from 5 years to 15 years, represented by circles) during the period from the second quarter of 1985 to the first quarter of 2007. We can see again that the long end of the real yield curve slopes slightly upwards, with the 5-year rate at around 2.96% and the 15-year
rate at around 3.08%. In the same graph, we also plot the means of the real interest rates, represented by the solid line, from the estimated term structure model. The model provides a very good fit of the average yield curve with small pricing errors. For example, for the 7-year, 10-year and 12-year yields, the root mean square pricing errors are 0.0643%, 0.0580% and 0.0379% per annum respectively.\(^9\) The lower panel of Figure 1 plots the sample standard deviations of the actual index-linked zero yields as well as the standard deviations of the real rates from the estimated model. We can see that the model also provides a good fit to the cross-section of the second moments of the real interest rates.

Most arbitrage-free term structure models with macroeconomic fundamentals assume Gaussian distributions with constant volatility. We find that allowing for stochastic volatility of consumption growth greatly improves the goodness-of-fit of the term structure model. Figure 2 plots the sample mean and sample standard deviations of the actual index-linked zero yields together with the mean yield curve and standard deviations of long-term real rates from an estimated term structure model. The terms structure model is same as the one in Section 3 except that it assumes homoskedasticity in consumption growth. The model has two state variables, short-run and long-run consumption growth. Estimates of the model parameters are also reported in Table (4). Compared to the model with stochastic volatilities in consumption growth, we can see that the constant-volatility model does a poor job in matching the average yield curve as well as the second moments of the interest rates. A formal likelihood ratio test also strongly favors the model with stochastic volatility.

The estimated term structure model in Section 3 allows us to extract time-varying levels of consumption growth volatility as well as the long-run component of consumption growth from inflation-indexed zero yields. Figure 3 plots the estimated state variable \(\hat{z}_t\) for the sample period of Q2.1985 -

\(^9\)Root mean square pricing errors are computed as \(\sqrt{\frac{\sum_{t=1}^{T}(R_{n,t} - \hat{R}_{n,t})^2}{T}}\), where \(R_{n,t}\) is the observed \(n\)-year index-link zero yield at time \(t\), \(\hat{R}_{n,t}\) is the fitted \(n\)-year real interest rate from the term structure model.
Q1.2007. $z_t$ drives the conditional variance of the consumption growth rate. Large variations in $\hat{z}_t$ indicate that the conditional variance of $\Delta c_{t+1}$ changes significantly over time. The correlation coefficient between $\Delta c_t$ and $\hat{z}_t$ is -0.3686, suggesting growth uncertainty tends to increase during low-growth periods and tends to decline during high-growth periods. Such time-varying growth volatilities are also very persistent. The autocorrelation coefficient for $\hat{z}_t$ is about 0.8199.

In Figure 4, we plot in the upper panel the estimated long-run component of consumption growth, $\hat{\mu}_t$, together with the quarterly consumption growth rate, $\Delta c_t$, for the period of Q2.1985 - Q1.2007. We have subtracted the mean growth rate from $\Delta c_t$ in our estimation. Mean consumption growth rate during this period is about 2.4% per year. Compared with $\Delta c_t$, the long-run component is much smaller but more persistent. While $\Delta c_t$ is negatively autocorrelated with a coefficient of -0.22, $\hat{\mu}_t$ exhibits strong positive autocorrelation with a coefficient of 0.9758. The ratio of the sample standard deviations of $\hat{\mu}_t$ and $\Delta c_t$ is about 0.1864, confirming that the long-run consumption growth is much less volatile than quarterly consumption growth.

