Analytic Moments for Conditional and Aggregated GARCH Variances and Returns

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Abstract
Knowledge of the dynamic properties and the higher moments of the distribution of returns on financial assets is of particular importance, since these processes exhibit volatility clustering and their distributions are also known to be non-normal. Thus the forecasting ability of GARCH models, which capture both of these effects, has been the subject of numerous research papers in the financial econometrics literature. However, most papers consider only the point variance forecasts generated by a GARCH process. This paper shows how approximate predictive distributions for forward and cumulative future returns and variances, in a generic asymmetric GARCH model, can be constructed. These are based on new analytic formulae for the first four moments, of the conditional returns and conditional variance distributions, and of the aggregated returns and variances.

Keywords: Approximate predictive distributions, GARCH, conditional moments.

JEL Classification: C53
Introduction

Mandelbrot (1963) and Fama (1965) observed that financial return time series may exhibit volatility clustering, so they are not independent, and they typically have non-normal distributions. The family of GARCH models has proved highly successful in capturing (at least partially) the leptokurtosis and volatility autocorrelation of these data. Following the seminal papers by Engle (1982) and Bollerslev (1986) numerous alternative specifications for the conditional variance equation have been suggested – see for example, the EGARCH model of Nelson (1991), the AGARCH introduced by Engle and Ng (1993), and the model of Glosten, Jagannathan and Runkle (1993), denoted by GJR from here onwards. Additionally GARCH models with non-normal conditional distributions for the innovation have been developed: see Bollerslev (1987), Nelson (1991), Haas, Mittnik, Paolella (2004) and many others. The time aggregation properties of GARCH models and their continuous time limits have been analysed by Nelson (1990), Drost and Nijman (1993), Corradi (2000) and Alexander and Lazar (2005). All of the above represent original theoretical contributions, and an equally large GARCH literature has examined the empirical in-sample and out-of-sample properties and performance of various GARCH models. The focus here is generally on the forecasting ability of alternative models: see, for example, Andersen and Bollerslev (1998) or Marcucci (2005)\(^1\). Thus both the theoretical properties as well as the empirical behaviour of various GARCH models have been researched at length. However, regarding the forecasting capability of GARCH models, the existing literature has focused almost exclusively on the accuracy of the point GARCH forecasts.

In this paper we derive analytical formulae for the first four conditional moments of forward one-period and future aggregated returns and variances, in the context of two different asymmetric GARCH models, with a generic distribution for the innovations. The moments are subsequently used to derive approximate predictive distributions for both returns and variances. We apply statistical tests to determine how well our approximate distributions serve their purpose: thus, we test whether they describe the conditional distributions of future returns and variances adequately, using real financial data. We also characterize the long term behaviour of the moments and derive their limits as the time horizon increases.

In an era when financial products become increasingly complex, density forecasting of portfolio returns becomes increasingly important. Forward looking returns and variances distributions have a number

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of important applications, and forecasts of the *entire* distribution (and not only of its first two moments) are very important if we believe that returns depart significantly from normality. Also, the new regulatory framework for risk management has made Value-at-Risk (VaR) the standard method for risk assessment and reporting. This adds further relevance to returns density forecasting since VaR is determined by a quantile of the returns distribution over a fixed risk horizon. Value-at-Risk estimation is only one example of the application of predictions of returns distributions; other applications include option pricing, hedging and portfolio management.

The reminder of this paper is organised as follows: Section 1 reviews the existing literature on derivations of conditional moments of various GARCH processes. Section 2 presents the original theoretical contribution of the paper, namely the computation of the first four moments of forward and cumulative returns and variances and their limits. Alternative methods for approximating distributions using these moments are described in Section 3. Section 4 deals with the evaluation methods used to assess the usefulness of the proposed predictive distributions. Section 5 constitutes the empirical section of the paper, while Section 6 concludes and proposes directions for future research.

1. **Existing Literature on Moments of GARCH Processes**

Whilst mean and variance dynamics are sufficient in a normal context, higher order dynamics are necessary when forecasting in the context of non-normality. Nelson (1991) was probably the first to consider the higher moments of a specific GARCH process, namely the EGARCH model that he proposed. Then Duan, Gauthier and Simonato (1999), and Wong and So (2003) derived the higher order conditional moments of returns in the framework of various established GARCH specifications; and Christoffersen, Jacobs and Wang (2005a and b) derived the variance of variance in the context of the NGARCH model, but only for one step ahead forecasts. Further details on these papers follow.

Duan, Gauthier and Simonato (1999) give analytical expressions for the first four moments of cumulative returns under the NGARCH model of Engle and Ng (1993) and under the risk neutral probability. Their formulae are not exact because some terms involve taking the expectation of variance raised to a non-integer power and these terms must be approximated. Their theoretical results are used to derive an approximate analytical GARCH option pricing formula using the Edgeworth expansion of the

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2 The moments are derived for the proposed EGARCH model and under various distributional assumptions for the innovations i.e. Normal, Student-t and GED respectively.
risk-neutral density function of returns. The method for approximating option prices – using Black-Scholes price adjusted for skewness and kurtosis of cumulative returns – is very general and can be applied to any type of asset returns dynamics, i.e. it is not restricted to GARCH-type models. However, the theoretical results and the empirical analysis that validates the usefulness of the method are carried out assuming a particular GARCH specification, i.e. NGARCH with a normal innovation process. The applicability of their contribution is limited by the fact that the derivations are carried out in a risk neutral framework, and hence their theoretical results are restricted to applications which are set in a risk neutral world.

Wong and So (2003) derive the exact conditional variance, and the third and fourth moments of the aggregated returns, in the context of a generic QGARCH model. This model encompasses the general GARCH \((p, q)\) process with almost any distribution on the error term, IGARCH, EWMA and the asymmetric GARCH model (A-GARCH), but it does not encompass GJR or NM-GARCH. The analytical expressions for these moments are then used to approximate the distribution of returns with a skewed Student \(t\) distribution whose moments match the derived moments of returns. The authors show how their theoretical results can be used to derive more efficient VaR estimation methods, in the sense that the accuracy/computational time ratio is relatively high. However, none of their proposed VaR methodologies proves satisfactory for long term horizons and low significance levels. Also, no measure of uncertainty in the forecasted VaR figure is considered and no test of conditional coverage in the sense of Christoffersen (1998) is performed; only unconditional coverage is considered.

2. Deriving Moments of Generic Asymmetric GARCH Returns and Variances

In this section we derive new analytic formulae for the first four moments of the predictive distributions of returns and variances in a general asymmetric GARCH process. We consider the first four conditional moments of both forward one-period and future cumulative (or aggregated) returns and variances. This is done for two established asymmetric GARCH models, namely the GJR model and the AGARCH model, and we employ a generic innovation distribution with zero mean, variance one and with finite higher moments.

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3 Actually, the analytic expressions for the moments are derived under restricted versions of the model: the formula for the variance of aggregated returns is computed for a QGARCH\((1, 1)\) only, while the third and fourth aggregate returns conditional moments are derived under the additional assumption that the error term follows a symmetric distribution.

4 The formulae for the symmetric GARCH \((1,1)\) model can be obtained from either the GJR or AGARCH derivations, by equating the asymmetry parameter \(\lambda\) with zero.
In this section we present new formulae for the mean, variance, skewness and kurtosis, and their limits as the time horizon increases, since these are the results that will be used when approximating the predictive distributions of the GARCH returns and variances. Also, since the procedure is similar for the two GARCH models, only the GJR formulae are discussed in this section. The derivations of these moments and their respective limits are rather lengthy and thus the proofs are only included in the appendix.

The mathematical specification of the generic GJR model is given by:

\[
\begin{align*}
\mu_t &= \mu + \theta_i \\
\epsilon_t &= \sigma_t \epsilon_i \\
z_t &\sim D(0, 1) \\
\sigma^2_t &= \omega + \alpha_1 \epsilon^2_{t-1} + \lambda \sigma^2_{t-1} I_{t-1} + \beta \sigma^2_{t-1}
\end{align*}
\]

where \( I_{t-1} \) is an indicator function which equals 1 if \( \epsilon_{t-1} < 0 \) and zero otherwise, \( z_t \) and \( \sigma_t \) are independent and \( D(0, 1) \) is a generic conditional distribution with mean zero, variance one, constant skewness \( \tau \) and kurtosis \( \kappa \) and constant higher moments.

It is worthwhile noting that for a \( D(0, 1) \) distribution the un-centred, centred and normalized moments are the same. If \( D(0, 1) \) is the standard normal distribution, Bollerslev (1986) gives a general formula for computing the even moments:

\[
\mu_{2i} = \prod_{j=1}^{i} (2j-1).
\]

The odd moments of a standard normal process are all equal to zero.

We use the following notation for the cumulative returns over \( T \) consecutive time periods, and for the centred moments of the forward and aggregated returns and variances:

\[
R_{t+T} = \sum_{s=1}^{T} r_{t+s},
\]

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5 Since the general focus in GARCH models is the conditional variance equation rather than the conditional mean equation, we specified a very simple mean equation; if returns exhibit autocorrelation, then we first de-autocorrelate the data and then estimate the GARCH parameters.

6 To be more precise, we have:

\[
\mu_i = E(z^i) = E\left( z^i - E\left( z^i \right) \right) = E\left( z^i \right)^{1/2}
\]

since un-centred, centred and standardized moments are all equal for a zero mean, unit variance distribution. Also, since the \( z \) process is i.i.d., conditional and unconditional moments of \( z \) are also identical. Actually, for the fourth conditional moment of returns to exist we need the first five moments of \( z \) to be finite, while we require up to the eighth moment of \( z \) to be finite in order to have a finite fourth conditional moment of future variances.
\[ \mu_{x,s}^i = E_i \left( x_{t+s}^i - E_i \left( x_{t+s}^i \right) \right) \]

\[ M_{x,T}^i = E_i \left[ \left( \sum_{t=1}^{n} (x_{t+s}^i - E(x_{t+s})) \right)^i \right]; \]

for \( x = r \) and \( \sigma' \), for \( s = 1, 2, \ldots, n \) and for \( i = 1, 2, 3, 4 \). Thus, the skewness and kurtosis of the forward returns and variances distributions are given by:

\[ \tau_{x,s} = \mu_{x,s}^3 \left( \mu_{x,s}^2 \right)^{-\frac{3}{2}} \quad \text{and} \quad \kappa_{x,s} = \mu_{x,s}^4 \left( \mu_{x,s}^2 \right)^{-2} \]

and the skewness and kurtosis of the aggregated returns and variances distributions are given by:

\[ \Sigma_{x,T} = M_{x,T}^3 \left( M_{x,T}^2 \right)^{-\frac{3}{2}} \quad \text{and} \quad \Kappa_{x,T} = M_{x,T}^4 \left( M_{x,T}^2 \right)^{-2}. \]

First we derive the first four un-centred moments of forward one-period returns and variances, and of cumulative returns and variances of our generic asymmetric GARCH process, i.e.

\[ E_i \left( x_{t+s}^i \right) \quad \text{and} \quad E_i \left[ \left( \sum_{t=1}^{n} x_{t+s} \right)^i \right], \]

for \( x = r \) and \( \sigma' \), for \( s = 1, 2, \ldots, n \) and for \( i = 1, 2, 3, 4 \). Subsequently, centred moments are derived, as well as formulae for the skewness and kurtosis of the generic GARCH process are obtained. Also in this section we consider the long term (limiting) behaviour of the derived moments.

For Theorem 1, which states the time-\( t \) conditional moments of forward one-period and aggregated returns and variances, we write:

\[ \bar{\mu}_{x,s}^{(1)} = \mu_{x,s}^2 = \sigma_0^2 + \varphi^{s-1} \left( \sigma_{r+1}^2 - \sigma_0^2 \right), \]

where \( \varphi = \alpha + \lambda F(0) + \beta \), \( F(0) \) is the distribution function for \( D(0,1) \) evaluated at zero, and \( \sigma_0^2 = \omega (1 - \varphi)^{-1} \). Also, the expressions for \( b_{x,s,t+u}, E_i \left( \varepsilon_{t+s}^2 \varepsilon_{t+u} \right) \) and \( E_i \left( \varepsilon_{t+u}^m \varepsilon_{t+u}^n \right) \) for \( m, n = 1,2,3,4 \) are rather long, and so they are given in the appendix.

**Theorem 1:**

(a) The conditional moments of forward one-period returns are:

Mean: \( \mu_{x,s}^{(1)} = \mu \)

Variance: \( \mu_{x,s}^2 = \sigma_0^2 + \varphi^{s-1} \left( \sigma_{r+1}^2 - \sigma_0^2 \right) \),
Skewness: \[ \tau_{r,s} = \frac{\tau}{8} \left(5 + 3 \frac{b_i}{(a_i)^2}\right), \]

Kurtosis: \[ \kappa_{r,s} = \kappa \frac{b_i}{(a_i)^2}. \]

(b) The conditional moments of the aggregated future returns are:

Mean:
\[ \bar{M}_{r,n}^{(1)} = T \mu \]

Variance
\[ M_{r,T}^{2} = T \sigma_0^2 + (1 - \varphi^{-1})(\sigma_{r+1}^2 - \sigma_0^2), \]

Skewness:
\[ \Sigma_{r,T} = \frac{\tau}{8} \left(5 \sqrt{\frac{a_i^2 + 3 \frac{b_i}{\sqrt{a_i}}}{a_i^2}}\right) + 3 \sum_{i=1}^{T-1} \sum_{j=1}^{T-1} E_i\left(\varepsilon_{t+i, t+i+j}^3\right) \left(M_{r,i}^{2}\right)^{1/2} \]

Kurtosis: \[ \kappa_{r,T} = \kappa \sum_{i=1}^{T} b_i + \sum_{i=1}^{T} \sum_{j=1}^{T-1} \left(4E_i\left(\varepsilon_{t+i, t+i+j}^3\right) + 6E_i\left(\varepsilon_{t+i, t+i+j}^2\right) + 12 \sum_{i=1}^{T} \sum_{j=1}^{T-1} \sum_{k=1}^{T-1-j} E_i\left(\varepsilon_{t+i, t+i+j, t+i+k+j}\right) \left(M_{r,i}^{2}\right)^{-2}\right) \]

(c) The conditional moments of forward one-period variances are:

Mean:
\[ \bar{\sigma}_{r,s}^{(1)} = \sigma_0^2 + \varphi^{-1}(\sigma_{r+1}^2 - \sigma_0^2), \]

Variance:
\[ \mu_{r,s}^2 = b_i - a_i^2, \]

Skewness:
\[ \tau_{r,s} = \frac{c_i - 3b_i a_i + 2a_i^3}{\left(b_i - a_i^2\right)^{1/2}}, \]

Kurtosis:
\[ \kappa_{r,s} = \frac{d_i - 4a_i c_i + 6a_i^2 b_i - 3a_i^4}{\left(b_i - a_i^2\right)^{1/2}}. \]

(d) The conditional moments of the aggregated future variances are:

Mean:
\[ \bar{M}_{r,T}^{(1)} = T \sigma_0^2 + \left(\sigma_{r+1}^2 - \sigma_0^2\right) \left(1 - \varphi^{-1}\right) \left(1 - \varphi^T\right) \]

Variance
\[ M_{r,T}^{2} = \sum_{i=1}^{T} \left(b_i - a_i^2\right) + 2 \sum_{i=1}^{T} \sum_{j=1}^{T-i} \left(b_i - a_i a_{i+j}\right), \]

Skewness:
\[ \Sigma_{r,T} = \left[ \sum_{i=1}^{T} \mu_{r,s}^3 + 3 \sum_{i=1}^{T} \left(c_i, r_{i+1}^3 + c_i, r_{i+1}^2 + 2 \left(a_i + a_{i+j}\right) \left(a_i a_{i+j} - b_i\right) - a_i b_{i+j} - a_{i+j}\right) \right] \left(M_{r,T}^{2}\right)^{-1/2} \]
\[ + 6 \sum_{i=1}^{T} \sum_{j=1}^{T-i} \left(c_i, r_{i+1}, r_{i+1+j} - a_i b_{i+j} - a_{i+j} b_i - a_{i+j} b_{i+j} + 2a_i a_{i+j} a_{i+j}\right) \left(M_{r,T}^{2}\right)^{-1/2} \]
Kurtosis:
\[ K_{\sigma^2, T} = \frac{M_{\sigma^4, T}^4}{(M_{\sigma^2, T}^2)^2}, \]
where all the quantities of interest in Theorem 1 a) - d) are given in the Appendices.

It is interesting to note that the conditional distribution of aggregated returns can exhibit skewness and excess kurtosis even when the conditional distribution of the one-period returns is normal. Cumulative returns have two sources of skewness: if the conditional distribution \( D(0, 1) \) is not skewed, the source of skewness in cumulative returns is the asymmetric response \( \lambda \) parameter in the model. However, forward one-period returns exhibit skewness if and only if \( D(0, 1) \) is itself a skewed distribution.

Compared with previous results, the main advantage of our approach is that the moments are expressed in terms of the model parameters only, which makes our formulae very straightforward to implement in practical applications. Moreover, the derivations are carried out in terms of a generic conditional distribution and for asymmetric GARCH specifications that are well established in the literature.

**Theorem 2:** If \( 0 < \gamma < 1, 0 < \varphi < 1 \) and \( \gamma \neq \varphi \), then the moments of the forward one-period and aggregated returns converge to the following limits when we increase the time horizon.

\[
\lim_{s \to \infty} \mu_{r,s} = \sigma_0^2 \\
\lim_{s \to \infty} \tau_{r,s} = \tau \left( \frac{5}{8} + \frac{3}{8} \left( \omega^2 + 2\omega\varphi^2 \right) (1 - \gamma)^{-1} \sigma_0^{-4} \right) \\
\lim_{s \to \infty} \kappa_{r,s} = \omega \left( \omega + 2\varphi^2 \right) (1 - \gamma)^{-1} \sigma_0^{-4}
\]

Also, the skewness and the excess kurtosis of the aggregated returns both converge to zero.

Theorem 2 shows that the conditional moments of the aggregated returns converge to those of a normal distribution provided that certain parameter conditions are met. It can be shown that these convergence conditions are the necessary conditions for the unconditional moments to exist – TO BE SHOWN. The classical central limit theorem does not apply here, because the variables are not independent. However, the result in Theorem 2 links to previous results of Diebold (1990?) who shows that the unconditional distribution of the aggregated returns for a conditionally normal GARCH converges to a normal distribution.

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7 The expression for the kurtosis of aggregated variance is not explicitly given here since it is particularly lengthy.
8 The expression for \( \varphi \) is given above, while \( \gamma \) is defined in the Appendix 1.1.
3. Distribution Approximations Methods

Given an infinite sequence of moments of a random quantity, one can determine its probability distribution. However, even with the first few moments the distribution can be approximated, and this is the approach that we take here: the distributions of the forward one-period returns and of the aggregated returns and are approximated using their first four moments. Several alternative approaches can be used to derive an approximate distribution from the moments of a random variable. Based on their relative merits and drawbacks, and also based on the frequency of their use in similar applications, we have selected three approximating methods for the empirical section of this paper. These are the Cornish-Fisher expansion, the Edgeworth expansion and the Johnson SU distribution. This section provides a very brief review of these methods.