To see how well $\hat{\mu}_t$ captures the variations of long-run consumption growth, we plot the 2-year moving average of quarterly consumption growth rate, $\overline{\Delta c_t}$, and the estimated long-run component, $\hat{\mu}_t$, in the lower panel of Figure 4. We can see that $\hat{\mu}_t$, especially since 1995, is closely correlated with the 2-year moving average. The correlation coefficient between $\overline{\Delta c_t}$ and $\hat{\mu}_t$ is around 0.2883 for the whole sample, and the correlation increases to 0.4204 for the period between 1995 to 2007. The ratio of the standard deviations of $\mu_t$ and $\overline{\Delta c_t}$ is 38% for the whole sample and is approximately 65% for the period between 1995 and 2007. Long-run risk models provide evidence that this small but persistent component of consumption growth together with time-varying volatilities go a long way to explain many features of financial markets. We show in the next section that these two state variables indeed account for most of the risk premiums in the U.K. inflation index-linked bond market.
Arbitrage-free term structure models with latent factors usually use the descriptive properties of the yield curve to characterize the state variables, such as the level, slope and curvature factors. However, in the current paper, all state variables, $X_t = (z_t, \mu_t, \Delta c_t)$, have clear economic interpretations in terms of consumption growth. Nonetheless, there is a close relationship between $X_t$ and the standard yield curve factors. Table 5 reports the correlation coefficients between the elements in $X_t$ and the level, slope and curvature of the real yield curve. The table shows that the long-run component in consumption growth, $\mu_t$, has the highest positive correlation with the level factor. Short-run (i.e. quarterly) consumption growth, $\Delta c_t$, has the highest positive correlation with the slope factor. The growth volatility, $z_t$, has the highest positive correlation with the curvature of the yield curve. $\Delta c_t$ and $z_t$ also have strong negative correlations with the curvature and the slope respectively. These results suggest that the long-run consumption growth may correspond to the level factor, while the short-run growth and consumption volatility may correspond to the slope or curvature factors.

4.2 Bond risk premiums

We define the expected continuously compounding holding period return, $hpr_t$, on a n-period bond by\textsuperscript{10}

$$e^{hpr_{n,t}} = E_t \left( \frac{P_{n,t+1} - P_{n-1,t+1}}{P_{n,t}} \right)$$

where the expectation is taken with respect to the physical probability measure. Let $r_t$ denote the one-period risk-free rate, and $rp_{n,t} \equiv hpr_t - r_t$ denote the expected excess return or risk premium. Using the solution to the term structure of interest rates in Section 3, we can easily show that the expected

\textsuperscript{10}Notice a slightly different definition of $hpr_t$ is $E_t(\log P_{n-1,t+1} - \log P_{n,t})$. We choose the current one because it folds the Jensen’s inequality term into $hpr_t$. 

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excess return or risk premium is given by

\[ rp_{n,t} = - B'_{n-1,y} \Omega_t \lambda_{y,t} \]

\[ - \{ [a(B_{n-1,z}) - a^*(B_{n-1,z})] + [b(B_{n-1,z}) - b^*(B_{n-1,z})z_t] \} \]  

(30)

where \( B_{n,z} \) and \( B_{y,n} = (B_{\mu,n}, B_{c,n})' \) are given in (26) and (27) respectively. \( \Omega_t \) is the conditional variance of \( Y_{t+1} = (\mu_{t+1}, \Delta c_{t+1})' \), which is given in (10). \( \lambda_{y,t} = (\lambda_{\mu,t}, \lambda_{c,t})' \) includes the market prices of risk associated with the state variables \( \mu_{t+1} \) and \( \Delta c_{t+1} \). And

\[ a(u_z) = \delta \log(1 + u_z c), \quad a^*(u_z) = \delta \log(1 + u_z c^*) \]

\[ b(u_z) = \frac{u_z \rho}{1 + u_z c}, \quad b^*(u_z) = \frac{u_z \rho^*}{1 + u_z c^*} \]

\[ c^* = \frac{c}{1 + \lambda_z c}, \quad \rho^* = \frac{\rho}{(1 + \lambda_z c)^2} \]

We can further write \( rp_{n,t} \) as

\[ rp_{n,t} = - \text{Cov}^{\varepsilon_1}_t (\log P_{n-1,t+1}, \log M_{t,t+1}) - \text{Cov}^{\varepsilon_2}_t (\log P_{n-1,t+1}, \log M_{t,t+1}) \]

\[ - \{ [a(B_{n-1,z}) - a^*(B_{n-1,z})] + [b(B_{n-1,z}) - b^*(B_{n-1,z})z_t] \} \]  

(31)

where \( \text{Cov}^{\varepsilon_1}_t (\log P_{n-1,t+1}, \log M_{t,t+1}) \) and \( \text{Cov}^{\varepsilon_2}_t (\log P_{n-1,t+1}, \log M_{t,t+1}) \) are simply the conditional covariance between the log bond return and the log of IMRS under, respectively, the long-run consumption shock \( \varepsilon_{1,t+1} \) and the short-run consumption shock \( \varepsilon_{2,t+1} \).\(^\text{11}\) They are given by

\[ \text{Cov}^{\varepsilon_1}_t (\log P_{n-1,t+1}, \log M_{t,t+1}) = (B_{n-1,u} + h B_{n-1,c}) \text{Var}_t (\varepsilon_{1,t+1})(\lambda_{\mu,t} + h \lambda_{c,t}) \]

and

\[ \text{Cov}^{\varepsilon_2}_t (\log P_{n-1,t+1}, \log M_{t,t+1}) = B_{n-1,c} \text{Var}_t (\varepsilon_{2,t+1}) \lambda_{c,t} \]

To understand the last term in (31), we notice that when \( u_z c \) and \( u_z c^* \)

\(^{11}\)See Section 3.1 for definitions of \( \varepsilon_{1,t+1} \) and \( \varepsilon_{2,t+1} \).
are small, therefore we have

\[ a(u_z) = \delta \log(1 + u_z c) \approx \delta u_z c, \quad a^*(u_z) = \delta \log(1 + u_z c^*) \approx \delta u_z c^* \]

and

\[ b(u_z) = \frac{u_z \rho}{1 + u_z c} \approx u_z \rho, \quad b^*(u_z) = \frac{u_z \rho^*}{1 + u_z c^*} \approx u_z \rho^* \]

Hence the last term in the risk premium equation (31) can be approximated by

\[ B_{z,n-1}[\delta(c - c^*) + (\rho - \rho^*)z_t] \]

Using the definition of \( c^* \) and \( \rho^* \) and the properties of autoregressive gamma process \( z_{t+1} \), this term can be written as (under similar approximations when \( \lambda_z c \) is small)

\[ \text{Cov}_t(-B_{z,n-1}z_{t+1}, -\lambda_z z_{t+1}) \]

or

\[ \text{Cov}_t^\delta(\log P_{n-1,t+1}, \log M_{t,t+1}) \]

i.e. the conditional covariance between the log bond return and the log of IMRS under the consumption volatility shock \((z_{t+1} - E_t z_{t+1})\).

These results show that, as in the standard long-run risk models, bond risk premiums can be decomposed into 3 distinct components: the long-run, short-run, and consumption volatility risk premiums

\[ rp_{n,t} \approx -\text{Cov}_t^\epsilon(\log P_{n-1,t+1}, \log M_{t,t+1}) - \text{Cov}_t^\delta(\log P_{n-1,t+1}, \log M_{t,t+1}) \]

\[ -\text{Cov}_t^\delta(\log P_{n-1,t+1}, \log M_{t,t+1}) \]

We plot these 3 distinct risk premiums estimated from the term structure model in Figure 5 (the upper panel) for the 15-year index-linked bond between the second quarter of 1985 and the first quarter of 2007. Their summary statistics are reported in Table 6. We find that both the long-run and consumption volatility risks are priced in the long-term index-linked bond yields and exhibit strong time variations. The long-run consumption growth
is in fact the primary source of risk and dominates the short-run and volatility risks. The standard deviations of the long-run, short-run, and volatility risk premiums (per quarter) are 0.0114, 0.0006 and 0.0011 respectively. The risk premium for the short-run consumption risk is quite small compared to the long-run and volatility risk premiums. The average (per quarter) long-run, short-run and volatility risk premiums are 0.0090, -0.0004 and -0.0044 respectively. The risk premium for growth volatility is negative throughout the sample period, indicating that long-term real bonds provide an effective hedge against the volatility risk in consumption growth. As we noted earlier, $\hat{z}_t$ is negatively correlated with consumption growth $\Delta c_t$. This implies that consumption volatility increases when consumption is low. Real bonds give investors a constant steam of consumption, therefore providing an insurance against the volatility risk. The estimated market prices of risk, $\hat{\lambda}_{\mu,t}$ and $\hat{\lambda}_{c,t}$ are plotted in Figure 6. Note that $\lambda_{z,t}$ is assumed to be constant in the model.