Cornish-Fisher expansion

In its most general specification, the Cornish-Fisher expansion approximates the quantiles of a probability distribution in terms of the quantiles of a base distribution; the base distribution that is usually used in empirical applications is the standard normal. Although popular in the finance literature, especially when associated with delta-gamma approximations for portfolios containing non-linear instruments (Jaschke (2001), Mina and Ulmer (1999)) the method has a number of disadvantages, some of which are shared with the Edgeworth expansion described below. Some of these limitations are outlined by Jaschke (2001). On the qualitative side, the method does not ensure convergence: when we increase the order of the expansion, i.e. the number of moments used, the approximation does not necessarily improve. Also the monotonicity of the approximated distribution function is not guaranteed. The method also has ‘wrong tail behaviour’ i.e. it becomes less reliable for extreme quantiles, which is a serious drawback for VaR calculations. On the quantitative side, the error of the approximation increases as the distribution function that we are

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9 To be precise, a distribution is uniquely determined by its moments only if the Carleman condition holds, i.e. only if \( \sum_{n=1}^{\infty} \frac{1}{\alpha_k} 2^n \rightarrow \infty \), where \( \{ \alpha_k \} \) is the moment sequence. Since we have derived only the first four moments, we can only apply these methods if we assume that this condition is met, so that the distribution that is approximated is indeed unique (see Serfling, 1980, p.46).

10 The standard normal distribution is also the base distribution used in the original Cornish-Fisher formulae presented in Cornish and Fisher (1937) and Fisher and Cornish (1960); Hill and Davis (1968) later proposed a generalised expansion of Cornish-Fisher type in which the base distribution is no longer restricted to being normal.

11 In a VaR application, this may lead to the 99% VaR estimates being lower than 95% VaR estimates!
approximating moves further away from the normal distribution (since the Cornish-Fisher is a local approximation). However, Jaschke (2001) claims that, in practice, the accuracy of the method is more than sufficient; this combined with its speed and relative simplicity motivates its choice.

Formally, the generalized Cornish-Fisher expansion for the $\alpha$ quantile $X_\alpha$ of a distribution $F$, where the base distribution is function is $\Phi$, is given by:

$$X_\alpha = \Phi^{-1}(\alpha) + \sum_{r=1}^{\infty} (-1)^{r+1} \frac{1}{r!} D^{(r)}[a']$$

where $D^{(r)} = \left(D + \frac{\varphi'}{\varphi}\right)\left(D + 2\frac{\varphi'}{\varphi}\right)\ldots\left(D + r\frac{\varphi'}{\varphi}\right)$ with $D$ denoting the differentiation operator and $\varphi$ the base density; and $a = (F - \Phi)/\varphi$.

We shall use a fourth order Cornish-Fisher expansion for the $\alpha$ quantile of the standardized aggregated returns $\hat{R}$, using the standard normal as the base distribution. This is given by:

$$\hat{R}_\alpha = \Phi^{-1}(\alpha) + \sum_{r=3}^{\infty} \frac{1}{r!} D^{(r)}[a'] = \Phi^{-1}(\alpha) + \frac{K_T - 3}{24} \Phi^{-1}(\alpha) - \frac{\Sigma_T^2}{36} \Phi^{-1}(\alpha) \left(2\Phi^{-1}(\alpha) - 5\right)$$

where $\Phi^{-1}(\alpha)$ is the standard normal $\alpha$ quantile, and $\Sigma_T$ and $K_T$ represent the skewness and kurtosis of the aggregated returns.

**Edgeworth expansion**

Somewhat similar to the Cornish-Fisher expansion, the Edgeworth expansion represents a method of approximating a density of interest around a base density, usually the standard normal density.\(^{12}\) It belongs to the class of Gram-Charlier expansions, being a rearrangement of a ‘Gram-Charlier A’ series. However, Gram-Charlier A series and Edgeworth series are only equivalent asymptotically, when an infinite number of terms enter the expansions. In empirical applications where finite order approximations are considered they can differ significantly, and the Edgeworth version is preferred since its error is controlled.\(^{13}\) The Edgeworth expansion shares the monotonicity and convergence problems of the Cornish-Fisher method described above. The first few terms of the Edgeworth expansion are:

$$f(x) = \varphi(x) - \frac{\lambda_3\varphi^{(3)}(x)}{6\sqrt{n}} + \frac{1}{n} \left[ \frac{\lambda_4\varphi^{(4)}(x)}{24} + \frac{\lambda_5\varphi^{(5)}(x)}{72} \right] + O\left(n^{-3/2}\right)$$

\(^{12}\) The Edgeworth expansion can be specified as either a density or a distribution function expansion.

\(^{13}\) The Edgeworth version of the expansion is actually a true asymptotic expansion, in the sense of Wallace (1958).
where $f$ is the density of the standardized cumulative returns, $\varphi$ is the standard normal density and $\varphi^{(k)}$ is its $k$th derivative; $\lambda_3 = \sqrt{n} \Sigma_T$, $\lambda_4 = n(K_T - 3)$ and $n$ is the sample size.

Johnson distributions

Johnson (1949) suggested the following generic transformation from a Johnson distributed variable $X$ to a standard normal variable $Z$:

$$z = \gamma + \delta g \left( \frac{x - \xi}{\lambda} \right)$$  \hspace{1cm} (7)

where $g$ is a monotonic function. Given there are four parameters, Johnson distributions are flexible and are likely to perform well in a wide range of empirical applications. Three different functional forms for $g$ give rise to three different types of Johnson distributions, namely the SU (unbounded), SL (lognormal) and SB (bounded) Johnson distributions respectively. The SU version, for which the $f$ function is the inverse of the hyperbolic sine function (i.e. $g(x) = \sinh^{-1}(x)$) is the most relevant for most financial applications, since it is leptokurtic.

Tuenter (2001) proposed the following algorithm for fitting an SU curve to a set of observed data by means of moment matching:

1. First compute the upper and lower bounds for $\omega = \exp\left(\delta^{-2}\right)$ which are as follows:

$$\omega_{\text{upper}} = \left(-1 + \left(2(K_T - 1)\right)^{1/2}\right)^{1/2}$$  \hspace{1cm} (8)

$$\omega_{\text{lower}} = \max(\omega_1, \omega_2)$$  \hspace{1cm} (9)

where $\omega_1$ and $\omega_2$ are the unique positive roots of $\omega^4 + 2\omega^3 + 3\omega^2 - 3 - K_T = 0$ and $(\omega - 1)(\omega + 2)^2 - \Sigma_T^2 = 0$, respectively.

2. Find $\omega$ in the interval $(\omega_1, \omega_2)$ such that

$$(\omega - 1 - m)\left(\omega + 2 + \frac{m}{2}\right) - \Sigma_T^2 = 0$$  \hspace{1cm} (10)

where

$$m = -2 + \left(4 + 2\left(\omega^2 - \frac{K_T + 3}{\omega^2 + 2\omega + 3}\right)^{1/2}\right)$$  \hspace{1cm} (11)
3. The estimates for the parameters of the Johnson distribution are:

\[ \hat{\delta} = \left( \ln(\omega) \right)^{1/2} \]

\[ \hat{\gamma} = -\text{sgn}(\Sigma_{\tau}) \sinh^{-1} \left[ \frac{(\omega + 1)(\omega - 1 - m)}{2\omega m} \right]^{1/2} \hat{\delta} \]

\[ \hat{\lambda} = \left( \frac{2mM_{\tau}^2}{(\omega + 1)(\omega - 1)^2} \right)^{1/2} \]

\[ \hat{\xi} = T_{\mu} - \text{sgn}(\Sigma_{\tau}) \frac{M_{\tau}^2 (\omega - m - 1)}{\omega - 1} \]  

(12)

4. Evaluation methods

The conditional distributions of returns do not exist in sample, since returns are observed in-sample and hence there is no uncertainty around their values. Since we consider an out-of-sample application all of the tests described below are out-of-sample performance tests.

**Testing the “goodness of fit” of the proposed approximate distribution**

We want to assess how well our proposed approximate distributions serve their purpose: do they adequately describe the observed conditional distributions of aggregated returns? Since the conditional distributions of returns are never observable, not even ex-post, we shall use simulated distributions as proxies. So in our case the null hypothesis is \( H_0: F_m = F_s \), where \( F_m \) is the distribution function for the approximate distribution, constructed using the first four moments, and \( F_s \) is the distribution function for the simulated distribution of future cumulative returns, using GARCH simulations.

Standard hypothesis tests whose null is the equality of the two distributions are the Kolmogorov-Smirnov (KS) and the Anderson Darling (AD) tests. The only difference between them is that the weighting given to different observations is different. Malevergne and Sornette (2003) argue that while the KS test focuses on the “body” of the distribution, the AD alternative is more accurate in the tails. The
Kolmogorov-Smirnov test (Kolmogorov (1933), Smirnov (1939), Massey (1951)) is, in effect, a simple hypothesis test. The test statistic is given by:

\[ KS = \sqrt{n}D, \]  

(13)

where \( n \) is the number of data points, i.e. the observed returns are \( \{r_1, ..., r_n\} \) and

\[ D = \max_{1 \leq i \leq n} |F_m(r_i) - F_s(r_i)|. \]  

(14)

When comparing alternative models, the models with the lowest \( D \) (or \( KS \)) value is deemed the most accurate (among the compared models) for predicting the returns distributions.

Anderson and Darling (1952) propose two distance measures, which are actually generalisations of the KS and Cramer von Mises statistics. The respective test statistics are given by:

\[ AD_1 = \max_{1 \leq i \leq n} |F_m(r_i) - F_s(r_i)| \left( n\psi(F_m(x)) \right)^{1/2} \]  

(15)

\[ AD_2 = n \sum_{i=1}^{n} \left[ (F_m(r_i) - F_s(r_i))^2 \psi(F_m(r_i)) \right] \]  

(16)

where \( \psi \) is a weighting function. The weighting function most generally used in empirical applications of the AD test is \( \psi(x) = (x(1-x))^{-1/2} \). Conducting these tests in our setting requires simulation of critical values.

The statistics only have standard distributions if the distribution under the null hypothesis is entirely pre-specified; however in our case the \( F_m \) distribution relies on estimated parameter values, so the theoretical critical values are no longer applicable.

Both the KS and AD tests are typically applied in their max, rather than average (or summation) form. However, Malevergne and Sornette (2003) argue that using the average as well as the max version of both tests can add value to the evaluation exercise, since the commonly used max versions of the tests are sensitive to outliers, since they are controlled by the point that maximizes the argument. Both the KS test and the AD test (in its two different versions proposed in the seminal paper) will be implemented in the empirical section.

Likelihood criteria

---

14 The KS as well as the AD tests are designed to compare a theoretical distribution, the distribution under the null, with an empirical, observed distribution. In our case the distribution based on the moment formulae is the theoretical and the (GARCH) simulated distribution is the (proxy for) the empirical.

15 The KS and Cramer von Misses tests are obtained when \( \psi(t) = 1 \), respectively, in the two formulations given above.

16 Malevergne and Sornette (2003) actually use the sum (integral) of absolute values, rather than the sum of squares in their average versions of both the KS and AD test.
We use the likelihood criterion for inter-model comparison, i.e. decide which approximation method works best for a particular time series

5: Empirical results

We consider three alternative GARCH models, namely the GARCH (1,1), GJR and AGARCH models. First, we use the estimated model parameters to compute the moments given by Theorem 1. Then we fit a distribution to these moments using each of the methods described in Section 3. The out-of-sample performance of our approximate distributions is tested using equity index (S&P 500), foreign exchange (euro/dollar), and interest rate (3-month Treasury Bills) daily data. The S&P data comprises 20 years of daily data from January 1989 to December 2008, totalling over 5000 observations. The forex data …etc…

We shall approximate the distribution of $h$-day cumulative returns with $h = 5, 10$ and 20 working days respectively. Then we apply three alternative approximation methods described above – the Cornish Fisher and Edgeworth expansions and Johnson $S_U$ distributions, respectively. Estimation as well as model performance is conducted in a rolling window format. A window of ten years of data (2528 observations) is rolled daily, so we have over 2500 estimations and out-of-sample predictions to assess. Each time the model parameters are estimated and the moments are computed using the formulae derived in Section 2, and predictive distributions are fitted using the three methods described in Section 3.

The distribution fitting methods are assessed using the goodness of fit criteria described above, namely the KS and the AD tests (in the two different versions) and the three approaches to fitting the moments are compared using the likelihood criterion described above. Thus, we base our analysis of the accuracy of the analytic approximations to GARCH returns and variance distributions on time series of about 2500 KS and AD test values, for each of the three GARCH models, and for three approximation methods.

Section 5: Conclusions and further research

The contribution of this paper is twofold. Firstly, we derive new analytic formulae for the moments of the conditional returns and variances, based on two asymmetric GARCH specifications which are well
established in the literature, and with a general innovation process. Secondly, we show how our theoretical results can be used to produce analytic approximations to the predictive returns and variances distributions. These distributions have a wide variety of applications to finance. For example, fast multi-period VaR forecasts may be derived, which take account of volatility clustering without recourse to either filtering or simulations.
References


Appendices:

Appendix 1.1 Conditional Moments for the generic GJR Future Aggregated Returns

Appendix 1.1 shows the derivations of the conditional moments for forward one-period returns and future aggregated returns in the context of a generic GJR model.

Model specification:

\[ r_t = \mu + \varepsilon_t \]  
\[ \varepsilon_t = \tau \sigma_t \]  
\[ z_t \sim t \text{ iid } D(0,1) \]  
\[ \sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \lambda \epsilon_{t-1}^2 I_{t-1} + \beta \sigma_{t-1}^2 \]

where \( z_t \) and \( \sigma_t \) are independent \( D \) is a generic conditional distribution with mean zero, variance 1, constant skewness \( \tau \), kurtosis \( \kappa \) and constant higher moments.

Notation: aggregated returns:

\[ R_{t:T} = \sum_{s=1}^{T} r_{t+s} \]

First we are going to compute the first four uncentred moments of forward one-period and of aggregated returns; then centred moments will be computed; finally formulae for standardized moments – skewness and kurtosis – will be obtained for the aggregated returns.

I. Un-centred Moments

First moment – forward returns

\[ E_t (r_{t+s}) = E_t (\mu + \varepsilon_{t+s}) = \mu + E_t \left( \frac{E_{t+s-1}(\varepsilon_{t+s})}{0} \right) = \mu \]

First moment – aggregated returns

Similar results have been derived for the generic AGARCH model as well, and can be obtained on request from the authors, but have been omitted here due to space considerations.
\[ E_t(R_{t+T}) = E_t \left( \sum_{s=1}^{T} r_{t+s} \right) = E_t \left( \sum_{s=1}^{T} (\mu + \varepsilon_{t+s}) \right) = T\mu \]  

(7)

**Second moment – forward returns**

\[ E_t(r_{t+s}^2) = E_t \left[ \left( \mu + \varepsilon_{t+s} \right)^2 \right] = E_t \left( \mu^2 + 2\mu\varepsilon_{t+s} + \varepsilon_{t+s}^2 \right) = \mu^2 + E_t (\sigma_{t+s}^2) = \mu^2 + \alpha_s \]  

(8)

where we use the notation \( \alpha_s = E_t \left( \sigma_{t+s}^2 \right) \).

The s-step ahead conditional variance forecasts in the context of the model are given by:

\[ \sigma_{t+s}^2 = \omega + \alpha \varepsilon_{t+s-1}^2 + \lambda \varepsilon_{t+s-1}^2 I_{t+s-1} + \beta \sigma_{t+s-1}^2 \]  

(9)

If we apply the expectation operator in the above expression for \( \sigma_{t+s}^2 \), and use the notations \( \varphi = \alpha + \lambda F(0) + \beta \) - where \( F(0) \) is the distribution function for \( D(0,1) \) evaluated at zero - and \( \sigma_0^2 = \frac{\omega}{1-\varphi} \), the final expression for \( \alpha_s \) is:

\[ \alpha_s = \sigma_0^2 + \varphi^{-1} \left( \sigma_{t+s-1}^2 - \sigma_0^2 \right) \]  

(10)

Thus,

\[ E_t \left( r_{t+s}^2 \right) = \mu^2 + \sigma_0^2 + \varphi^{-1} \left( \sigma_{t+s-1}^2 - \sigma_0^2 \right) \]  

(11)

**Second moment – aggregated returns**

\[ E_t \left( R_{t+T}^2 \right) = E_t \left( \sum_{s=1}^{T} r_{t+s} \right)^2 = E_t \left( \sum_{s=1}^{T} r_{t+s} + 2 \sum_{i=1}^{T} \sum_{j=1}^{T-i} r_{t+i} r_{t+j} \right) \]  

(12)

\[ E_t \left( \sum_{s=1}^{T} r_{t+s}^2 \right) = \sum_{s=1}^{T} \left( \mu^2 + \sigma_0^2 + \varphi^{-1} \left( \sigma_{t+s-1}^2 - \sigma_0^2 \right) \right) = T \left( \mu^2 + \sigma_0^2 + \left( \frac{1}{1-\varphi} - 1 \right) \sigma_{t+s-1}^2 \right) \]  

(13)

\[ E_t \left( \sum_{i=1}^{T} \sum_{j=1}^{T-i} r_{t+i} r_{t+j} \right) = \sum_{i=1}^{T} \sum_{j=1}^{T-i} E_t \left( r_{t+i} r_{t+j} \right) = \sum_{i=1}^{T} \sum_{j=1}^{T-i} E_t \left( (\mu + \varepsilon_{t+i})(\mu + \varepsilon_{t+j}) \right) = \frac{T(T-1)}{2} \mu^2 \]  

(14)

Hence, the expression for the second moment of aggregated returns becomes:

\[ E_t \left( R_{t+T}^2 \right) = T^2 \mu^2 + T \sigma_0^2 + \left( \sigma_{t+s-1}^2 - \sigma_0^2 \right) \frac{1}{1-\varphi} \]  

(15)
Third moment – forward returns

\[ E_t \left( r^3_{t+s} \right) = E_t \left[ \left( \mu + \varepsilon_{t+s} \right)^3 \right] = E_t \left( \mu^3 + 3\mu^2 \varepsilon_{t+s} + 3\mu \varepsilon_{t+s}^2 + \varepsilon_{t+s}^3 \right) = \mu^3 + 3\mu \sigma_t + \varepsilon \left( \sigma^3_{t+s} \right) \]  \hspace{1cm} (16)

We can only get \( E_t \left( \sigma^3_{t+s} \right) \) approximately, by using a second order Taylor series expansion for \( f(X) = X^{3/2} \) around \( E_t \left( \sigma^2_{t+s} \right) \):

\[ f(X) \approx f \left( E_t(X) \right) + f' \left( E_t(X) \right) \left( X - E_t(X) \right) + 0.5 f'' \left( E_t(X) \right) \left( X - E_t(X) \right)^2 \]

where

\[ f'(X) = \frac{3}{2} X^{1/2} \]
\[ f''(X) = \frac{3}{4} X^{-1/2} \]

Applying the expectation operator in the above, we get:

\[ E_t \left( f(X) \right) = f \left( E_t(X) \right) + 0.5 f'' \left( E_t(X) \right) V_t(X) \]

Now, replacing \( X \) by \( \sigma_{t+s}^2 \), we get:

\[ E_t \left( \sigma^3_{t+s} \right) = \frac{1}{8} \left( 5\sigma^{3/2}_t + \frac{b_s}{\sqrt{a_s}} \right) \hspace{1cm} (17) \]

where we used the notation \( b_s = E_t \left( \sigma^4_{t+s} \right) \).