The risk premium associated with long-run growth prospects is by far the most important component of bond risk premiums. There is a big swing in this long-run risk premium. The upper panel of Figure 5 shows that the long-run risk premium is positive during the early sample period (from 1985 to 1998). Since 1998, the risk premium is mostly negative. This major movement in the long-run risk premium is responsible for the change of the yield curve slope that we observe in the data. In the lower panel of Figure 5, we plot the observed average yield curve for the periods of 1985-1997 and 1998-2007 respectively. In the early period the yield curve slopes upwards, consistent with the positive long-run risk premiums. The slope turns negative in the latter period when the long-run risk premium becomes negative. When the risk premium is positive, investors demand a higher expected rate of return on longer-term bonds than shorter-term bonds, driving down (up) the prices (the yields) of longer-term bonds relative to those of shorter-term bonds, therefore resulting in an upward sloping yield curve. On the contrary, when the risk premium is negative (together with the negative volatility risk premium), we observe a downward sloping yield curve. Notice that during
the whole sample period, the volatility risk premium remains negative, and the short-run risk premium, though fluctuating between positive and negative values, is very small in magnitude. Time variations in these two risk premiums can not be the driving force for the major change of the yield curve slope during the sample period.

The upward sloping real yield curve (or the positive long-run risk premiums) during the first sample period is puzzling. Since real interest rates are positively correlated with consumption growth, as pointed out by Piazzesi and Schneider (2006), returns on long-term real bonds increase if future real interest rates - therefore future expected consumption growth - are low. Moreover, long-run consumption growth is persistent. These together imply that long-term real bonds can actually provide an insurance against the long-run consumption risk, and long-run risk premiums should be negative, therefore resulting a downward sloping yield curve. One potential explanation for the positive long-run risk premiums is that \( \hat{\mu}_t \) might be a poor estimate of the long-run component of consumption growth and has picked up the effects of other state variables, such as liquidity premiums in the index-linked bond market, in the early sample period. In fact, as we discussed before, the correlation between \( \hat{\mu}_t \) and the two-year moving average consumption growth rate is much lower in the early sample period (28.83\%) than that of the latter period (42.04\%). We leave it to future research to understand fully the causes for the positive slope of the real yield curve during 1985-1997.

In long-run risk models with constant market prices of risk, time variations in risk premiums are driven solely by the stochastic volatility of consumption growth, see Equation (14). In our empirical exercise, we relax this restriction on the market prices of risk so that both \( \lambda_{\mu,t} \) and \( \lambda_{c,t} \) are functions of the state variables, hence are time-varying, see Equation (15). Bond risk premiums are given by

\[
rp_{n,t} = - B'_{n-1,y} \Omega_t \lambda_{y,t} - \left\{ [a(B_{n-1,z}) - a^*(B_{n-1,z})] + [b(B_{n-1,z}) - b^*(B_{n-1,z}) z_t] \right\}
\]  

(33)
where \( \lambda_{y,t} = (\lambda_{\mu,t}, \lambda_{c,t})' \). The estimated risk premiums of the 15-year real bond, \( \hat{r}_{p_{n,t}} \), are plotted in Figure 7. To evaluate the contribution of the stochastic growth volatilities to the time variations in \( \hat{r}_{p_{n,t}} \), we also plot in Figure 7 the estimated risk premiums of the 15-year real bond with constant market prices of risk by fixing both \( \hat{\lambda}_{\mu,t} \) and \( \hat{\lambda}_{c,t} \) at their sample averages (the dotted line) so that the time variations in \( \hat{r}_p \) are driven by \( \hat{z}_t \) alone, where \( \hat{z}_t \) is plotted in Figure 3. Note that \( \lambda_{z,t} \) is assumed to be constant in our empirical model. The figures clearly show that, while the consumption growth volatility, driven by \( \hat{z}_t \), changes a lot during the sample period, most of the movements in \( \hat{r}_{p_{n,t}} \) are caused by time-varying market prices of risk. \( \hat{z}_t \) fails to generate enough volatility in \( \hat{r}_{p_{n,t}} \) when both market prices of risk, \( \hat{\lambda}_{\mu,t} \) and \( \hat{\lambda}_{c,t} \), are fixed at constant values. The standard deviation of \( \hat{r}_{p_{n,t}} \) with constant market prices of risk is less than 10% of the standard deviation of the estimated risk premiums. Moreover \( \hat{z}_t \) is unable to account for the major movement in the bond risk premiums in the late 1990s when \( \hat{r}_{p_{n,t}} \) switches signs, causing the yield curve to slope downwards.

5 Conclusions

Latent-factor term structure models based on the no-arbitrage condition have been the most popular framework for studying the joint dynamics of interest rates of different maturities. These models have rich specifications of time-varying risk premiums and are able to account for many salient features of bond yields. Compared to other economically grounded asset pricing models, the empirical success of these dynamic factor models mainly is due to that they only impose the no-arbitrage condition on the cross-section of bond yields while relaxing other general equilibrium restrictions. The cost of the added econometric flexibilities, however, is that the factors or state variables in these models usually lack clear economic interpretations. They only summarize the statistical properties of the yield curve.

We attempt to bridge this gap by estimating a consumption-based, no-
The arbitrage model of the term structure of real interest rates. The empirical exercise conducted in this paper is based on equilibrium long-run risk models. On one hand, we retain the same econometric flexibilities of the latent-factor models and are able to obtain a tractable solution of the term structure of real interest rates with time-varying risk premiums and stochastic volatilities. On the other hand, the state variables in our model are linked directly to the long-run and short-run expected consumption growth as well as time-varying levels of the growth volatility.

The model allows us to examine empirically the roles of different consumption risks in determining long-term real interest rates. We extract a small but persistent long-run component in consumption growth as well as time-varying levels of growth volatility from long-term, index-linked bond yields. Consistent with the calibration results of equilibrium long-run risk models, we find that both risks are priced in the bond market. The long-run consumption risk, in fact, dominates the short-run and consumption volatility risks and drives most of the variations in the risk premiums of long-term real bonds. The risk premium for consumption volatility is negative, suggesting that long-term real bonds provide an effective hedge against the volatility risk in consumption growth. The short-term consumption risk commands a very small risk premium. In contrast to the standard long-run risk model, however, we find that the stochastic growth volatility alone is not sufficient to account for the time variations in bond risk premiums. Movements of the risk premiums seem to be primarily driven by time-varying market prices of risk.

In this paper, the more general econometric specifications relative to the standard long-run risk model, however, come at the expense of the tight connection between the market prices of risk and the deep structural parameters that characterize investor’s preferences. A nature extension of the current paper is to derive time-varying market prices of risk from a general equilibrium model. Such a structural model may help us further understand the economic forces underlying the dynamic behavior of the real yield curve and its relation to various consumption risks. Our empirical exercises is an
intermediate step toward achieving this goal. Another extension is to include other long-lived assets such as stocks and nominal bonds in the study along the line of Lettau and Wachter (2007) and Lustig et al. (2008) among other. The expanded asset space can better capture all important sources of aggregate risk that affect investor’s stochastic discount factor. These extensions are left for future research.