\[ b_s = E_t \left( \left( \omega + \left( \alpha + \lambda I_{t+s-1} \right) \varepsilon_{t+s-1}^4 + \beta \sigma_{t+s-1}^2 \right)^2 \right) \]
\[ = E_t \left( \omega^2 + \left( \alpha + \lambda I_{t+s-1} \right)^2 \varepsilon_{t+s-1}^4 + \beta^2 \sigma_{t+s-1}^4 + 2\omega \left( \alpha + \lambda I_{t+s-1} \right) \varepsilon_{t+s-1}^2 + 2\omega \beta \sigma_{t+s-1}^2 + 2\beta \left( \alpha + \lambda I_{t+s-1} \right) \varepsilon_{t+s-1} \sigma_{t+s-1} \right) \]
\[ = \omega^2 + 2\omega \left( \alpha + \lambda F(0) + \beta \right) a_{s-1} + \left( x \left( \alpha^2 + \left( \lambda^2 + 2\alpha\lambda \right) F(0) \right) + \beta^2 + 2\beta \left( \alpha + \lambda F(0) \right) \right) b_{s-1} \]
\[ = \omega^2 + 2\omega \varphi a_{s-1} + \left( \varphi^2 + \left( \alpha - 1 \right) \left( \alpha + \lambda F(0) \right)^2 + \chi \lambda^2 F(0) \left( 1 - F(0) \right) \right) b_{s-1} \]

where

\[ \chi = \varphi^2 + \left( \alpha - 1 \right) \left( \alpha + \lambda F(0) \right)^2 + \chi \lambda^2 F(0) \left( 1 - F(0) \right) > 0 \]
\[ b_1 = \sigma^4_{t+1} \]

Solving the above recursion, the expression for \( b_s \) becomes:
\[ b_s = \sum_{i=1}^{n} \gamma^{j-1}\left(\omega^2 + 2\omega\varphi(\sigma^2_0 + \varphi^{-i-1}(\sigma^2_{t+1} - \sigma^2_0))\right) + \gamma^{-i}b_i \]

\[ = \sum_{i=1}^{n} \gamma^{j-1}\left(\omega^2 + 2\omega\varphi(\sigma^2_0 + \varphi^{-i-1}(\sigma^2_{t+1} - \sigma^2_0))\right) + \gamma^{-i}b_i \]

\[ = (\omega^2 + 2\omega\varphi_0)(1 - \gamma)^{-1}(1 - \gamma^{-1}) + 2\omega\varphi^{-i-1}(\sigma^2_{t+1} - \sigma^2_0)\left(1 - \frac{\gamma}{\varphi}\right)^{-1}\left(1 - \frac{\gamma}{\varphi}\right) + \gamma^{-i}b_i \]

\[ = C_1(1 - \gamma^{-i}) + C_2(\varphi^{-i-1} - \gamma^{-1}) + \gamma^{-i}b_i \]

\[ = C_1 + (-C_1 - C_2 + b_i)\gamma^{-i} + C_2\varphi^{-i} \]

where

\[ C_1 = (\omega^2 + 2\omega\varphi_0)(1 - \gamma)^{-2} \]

\[ C_2 = 2\omega\varphi(\sigma^2_{t+1} - \sigma^2_0)(\varphi - \gamma)^{-1} \]

Hence, the expression for the third moment of returns becomes:

\[ E_t\left(\tau^{3}\right) = \mu^3 + 3\mu\alpha_i + \tau\left(5\alpha_i^2 + \frac{5}{8}b_i\right) \]

(18)

**Third moment – aggregated returns**

\[ E_t(R_{1:T}^{3}) = E_t\left(\sum_{i=1}^{T} r_{t+1}^{3}\right) = \sum_{i=1}^{T} E_t\left(r_{t+1}^{3}\right) + 3\sum_{i=1}^{T-1} \sum_{j=i+1}^{T} E_t\left(r_{t+1}^{2}r_{t+2}^{2}\right) + E_t\left(r_{t+1}^{2}r_{t+2}^{2}\right) + 6\sum_{i=1}^{T-1} \sum_{j=i+1}^{T} E_t\left(r_{t+1}^{2}r_{t+2}^{2}\right) \]

(19)

\[ \sum_{i=1}^{T} E_t\left(r_{t+1}^{3}\right) = \sum_{i=1}^{T} \left[\mu^3 + 3\mu\alpha_i + \tau E_t\left(\sigma^3_{t+1}\right)\right] = \mu T\left(\mu^2 + 3\sigma^2_0\right) + 3\mu(1 - \varphi)^{-1}(1 - \varphi^T)(\sigma^2_{t+1} - \sigma^2_0) + \tau\sum_{i=1}^{T} E_t\left(\sigma^3_{t+1}\right) \]

(20)

\[ \sum_{i=1}^{T} \sum_{j=i}^{T} E_t\left(r_{t+1}^{2}r_{t+2}^{2}\right) = \sum_{i=1}^{T} \sum_{j=i}^{T} E_t\left(\left(\mu^2 + 2\mu\varepsilon_{t+1} + \varepsilon_{t+1}^2\right)(\mu + \varepsilon_{t+1})\right) = \frac{T(T-1)}{2}\mu^3 + \mu\sum_{i=1}^{T}(T - i)\alpha_i \]

\[ \sum_{i=1}^{T} (T - i)\alpha_i = \sum_{i=1}^{T} (T - i)(\sigma^2_0 + \varphi^{-i-1}(\sigma^2_{t+1} - \sigma^2_0)) = \frac{T(T-1)}{2}\sigma^2_0 + (\sigma^2_{t+1} - \sigma^2_0)\sum_{i=1}^{T} (T - i)\varphi^{-i-1} \]

\[ \sum_{i=1}^{T} (T - i)\varphi^{-i-1} = T\frac{T\varphi^{-1} - \sum_{i=1}^{T} \varphi^{-i-1}}{1 - \varphi} = T\frac{1 - \varphi^T}{1 - \varphi} - \frac{\sum_{i=1}^{T} \varphi^{-i-1} - T\varphi^T}{1 - \varphi} = T\frac{1 - \varphi^T}{1 - \varphi} = (1 - \varphi)^{-1}\left(T - (1 - \varphi)^{-1}(1 - \varphi^T)\right) \]

We get:
Hence, the final expression for \( \sum_{i=1}^{T} \sum_{j=1}^{T-i} E_i (r_{i+1}^2) \) becomes:

\[
\sum_{i=1}^{T} \sum_{j=1}^{T-i} E_i (r_{i+1}^2) = \frac{T(T-1)}{2} (\mu^3 + \mu \sigma_0^2) + \left(1 - \varphi^{-1}\right) \left( T - (1 - \varphi^{-1}) (1 - \varphi^T) \right) \mu \left(\sigma_{i+1}^2 - \sigma_0^2\right) (21)
\]

Solving the recursion above, we get:

\[
E_t (z_{t+1}^2) = \varphi^{-1} E_t (z_{t+1}^2) + \varphi^{-1} E_t (z_{t+1} \sigma_{t+1}^2) = \varphi^{-1} E_t (z_{t+1} (\omega + (\alpha + \lambda z_{t+1}) z_{t+1}^2 + \beta \sigma_{t+1}^2))
\]

\[
= \varphi^{-1} \left[ E_t (\sigma_{t+1}^3) + \lambda E_t \left( z_{t+1}^3 \right) \right] + \beta E_t (z_{t+1} \sigma_{t+1}^3)
\]

\[
= \varphi^{-1} \left[ \alpha \tau E_t (\sigma_{t+1}^3) + \lambda E_t \left( \sigma_{t+1}^3 \int_{x=-\infty}^{0} x^3 f(x) dx \right) \right] + \beta E_t (z_{t+1} \sigma_{t+1}^3)
\]

\[
= \varphi^{-1} \left[ \alpha \tau + \lambda \int_{x=-\infty}^{0} x^3 f(x) dx \right] E_t (\sigma_{t+1}^3)
\]

\[
= \frac{\varphi^{-1}}{8} \left( \alpha \tau + \lambda \int_{x=-\infty}^{0} x^3 f(x) dx \right) \left( 5a_i \frac{b}{\sqrt{a_i}} \right)
\]

since \( E_t (z_{t+1} \sigma_{t+1}^3) = E_t (E_{t+1} (z_{t+1} \sigma_{t+1}^3)) = E_t (\sigma_{t+1}^3 E_{t+1} (z_{t+1})) = 0 \); \( f \) is the pdf of the conditional distribution \( D(0,1) \).
\[
\sum_{i=1}^{T} \sum_{j=1}^{T-i} a_{ij} = \sum_{i=1}^{T} \sum_{j=1}^{T-i} \left( \sigma_{ij}^2 + \varphi^{i-j} \left( \sigma_{ij}^2 - \sigma_0^2 \right) \right) = \sigma_0^2 \frac{T(T-1)}{2} + \left( \sigma_{ij}^2 - \sigma_0^2 \right) \sum_{i=1}^{T} \sum_{j=1}^{T-i} \varphi^{i-j} \\
= \sigma_0^2 \frac{T(T-1)}{2} + \left( 1 - \varphi^{-1} \right) \left( \sigma_{ij}^2 - \sigma_0^2 \right) \sum_{i=1}^{T} \varphi \left[ 1 - \varphi^{-i} \right] \\
= \sigma_0^2 \frac{T(T-1)}{2} + \left( 1 - \varphi^{-1} \right) \left( \sigma_{ij}^2 - \sigma_0^2 \right) \sum_{i=1}^{T} \varphi^{-i} - \varphi^{-i} \right] \\
= \frac{T(T-1)}{2} \sigma_0^2 + \left( 1 - \varphi^{-1} \right) \left[ \varphi^{-1} \left( 1 - \varphi^{-1} \right) - T \varphi^{-i} \right] \left( \sigma_{ij}^2 - \sigma_0^2 \right)
\]

\[
\sum_{i=1}^{T} \sum_{j=1}^{T-i} E_i \left( e_{ij}^2 \right) = \frac{\alpha_T + \lambda}{\varphi} \int_{x=0}^{\infty} x^3 f(x) dx \left[ \sum_{i=1}^{T} \sum_{j=1}^{T-i} \left( 5a_i \gamma^2 + 3 \frac{b_i}{\sqrt{a_i}} \right) \right] \\
= \frac{\alpha_T + \lambda}{\varphi} \int_{x=0}^{\infty} x^3 f(x) dx \left[ 1 - \varphi^{-1} \right] \left( \sum_{i=1}^{T} \left( 1 - \varphi^{-i} \right) \left( 5a_i \gamma^2 + 3 \frac{b_i}{\sqrt{a_i}} \right) \right)
\]

Hence the final expression for \( \sum_{i=1}^{T} \sum_{j=1}^{T-i} E_i \left( r_{ij}^2 \right) \) becomes:

\[
\sum_{i=1}^{T} \sum_{j=1}^{T-i} E_i \left( r_{ij}^2 \right) = \mu \frac{T(T-1)}{T} \left( \mu^2 + \sigma_0^2 \right) + \left( 1 - \varphi^{-1} \right) \left[ \mu \left( 1 - \varphi^{-1} \right) - T \varphi^{-i} \right] \left( \sigma_{ij}^2 - \sigma_0^2 \right) + \sum_{i=1}^{T} \varphi^{-i} \left( \alpha_T E_i \left( \sigma_{ij}^2 \right) + \lambda E_i \left( 0 \right) \right)
\]

(22)

Fourth moment – forward returns

\[
E_i \left( r_{ij}^4 \right) = \mu \left( \sigma_i \right)^4 + 4\mu \sigma_i \left( \sigma_i \right)^3 + 6\left( \sigma_i \right)^2 \sigma_i \left( \sigma_i \right)^2 + 4\sigma_i \left( \sigma_i \right)^2 + \left( \sigma_i \right)^4
\]

(23)

Fourth moment – aggregated returns

\[
E_i \left( R_{ij}^4 \right) = \sum_{i=1}^{T} E_i \left( r_{ij}^4 \right) + \sum_{i=1}^{T} \left[ 4 \left( E_i \left( r_{ij}^3 \right) + E_i \left( r_{ij}^2 \right) \right) + 6E_i \left( r_{ij}^2 \right) \right]
\]

(24)

\[
\sum_{i=1}^{T} E_i \left( r_{ij}^2 \right) = \sum_{i=1}^{T} \left( \mu^2 + 6\mu^2 a_i + 4\mu^2 \sigma_i \left( \sigma_i \right)^2 \right) + \lambda b_i
\]

(25)
\[
\sum_{i=1}^{T} \sum_{j=1}^{T-i} E_i \left(r_{i,j} r_{i+i,j}^{*}\right) = \sum_{i=1}^{T} \sum_{j=1}^{T-i} \left(\mu^3 + 3\left(\mu^2 \varepsilon_{i+j} + \mu \varepsilon_{i+i}^2\right) + \varepsilon_{i+j}^3\right) \left(\mu + \varepsilon_{i+i}\right) = \sum_{i=1}^{T} \sum_{j=1}^{T-i} \left(\mu^3 + 3\mu^2 a_i + \mu \varepsilon_i \left(\sigma_{i+i}^3\right)\right)
\]
\[
= \frac{T(T-1)}{2} \mu^2 \left(\mu^2 + 3\sigma_0^2\right) + 3\mu^2 (1-\varphi)^{-1} \left[T - (1-\varphi)^{-1} \left(1-\varphi^T\right)\right] \left(\sigma_{i+i}^2 - \sigma_0^2\right) + \mu \sum_{i=1}^{T} (T-i) E_i \left(\sigma_{i+i}^3\right)
\]
since
\[
\sum_{i=1}^{T} \sum_{j=1}^{T-i} a_i = \sum_{i=1}^{T} \sum_{j=1}^{T-i} \left(\sigma_0^2 + \varphi^{-1} \left(\sigma_{i+i}^2 - \sigma_0^2\right)\right)
\]
\[
= \frac{T(T-1)}{2} \sigma_0^2 + \left(\sigma_{i+i}^2 - \sigma_0^2\right) (1-\varphi)^{-1} \left[T - (1-\varphi)^{-1} \left(1-\varphi^T\right)\right]
\]

Using a second order Taylor expansion around \(a_{i+i} = E_i \left(\sigma_{i+i}^3\right)\), we get:
\[
\sigma_{i+i}^3 \approx \left(a_{i+i}^3 + \frac{3}{2} \left(a_{i+i}^3\right) \left(\sigma_{i+i}^2 - a_{i+i}\right) + \frac{3}{8} \left(a_{i+i}^3\right) \left(\sigma_{i+i}^2 - a_{i+i}^2\right)^2\right)
\]
where \(a_{i+i}\) is known at time \(t\); replacing this expression into the above we get:
\[
E_i \left(\varepsilon_{i+i} \varepsilon_{i+i}^3\right) = \tau E_i \left(\varepsilon_{i+i} \left(a_{i+i}^3 + \frac{3}{2} \left(a_{i+i}^3\right) \left(\sigma_{i+i}^2 - a_{i+i}\right) + \frac{3}{8} \left(a_{i+i}^3\right) \left(\sigma_{i+i}^2 - a_{i+i}^2\right)^2\right)\right)
\]
\[
= \tau \left(\frac{3}{2} \left(a_{i+i}^3\right) E_i \left(\varepsilon_{i+i} \sigma_{i+i}^2\right) + \frac{3}{8} \left(a_{i+i}^3\right) \left(E_i \left(\varepsilon_{i+i} \sigma_{i+i}^4\right) - \frac{3}{4} \left(a_{i+i}^3\right) E_i \left(\varepsilon_{i+i} \sigma_{i+i}^2\right)\right)\right)
\]
\[
= \tau \left(\frac{3}{4} \left(a_{i+i}^3\right) \left(E_i \left(\varepsilon_{i+i} \sigma_{i+i}^2\right) + \frac{1}{2} \left(a_{i+i}^3\right) E_i \left(\varepsilon_{i+i} \sigma_{i+i}^4\right)\right)\right)
\]
\[
E_i \left(\varepsilon_{i+i} \sigma_{i+i}^4\right) = E_i \left(\varepsilon_{i+i} \left(\omega + \alpha \left(\varepsilon_{i+i+1} - \lambda\right)^2 + \beta \sigma_{i+i+1}^2\right)\right)^2
\]
\[
= E_i \left(\varepsilon_{i+i} \left(\omega^2 + \left(\alpha + \lambda \Gamma_{i+i+1}\right) \varepsilon_{i+i+1}^2 + \beta^2 \sigma_{i+i+1}^2 + 2 \omega \left(\alpha + \lambda \Gamma_{i+i+1}\right) \varepsilon_{i+i+1}^2 + 2 \omega \beta \sigma_{i+i+1}^2\right)\right)
\]
\[
= \left(\left(\alpha^2 + \lambda^2 F(0)\right) \varepsilon_{i+i+1}^2 + \beta^2 \left(\alpha + \lambda F(0)\right) \varepsilon_{i+i+1}^2 + 2 \omega \left(\alpha + \lambda F(0) + \beta\right) E_i \left(\varepsilon_{i+i} \sigma_{i+i+1}^2\right)\right)
\]
\[ E \left( \varepsilon_{z}, E_{z+i+j-2} \left( \left( x^2 + 2 \alpha \lambda I_{z+i+j-1} + \lambda^2 I_{z+i+j-1} \right) z^4_{z+i+j-1} \varepsilon z_{z+i+j-1}^4 \right) \right) + \beta^2 E \left( \varepsilon_{z}, \varepsilon_{z+i+j-1} \varepsilon z_{z+i+j-1}^4 \right) + 2 \omega \left( \alpha + \lambda F(0) \right) E \left( \varepsilon_{z}, \varepsilon z_{z+i+j-1}^2 \right) + 2 \omega \varepsilon_{z} E \left( \varepsilon_{z}, \varepsilon z_{z+i+j-1}^2 \right) + 2 \beta \left( \alpha + \lambda F(0) \right) E \left( \varepsilon_{z}, \varepsilon z_{z+i+j-1}^4 \right) \]

where we have used that conditional on the filtration \( \Phi \), the indicator function \( I_{z+i} \) is independent of all (contemporaneous) \( \varepsilon z_{2k} \), for any natural number \( k \).

Notation:

\[ b_{i,d(z)} = E_{i} \left( \varepsilon_{z}, \varepsilon_{z+i+j}^2 \right) \]
\[ c_{i,d(z)} = E_{i} \left( \varepsilon_{z}, \varepsilon_{z+i+j}^3 \right) \]
\[ d_{i,d(z)} = E_{i} \left( \varepsilon_{z}, \varepsilon_{z+i+j}^4 \right) \]

\[ \gamma = (\alpha x^2 + \beta^2 + 2x \beta + \left( \alpha \lambda^2 + 2 \lambda \left( \alpha x + \beta \right) \right) F(0)) \]

\[ b_{i,d(z)} = \varphi^{-1} \left( \alpha x + \lambda \int_{x=-\infty}^{0} x^3 f(x) \, dx \right) E_{i} \left( \sigma_{z+i}^3 \right) = C_{4} \varphi^{-1} E_{i} \left( \sigma_{z+i}^3 \right) \]

where

\[ C_{4} = \left( \alpha x + \lambda \int_{x=-\infty}^{0} x^3 f(x) \, dx \right) \]

\[ c_{i,d(z)} = \frac{3}{4} \left( a_{i,z+i} \right)^{1/2} \left( b_{i,d(z)} + \frac{1}{2} \left( a_{i,z+i} \right)^{-1} d_{i,d(z)} \right) \]

\[ d_{i,d(z)} = \gamma d_{i,d(z)} + 2 \omega \varphi b_{i,d(z)} \]

or, in non-recursive form:

\[ d_{i,d(z)} = \gamma^{-1} d_{i,d(z)} + 2 \omega \varphi \sum_{j=1}^{z-i} \gamma^{-j} b_{i,d(z-j)} \]

\[ \sum_{j=1}^{z-i} \gamma^{-j} b_{i,d(z-j)} = C_{4} \sum_{j=1}^{z-i} \gamma^{-j} \varphi \left( \gamma^{-j} \right) E_{i} \left( \sigma_{z+i}^3 \right) = C_{4} \varphi^{-2} E_{i} \left( \sigma_{z+i}^3 \right) \sum_{j=1}^{z-i} \left( \frac{\gamma}{\varphi} \right)^{j-1} \]

\[ = C_{4} \varphi^{-2} E_{i} \left( \sigma_{z+i}^3 \right) \frac{1 - \left( \frac{\gamma}{\varphi} \right)^{z-i}}{1 - \frac{\gamma}{\varphi}} = C_{4} \left( \varphi^{-1} - \gamma^{-1} \right) E_{i} \left( \sigma_{z+i}^3 \right) \left( \gamma^{-z} - \gamma^{-1} \right) \]

or

\[ \sum_{j=1}^{z-i} \gamma^{-j} b_{i,d(z-j)} = C_{4} \left( \varphi^{-1} - \gamma^{-1} \right) E_{i} \left( \sigma_{z+i}^3 \right) \]

27
where

\[ C_4 = \left( \alpha t + \lambda \int_{x=-\infty}^{0} x^3 f(x) \, dx \right) (\varphi - \gamma)^{-1} = C_3 (\varphi - \gamma)^{-1} \]

\[ d_{i,i+s} = E_i \left( \varepsilon_{i,s} \sigma_{i+1+s} \right) = E_i \left( \varepsilon_{i,s} \left( \omega + (\alpha + \lambda I_{i+1+3}) \varepsilon_{i+1+3} + \beta \sigma_{i+1+3} \right) \right) \]

\[ E_i \left( \varepsilon_{i+1+3} \left( \omega^2 + (\alpha + \lambda I_{i+1+3}) \varepsilon_{i+1+3}^2 + \beta^2 \sigma_{i+1+3}^2 + 2 \omega \left( \alpha + \lambda I_{i+1+3} \right) \varepsilon_{i+1+3}^2 + 2 \omega \beta \varepsilon_{i+1+3} \right) \right) \]

\[ = E_i \left( \left( \varepsilon_{i+1+3}^2 + \lambda (2 \alpha + \lambda) I_{i+1+3} \varepsilon_{i+1+3} + \beta E_i \left( \sigma_{i+1+3}^3 E_{i+1+3} \left( z_{i+1+3} \right) + 2 \omega E_i \left( \sigma_{i+1+3}^3 \varepsilon_{i+1+3} \right) \right) + 2 \omega \beta E_i \left( \sigma_{i+1+3}^3 \varepsilon_{i+1+3} \right) \right) \]

\[ = E_i \left( \varepsilon_{i+1+3}^2 + \lambda (2 \alpha + \lambda) I_{i+1+3} \varepsilon_{i+1+3} + \beta \sigma_{i+1+3} \sigma_{i+1+3} \right) \]

\[ = \alpha^2 \mu E_i \left( \sigma_{i+1+3}^5 \right) + \lambda (2 \alpha + \lambda) E_i \left( \sigma_{i+1+3}^5 E_{i+1+3} \left( I_{i+1+3} z_{i+1+3} \right) \right) + 2 \omega \alpha \varepsilon_i \left( \sigma_{i+1+3}^3 \right) + 2 \omega \lambda E_i \left( \sigma_{i+1+3}^3 E_{i+1+3} \left( I_{i+1+3} z_{i+1+3} \right) \right) \]

\[ + 2 \alpha \beta \varepsilon_i \left( \sigma_{i+1+3}^3 \right) + 2 \lambda \varepsilon_i \left( \sigma_{i+1+3}^5 E_{i+1+3} \left( I_{i+1+3} z_{i+1+3} \right) \right) \]

\[ E_{i+1+3} \left( I_{i+1+3} z_{i+1+3} \right) = \int_{x=-\infty}^{0} x^3 f(x) \, dx \quad \text{and} \quad E_{i+1+3} \left( I_{i+1+3} z_{i+1+3} \right) = \int_{x=-\infty}^{0} x^3 f(x) \, dx \]

Hence the expression above becomes:

\[ d_{i,i+s} = \left[ \alpha (x \mu + 2 \omega \beta) + \lambda (2 \alpha + \lambda) \int_{x=-\infty}^{0} x^3 f(x) \, dx \right] E_i \left( \sigma_{i+1+3}^5 \right) + 2 \left[ \omega \alpha + \lambda (\omega + \beta) \int_{x=-\infty}^{0} x^3 f(x) \, dx \right] E_i \left( \sigma_{i+1+3}^3 \right) \]

\[ \text{where} \]

\[ C_5 = \alpha (x \mu + 2 \omega \beta) + \lambda (2 \alpha + \lambda) \int_{x=-\infty}^{0} x^3 f(x) \, dx \]

\[ C_6 = 2 \left[ \omega \alpha + \lambda (\omega + \beta) \int_{x=-\infty}^{0} x^3 f(x) \, dx \right] \]

Hence, the expression for \( d_{i,i+s} \) becomes:

\[ d_{i,i+s} = g^{-1} \left( C_5 E_i \left( \sigma_{i+1+3}^5 \right) + C_6 E_i \left( \sigma_{i+1+3}^3 \right) \right) + 2 \omega \varphi C_4 \left( g^{-1} - g^{-1} \right) E_i \left( \sigma_{i+1+3}^3 \right) \]

\[ = C_5 g^{-1} E_i \left( \sigma_{i+1+3}^5 \right) + \left( C_6 - 2 \omega \varphi C_4 \right) g^{-1} + 2 \omega \varphi C_4 g^{-1} \right) E_i \left( \sigma_{i+1+3}^3 \right) \]

Consequently, the expression for \( c_{i,i+s} \) becomes:
\[ c_{i(i+1)} = \frac{3}{4} \left( a_{i(i+1)} \right)^{1/2} \left( C_i \varphi^{-1} E_i \left( \sigma_{i(i+1)}^3 \right) + \frac{1}{2} \left( a_{i(i+1)} \right)^{-1} \left( C_i \gamma^{-1} E_i \left( \sigma_{i(i+1)}^3 \right) + \left( 2 \omega \varphi \rho_i \left( \sigma_{i(i+1)}^3 \right) + \left( C_i - 2 \omega \varphi \rho_i \right) \gamma^{-1} \right) E_i \left( \sigma_{i(i+1)}^3 \right) \right) \]

\[ = \frac{3}{4} \left( C_i + \omega \varphi \rho_i \right) \left( a_{i(i+1)} \right)^{1/2} \varphi^{-1} E_i \left( \sigma_{i(i+1)}^3 \right) + \frac{3}{8} C_i \left( a_{i(i+1)} \right)^{-1/2} \gamma^{-1} E_i \left( \sigma_{i(i+1)}^3 \right) + \frac{3}{8} \left( C_i - \omega \varphi \rho_i \right) \left( a_{i(i+1)} \right)^{-1/2} \gamma^{-1} E_i \left( \sigma_{i(i+1)}^3 \right) \]

**Method 1 - \( E_i \left( \sigma_{i(i+1)}^3 \right) \)**

We get \( E_i \left( \sigma_{i(i+1)}^3 \right) \) approximately only, using a second order Taylor expansion of \( g(X) = X^{3/2} \) around \( E_i \left( \sigma_{i(i+1)}^3 \right) \)

\[ g(X) \approx g \left( E_i(X) \right) + g' \left( E_i(X) \right) \left( X - E_i(X) \right) + 0.5 g'' \left( E_i(X) \right) \left( X - E_i(X) \right)^2 \]

where

\[ g'(X) = \frac{5}{2} X^{3/2} \]

\[ g''(X) = \frac{15}{4} X^{1/2} \]

Hence, for \( X = \sigma_{i(i+1)}^3 \), we get:

\[ \sigma_{i(i+1)}^3 = \left( a_i \right)^{3/2} + \frac{5}{2} \left( a_i \right)^{1/2} \left( \sigma_{i(i+1)}^3 - a_i \right) + \frac{15}{8} \left( a_i \right)^{1/2} \left( \sigma_{i(i+1)}^3 - a_i \right)^2 \]

Applying the expectation operator in the above, we get:

\[ E_i \left( \sigma_{i(i+1)}^3 \right) = \left( a_i \right)^{3/2} + \frac{15}{8} \left( a_i \right)^{1/2} E_i \left( \left( \sigma_{i(i+1)}^3 - a_i \right)^2 \right) = \left( a_i \right)^{3/2} + \frac{15}{8} \left( a_i \right)^{1/2} \left( b_i - \left( a_i \right)^2 \right) = \frac{1}{8} \left( a_i \right)^{3/2} \left(-7 \left( a_i \right)^2 + 15b_i \right) \]

**Method 2 - \( E_i \left( \sigma_{i(i+1)}^3 \right) \)**

We can also get \( E_i \left( \sigma_{i(i+1)}^3 \right) \) (approximately) using a second order Taylor expansion of \( g(X) = X^{3/4} \) around \( E_i \left( \sigma_{i(i+1)}^3 \right) \)

\[ g(X) \approx g \left( E_i(X) \right) + g' \left( E_i(X) \right) \left( X - E_i(X) \right) + 0.5 g'' \left( E_i(X) \right) \left( X - E_i(X) \right)^2 \]

where

\[ g'(X) = \frac{5}{4} X^{1/4} \]

\[ g''(X) = \frac{5}{16} X^{-3/4} \]

Hence, for \( X = \sigma_{i(i+1)}^3 \), we get:
Applying the expectation operator in the above, we get:

$$E_i \left( \sigma_{ri}^4 \right) = (b_i)^{3/4} + \frac{5}{4} (b_i)^{3/4} \sigma_i^4 + \frac{5}{16} (b_i)^{3/4} \sigma_i^4 + \frac{5}{16} (b_i)^{3/4} \sigma_i^4 + \frac{5}{16} (b_i)^{3/4} \sigma_i^4$$

where the expression for $E_i \left( \sigma_{ri}^4 \right)$ is given in Appendix 2.1 of this paper, which comprises the derivations for the moments of variances.
\[\sum_{i=1}^{T} \sum s_i a_i = \sum_{i=1}^{T} (T - i) a_i = \frac{T(T-1)}{2} \sigma_0^2 + (1 - \varphi)^{-1} \left[ T - (1 - \varphi)^{-1} (1 - \varphi^T) \right] (\sigma_{r+1}^2 - \sigma_0^2) \]

\[\sum_{i=1}^{T} \varphi^i a_i = \varphi (1 - \varphi)^{-1} \sum_{i=1}^{T} (1 - \varphi^T) a_i = \varphi (1 - \varphi)^{-1} \left( \sum_{i=1}^{T} a_i - \sum_{i=1}^{T} \varphi^i - (\sigma_0^2 + \varphi^{-1} (\sigma_{r+1}^2 - \sigma_0^2)) \right) = \varphi (1 - \varphi)^{-1} \left( \sum_{i=1}^{T} a_i - S \right) \]

\[\sum_{i=1}^{T} a_i = T\sigma_0^2 + (1 - \varphi)^{-1} (1 - \varphi^T) (\sigma_{r+1}^2 - \sigma_0^2) \]

\[S = \sum_{i=1}^{T} \varphi^{T-i} \left( \sigma_0^2 + \varphi^{-1} (\sigma_{r+1}^2 - \sigma_0^2) \right) = \sigma_0^2 \sum_{i=1}^{T} \varphi^i + T (\sigma_{r+1}^2 - \sigma_0^2) \varphi^{T-i} \]

\[= \sigma_0^2 (1 - \varphi)^{-1} (1 - \varphi^T) + T \varphi^{T-i} (\sigma_{r+1}^2 - \sigma_0^2) \]

Hence, the expression for \( \sum_{i=1}^{T} \sum s_i (1 - \varphi^i) a_i \) becomes:

\[\sum_{i=1}^{T} \sum s_i (1 - \varphi^i) a_i \]

\[= \sigma_0^2 \left[ \frac{T(T-1)}{2} \sigma_0^2 + (\sigma_{r+1}^2 - \sigma_0^2) \varphi^{T-i} \left( T - (1 - \varphi)^{-1} (1 - \varphi^T) \right) - (1 - \varphi)^{-1} \right] \]

\[= \left( T\sigma_0^2 - (1 - \varphi)^{-1} (1 - \varphi^T) (\sigma_{r+1}^2 - \sigma_0^2) + \sigma_0^2 (1 - \varphi)^{-1} (1 - \varphi^T) + T (\sigma_{r+1}^2 - \sigma_0^2) \varphi^{T-i} \right) \]

Last sum to be computed is:

\[\sum_{i=1}^{T} \varphi^i b_i = (1 - \varphi)^{-1} \sum_{i=1}^{T} (1 - \varphi^T) b_i = (1 - \varphi)^{-1} \left( \sum_{i=1}^{T} b_i - \sum_{i=1}^{T} \varphi^i b_i \right) \]

\[= C_i T + (-C_i - C_2 + b_i)(1 - \gamma)^{-1} (1 - \gamma^T) + C_2 (1 - \gamma)^{-1} (1 - \gamma^T) \]

\[= C_i T + (-C_i - C_2 + b_i)(1 - \gamma)^{-1} (1 - \gamma^T) + C_2 (1 - \gamma)^{-1} (1 - \gamma^T) \]

\[= C_i T + (-C_i - C_2 + b_i)(1 - \gamma)^{-1} (1 - \gamma^T) + C_2 (1 - \gamma)^{-1} (1 - \gamma^T) \]

\[= C_i T + (-C_i - C_2 + b_i)(1 - \gamma)^{-1} (1 - \gamma^T) + C_2 (1 - \gamma)^{-1} (1 - \gamma^T) \]

Hence, the final expression for \( \sum_{i=1}^{T} \sum E_i \left( \sigma_{r+1}^2 \sigma_{r+1}^2 \right) \) becomes:
II. Centred moments

Second Conditional Centred Moment (Variance) - forward returns

\[ \mu_{r,a}^2 = E_i \left( \left( r_{t+a} - E \left( r_{t+a} \right) \right)^2 \right) = E_i \left( \varepsilon_{t+a}^2 \right) = a_i = \sigma_0^2 + \varphi^{-1} \left( \sigma_{t+1}^2 - \sigma_0^2 \right) \]  

(25)

Second Conditional Centred Moment (Variance) – aggregated returns

\[ M_{r,T}^2 = E_i \left( \left( \sum_{t=1}^{T} \left( r_{t+a} - \mu \right) \right)^2 \right) = E_i \left( \sum_{t=1}^{T} \varepsilon_{t+a}^2 \right) = \sum_{t=1}^{T} a_i + 2 \sum_{i=1}^{T} \sum_{t=1}^{T} E_i \left( \varepsilon_{t+a} \varepsilon_{t+i+a} \right) = \sum_{t=1}^{T} a_i \]  

(30)

Third Conditional Centred Moment - forward returns

\[ \nu_{r,a}^3 = E_i \left( \varepsilon_{t+a}^3 \right) = \tau E_i \left( \sigma_{t+1}^3 \right) \approx \frac{\varsigma}{8} \left( 5 \hat{a}_i \frac{\gamma}{\varphi} + 3 \frac{b_i}{\sqrt{a_i}} \right) \]  

(32)

Third Conditional Centred Moment - aggregated returns

\[ M_{r,T}^3 = E_i \left( \left( \sum_{t=1}^{T} \varepsilon_{t+a} \right)^3 \right) \]

\[ = \sum_{t=1}^{T} E_i \left( \varepsilon_{t+a}^3 \right) + 3 \sum_{i=1}^{T} \sum_{t=1}^{T} E_i \left( \varepsilon_{t+a}^2 \varepsilon_{t+i+a} \right) + \sum_{i=1}^{T} \sum_{t=1}^{T} \sum_{j=1}^{T} E_i \left( \varepsilon_{t+a} \varepsilon_{t+i+a} \varepsilon_{t+j+a} \right) \]  

(31)
\[
\tau_{r,s} = \frac{\mu_{r,s}^4}{\left(\mu_{r,s}^2 \right)^{\frac{3}{2}}} = \frac{\tau}{8} \left(\frac{5a_s^3 + 3b_s}{\sqrt{a_s}}\right) = \frac{\tau}{8} \left(\frac{5 + 3\frac{b_s}{a_s}}{(a_s)^{3/2}}\right)
\]
Kurtosis - forward returns

\[ \chi_{r,s} = \frac{\mu_{r,s}^4}{(\mu_{r,s}^2)^2} = \frac{b_s}{(a_s)^2} \]  
(37)

Skewness - aggregated returns

\[ \Sigma_{r,T} = \frac{M_{r,T}^3}{(M_{r,T}^2)^{3/2}} \]  
(38)

Kurtosis - aggregated returns

\[ K_{r,T} = \frac{M_{r,T}^4}{(M_{r,T}^2)^2} \]  
(39)
Appendix 1.2 Limits for the Central Moments of GJR Future Returns

Appendix 1.2 derives the limits for the central moments of forward one-period returns and aggregated returns, in the context of a generic GJR model, as specified in Appendix 1.1.

Forward Variance limit

\[ \lim_{s \to \infty} \mu_{s,s}^2 = \lim_{s \to \infty} \left( \sigma_0^2 + \varphi^{-1} \left( \sigma_{s+1}^2 - \sigma_0^2 \right) \right) = \sigma_0^2 \text{ (since } 0 < \varphi < 1) . \]

Aggregated variance limit – expressed in daily units

\[ \lim_{T \to \infty} \frac{M_T^2}{T} = \lim_{T \to \infty} \left[ T \sigma_0^2 + \left( 1 - \varphi \right)^{-1} \left( 1 - \varphi^T \right) \left( \sigma_{s+1}^2 - \sigma_0^2 \right) \right] = \sigma_0^2 \]

Forward Skewness limit

\[ \lim_{s \to \infty} \tau_{s,s} = \lim_{s \to \infty} \left[ \frac{\tau}{8} \left( 5 + 3 \frac{b_s}{(a_s)_{\gamma}} \right) \right] = \left[ \frac{\tau}{8} \left( 5 + 3 \frac{\lim b_s}{(\lim a_s)_{\gamma}} \right) \right] \]

\[ \lim b_s = \lim_{s \to \infty} \left[ C_s + (-C_s - C_2 + b_s)_{\gamma}^{-1} + C_2 \varphi^{-1} \right] \]

\[
\begin{align*}
C_s & \quad \gamma \in (0,1) \\
\text{sgn}(-C_s - C_2 + b_s) & \quad \gamma \in (1,\infty) \\
\infty & \quad \gamma = 1
\end{align*}
\]

Since the expectation of any power of the variance cannot be negative, we have the following parameter condition: \( \text{sgn}(-C_s - C_2 + b_s) = 1 \).