References


Table 1 Summary Statistics: consumption and real yield curve

<table>
<thead>
<tr>
<th></th>
<th>$\Delta c_t$</th>
<th>$R_{5,t}$</th>
<th>$R_{7,t}$</th>
<th>$R_{10,t}$</th>
<th>$R_{12,t}$</th>
<th>$R_{15,t}$</th>
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<tr>
<td>Mean</td>
<td>2.4269</td>
<td>2.9577</td>
<td>3.0039</td>
<td>3.0536</td>
<td>3.0741</td>
<td>3.0872</td>
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<td>Std. Dev.</td>
<td>3.3779</td>
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<td>0.8586</td>
<td>0.9206</td>
<td>0.9535</td>
<td>0.9934</td>
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<tr>
<td>Skewness</td>
<td>0.3509</td>
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<td>-0.2827</td>
<td>-0.2847</td>
<td>-0.2981</td>
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<tr>
<td>Kurtosis</td>
<td>1.9432</td>
<td>1.7450</td>
<td>1.6637</td>
<td>1.6571</td>
<td>1.6629</td>
<td>1.6824</td>
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<td>0.9040</td>
<td>0.9250</td>
<td>0.9420</td>
<td>0.9480</td>
<td>0.9530</td>
</tr>
</tbody>
</table>

$\Delta c_t$ is annualized quarterly growth rate of real per-capita consumption on non-durable goods and services. $R_{i,t}$ ($i = 5, 7, 10, 12, 15$) is $i$-year real interest rate extracted from prices of inflation-indexed government bonds. Interest rates and consumption growth rates are all in percentage points. The sample period is Q2.1985 - Q1.2007.

Table 2 Cross Correlations: consumption and real interest rates

<table>
<thead>
<tr>
<th></th>
<th>$\Delta c_{t-8}$</th>
<th>$\Delta c_{t-4}$</th>
<th>$\Delta c_{t-1}$</th>
<th>$\Delta c_t$</th>
<th>$\Delta c_{t+1}$</th>
<th>$\Delta c_{t+4}$</th>
<th>$\Delta c_{t+8}$</th>
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</thead>
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<td>$R_{5,t}$</td>
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<td>0.0163</td>
<td>-0.0014</td>
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<td>$R_{10,t}$</td>
<td>0.0958</td>
<td>0.0507</td>
<td>0.0469</td>
<td>0.1303</td>
<td>0.1350</td>
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<td>$R_{15,t}$</td>
<td>0.0966</td>
<td>0.0669</td>
<td>0.0620</td>
<td>0.1367</td>
<td>0.1361</td>
<td>0.1720</td>
<td>0.1011</td>
</tr>
</tbody>
</table>

$\Delta c_t$ is annualized quarterly growth rate of real per-capita consumption on non-durable goods and services. $R_{i,t}$ ($i = 5, 10, 15$) is $i$-year real interest rate extracted from prices of inflation-indexed government bonds. The sample period is Q2.1985 - Q1.2007.
Table 3 Forecasting Consumption Growth