For \( \gamma=1 \), \( b_s \) becomes:

---

\[ ^{18} \text{The convergence of aggregated skewness and kurtosis relies on the fact that the error terms in the 2nd order Taylor approximations for the odd moments of the variance also converge (when } \gamma \text{ is between 0 and 1). \]
\[ b_i = \sum_{i=1}^{\omega} \left( \omega^2 + 2\omega \varphi \sigma_{i-1} \right) + b_i = (s - 1) \omega^2 + 2\omega \varphi \sum_{i=1}^{\omega} a_i + b_i \]

\[ = (s - 1) \omega^2 + 2\omega \varphi \left[ (s - 1) \sigma_0^2 + (1 - \varphi)^{-1} \left( 1 - \varphi^{-1} \right) \left( \sigma_{i-1}^2 - \sigma_0^2 \right) \right] + b_i \]

\[ = (s - 1) \omega^2 + 2\omega \varphi \sigma_0^2 + 2 \omega (1 - \varphi)^{-1} \varphi^2 \left( 1 - \varphi^{-1} \right) \left( \sigma_{i-1}^2 - \sigma_0^2 \right) + b_i \]

\[ = (s - 1) \left( \omega^2 + 2\omega \varphi \sigma_0^2 \right) + 2 \omega \varphi \sigma_0^2 (1 - \varphi^{-1}) \left( \sigma_{i-1}^2 - \sigma_0^2 \right) + b_i \]

Hence, \( \lim_{s \to \infty} b_s = \lim_{s \to \infty} \left( (s - 1) \left( \omega^2 + 2\omega \varphi \sigma_0^2 \right) + 2 \omega \varphi \sigma_0^2 (1 - \varphi^{-1}) \left( \sigma_{i-1}^2 - \sigma_0^2 \right) + b_i \right) = \infty \)

Hence, the limit of the forward skewness becomes:

\[ \lim_{\gamma \to \infty} \tau_{\gamma,s} = \begin{cases} \
\frac{\tau}{8} \left( 5 + 3 \frac{\omega^2 + 2\omega \varphi \sigma_0^2}{(1 - \gamma) \sigma_0^4} \right) & \gamma \in (0,1) \\
\sgn(\tau) \sgn(-C_1 - C_2 + b_1) & s \in (1, \infty) \\
\sgn(\tau) \infty & \gamma \in [1, \infty) 
\end{cases} \]

**Aggregated Skewness limit**

\[ \lim_{T \to \infty} \sum_{r,T} = \lim_{T \to \infty} M_{r,T}^3 \left( \frac{M_{r,T}^2}{T} \right)^{\frac{3}{2}} = \lim_{T \to \infty} \frac{M_{r,T}^3}{T^{\frac{3}{2}}} = \left( \lim_{T \to \infty} \frac{M_{r,T}^3}{T^{\frac{3}{2}}} \right)^{\frac{3}{2}} = \frac{1}{\sigma_0^3} \lim_{T \to \infty} \frac{M_{r,T}^3}{T^{\frac{3}{2}}} \]

\[ M_{r,T}^3 = \sum_{i=1}^{T} E_i \left( \sigma_{i-1}^3 \right) + 3 \sum_{i=1}^{T} E_i \left( \varepsilon_{i-1} \varepsilon_{i+1}^2 \right) = \frac{\tau}{8} \sum_{i=1}^{T} \left( 5a_i^{\frac{3}{2}} + 3 \frac{b_i}{\sqrt{a_i}} \right) + 3 \sum_{i=1}^{T} E_i \left( \varepsilon_{i-1} \varepsilon_{i+1}^2 \right) \]

\[ = \frac{\tau}{8} \sum_{i=1}^{T} \left( 5a_i^{\frac{3}{2}} + 3 \frac{b_i}{\sqrt{a_i}} \right) + 3 \left( \frac{\alpha \tau + \lambda \int_{-\infty}^{0} x^3 f(x) dx}{8} \right) \left( 1 - \varphi^{-1} \right) \sum_{i=1}^{T} \left( 5a_i^{\frac{3}{2}} + 3 \frac{b_i}{\sqrt{a_i}} \right) \]

\[ = \sum_{i=1}^{T} \left( C - \varphi^{-1} \right) \left( 5a_i^{\frac{3}{2}} + 3 \frac{b_i}{\sqrt{a_i}} \right) \]
where \( C = \frac{\tau + 3 (\alpha + \lambda \int_{x=0}^{0} x^3 f(x) \, dx) (1 - \varphi)}{8} \)

Hence, the limit of the aggregated skewness is given by:

\[
\lim_{T \to \infty} \sum_{i=1}^{T} (C - \varphi^{-i}) \left( 5\alpha_{i} \frac{3}{\sqrt{a_{i}}} + \frac{b_{i}}{\sqrt{a_{i}}} \right) = \frac{1}{\sigma_{0}^2} \cdot \frac{T^{3/2}}{T^{3/2}}
\]

Consider the limit of \( \lim_{T \to \infty} \frac{\sum_{i=1}^{T} (C - \varphi^{-i}) \left( 5\alpha_{i} \frac{3}{\sqrt{a_{i}}} + \frac{b_{i}}{\sqrt{a_{i}}} \right)}{T^{3/2}} \)

\textit{Case 1:} \( C \leq 0 \) then \((C - \varphi^{-i}) \leq 0\) for any \( i \).

Notation:

\[
L = \lim_{T \to \infty} \frac{\sum_{i=1}^{T} (C - \varphi^{-i}) \left( 5\alpha_{i} \frac{3}{\sqrt{a_{i}}} + \frac{b_{i}}{\sqrt{a_{i}}} \right)}{T^{3/2}} = \lim_{T \to \infty} S
\]

\[
L_1 = \lim_{T \to \infty} \frac{\sum_{i=1}^{T} (C - \varphi^{-i}) \left( 5\left( \max_{\text{all}} (a_{i}) \right) \frac{3}{\sqrt{a_{i}}} + \frac{b_{i}}{\sqrt{a_{i}}} \right)}{T^{3/2}} = \lim_{T \to \infty} S_1
\]

\[
L_2 = \lim_{T \to \infty} \frac{\sum_{i=1}^{T} (C - \varphi^{-i}) \left( 5\left( \min_{\text{all}} (a_{i}) \right) \frac{3}{\sqrt{a_{i}}} + \frac{b_{i}}{\sqrt{a_{i}}} \right)}{T^{3/2}} = \lim_{T \to \infty} S_2
\]

Since \( S_1 \leq S \leq S_2 \), if \( L_1 = L_2 \), then \( L = L_1 = L_2 \) by the squeeze theorem.
\[
S_i = \frac{\sum_{i=1}^{T} (C - \varphi^{-i}) \left( 5 \left( \max_{\text{acst}} (a_i) \right)^{\frac{1}{2}} + 3 \left( \min_{\text{acst}} (a_i) \right)^{\frac{1}{2}} b_i \right)}{T^{\frac{1}{2}}}
\]

Hence, \(L_i\) becomes:

\[
L_i = 5 \left( L_{\text{max}} \right)^{\frac{3}{2}} \lim_{T \to \infty} \frac{1}{T} \left( \frac{1}{T^{\frac{1}{2}}} \right) \sum_{i=1}^{T} (C - \varphi^{-i}) b_i = 3 \left( L_{\text{min}} \right)^{\frac{1}{2}} \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{T} (C - \varphi^{-i}) b_i
\]

For \(\gamma \neq 1\), \(\sum_{i=1}^{T} (C - \varphi^{-i}) b_i\) is given by:

\[
\sum_{i=1}^{T} (C - \varphi^{-i}) b_i = \sum_{i=1}^{T} (C - \varphi^{-i}) \left( C_i + (-C_i - C_{i-1} + b_i) \varphi^{-i+1} + C_{2i+1} \varphi^{i+1} \right)
\]

\[
= CC_i T + C(-C_i - C_{i-1} + b_i)(1 - \gamma)^{-i} (1 - \gamma^T) + CC_i (1 - \varphi^{-i}) (1 - \varphi^T)
\]

\[
- C_i (1 - \varphi^{i})(1 - \varphi^T) - (-C_i - C_{i-1} + b_i)(\varphi - \gamma)^{-i} (\varphi^T - \gamma^T) - TC_{2i} \varphi^{i-1}
\]

\[
= CC_i T + C(-C_i - C_{i-1} + b_i)(1 - \gamma)^{-i} (1 - \gamma^T) + CC_i (1 - \varphi^{-i}) (1 - \varphi^T)
\]

\[
- (-C_i - C_{i-1} + b_i)(\varphi - \gamma)^{-i} (\varphi^T - \gamma^T) - TC_{2i} \varphi^{i-1}
\]

For \(\gamma = 1\), \(\sum_{i=1}^{T} (C - \varphi^{-i}) b_i\) becomes:

\[
\sum_{i=1}^{T} (C - \varphi^{-i}) b_i = \sum_{i=1}^{T} (C - \varphi^{-i}) \left[ (i - 1)(\omega^2 + 2\omega \varphi \sigma_0^2) + 2\omega \varphi \sigma_0^2 (1 - \varphi^{-i+1}) (\sigma_{i+1}^2 - \sigma_0^2) + b_i \right]
\]

\[
= C(\omega^2 + 2\omega \varphi \sigma_0^2) \frac{T(T-1)}{2} - \varphi(\omega^2 + 2\omega \varphi \sigma_0^2)(1 - \varphi)^{-i}(1 - \varphi^T - \varphi^{-1}(T - 1))
\]

\[
+ 2C \omega \varphi \sigma_0^2 (\sigma_{i+1}^2 - \sigma_0^2) (T - (1 - \varphi)^{-i}(1 - \varphi^T)) - 2\omega \varphi \sigma_0^2 (\sigma_{i+1}^2 - \sigma_0^2) [(1 - \varphi)^{-i}(1 - \varphi^T) - T \varphi^{i-1}]
\]

\[
+ b_i \left( T \varphi^{-1} (1 - \varphi^T) \right)
\]

Hence,

\[
L_i = \lim_{T \to \infty} \frac{\sum_{i=1}^{T} (C - \varphi^{-i}) b_i}{T^{\frac{1}{2}}} = \begin{cases} 
0 & \gamma \in (0,1) \\
\text{sgn} \left( -C(1 - \gamma)^{-i} (\varphi - \gamma)^{-i} (-C_i - C_{i-1} + b_i) \right)^{\infty} & \gamma \in (1, \infty) \\
\text{sgn} \left( C(\omega^2 + 2\omega \varphi \sigma_0^2) \right)^{\infty} & \gamma = 1
\end{cases}
\]
Since $\text{sgn}(-C_1-C_2+b_i)$ needs to be 1, see above and it can easily be noted that $\text{sgn}\left(-C(1-\gamma)^{-1}+(\varphi-\gamma)^{-1}\right) = -1$ and that $\text{sgn}\left(\omega^2+2\omega\varphi\sigma^2\right) = 1$. Hence the above limit becomes:

$$L_1 = \begin{cases} 0 & \gamma \in (0,1) \\ -\infty & \gamma \in [1,\infty) \end{cases} = \begin{cases} 0 & \gamma \in (0,1) \\ \text{sgn}(C) & \gamma \in [1,\infty) \end{cases}$$

Analogously it can be shown that $L_1 = L_2$ and hence $\lim_{T \to \infty} \sum_T = L = L_1 = L_2 = \begin{cases} 0 & \gamma \in (0,1) \\ -\infty & \gamma \in [1,\infty) \end{cases}$

Actually, it can be noticed that for $C < 0$ the aggregated skewness is always negative (and hence its limit can never be positive or plus infinity).

Case 2: $C > 0$ then $\exists \ i^*$ such that $(C - \varphi^{T_i}) > 0$, for any $i < i^*$ and $(C - \varphi^{T_i}) \leq 0$ for any $i > i^*$. It can be easily seen that if $C > \varphi$, then $i^* = 1$.

$$\lim_{T \to \infty} \frac{T^{\gamma/2}}{\text{T}} \sum_{i=1}^{i^*}(C - \varphi^{T_i})\left(5a_i^{\gamma/2} + 3\frac{b_i}{\sqrt{a_i}}\right) = \lim_{T \to \infty} S$$

$$S = \frac{T^{\gamma/2}}{\text{T}} \sum_{i=1}^{i^*}(C - \varphi^{T_i})\left(5a_i^{\gamma/2} + 3\frac{b_i}{\sqrt{a_i}}\right) + \frac{T^{\gamma/2}}{\text{T}} \sum_{i=i^*+1}^{\infty}(C - \varphi^{T_i})\left(5a_i^{\gamma/2} + 3\frac{b_i}{\sqrt{a_i}}\right)$$

$$S_i = \frac{T^{\gamma/2}}{\text{T}} \sum_{i=1}^{i^*}(C - \varphi^{T_i})\left(5\left(\max_{\text{hist}}(a_i)\right)^{\gamma/2} + 3\left(\min_{\text{hist}}(a_i)\right)^{1/2} b_i\right) + \frac{T^{\gamma/2}}{\text{T}} \sum_{i=i^*+1}^{\infty}(C - \varphi^{T_i})\left(5\left(\min_{\text{hist}}(a_i)\right)^{\gamma/2} + 3\left(\max_{\text{hist}}(a_i)\right)^{1/2} b_i\right)$$

$$S_2 = \frac{T^{\gamma/2}}{\text{T}} \sum_{i=1}^{i^*}(C - \varphi^{T_i})\left(5\left(\min_{\text{hist}}(a_i)\right)^{\gamma/2} + 3\left(\max_{\text{hist}}(a_i)\right)^{1/2} b_i\right) + \frac{T^{\gamma/2}}{\text{T}} \sum_{i=i^*+1}^{\infty}(C - \varphi^{T_i})\left(5\left(\max_{\text{hist}}(a_i)\right)^{\gamma/2} + 3\left(\min_{\text{hist}}(a_i)\right)^{1/2} b_i\right)$$
Since $S_2 \leq S \leq S_1$, if $L_1 = \lim_{T \to \infty} S_1 = L_2 = \lim_{T \to \infty} S_2$, then $L = L_1 = L_2$ by the squeeze theorem.

\[
S_{11} = 5 \left( \max_{i \in T} (a_i) \right)^{3/2} \left[ \frac{C_i^*}{T^{3/2}} - \frac{\varphi^{T-i+1}}{T^{3/2}} \sum_{i=0}^{T-1} \varphi^{i-1} \right] + 3 \left( \min_{i \in T} (a_i) \right)^{1/2} \left[ \frac{\varphi^{T-i}}{T^{3/2}} - \frac{\sum_{i=1}^{T-1} \varphi^{i-1} b_i}{T^{3/2}} \right]
\]

Treating $i$ as a constant, $S_{11}$ becomes:

\[
S_{11} = 5 \left( \max_{i \in T} (a_i) \right)^{3/2} \left[ \frac{C_i^*}{T^{3/2}} - \frac{\varphi^{T-i+1} f_i(i^*)}{T^{3/2}} \right] + 3 \left( \min_{i \in T} (a_i) \right)^{1/2} \left[ \frac{f_i(i^*)}{T^{3/2}} - \frac{\varphi^{T-i} f_i(i^*)}{T^{3/2}} \right]
\]

where $f_j = 1, 2, 3$ will be treated as constants.

\[
S_{12} = \frac{\sum_{i=1}^{T} (C - \varphi^{T-i}) \left( 5 \left( \min_{i \in T} (a_i) \right)^{3/2} + 3 \left( \max_{i \in T} (a_i) \right)^{1/2} b_i \right)}{T^{3/2}}
\]

\[
S_{12} = 5 \left( \min_{i \in T} (a_i) \right)^{3/2} \left[ \frac{T^{3/2} \left( (T - i^* + 1) C - \varphi \sum_{i=0}^{T-1} \varphi^{i-1} \right)}{T^{3/2}} \right] + 3 \left( \max_{i \in T} (a_i) \right)^{1/2} \left[ \frac{T^{3/2} \left( \sum_{i=1}^{T} b_i - \sum_{i=1}^{T} \varphi^{T-i} \right)}{T^{3/2}} \right]
\]

\[
\sum_{i=1}^{T} b_i = \sum_{i=1}^{T} \left( C_i + (-C_i - C_2 + b_i) \gamma^{i-1} + C_2 \gamma^{i-1} \right)
\]

\[
= (T - i^* + 1) C_i + (-C_i - C_2 + b_i) \gamma^{i-1} \sum_{i=0}^{T-1} \varphi^{i} + C_2 \gamma^{i-1} \sum_{i=0}^{T-1} \varphi^{i}
\]

\[
= (T - i^* + 1) C_i + (-C_i - C_2 + b_i) \gamma^{i-1} \left( 1 - \gamma^{-1} \right) \left( 1 - \gamma^{T-i*+1} \right) + C_2 \gamma^{i-1} \left( 1 - \gamma^{-1} \right) \left( 1 - \gamma^{T-i*+1} \right)
\]
\[
\sum_{i=1}^{T} b_i \varphi^{T-i} = \sum_{i=1}^{T} \varphi^{T-i} \left( C_i + (-C_1 - C_2 + b_1) \gamma^{i-1} + C_3 \phi^{i-1} \right) \\
= C_i \sum_{i=1}^{T-i} \varphi^i + (-C_1 - C_2 + b_1) \varphi^{T-i} \sum_{i=1}^{T-i} \left( \gamma \varphi^i \right)^{-1} + C_3 (T - i^* + 1) \varphi^{T-i} \\
= C_i (1 - \varphi)^{-1} \left( 1 - \varphi^{T-i^*+1} \right) + (-C_1 - C_2 + b_1) \gamma^{i-1} (\varphi - \gamma)^{-1} \left( \varphi^{T-i^*+1} - \gamma^{T-i^*+1} \right) + C_3 (T - i^* + 1) \varphi^{T-i} \\
S_i = 5 \max_{i \in \mathbb{N}} (a_i) \gamma^{1/2} \left[ \frac{C_i \gamma^{i-1} f_i(i^*)}{T^{1/2}} - \frac{\varphi^{T-i} f_i(i^*)}{T^{1/2}} \right] + 3 \min_{i \in \mathbb{N}} (a_i) \gamma^{1/2} \left[ \frac{f_i(i^*)}{T^{1/2}} - \frac{\varphi^{T-i} f_i(i^*)}{T^{1/2}} \right] + \\
\left[ C C_i (T - i^* + 1) + C(-C_1 - C_2 + b_1) \gamma^{i-1} (1 - \gamma)^{-1} \left( 1 - \varphi^{T-i^*+1} \right) \right] \\
3 \max_{i \in \mathbb{N}} (a_i) \gamma^{1/2} T^{-3/2} \left[ (T - i^* + 1) C - \gamma^{1} (1 - \varphi^{T-i^*+1}) \right] \\
= \left\{ \begin{array}{ll}
0 & \gamma \in (0, 1) \\
\gamma & \gamma = 1 \\
\end{array} \right.
\]

For \( \gamma = 1 \), \( \sum_{i=1}^{T} (C - \varphi^{T-i}) b_i \) and \( \sum_{i=1}^{T} (C - \varphi^{T-i}) b_i \) are given by:

\[
\sum_{i=1}^{T} (C - \varphi^{T-i}) b_i = \sum_{i=1}^{T} (C - \varphi^{T-i}) \left[ (i-1)(\omega^2 + 2\omega \sigma^2) + 2\omega \sigma^2 (1 - \varphi^{i-1}) \left( \sigma_{i+1}^2 - \sigma^2 \right) + b_i \right] = O(T) \\
\sum_{i=1}^{T} (C - \varphi^{T-i}) b_i = \sum_{i=1}^{T} (C - \varphi^{T-i}) \left[ (i-1)(\omega^2 + 2\omega \sigma^2) + 2\omega \sigma^2 (1 - \varphi^{i-1}) \left( \sigma_{i+1}^2 - \sigma^2 \right) + b_i \right] \\
= C(\omega^2 + 2\omega \sigma^2) \left[ \frac{T(T-1)}{2} - \frac{(i^*-1)(i^*-2)}{2} \right] + O(T)
\]

Replacing the sums above in the expression for \( S_i \), we get \( L_i = \text{sgn}(C(\omega^2 + 2\omega \sigma^2)) \frac{T}{\infty} \) for \( \gamma = 1 \) (and \( C > 0 \)) as written above.
Analogously it can be shown that

\[
L_2 = \lim_{T \to \infty} S_2 = L_1 = \begin{cases} 
0 & \gamma \in (0, 1) \\
\sgn\left(\left(-C(1 - \gamma)^{-1} + (\varphi - \gamma)^{-1}\right)\right) & \gamma \in (1, \infty) \\
\infty & \gamma = 1
\end{cases}
\]

Finally, using the squeeze theorem we get that:

\[
\lim_{T \to \infty} \Sigma_{r,T} = L = \begin{cases} 
0 & \gamma \in (0, 1) \\
\sgn\left(\left(-C(1 - \gamma)^{-1} + (\varphi - \gamma)^{-1}\right)\right) & \gamma \in (1, \infty) \\
\infty & \gamma = 1
\end{cases}
\]

**Forward Kurtosis limit**

\[
lm_{r,T} = \lim_{s \to \infty} \frac{\mu}{\sigma} = \lim_{s \to \infty} \frac{b_s}{\mu(a_s)} = \left\{ \begin{array}{ll}
\omega \left(\omega + 2\varphi \sigma_0^2\right)(1 - \gamma)^{-1} & \gamma \in (0, 1) \\
\sgn(-C_1 - C_2 + b_1) & \gamma \in (1, \infty) \\
\infty & \gamma = 1
\end{array} \right.
\]

Since kurtosis cannot be negative, we have the following parameter condition: \(\sgn(-C_1 - C_2 + b_1) = 1\).

**Aggregated Kurtosis limit**

\[
\lim_{T \to \infty} K_{r,T} = \lim_{T \to \infty} \frac{M_{r,T}}{\left(M_{r,T}^2\right)^{\frac{3}{2}}} = \lim_{T \to \infty} \frac{\left(\frac{M_{r,T}}{T}\right)^{\frac{3}{2}}}{\left(\frac{M_{r,T}^2}{T}\right)^{\frac{3}{2}}} = \lim_{T \to \infty} \frac{\left(\frac{M_{r,T}}{T}\right)^{\frac{3}{2}}}{\sigma_0^2}
\]

\[
\lim_{T \to \infty} \frac{M_{r,T}}{T^2} = \mu \lim_{T \to \infty} \frac{b_s}{T^2} + \lim_{T \to \infty} \frac{\sum_{i=1}^{T} \left(\sum_{k=1}^{T-i} \left(4E_i(t_{i+k}) + 6E_i(t_{i+k})^2\right)\right)}{T^2}
\]
\[
\lim_{T \to \infty} \frac{\sum_{i=1}^{T} b_i}{T^2} = \lim_{T \to \infty} \frac{1}{T^2} \left[ C_i T + (C_i - C_2 + b_1)(1 - \gamma)^{-1} \right] \left( 1 - \gamma^T \right) + C_2 (1 - \varphi)^{-1} \left( 1 - \varphi^T \right) \\
= \lim_{T \to \infty} \left\{ \frac{C_i}{T} + (C_i - C_2 + b_1) \frac{(1 - \gamma)^{-1}}{T^2} \right. \left( 1 - \gamma^T \right) + C_2 (1 - \varphi)^{-1} \frac{(1 - \varphi^T)}{T^2} \right. \\
\left. \text{if } \gamma \neq 1 \right. \\
\left. \lim_{T \to \infty} (C_i - C_2 + b_1) \frac{(1 - \gamma)^{-1}}{T} \right. + C_2 (1 - \varphi)^{-1} \frac{(1 - \varphi^T)}{T^2} \right. \\
\left. \text{if } \gamma = 1 \right. \\
= \left\{ \begin{array}{ll}
0 & \gamma \in (0, 1] \\
\text{sgn}(-C_i - C_2 + b_1) \infty & \gamma \in (1, \infty) 
\end{array} \right.
\]

If \( \gamma = 1 \), then \( \sum_{i=1}^{T} b_i \) becomes

\[
\sum_{i=1}^{T} b_i = \sum_{i=1}^{T} (C_i + (C_i - C_2 + b_1) \gamma^{-1} + C_2 \gamma^{-1}) = (C_i + b_1) T + C_2 (1 - \varphi^{-1}) \left( 1 - \varphi^T \right)
\]

and

\[
\lim_{T \to \infty} \frac{\sum_{i=1}^{T} b_i}{T^2} = \lim_{T \to \infty} \left( \frac{C_i + b_1}{T} \right) + C_2 (1 - \varphi^{-1}) \frac{(1 - \varphi^T)}{T^2} = 0
\]

\[
\lim_{T \to \infty} \frac{\sum_{i=1}^{T} c_{i,i,i})}{T^2} = \tau \lim_{T \to \infty} \frac{\sum_{i=1}^{T} \sum_{j=1}^{T} E_i \left( \sigma_{i,i,i} \right)}{T^2} = \tau \lim_{T \to \infty} \frac{\sum_{i=1}^{T} \sum_{j=1}^{T} c_{i,j,i})}{T^2}
\]

We consider \( \sum_{i=1}^{T} c_{i,i,i}) \) firstly:

\[
\sum_{i=1}^{T} c_{i,i,i}) = \sum_{i=1}^{T} \left[ \frac{3}{4} \left( C_3 + \omega \varphi C_4 \right) (a_{i,i}) \frac{1}{2^2} \varphi^{-1} E_i \left( \sigma_{i,i,i} \right) + \frac{3}{8} C_2 \left( a_{i,i} \right) \frac{1}{2^2} \varphi^{-1} E_i \left( \sigma_{i,i,i} \right) + \frac{3}{8} \left( C_0 - 2 \omega \varphi C_4 \right) \left( a_{i,i} \right) \frac{1}{2^2} \varphi^{-1} E_i \left( \sigma_{i,i,i} \right) \right]
\]

\[
= \frac{3}{4} \left( C_3 + \omega \varphi C_4 \right) E_i \left( \sigma_{i,i,i} \right) \sum_{i=1}^{T} \left( a_{i,i} \right) \frac{1}{2^2} \varphi^{-1} E_i \left( \sigma_{i,i,i} \right) + \frac{3}{8} C_2 E_i \left( \sigma_{i,i,i} \right) \sum_{i=1}^{T} \left( a_{i,i} \right) \frac{1}{2^2} \varphi^{-1} E_i \left( \sigma_{i,i,i} \right) + \frac{3}{8} \left( C_0 - 2 \omega \varphi C_4 \right) E_i \left( \sigma_{i,i,i} \right) \sum_{i=1}^{T} \left( a_{i,i} \right) \frac{1}{2^2} \varphi^{-1} E_i \left( \sigma_{i,i,i} \right)
\]

\[
c_{i,min} \leq \sum_{i=1}^{T} c_{i,i,i}) \leq c_{i,max}
\]

where
\[
c_i,\text{max} = \frac{3}{4}(C_3 + \omega \varphi C_4) E_i\left(\sigma_{i,r}^3\right) \left[h^+(a_{i,r}, (C_3 + \omega \varphi C_4))\right]^{\frac{1}{2}} \sum_{i=1}^{T} \gamma^{i-1}
\]
\[
+ \frac{3}{8} C_5 E_i\left(\sigma_{i,r}^5\right) \left[h^-(a_{i,r}, C_5)\right]^{\frac{1}{2}} \sum_{i=1}^{T} \gamma^{i-1}
\]
\[
+ \frac{3}{8}(C_6 - 2\omega \varphi C_4) \left[h^-(a_{i,r}, (C_6 - 2\omega \varphi C_4))\right]^{\frac{1}{2}} E_i\left(\sigma_{i,r}^3\right) \sum_{i=1}^{T} \gamma^{i-1}
\]

where
\[
h^+(x_{i,r}, y) = \begin{cases} 
\max_{1 \leq s \leq T-i} (x_{i,s}) & y \geq 0 \\
\min_{1 \leq s \leq T-i} (x_{i,s}) & y < 0
\end{cases}
\]
and
\[
h^-(x_{i,r}, y) = \begin{cases} 
\min_{1 \leq s \leq T-i} (x_{i,s}) & y \geq 0 \\
\max_{1 \leq s \leq T-i} (x_{i,s}) & y < 0
\end{cases}
\]

\[
c_i,\text{max} = \frac{3}{4}(1 - \varphi)^{-1}(C_3 + \omega \varphi C_4) E_i\left(\sigma_{i,r}^3\right) \left[h^+(a_{i,r}, (C_3 + \omega \varphi C_4))\right]^{\frac{1}{2}} \left(1 - \varphi^{T-i}\right)
\]
\[
+ \frac{3}{8}(1 - \gamma)^{-1} C_5 E_i\left(\sigma_{i,r}^5\right) \left[h^-(a_{i,r}, C_5)\right]^{\frac{1}{2}} \left(1 - \gamma^{T-i}\right)
\]
\[
+ \frac{3}{8}(1 - \gamma)^{-1}(C_6 - 2\omega \varphi C_4) \left[h^-(a_{i,r}, (C_6 - 2\omega \varphi C_4))\right]^{\frac{1}{2}} E_i\left(\sigma_{i,r}^3\right) \left(1 - \gamma^{T-i}\right)
\]

and

\[
c_i,\text{min} = \frac{3}{4}(1 - \varphi)^{-1}(C_3 + \omega \varphi C_4) E_i\left(\sigma_{i,r}^3\right) \left[h^+(a_{i,r}, -(C_3 + \omega \varphi C_4))\right]^{\frac{1}{2}} \left(1 - \varphi^{T-i}\right)
\]
\[
+ \frac{3}{8}(1 - \gamma)^{-1} C_5 E_i\left(\sigma_{i,r}^5\right) \left[h^-(a_{i,r}, -(C_5))\right]^{\frac{1}{2}} \left(1 - \gamma^{T-i}\right)
\]
\[
+ \frac{3}{8}(1 - \gamma)^{-1}(C_6 - 2\omega \varphi C_4) \left[h^-(a_{i,r}, -(C_6 - 2\omega \varphi C_4))\right]^{\frac{1}{2}} E_i\left(\sigma_{i,r}^3\right) \left(1 - \gamma^{T-i}\right)
\]

Hence,
\[
\frac{\sum_{i=1}^{T} c_{\text{min}}}{T^2} \leq \frac{\sum_{i=1}^{T} c_{i,(1+1)}}{T^2} \leq \frac{\sum_{i=1}^{T} c_{\text{max}}}{T^2}
\]

Now consider
\[
c_{\text{min}} \leq \frac{\sum_{i=1}^{T} c_{i,\text{max}}}{T} \leq c_{\text{max}},
\]
Analogously,

\[
\begin{align*}
\epsilon_{\min} &= \frac{3}{4} (1 - \varphi)^{-1} (C_3 + \omega \varphi C_4) h_i^+ \left( E_i \left( \sigma_{\varphi i}^3 \right), (C_3 + \omega \varphi C_4) \right) \\
&\quad \cdot \left\{ h_i^+ \left[ h_i^+ \left( a_{\varphi i i}, (C_3 + \omega \varphi C_4) \right) \right] \left( C_3 + \omega \varphi C_4 \right) \right\} \frac{1}{2} \left( T - \frac{1 - \varphi^T}{1 - \varphi} \right) \\
&\quad + \frac{3}{8} (1 - \gamma)^{-1} C h_i^+ \left( E_i \left( \sigma_{\varphi i}^3 \right), C_3 \right) \left\{ h_i^+ \left[ h_i^+ \left( a_{\varphi i i}, C_3 \right) \right] \left( C_3 \right) \right\} \frac{1}{2} \left( T - \frac{1 - \gamma^T}{1 - \gamma} \right) \\
&\quad + \frac{3}{8} (1 - \gamma)^{-1} (C_0 - 2 \omega \varphi C_4) h_i^+ \left( E_i \left( \sigma_{\varphi i}^3 \right), (C_0 - 2 \omega \varphi C_4) \right) \\
&\quad \cdot \left\{ h_i^+ \left[ h_i^+ \left( a_{\varphi i i}, (C_0 - 2 \omega \varphi C_4) \right) \right] \left( C_0 - 2 \omega \varphi C_4 \right) \right\} \frac{1}{2} \left( T - \frac{1 - \gamma^T}{1 - \gamma} \right)
\end{align*}
\]

\[
\lim_{i, n}(E_i \left( \sigma_{\varphi i}^3 \right)) = \lim_{i, n} \left( \frac{5 a_i^3}{8} + \frac{b_i}{\sqrt{a_i}} \right) = \frac{1}{8} \left( 5 \lim_{i, n} (a_i)^3 + 3 \lim_{i, n} b_i \right)
\]

\[
= \frac{1}{8} \left( 5 a_0^3 + 3 \frac{C_1 + (b_1 - C_1 - C_2) \lim_{i, n} \gamma^{-1} + C_2 \lim_{i, n} \varphi^{-1}}{\sigma_0} \right)
\]

\[
= \frac{1}{8} \left( 5 a_0^3 + 3 \frac{C_1}{\sigma_0} \right) = \frac{1}{8} \left( 5 a_0^3 + 3 \left[ \sigma_0 \left( 1 - \gamma \right) \right]^{-1} \left( \omega^2 + 2 \omega \varphi a_0^2 \right) \right)
\]

\( \gamma \in (0,1) \)
Hence, \( \lim_{T \to \infty} \left[ \min_{i=1} T \left( E_i \left( \sigma_{i}^3 \right) \right) \right] \) and \( \lim_{T \to \infty} \left[ \max_{i=1} T \left( E_i \left( \sigma_{i}^3 \right) \right) \right] \) exist and are finite (and equal to \( \lim_{i \to \infty} \left( E_i \left( \sigma_{i}^3 \right) \right) \)) as long as \( \gamma < 1 \).

Analogously, \( \lim_{i \to \infty} \left[ \max_{i=1} T \left( E_i \left( \sigma_{i}^4 \right) \right) \right] \) and \( \lim_{i \to \infty} \left[ \min_{i=1} T \left( E_i \left( \sigma_{i}^4 \right) \right) \right] \) exist and are finite as long as \( \gamma < 1 \).

Also, since \( \lim_{i \to \infty} \left( a_i^2 \right) = \sigma_0^2 \), then \( \lim_{i \to \infty} f_i^2 \left( a_{i+1} \right) \) where \( j,k = 1,2 \) \( f_1 = \max \) and \( f_2 = \min \) all exist and are finite.

Hence,

\[
\lim_{T \to \infty} \frac{\sigma_{i}^2}{T^2} = L_3 \lim_{T \to \infty} \left( \frac{1}{T} - \left( 1 - \varphi \right)^{-1} \left( 1 - \varphi T \right) \right) + L_4 \lim_{T \to \infty} \left( \frac{1}{T} - \left( 1 - \gamma \right)^{-1} \left( 1 - \gamma T \right) \right) = 0 \quad \gamma \in (0,1)
\]

where

\[
L_3 = \frac{3}{4} \left( 1 - \varphi \right)^{-1} \left( C_3 + \omega \varphi C_4 \right) \lim_{T \to \infty} \left[ h_i \left( E_i \left( \sigma_{i}^3 \right), \left( C_3 + \omega \varphi C_4 \right) \right) \right] \]

\[
\lim_{T \to \infty} \left[ h_i \left( a_{i+1}, \left( C_3 + \omega \varphi C_4 \right), \left( C_3 + \omega \varphi C_4 \right) \right) \right]^{\frac{1}{2}}
\]

\[
L_4 = \frac{3}{8} \left( 1 - \gamma \right)^{-1} \left[ C_4 h_i \left( E_i \left( \sigma_{i}^4 \right), C_4 \right) \left[ h_i \left( a_{i+1}, C_3 \right) \right] \right]^{\frac{1}{2}} + \left( C_6 - 2 \omega \varphi C_4 \right)
\]

\[
\lim_{T \to \infty} \sum_{i=1}^{T} \sum_{i=1}^{T} E_i \left( \tilde{s}_i^2, \tilde{s}_{i+1}^2 \right)
\]

\[
\frac{\sigma_{i}^2}{T^2} \lim_{T \to \infty} \left[ \left( \frac{T(T-1)}{2} \right) \left( C_3 + \omega \varphi C_4 \right) - (1 - \varphi)^{-1} \left( 1 - \varphi T \right) \left( \sigma_{i}^2 - \sigma_0^2 \right) \right]
\]

\[
\left[ T \sigma_0^2 \left( 1 - \gamma \right)^{-1} \left( 1 - \gamma T \right) \left( \sigma_{i}^2 - \sigma_0^2 \right) \right]
\]

\[
\left[ \left( \tau \lambda \left( \varphi + \beta \right) \right) \left( 1 - \varphi \right)^{-1} \left[ T C_i + \left( b_i - C_i - C_i \right) (1 - \varphi)^{-1} \left( 1 - \varphi^2 \right) + C_2 (1 - \varphi)^{-1} \left( 1 - \varphi^2 \right) \right]
\]

\[
\left[ -C_1 (1 - \varphi)^{-1} \left( 1 - \varphi^2 \right) - \left( b_i - C_i + b_i \right) (1 - \varphi)^{-1} \left( 1 - \varphi^2 \right) \right]
\]

Finally, the aggregated kurtosis limit is:
\[
\lim_{T \to \infty} K_{*T} = \frac{1}{\sigma_0^2} \left[ \sum_{i=1}^{T} \frac{b_i}{T^2} + 4 \lim_{T \to \infty} \frac{\sum_{i=1}^{T-1} \left( E_i \left( \varepsilon_{i,i+1}^3 \right) \right)}{T^2} + 6 \lim_{T \to \infty} \frac{\sum_{i=1}^{T-1} \left( E_i \left( \varepsilon_{i,i+1}^2 \varepsilon_{i,i+2}^2 \right) \right)}{T^2} \right]
\]
\[
= \frac{1}{\sigma_0^2} \left[ x \times 0 + 4 \times 0 + 6 \times \frac{\sigma_0^4}{2} \right] = 3 \quad \text{for} \quad \gamma \in (0,1)
\]
Appendix 2.1 Conditional Moments for the generic GJR Future Aggregated Variances

Appendix 2.1 shows the derivations of the conditional moments for forward one-period variances and future aggregated variances in the context of a generic GJR model, as specified in Appendix 1.1.