<table>
<thead>
<tr>
<th>$\Delta c_{t+1}$</th>
<th>$R_{5,t-3}$</th>
<th>$SR_{15,t-7}$</th>
<th>$\Delta c_{t-2}$</th>
<th>$\Delta c_{t-3}$</th>
<th>$\Delta c_{t-4}$</th>
<th>$\Delta c_{t-6}$</th>
<th>$R^2$</th>
<th>DW</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3100</td>
<td>-1.7635</td>
<td>-0.1901</td>
<td>0.2909</td>
<td>-0.3161</td>
<td>0.1284</td>
<td>0.38</td>
<td>2.07</td>
<td></td>
</tr>
<tr>
<td>(0.3709)</td>
<td>(0.6149)</td>
<td>(0.1007)</td>
<td>(0.0869)</td>
<td>(0.0745)</td>
<td>(0.0920)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\overline{\Delta c_{t-t+4}}$</th>
<th>$R_{5,t}$</th>
<th>$SR_{15,t}$</th>
<th>$\overline{\Delta c_{t-1-t+3}}$</th>
<th>$R^2$</th>
<th>DW</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2916</td>
<td>-1.8609</td>
<td>0.7966</td>
<td></td>
<td>0.73</td>
<td>1.47</td>
</tr>
<tr>
<td>(0.5003)</td>
<td>(0.6844)</td>
<td>(0.0758)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.9789</td>
<td>-1.5871</td>
<td></td>
<td></td>
<td>0.15</td>
<td>0.37</td>
</tr>
<tr>
<td>(0.8910)</td>
<td>(2.3278)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\overline{\Delta c_{t-t+8}}$</th>
<th>$R_{5,t}$</th>
<th>$SR_{15,t}$</th>
<th>$\overline{\Delta c_{t-1-t+7}}$</th>
<th>$R^2$</th>
<th>DW</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4508</td>
<td>-2.3851</td>
<td>0.8399</td>
<td></td>
<td>0.85</td>
<td>2.13</td>
</tr>
<tr>
<td>(0.5369)</td>
<td>(0.9056)</td>
<td>(0.0530)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.7358</td>
<td>-4.7357</td>
<td></td>
<td></td>
<td>0.28</td>
<td>0.31</td>
</tr>
<tr>
<td>(1.7083)</td>
<td>(2.6831)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\Delta c_t$ is annualized quarterly growth rate of real per-capita consumption on non-durable goods and services. $\overline{\Delta c_{t-t+i}}$ is the average consumption growth rate between quarter $t$ and quarter $t+i$. $R_{5,t}$ and $R_{15,t}$ are the 5-year and 15-year real interest rates, respectively, extracted from prices of inflation-indexed government bonds. $SP_{15,t}$ is the yield spread obtained as $SP_{15,t} = R_{15,t} - R_{5,t}$. $R^2$ is the adjusted R-square. DW is the Dubin-Watson statistics. Numbers in parentheses are heteroskedasticity and autocorrelation consistent standard errors. The sample period is Q2.1985 - Q1.2007.
Table 4 Parameter Estimates of the Term Structure Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>With Stochastic Volatility</th>
<th>Without Stochastic Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>5.27310 (0.00008)</td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.56822 (0.00001)</td>
<td></td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>0.93398 (0.00600)</td>
<td>0.90362 (0.00005)</td>
</tr>
<tr>
<td>$k$</td>
<td>1.12265 (0.00006)</td>
<td>1.15272 (0.00001)</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>0.00622 (0.00292)</td>
<td>0.00779 (0.00001)</td>
</tr>
<tr>
<td>$h$</td>
<td>4.83654 (0.00028)</td>
<td>3.66286 (0.00092)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.00188 (0.02097)</td>
<td></td>
</tr>
<tr>
<td>$A_1$</td>
<td>-0.03111 (0.00326)</td>
<td>0.00612 (0.00231)</td>
</tr>
<tr>
<td>$B_{z,1}$</td>
<td>3.36817 (0.00187)</td>
<td></td>
</tr>
<tr>
<td>$B_{\mu,1}$</td>
<td>0.90520 (0.00496)</td>
<td>2.22600 (0.00003)</td>
</tr>
<tr>
<td>$B_{c,1}$</td>
<td>0.89844 (0.00616)</td>
<td>-0.26755 (0.00015)</td>
</tr>
<tr>
<td>$\lambda_z$</td>
<td>31.51876 (0.00030)</td>
<td></td>
</tr>
<tr>
<td>$\psi_0^*$</td>
<td>0.00008 (0.00000)</td>
<td>0.00005 (0.00000)</td>
</tr>
<tr>
<td>$\psi_1^*$</td>
<td>0.99999 (0.00338)</td>
<td>0.99999 (0.01299)</td>
</tr>
<tr>
<td>$k^*$</td>
<td>1.06191 (0.00069)</td>
<td>0.43577 (0.00001)</td>
</tr>
<tr>
<td>$\sigma_7$</td>
<td>0.00016 (0.00007)</td>
<td>0.00041 (0.00003)</td>
</tr>
<tr>
<td>$\sigma_{10}$</td>
<td>0.00015 (0.00004)</td>
<td>0.00075 (0.00005)</td>
</tr>
<tr>
<td>$\sigma_{12}$</td>
<td>0.00010 (0.00006)</td>
<td>0.00089 (0.00001)</td>
</tr>
<tr>
<td>Likelihood Function</td>
<td>3623.75</td>
<td>3307.86</td>
</tr>
</tbody>
</table>
Table 5 Correlations between Consumption Growth and Yield Curve Factors