Recall that the $s$-step ahead conditional variance forecasts in the context of a GJR model are given by:

$$\sigma_{t+s}^2 = \omega + (\alpha + \lambda I_{t+s-1})\varepsilon_{t+s-1}^2 + \beta \sigma_{t+s-1}^2$$  \hspace{1cm} (1)

I. Un-centred Conditional Moments of Variances

First moment – forward variance

Applying the conditional expectation operator in (1) above and using previously defined notation, we get:

$$a_s = \sigma_0^2 + \varphi^{s-1}\left(\sigma^2_{t+1} - \sigma_0^2\right)$$  \hspace{1cm} (2)

First moment – aggregated variance

$$A_T = E\left(\sum_{t=1}^{T} \sigma_{t+s}^2\right) = \sum_{t=1}^{T} a_s = T\sigma_0^2 + \frac{1 - \varphi^T}{1 - \varphi}\left(\sigma^2_{t+1} - \sigma_0^2\right)$$  \hspace{1cm} (3)

Or, in recursive form:

$$A_T = A_{T-1} + \sigma_0^2 + \varphi^{T-1}\left(\sigma^2_{t+1} - \sigma_0^2\right)$$

Second moment – forward variance

$$b_s = E\left[\left(\sigma^2_{t+s}\right)^2\right] = E\left((\omega + (\alpha + \lambda I_{t+s-1})\varepsilon_{t+s-1}^2 + \beta \sigma_{t+s-1}^2)^2\right)$$

This was solved for in Appendix 1.1 and the final expression was found to be:

$$b_s = C_1 + (-C_1 - C_2 + b_1)\gamma^{s-1} + C_2\varphi^{s-1}$$  \hspace{1cm} (4)

where

$\varphi$ This is the same expression as per Appendix 1.1 equation (10); it is included again here for completeness.
\[ \gamma = \varphi^2 + (\lambda - 1)(\varphi + \lambda F(0))^2 + \kappa \lambda^2 F(0)(1 - F(0)) > 0 \]

\[ C_i = (\omega^2 + 2\omega \varphi \sigma_0^2)(1 - \gamma)^i \]

\[ C_2 = 2\omega \varphi (\sigma_{i+1}^2 - \sigma_0^2)(\varphi - \gamma)^{-1} \]

\[ b_1 = \sigma_{i+1}^2 \]

**Second moment – aggregated variance**

\[ B_T = E_i \left( \left( \sum_{i=1}^{T} \sigma_{i+1}^2 \right)^2 \right) = \sum_{i=1}^{T} b_i + 2 \sum_{i=1}^{T} b_{i+1} \]

where

\[ b_{i+1} = E_i \left( \sigma_{i+1}^2 \sigma_{i+1}^2 \right) = E_i \left( \sigma_{i+1}^2 \left( \omega + (\lambda \varphi \sigma_0^2) \varphi_{i+1}^2 + \beta \varphi_{i+1}^2 \right) \right) = \sigma_{i+1}^2 + \varphi^2 (b_i - \sigma_0^2 a_i) \]

Hence,

\[ B_T = \sum_{i=1}^{T} \left( b_i + 2\sigma_0^2 (T - i) a_i \right) + 2 \sum_{i=1}^{T} \left( \varphi^2 (b_i - \sigma_0^2 a_i) \right) \]

\[ \sum_{i=1}^{T} b_i = \sum_{i=1}^{T} \left( C_i + (-C_i - C_2 + b_i) \varphi_{i+1}^2 + C_2 \varphi_{i+1}^2 \right) = TC_i + (-C_i - C_2 + b_i)(1 - \gamma)^{-1}(1 - \gamma^T) + C_2 (1 - \varphi^T) \]

\[ \sum_{i=1}^{T} (T - i) a_i = \sum_{i=1}^{T} \left( \sigma_0^2 + \varphi_{i+1}^2 (\sigma_{i+1}^2 - \sigma_0^2) \right) = \frac{T(T-1)}{2} \sigma_0^2 + (\sigma_{i+1}^2 - \sigma_0^2) \sum_{i=1}^{T} (T - i) \varphi_{i+1}^2 \]

\[ \sum_{i=1}^{T} (T - i) \varphi_{i+1}^2 = T \sum_{i=1}^{T} \varphi_{i+1}^2 - \sum_{i=1}^{T} \varphi_{i+1}^2 = T(1 - \varphi)(1 - \varphi^T) - (1 - \varphi)^{-1} \left( (1 - \varphi^{-1}) (1 - \varphi^{-1}) - T \varphi^{-T} \right) \]

\[ = (1 - \varphi)^{-1} \left( T - (1 - \varphi)^{-1}(1 - \varphi^T) \right) \]

Hence,

\[ \sum_{i=1}^{T} (T - i) a_i = \frac{T(T-1)}{2} \sigma_0^2 + (1 - \varphi)^{-1} \left( T - (1 - \varphi)^{-1}(1 - \varphi^T) \right) \left( \sigma_{i+1}^2 - \sigma_0^2 \right) \]

\[ \sum_{i=1}^{T} \varphi b_i = \varphi (1 - \varphi)^{-1} \sum_{i=1}^{T} b_i \left( (1 - \varphi^{-1}) = \varphi (1 - \varphi)^{-1} \sum_{i=1}^{T} \left( C_i + (-C_i - C_2 + b_i) \varphi_{i+1}^2 + C_2 \varphi_{i+1}^2 \right) (1 - \varphi^{-1}) \right) \]

\[ = \varphi (1 - \varphi)^{-1} \left[ T \left( C_i - C_2 \varphi_{i+1}^2 + \varphi_{i+1}^2 \right) + (C_2 - C_1)(1 - \varphi^{-1}) (1 - \varphi^T) \right] \]

\[ + (1 - \varphi)^{-1} \left( (1 - \varphi^{-1}) (1 - \varphi^{-1}) - (1 - \varphi)^{-1} (1 - \varphi^{-1}) \right) \]

\[ \sum_{i=1}^{T} \sum_{i=1}^{T} a_i = \varphi (1 - \varphi)^{-1} \sum_{i=1}^{T} \left( \sigma_0^2 + \varphi_{i+1}^2 (\sigma_{i+1}^2 - \sigma_0^2) \right) = \varphi (1 - \varphi)^{-1} \left( T \sigma_0^2 - (\varphi - \sigma_{i+1}^2 - \sigma_0^2) (1 - \varphi^T) \right) \]
Hence, the final expression for $B_T$ becomes:

$$B_T = TC_i + (-C_i - C_2 + b_i)(1 - \gamma^{-1}(1 - \gamma^T) + C_2(1 - \varphi^{-1}(1 - \varphi^T))$$

$$+ 2\sigma_0^2 \left(\frac{T(T-1)}{2} \sigma_0^2 + (1 - \varphi)^{-1} \left[T - (1 - \varphi)^{-1}(1 - \varphi^T)\right] \left(\sigma_{i+1}^2 - \sigma_0^2\right)\right)$$

$$+ 2\varphi (1 - \varphi)^{-1} \left[T \left(C_i - C_2 \gamma^T \right) + (C_2 - C_i) \left(1 - \varphi^{-1}(1 - \varphi^T)\right) + (-C_i - C_2 + b_i)\right] \left(1 - \gamma^{-1}(1 - \gamma^T) - (\varphi - \gamma)^{-1}(\varphi^T - \gamma^T)\right)$$

$$- 2\sigma_0^2 \varphi (1 - \varphi)^{-1} T \left(\sigma_0^2 - \varphi^{-2}(\sigma_{i+1}^2 - \sigma_0^2)\right) + (1 - \varphi)^{-1} \left(\sigma_{i+1}^2 - 2\sigma_0^2\right) (1 - \varphi^T)$$

Third moment – forward variance

$$c_s = E_i \left(\sigma_{i+s}^2\right)^3 = E_i \left(\omega + \left(x + \lambda I_{i+s-1}\right) x^2_{i+s-1} + \beta \sigma_{i+s-1}^2\right)^3$$

$$= \omega^3 + 3\omega^2 x^2_{i+s-1} + 3\omega^3 \left(x^2 + 2\alpha x F(0) + \lambda^2 F(0)\right) + \beta^2 + 2\beta \left(\alpha + \lambda F(0)\right) b_{i+s-1}$$

$$+ \left[\mu_s \left(x^3 + 3x^2 \lambda F(0) + 3x \lambda^2 F(0) + \lambda^3\right) + 3\beta \mu_x \left(x^2 + 2\alpha x F(0) + \lambda^2 F(0)\right) + 3\beta^2 \left(x + \lambda F(0)\right) + \beta^3\right] c_{i+s-1}$$

Solving the recursion in (6) above, we get:

$$c_s = \sum_{i=0}^{i_s} C_i \left(\omega^3 + 3\omega^2 x^2_{i+s-1} + 3\omega x b_{i+s-1}\right) + C_{i}^{-1} c_i$$

(7)

where

$$c_i = \sigma_{i+1}^2$$

$$C_i = \mu_s \left(x^3 + 3x^2 \lambda F(0) + 3x \lambda^2 F(0) + \lambda^3\right) + 3\beta \mu_x \left(x^2 + 2\alpha x F(0) + \lambda^2 F(0)\right) + 3\beta^2 \left(x + \lambda F(0)\right) + \beta^3$$

Third moment – aggregated variance

$$E_i \left(\sum_{i=1}^{i_s} \sigma_{i+s}^2\right)^3 = \sum_{i=1}^{i_s} c_i + 3\sum_{i=1}^{i_s} \sum_{i=1}^{i_s} E_i \left(\sigma_{i+s}^2 \sigma_{i+s}^2 \sigma_{i+s}^2\right) + E_i \left(\sigma_{i+s}^2 \left(\sigma_{i+s}^2 \sigma_{i+s}^2\right)\right) + 6 \sum_{i=1}^{i_s} \sum_{i=1}^{i_s} \sum_{i=1}^{i_s} E_i \left(\sigma_{i+s}^2 \sigma_{i+s}^2 \sigma_{i+s}^2 \right)$$

Notation:

$$C_T = E_i \left(\sum_{i=1}^{i_s} \sigma_{i+s}^2\right)^3$$

$$c_{i+s} = E_i \left(\sigma_{i+s}^2 \sigma_{i+s}^2 \right)$$
\[ c_{i,t+1} = E_t \left( \sigma_{i+1}^2 \left( \sigma_{i+2}^2 \right)^2 \right) \]

\[ c_{i,i+1} = E_t \left( \sigma_{i+1}^2 \sigma_{i+2} \sigma_{i+3} \sigma_{i+4} \right) \]

\[ C_T = \sum_{i=1}^{T} c_i + 3 \sum_{i=1}^{T-i} \left( c_{i,i+1} + c_{i,i+2} \right) + 6 \sum_{i=1}^{T-i} \sum_{j=1}^{i-1} \sum_{j=1}^{i+1} c_{i,j+3}\]

\[ c_{i,j+3} = E_t \left( \left( \sigma_{i+1}^2 \right)^2 \left( \omega + \lambda I_{i+1} + \beta \sigma_{i+1} \right) \right) = \omega b_i + \varphi c_{i,j+1} \]

Solving the recursion above, we get:

\[ c_{i,j+3} = \omega \left( 1 - \varphi \right)^{-1} \left( 1 - \varphi^{-1} \right) b_i + \varphi^{-1} c_{i,j+1} = \omega \left( 1 - \varphi \right)^{-1} \left( 1 - \varphi^{-1} \right) b_i + \varphi c_i = \sigma_{i+1}^2 b_i + \varphi \left( c_i - \sigma_{i+1}^2 b_i \right) \]

\[ c_{i,j+3} = E_t \left( \left( \sigma_{i+1}^2 \right)^2 \left( \omega + \lambda I_{i+1} + \beta \sigma_{i+1} \right) \right) = \omega a_i + \lambda \left( \alpha^2 + 2 \lambda F \left( 0 \right) \right) c_{i,j+1} + \beta^2 a_{i+1} + 2 \omega \left( \alpha + \lambda F \left( 0 \right) \right) b_{i,j+1} + 2 \beta b_{i,j+1} + 2 \beta^2 \left( \alpha + \lambda F \left( 0 \right) \right) c_{i,j+1} \]

Solving the recursion above, we get:

\[ c_{i,j+3} = \sum_{j=0}^{\infty} \gamma^j \left( \omega^2 a_i + 2 \omega \beta b_{i,j+1} \right) + \gamma^j c_{i,j+1} = \sum_{j=0}^{\infty} \gamma^j \left( \omega^2 a_i + 2 \omega \beta b_{i,j+1} \right) + \gamma^j c_i \]

\[ c_{i,i+1} = E_t \left( \sigma_{i+1}^2 E_{i+1} \left( \sigma_{i+2}^2 \sigma_{i+3}^2 \right) \right) = E_t \left( \sigma_{i+1}^2 E_{i+1} \left( \sigma_{i+2}^2 \sigma_{i+3}^2 \right) \right) \text{ where } t_i = t + i \]

We showed before that \( E_t \left( \sigma_{i+1}^2 \right) = \sigma_{i+1}^2 a_i + \varphi \left( b_i - \sigma_{i+1}^2 a_i \right) \), hence: \( E_t \left( \sigma_{i+2}^2 \sigma_{i+3}^2 \right) = \sigma_{i+2}^2 a_i + \varphi \left( b_i - \sigma_{i+2}^2 a_i \right) \)

where \( a_i = E_t \left( \sigma_{i+1}^2 \right) \) and \( b_i = E_t \left( \sigma_{i+1}^2 \right) \)
Since

\[ \sigma_0^2 a_i + \varphi^i (b_i - \sigma_0^2 a_i) = \sigma_0^4 + \sigma_0^2 (\sigma_{i+1}^2 - \sigma_0^2) \varphi^{i-1} + \varphi^i \left( C_i + (-C_i - C_2 + b_i) \gamma^{i-1} + C_2 \varphi^{i-1} - \sigma_0^4 (\sigma_{i+1}^2 - \sigma_0^2) \varphi^{i-1} \right) \]

\[ = \sigma_0^4 + \sigma_0^2 (\sigma_{i+1}^2 - \sigma_0^2) \varphi^{i-1} + \varphi^i \left( C_i + \left( -C_i - 2 \omega \varphi (\sigma_{i+1} - \sigma_0) (\gamma^{i-1} + \sigma_{i+1}^4) \right) \gamma^{i-1} + 2 \omega \varphi^i (\sigma_{i+1}^2 - \sigma_0^2) (\varphi - \gamma^{i-1}) - \sigma_0^4 (\sigma_{i+1}^2 - \sigma_0^2) \varphi^{i-1} \right) \]

\[ = \left( \varphi^{i-1} \gamma^i \right) \sigma_{i+1}^4 + \left( \sigma_0^4 (1 - \varphi^{i-1}) + 2 \omega \varphi^{i+1} (\varphi - \gamma^{i-1}) \right) \sigma_{i+1}^2 + \sigma_0^4 (1 - \varphi^{i-1}) \]

\[ + \varphi^i \left( C_i + \left( -C_i + 2 \omega \varphi \sigma_0^2 (\varphi - \gamma^{i-1}) \right) \gamma^{i-1} - 2 \omega \varphi \sigma_0^2 (\varphi - \gamma^{i-1}) - \sigma_0^4 (1 - \varphi^{i-1}) \right) \]

then:

\[ E_i \left( \sigma_{i+1}^2, \sigma_{i+1}^4 \right) = \left( \varphi^{i-1} \gamma^i \right) \sigma_{i+1}^4 + \left( \sigma_0^4 (1 - \varphi^{i-1}) + 2 \omega \varphi^{i+1} (\varphi - \gamma^{i-1}) \right) \sigma_{i+1}^2 + \sigma_0^4 (1 - \varphi^{i-1}) \]

\[ + \varphi^i \left( C_i + \left( -C_i + 2 \omega \varphi \sigma_0^2 (\varphi - \gamma^{i-1}) \right) \gamma^{i-1} - 2 \omega \varphi \sigma_0^2 (\varphi - \gamma^{i-1}) - \sigma_0^4 (1 - \varphi^{i-1}) \right) \]

Hence, the expression for \( c_{i,i+1,i+1+1} \) becomes:

\[ c_{i,i+1,i+1+1} = \left( \varphi^{i-1} \gamma^i \right) E_i \left( \sigma_{i+1}^2, \sigma_{i+1}^4 \right) + \left( \sigma_0^4 (1 - \varphi^{i-1}) + 2 \omega \varphi^{i+1} (\varphi - \gamma^{i-1}) \right) \sigma_{i+1}^2 + \sigma_0^4 (1 - \varphi^{i-1}) \]

\[ + \varphi^i \left( C_i + \left( -C_i + 2 \omega \varphi \sigma_0^2 (\varphi - \gamma^{i-1}) \right) \gamma^{i-1} - 2 \omega \varphi \sigma_0^2 (\varphi - \gamma^{i-1}) - \sigma_0^4 (1 - \varphi^{i-1}) \right) \]

Hence, the final expression for \( c_{i,i+1,i+1+1} \) is:

\[ c_{i,i+1,i+1+1} = \left( \varphi^{i-1} \gamma^i \right) \left( \omega^2 a_i + 2 \omega \varphi b_i + \gamma c_i \right) + \left( \sigma_0^4 (1 - \varphi^{i-1}) + 2 \omega \varphi^{i+1} (\varphi - \gamma^{i-1}) \right) \left( \omega a_i + \varphi b_i \right) \]

\[ + \left[ \sigma_0^4 (1 - \varphi^{i-1}) + \varphi^i \left( C_i + \left( -C_i + 2 \omega \varphi \sigma_0^2 (\varphi - \gamma^{i-1}) \right) \gamma^{i-1} - 2 \omega \varphi \sigma_0^2 (\varphi - \gamma^{i-1}) - \sigma_0^4 (1 - \varphi^{i-1}) \right) \right] a_i \]
Fourth moment – forward variance

\[ d_i = E_i \left( \left( \sigma_{t+i}^2 \right)^4 \right) = E_i \left[ \left( \omega + (\alpha + \lambda F_{t+i-1}) \epsilon_{t+i-1}^2 + \beta \sigma_{t+i-1}^2 \right)^4 \right] = \omega^4 + \mu_4 \left( \alpha^4 + F(0)\left( \lambda^4 + 4(\alpha^2 \lambda + \alpha \lambda^3) + 6\alpha^2 \lambda^2 \right) \right) d_{i-1} + \beta^4 d_{i-1} \]

\[ + 4 \left[ \omega^3 \left( \alpha + \lambda F(0) \right) a_{t+i-1} + \omega^3 \mu_3 \left( \alpha^3 + F(0)\left( \lambda^3 + 3(\alpha^2 \lambda + \alpha \lambda^2) \right) \right) \right] c_{t+i-1} \]

\[ + 4 \left[ \mu_4 \beta \left( \alpha^4 + F(0)\left( \lambda^4 + 3(\alpha^2 \lambda + \alpha \lambda^2) \right) \right) d_{t+i-1} + \omega \beta^3 c_{t+i-1} + \beta^4 (\alpha + \lambda F(0)) d_{i-1} \right] \]

\[ + 6 \left[ \omega^2 \alpha (\alpha^2 + \lambda^2 F(0) + 2\alpha \lambda F(0)) b_{t+i-1} + \omega \beta^2 \sigma_{t+i-1} + 4 \left( \alpha^2 + \lambda^2 F(0) + 2\alpha \lambda F(0) \right) c_{t+i-1} + \omega \beta^2 (\alpha + \lambda F(0)) d_{i-1} \right] \]

\[ + 12 \left[ \omega^2 \beta (\alpha + \lambda F(0)) d_{t+i-1} + \omega \beta (\alpha^2 + \lambda^2 F(0) + 2\alpha \lambda F(0)) c_{t+i-1} + \omega \beta^3 (\alpha + \lambda F(0)) \right] \]

\[ = C_{i+0} + C_{i+1} a_{t+i-1} + C_{i+2} b_{t+i-1} + C_{i+3} c_{t+i-1} + C_{i+4} d_{i-1} \]

Solving the recursion, we get:

\[ d_{i+1} = \sum_{j=0}^{i} C_j \left( \omega^4 + \omega^3 \alpha a_{t+i-j-1} + \omega^2 \beta \sigma_{t+i-j-1} + \omega \beta^2 \sigma_{t+i-j-1} + \beta^3 \right) \]

where

\[ C_0 = 6\omega^2 \left( \alpha^2 + \lambda^2 F(0) + 2\alpha \lambda F(0) \right) + \beta^2 \]

\[ C_1 = \omega \left[ \mu_4 \left( \alpha^4 + F(0)\left( \lambda^4 + 3(\alpha^2 \lambda + \alpha \lambda^2) \right) \right) \right] + \omega \beta^2 \left( \alpha^2 + \lambda^2 F(0) + 2\alpha \lambda F(0) \right) + \beta^2 (\alpha + \lambda F(0)) \]

\[ C_2 = \mu_4 \left( \alpha^4 + F(0)\left( \lambda^4 + 4(\alpha^2 \lambda + \alpha \lambda^3) + 6\alpha^2 \lambda^2 \right) \right) + \beta^4 + \omega \beta^2 \left( \alpha^2 + \lambda^2 F(0) + 2\alpha \lambda F(0) \right) + \beta^2 (\alpha + \lambda F(0)) \]

\[ + 6\omega \beta^2 \left( \alpha^2 + \lambda^2 F(0) + 2\alpha \lambda F(0) \right) + 12 \omega \beta^2 (\alpha + \lambda F(0)) \]

Fourth moment – aggregated variance

\[ E_i \left( \left( \sum_{t+i}^{T} \sigma_{t+i}^2 \right)^4 \right) = \sum_{t+i}^{T} d_{t+i} + \sum_{t+i}^{T} \sum_{j=0}^{i-1} \left[ 4 \left( E_i \left( \sigma_{t+i}^2 \right) \sigma_{t+i+j}^2 \left( \sigma_{t+i+j}^2 \right) \right) + 6E_i \left( \left( \sigma_{t+i}^2 \right)^2 \left( \sigma_{t+i+j}^2 \right)^2 \right) \right] \]

\[ + 12 \sum_{t+i}^{T} \sum_{j=0}^{i-1} \sum_{k=0}^{j} \left[ E_i \left( \sigma_{t+i}^2 \sigma_{t+i+j+k}^2 \right) \right] + \sum_{t+i}^{T} \sum_{j=0}^{i-1} \sum_{k=0}^{j} \left[ E_i \left( \sigma_{t+i}^2 \sigma_{t+i+j+k}^2 \right) \right] \]

\[ + 24 \sum_{t+i}^{T} \sum_{j=0}^{i-1} \sum_{k=0}^{j} \sum_{l=0}^{k} \left[ E_i \left( \sigma_{t+i}^2 \sigma_{t+i+j+k+l}^2 \right) \right] \]

Notation:

\[ D_i = E_i \left( \left( \sum_{t+i}^{T} \sigma_{t+i}^2 \right)^4 \right) \]

\[ d_{i,j+k} = E_i \left( \sigma_{t+i}^2 \sigma_{t+i+j+k}^2 \right) \]

\[ d_{i,j+k} = E_i \left( \sigma_{t+i}^2 \left( \sigma_{t+i+j+k}^2 \right)^2 \right) \]

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\[
\begin{aligned}
d_{r,i\rightarrow s,j} &= E_i \left( \left( \sigma_{r,i}^2 \right) \left( \sigma_{s,j}^2 \right) \right) \\
d_{r,i\rightarrow s,j\rightarrow r} &= E_i \left( \left( \sigma_{r,i}^2 \right) \sigma_{s,j}^2 \sigma_{r,i}^2 \right) \\
d_{l,(i\rightarrow s,j)} &= E_i \left( \sigma_{r,i}^2 \left( \sigma_{s,j}^2 \right) \sigma_{r,i}^2 \right) \\
d_{l,(i\rightarrow s,j\rightarrow k)} &= E_i \left( \sigma_{r,i}^2 \sigma_{s,j}^2 \sigma_{r,i}^2 \right)
\end{aligned}
\]

\[
D_r = \sum_{i=1}^{T} d_i + \sum_{i=1}^{T-j} \left( 4 \left( d_{r,i\rightarrow s} + d_{l,(i\rightarrow s)} \right) + 6 d_{r,i\rightarrow s} \right) + 12 \sum_{i=1}^{T-j} \sum_{j=1}^{T-j-i} (d_{r, i\rightarrow s, j\rightarrow r} + d_{l,(i\rightarrow s, j\rightarrow r)} + d_{l,i\rightarrow s,j\rightarrow k}) + 24 \sum_{i=1}^{T-1} \sum_{j=1}^{T-j-1} \sum_{k=1}^{T-k-i} d_{l,i\rightarrow s,j\rightarrow k}
\]

\[
\begin{aligned}
d_{r,i\rightarrow s} &= E_i \left( \left( \sigma_{r,i}^2 \right)^3 \left( \omega + (\lambda + \lambda \Gamma_{z_{s-1}}) \epsilon_{z_{s-1}}^2 + \beta \sigma_{z_{s-1}}^2 \right) \right) = \omega c_i + \varphi d_{r,i\rightarrow s} = \omega (1 - \varphi) \left( 1 - \varphi' \right) c_i + \varphi' d_i \\
&= \sigma_i^2 c_i + \varphi' \left[ d_i - \sigma_i^2 c_i \right]
\end{aligned}
\]

\[
\begin{aligned}
d_{l,(i\rightarrow s)} &= E_i \left( \sigma_{r,i}^2 \right) \left( \omega + (\lambda + \lambda \Gamma_{z_{s-1}}) \epsilon_{z_{s-1}}^2 + \beta \sigma_{z_{s-1}}^2 \right) + 3 \omega \left( \left( \alpha^2 + 2 \alpha \lambda \Gamma_{z_{s-1}} + \lambda^2 \Gamma_{z_{s-1}} \right) \epsilon_{z_{s-1}}^4 + 2 \left( \alpha + \lambda \Gamma_{z_{s-1}} \right) \beta \epsilon_{z_{s-1}}^2 \sigma_{z_{s-1}}^2 + \beta^2 \sigma_{z_{s-1}}^4 \right) \\
&+ \left( \alpha^3 + 3 \alpha^2 \lambda \Gamma_{z_{s-1}} + 3 \alpha \lambda^2 \Gamma_{z_{s-1}} + \lambda^3 \Gamma_{z_{s-1}} \right) \epsilon_{z_{s-1}}^6 + 3 \left( \alpha^2 + 2 \alpha \lambda \Gamma_{z_{s-1}} + \lambda^2 \Gamma_{z_{s-1}} \right) \beta \epsilon_{z_{s-1}}^4 \sigma_{z_{s-1}}^2 \\
&+ 3 \left( \alpha + \lambda \Gamma_{z_{s-1}} \right) \beta^2 \epsilon_{z_{s-1}} \sigma_{z_{s-1}}^4 + \beta^3 \sigma_{z_{s-1}}^6 \right) \right)
\end{aligned}
\]

\[
\begin{aligned}
&= \omega a_i + \omega b_{i\rightarrow s-1} + 3 \omega \left( \left( \alpha^2 + 2 \alpha \lambda F(0) + \lambda^2 F(0) \right) \beta + 3 \left( \alpha + \lambda F(0) \right) \beta^2 + \beta^3 \right) c_{l,i\rightarrow s-1} \\
&+ \left( \mu \left( \alpha^3 + F(0) \left( 3 \alpha^2 \lambda + 3 \alpha \lambda^2 + \lambda^3 \right) \right) + 3 \alpha \left( \alpha^2 + 2 \alpha \lambda F(0) + \lambda^2 F(0) \right) \beta + 3 \left( \alpha + \lambda F(0) \right) \beta^2 + \beta^3 \right) d_{l,i\rightarrow s-1} \\
&= \sum_{j=1}^{T} C_i \left( \omega a_i + 3 \omega b_{i\rightarrow j-1} + 3 \omega c_{l,i\rightarrow j-1} \right) + C_i d_i.
\end{aligned}
\]

where
\[
C_r = \mu \left( \alpha^3 + F(0) \left( 3 \alpha^2 \lambda + 3 \alpha \lambda^2 + \lambda^3 \right) \right) + 3 \alpha \left( \alpha^2 + 2 \alpha \lambda F(0) + \lambda^2 F(0) \right) \beta + 3 \left( \alpha + \lambda F(0) \right) \beta^2 + \beta^3.
\]

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\[d_{i,j} = E_i \left( \sigma_{i+1}^2 \left( \omega^2 + \left( \alpha^2 + 2\lambda I_{i+1}^{-1} + \lambda^2 I_{i+1}^{-1} \right) \varepsilon_{i+1}^2 \right) \right) \]

\[= \omega^2 b_i + 2\omega \phi \varepsilon_{i} d_{i,j} + \left( x^2 + 2\lambda \phi F(0) + \lambda^2 \phi F(0) \right) + 2\beta (\alpha + \lambda \phi F(0)) + \beta^2 \sigma_{i+1}^2 \]

\[= \sum_{j=0}^{\infty} \gamma^j \left( \omega^2 b_i + 2\omega \phi \varepsilon_{i} d_{i,j-1} \right) + \gamma^j d_i \]

\[d_{i,j} = E_i \left( \sigma_{i+1}^2 \left( \sigma_{i+1}^2 \varepsilon_{i+1}^2 \right) \right) = E_i \left( \sigma_{i+1}^2 \left( \sigma_{i+1}^2 \varepsilon_{i+1}^2 \right) \right) \]

By analogy with \( c_{i,i+1,j} = E_i \left( \sigma_{i+1}^2 \left( \sigma_{i+1}^2 \varepsilon_{i+1}^2 \right) \right) \), we get that the expression for \( d_{i,j} \) is:

\[d_{i,j} = \left( \sigma_{i+1}^2 \left( \sigma_{i+1}^2 \varepsilon_{i+1}^2 \right) \right) + \left( \sigma_{i+1}^2 \left( \sigma_{i+1}^2 \varepsilon_{i+1}^2 \right) \right) + \sum_{j=0}^{\infty} \gamma^j \left( \omega^2 b_i + 2\omega \phi \varepsilon_{i} d_{i,j-1} \right) + \gamma^j d_i \]

\[d_{i,j} = E_i \left( \sigma_{i+1}^2 \left( \sigma_{i+1}^2 \varepsilon_{i+1}^2 \right) \right) \]

We showed before that:

\[c_{i,j} = E_i \left( \sigma_{i+1}^2 \right) = \sigma_{i+1}^2 b_i + \phi \left( c_i - \sigma_{i}^2 b_i \right) \]

Hence

\[c_{i,j} = E_i \left( \sigma_{i+1}^2 \right) = \sigma_{i+1}^2 b_i + \phi \left( c_i - \sigma_{i}^2 b_i \right) \]

Using the tower law, we can solve for \( d_{i,j} \) and \( c_{i,j} \) also.

II. Centred Moments

Second Centred Moment (Variance) – forward variance

\[\mu_{\sigma_{i+1}^2} = E_i \left( \sigma_{i+1}^2 - a_i \right) = E_i \left( \sigma_{i+1}^2 + a_i^2 - 2a_i \sigma_{i+1}^2 \right) = b_i - a_i^2 \]
Second Centred Moment (Variance) – aggregated variance

\[ M^2_{\sigma^2,T} = E_i \left( \sum_{t=1}^{T} (\sigma^2_{t,i,s} - a_i) \right)^2 = \sum_{i=1}^{T} \mu^2_{\sigma^2,a} + 2 \sum_{j=1}^{T-1} \sum_{i=1}^{T-j} E_i \left( (\sigma^2_{t,i,s} - a_i)(\sigma^2_{t+j,i,s} - a_i) \right) = \sum_{i=1}^{T} (b^2_i - a^4_i) + 2 \sum_{j=1}^{T-j} (b_{j,i,s} - a_i a_{j,i,s}) \]

Or

\[ M^2_{\sigma^2,T} = B_T - \sum_{i=1}^{T} a_i a_j \]

Third Centred Moment – forward variance

\[ \mu^3_{\sigma^3,a} = E_i \left( (\sigma^3_{t,i,s} - a_i)^3 \right) = E_i \left( \sigma^6_{t,i,s} - 3a_i \sigma^4_{t,i,s} + 3a_i^2 \sigma^2_{t,i,s} - a^3_i \right) = c_i - 3b_j a_i + 2a^3_i \]

Third Centred Moment – aggregated variance

\[ M^3_{\sigma^3,T} = E_i \left( \left( \sum_{t=1}^{T} (\sigma^3_{t,i,s} - a_i) \right)^3 \right) = \sum_{i=1}^{T} \mu^3_{\sigma^3,a} + 3 \sum_{j=1}^{T-1} \sum_{i=1}^{T-j} E_i \left( (\sigma^3_{t,i,s} - a_i)(\sigma^3_{t+j,i,s} - a_i)(\sigma^3_{t+j+i,s} - a_i) \right) + 6 \sum_{j=1}^{T-j} \sum_{i=1}^{T-j} \sum_{j} E_i \left( (\sigma^3_{t+i,j,s} - a_i)(\sigma^3_{t+j+i,s} - a_i)(\sigma^3_{t+j+i,s} - a_i) \right) \]

\[ M^3_{\sigma^3,T} = \sum_{i=1}^{T} \mu^3_{\sigma^3,a} + 3 \sum_{j=1}^{T-1} \sum_{i=1}^{T-j} \left( c_{j,i,s} + c_{t+j,i,s} + 2(a_i + a_{i+s})(a_i a_{i+s} - b_{i,s} - b_{i,s} - a_i a_{j+i,s}) - a_i a_{j+i,s} b_{j,s} \right) + 6 \sum_{j=1}^{T-j} \sum_{i=1}^{T-j} \sum_{j} \left( c_{i,j,i+s+j} - a_i b_{i+j,i+s+j} - a_i b_{i+j,i+s+j} - a_i b_{i+j,i+s+j} + 2a_i a_{i+s} a_{i+s+j} \right) \]

Fourth Centred Moment – forward variance

\[ \mu^4_{\sigma^4,a} = E_i \left( (\sigma^4_{t,i,s} - a_i)^4 \right) \]

\[ = E_i \left( \sigma^8_{t,i,s} - 4a_i \sigma^6_{t,i,s} + 6a_i^2 \sigma^4_{t,i,s} - 4a_i^3 \sigma^2_{t,i,s} + a^4_i \right) \]

\[ \mu^4_{\sigma^4,a} = d_i - 4a_i c_i + 6a_i^2 b_i - 3a^4_i \]

Fourth Centred Moment – aggregated variance
III. Standardized moments

Skewness - forward variance

\[ \tau_{\alpha, \beta} = \frac{\mu_{\alpha, \beta}^2}{\left( \sigma_{\alpha, \beta}^2 \right)^2} = \frac{c_\alpha - 3b_\alpha a_\beta + 2a_\beta^3}{\left( b_\beta - a_\beta^2 \right)^2} \]
Skewness - aggregated variance

\[ \Sigma_{\sigma^2,T} = \frac{M_{\sigma^2,T}^3}{\left(M_{\sigma^2,T}^2\right)^{\frac{3}{2}}} \]

Kurtosis – forward variance

\[ \kappa_{\sigma^2,t} = \frac{\mu_{\sigma^2,t}^4}{\left(\mu_{\sigma^2,t}^2\right)^2} = \frac{d_s - 4a_s c_s + 6a_s^2 b_s - 3a_s^4}{(b_s - a_s^2)^2} \]

Kurtosis – aggregated variance

\[ K_{\sigma^2,T} = \frac{M_{\sigma^2,T}^4}{\left(M_{\sigma^2,T}^2\right)^2} \]

Appendix 2.2 Limits for the Central Moments of GJR Future Variances

Variance of Variance limit

\[ \lim_{s \to \infty} \mu_{\sigma^2,s}^2 = \lim_{s \to \infty} \left(b_s - a_s^2\right) \]

For \( \gamma \neq 1 \), we have:

\[ \lim_{s \to \infty} \mu_{\sigma^2,s}^2 = \lim_{s \to \infty} \left[ C_i + \left(-C_i - C_s + b_i\right) \gamma^{s-1} + C_s \gamma^{s-1} \left(\sigma_0^4 + \gamma^{2(s-1)} \left(\sigma_{t+1}^2 - \sigma_0^2\right)^2 + 2\sigma_0^2 \gamma^{s-1} \left(\sigma_{t+1}^2 - \sigma_0^2\right)\right)\right] \]

\[ = \lim_{s \to \infty} \left[ \left(C_i - \sigma_0^4\right) + \left(-C_i - C_s + b_i\right) \gamma^{s-1} + \left[C_s - 2\sigma_0^2 \left(\sigma_{t+1}^2 - \sigma_0^2\right)\right] \gamma^{s-1} - \gamma^{2(s-1)} \left(\sigma_{t+1}^2 - \sigma_0^2\right)^2 \right] \]

\[ = \begin{cases} 
\left(C_i - \sigma_0^4\right) & \gamma \in (0,1) \\
\text{sgn}(-C_i - C_s + b_i) \infty & \gamma \in (1, \infty) 
\end{cases} \]

Since the variance cannot be negative, we have the following parameter condition: \( \text{sgn}(-C_i - C_s + b_i) = 1 \).

Hence,

\[ \lim_{s \to \infty} \mu_{\sigma^2,s}^2 = \begin{cases} 
\left(C_i - \sigma_0^4\right) & \gamma \in (0,1) \\
\infty & \gamma \in (1, \infty) 
\end{cases} \]

For \( \gamma = 1 \), the above limit becomes:

---

20 This section currently represents work in progress.
Hence, \( \lim_{s \to \infty} b_s = \lim_{s \to \infty} (s - 1)(\omega^2 + 2\omega \varphi \sigma_0^2) + 2\omega \varphi \sigma_0^2 (1 - \varphi^{-1})(\sigma_{s+1} - \sigma_0^2) + b_s = \infty \)

\[
\lim_{s \to \infty} \mu_s^2 = \lim_{s \to \infty} \left[ (s - 1)(\omega^2 + 2\omega \varphi \sigma_0^2) + 2\omega \varphi \sigma_0^2 (1 - \varphi^{-1})(\sigma_{s+1} - \sigma_0^2) + b_s - \left( \sigma_0^4 + \varphi_0^2 \varphi_0^{-1}(\sigma_{s+1} - \sigma_0^2)^2 + 2\sigma_0^2 \varphi_0^{-1}(\sigma_{s+1} - \sigma_0^2) \right) \right]
\]

\[
= \lim_{s \to \infty} \left[ (2\omega \varphi \sigma_0^2 (\sigma_{s+1} - \sigma_0^2) + b_s - \sigma_0^2) + (s - 1)(\omega^2 + 2\omega \varphi \sigma_0^2) - 2\sigma_0^2 (\omega \varphi - 1)(\sigma_{s+1} - \sigma_0^2) \varphi^{-1} - \varphi_0^2 (\sigma_{s+1} - \sigma_0^2)^2 \right]
\]

\[= \infty \]

Hence, the final expression for \( \lim_{s \to \infty} \mu_s^2 \) is:

\[
\lim_{s \to \infty} \mu_s^2 = \begin{cases} 
(C_i - \sigma_0^2) & \gamma \in (0, 1) \\
\infty & \gamma \in [1, \infty) 
\end{cases}
\]

**Skewness of forward variance limit**

\[
\lim_{s \to \infty} \tau_{x_s, m} = \lim_{s \to \infty} \left( c_s - 3b_s a_s + 2a_s^3 \right) \left( b_s - a_s^2 \right)^{\frac{1}{2}} = \lim_{s \to \infty} \left( c_s - 3b_s a_s + 2a_s^3 \right) \lim_{s \to \infty} \left( b_s - a_s^2 \right)^{\frac{1}{2}} = \lim_{s \to \infty} \left( c_s - 3b_s a_s + 2\left[ \lim_{s \to \infty} a_s \right]^3 \right) \left[ \lim_{s \to \infty} \left( b_s - a_s^2 \right)^{\frac{1}{2}} \right]^2
\]

if all limits exist and are finite and the denominator is non-zero.

\[
\lim_{s \to \infty} a_s = \lim_{s \to \infty} \left( \sigma_0^2 + \varphi_0^{-1}(\sigma_{s+1} - \sigma_0^2) \right) = \sigma_0^2 \quad \text{(since } 0 < \varphi < 1)\]

\[
\lim_{s \to \infty} b_s = \lim_{s \to \infty} \left[ C_i + \left( -C_i - C_2 + b_s \right) \varphi^{-1} + C_2 \varphi^{-1} \right] = C_i = \left( \omega^2 + 2\omega \varphi \sigma_0^2 \right)(1 - \gamma)^{-1} \quad \text{for } \gamma \in (0, 1)
\]

---

21 It can be easily seen that a sufficient condition for \( \gamma > 0 \) is \( \varphi > 1 \); however, most of the financial time series are leptokurtic i.e. \( k > 3 \), hence the condition \( \gamma > 0 \) is almost surely met in all likely financial applications.