<table>
<thead>
<tr>
<th></th>
<th>Yield Curve Level</th>
<th>Yield Curve Slope</th>
<th>Yield Curve Curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_t$</td>
<td>.9400</td>
<td>.7134</td>
<td>-.5916</td>
</tr>
<tr>
<td>$\Delta c_t$</td>
<td>.1141</td>
<td>.1644</td>
<td>-.2037</td>
</tr>
<tr>
<td>$z_t$</td>
<td>-.4044</td>
<td>-.9771</td>
<td>.8368</td>
</tr>
</tbody>
</table>

Yield curve level is measured by $(R_5 + R_{15})/2$, the yield curve slope is measured by $(R_{15} - R_5)$, the curvature is measured by $(R_5 - 2 \times R_{10} + R_{15})$, where $R_5$, $R_{10}$, $R_{15}$ are 5-year, 10-year and 15-year inflation-indexed real yields respectively. $\Delta c_t$ is the quarterly consumption growth rate, $\mu_t$ and $z_t$ represent the long-run consumption growth and the stochastic growth volatility respectively.

Table 6 Decomposing Bond Risk Premiums

<table>
<thead>
<tr>
<th></th>
<th>Long-run Risk</th>
<th>Short-run Risk</th>
<th>Volatility Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0090</td>
<td>-0.0004</td>
<td>-0.0044</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.0114</td>
<td>0.0006</td>
<td>0.0011</td>
</tr>
<tr>
<td>Autocorrelation</td>
<td>0.9758</td>
<td>0.4235</td>
<td>0.8199</td>
</tr>
</tbody>
</table>

The risk premiums are the expected excess holding-period return (per quarter) defined in (31). Maturity of the real bond is 15 years.
Figure 1: The Estimated Long-end of the Real Yield Curve: with stochastic volatility in consumption growth

Upper panel of the figure plots the average of yield curve. The lower panel of the figure plots the standard deviations of real rates. The maturities are in years. The solid lines are from the model. Circles represent data. The term structure model assumes stochastic volatility in consumption growth.
Figure 2: The Estimated Long-end of the Real Yield Curve: with constant volatility in consumption growth

Upper panel of the figure plots the average of yield curve. The lower panel of the figure plots the standard deviations of real rates. The maturities are in years. The solid lines are from the model. Circles represent data. The term structure model assumes constant volatility in consumption growth.
The figure plots the estimated state variable $z_t$ for the period Q2.1985 - Q1.2007. $z_t$ drives the stochastic volatility of consumption growth.
The figure plots (de-meaned) consumption growth rate (the dotted line, denoted as $gc$ or $lgc$) together with its estimated long-run component (the solid line, denoted as $ut$) for the sample period Q2.1985 - Q1.2007. The upper panel includes the quarterly consumption growth rate (dotted line). The lower panel includes the 2-year moving average (dotted line) of quarterly growth rates. In both panels, the solid line is the estimated long-run component, $\hat{\mu}_t$. 
The upper panel plots the estimated risk premiums (per quarter) for the 15-year inflation index-linked government bond during Q2.1985 - Q1.2007. RiskL plots the estimated long-run consumption risk premiums. RiskS plots the estimated short-run consumption risk premiums. RiskV plots the estimated growth volatility risk premiums. The lower panel display the observed average inflation index-linked real yield curves during 1985-1997 (with diamond signs) and 1998-2007 (with circle signs) respectively.
Figure 6: Market Prices of Risk: $\hat{\lambda}_{u,t}$ and $\hat{\lambda}_{c,t}$

The upper panel plots the estimated market price of long-run consumption risk $\hat{\lambda}_{u,t}$. The lower panel plots the estimated market price of short-run consumption risk $\hat{\lambda}_{c,t}$. 
The solid line plots the estimated risk premiums for the 15-year real bond with time-varying market prices of risks, $\hat{\lambda}_{\mu,t}$ and $\hat{\lambda}_{c,t}$. The dotted line plots the same bond risk premiums with constant market prices of risks by fixing both $\hat{\lambda}_{\mu,t}$ and $\hat{\lambda}_{c,t}$ at their sample averages.