Persistent Deficit, Growth and Indeterminacy:
The "Golden Rule of Public Finance" Revisited

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Abstract: In this paper, we look for long run and short run effects of fiscal deficits on economic growth. We extend the Barro (1990) endogenous growth model with productive public spending to public deficit and debt. The model shows a multiplicity of long run balanced growth paths (a high-growth and a low-growth steady states), and a possible indeterminacy of the transition path, which may be consistent with empirical literature that exhibits strong non-linear responses of economic growth to fiscal deficits. Under very general hypothesis, our model shows that the “Golden Rule of Public Finance”, which allows a government to run a deficit, only if this deficit is devoted to public investment, leads to a lower balanced growth path in the long run. In the short run, on the other hand, according to multiplicity, the effect of public deficit impulses depends on the initial level of public debt. Starting from the high-growth steady state, the rate of economic growth may initially increase; thus the welfare effect of a deficit-financed increase in productive public expenditures depends on parameters. Starting from the low-growth steady state, the effect of a public deficit impulse is subjected to expectations on public debt sustainability.

JEL classification: H62, H63, E62

Keywords: deficit, productive government spending, endogenous growth, non-linear effects of fiscal policy, golden rule of public finance

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Since the mid-seventies, OECD countries have been characterized by persistent deficits (see Table 1). This persistent attempt to run fiscal deficits over such a long period has probably affected the balanced growth path of these countries, but there is no unanimous theoretical or empirical answer to this question. The “Ricardian Equivalence” advocates claim that the financing method of government expenditures is neutral, and that only the present value of the intertemporal flow of public spending may disrupt individual behaviors. But the conditions for the “Ricardian Equivalence” to operate are severe, and exclude in particular productive public expenditures (or public investment), i.e., expenditures which may improve private productivity, such as public infrastructures. In a non-Ricardian set-up with productive expenditures, one must wonder about short run and long run effects of fiscal deficit on the rate of economic growth.

Table 1 – Public deficit (percentage of GDP) in OECD countries (1971 – 2005)

<table>
<thead>
<tr>
<th>Year</th>
<th>Deficit (%)</th>
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<tbody>
<tr>
<td>1971</td>
<td>-1.2</td>
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<tr>
<td>1972</td>
<td>-1.5</td>
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<tr>
<td>1973</td>
<td>-2.1</td>
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<tr>
<td>1974</td>
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<td>2004</td>
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Source: OECD.

Considering the long run effect of fiscal deficits, two points of view emerge. The first postulates that a higher deficit today means a higher debt interest burden tomorrow, with a crowding out effect on the balanced growth path. The second argues that a higher deficit may provide resources for productive public expenditures, which may improve the balanced growth path in the long run. In discussing the fiscal rules set in the Treaty of Maastricht and in the Stability and Growth Pact (hereafter SGP), some authors have pointed out the risk that these rules may permanently reduce public investment and the rate of economic growth, and have suggested the adoption of a “Golden Rule for Public Finance” (hereafter GRPF), namely that productive public expenditures may be excluded from the SGP deficit ceiling, so that a government is allowed to run a budget deficit as long as this deficit is used to finance increases in infrastructures\(^1\). Since achieving the deficit target by a cut in public investment may be easier than by a cut in “unproductive” public expenditures (like wages and transfer payments), but to the detriment of economic growth (Alesina & Perotti, 1997), public debt might be issued to finance productive expenditures.

As regards the short run perspective, a number of recent empirical and theoretical papers\(^2\) try to identify so-called “Neoclassical”, “Ricardian” or “Neokeynesian” effects of fiscal deficit on economic growth. These studies, initiated by Feldstein (1982), Giavazzi & Pagano (1990) and Blanchard (1990), indicate that fiscal deficits seem associated with strong non-linear effects on growth, probably in line with expectations switches and initial debt level. Fiscal impulses may have traditional “Keynesian” effects or reversed effects, depending on...
on government financial stance. For example, in high-debt contexts, a fiscal correction may reduce the probability of public sector default, thus improving confidence and increasing consumption and investment\(^3\) (Perotti, 1999).

This paper attempts to join these two streams of literature (short run and long run) about the effects of fiscal deficits on economic growth. To study the long run implications of persistent deficits, a model should respond to at least two constraints: first, it must somehow allow for growth in the long run, because considering persistent deficits implies a perpetual public debt growth, which will conduct to an explosion of the public debt to output ratio in the long-run, if steady-state output is constant\(^4\). Second, issuing debt is costly, since government must pay interest payments, thus one must wonder about the allocation of these resources. Since, as the related literature emphasizes, government consumption and wasteful expenditures are long run growth reducing, it clearly appears that one should consider productive government growth-enhancing spending (public investment) in order to hope for some positive effects of fiscal deficits on long run growth\(^5\). The simplest way to fulfill these two constraints is to consider the Barro (1990) endogenous growth model with productive public spending. In Barro (1990), neither public debt, nor public deficits are allowed, thus all public expenditures are productive and growth-enhancing\(^6\). To come close to the data and our purpose, we introduce persistent deficits, one immediate consequence being the appearance of unproductive public spending, in the form of interest payments on public debt. By so doing, the model is able to deal with the growth effect of public debt burden in the long run, offering a new perspective on the GRPF. Moreover, introducing public debt into Barro model generates multiplicity of balanced long-run growth paths, thus non-linearities of fiscal policy.

Compared with the large number of empirical studies, there are, to our knowledge, only a few papers that deal with persistent deficits in growth models. Non-linear effects of fiscal policy are studied by Sutherland (1997) in a stochastic continuous time model, adding to the previous work of Bertola & Drazen (1993), but in a no-growth context. In an interesting paper, Park & Philippopoulos (2004) address the question of indeterminacy of fiscal policy in an endogenous growth model, but their model presents no large interest in the deficit-to-growth relation. In a growth model close to our approach, Greiner & Semmler (2000), regarding the rate of economic growth, and Ghosh & Mourmouras (2004), regarding welfare, remove the balanced budget assumption of Barro (1990) and study different “debt rules”, including the GRPF. Greiner & Semmler (2000) claim in particular that the growth effect of an increase in public investment depends on the “budgetary regime” the government operates within. Nevertheless, they do not compare balanced budget regime with deficit regimes (and notably with the GRPF), and their deficit rules are rather complicated. Moreover, they do not study transitional dynamics and their model does not exhibit any kind of non-linearity.

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\(^3\) Although Giavazzi, Jappelli & Pagano (2000) do not find any link between the initial level of public debt and non-linear effects of fiscal policy, in contrast with Perotti (1999).

\(^4\) Precisely, Kamps (2004, chapter 4) studies deficits in a RBC no-growth model and shows that the lack of growth in the long run needs for future adjustments in taxes, for the public finance stance to be sustainable in the long term. However, this scenario does not allow studying ceteris paribus changes in deficits, for example, while such a policy experiment constitutes an important issue that we want to address.

\(^5\) Consider that government deficit is zero and the government decides to issue debt in order to finance spending. With persistent deficits, public debt persists in the long run and its unit cost is the real interest rate. With other things (in particular taxes) unchanged, this crowds-out capital accumulation and some positive effects must be considered in order to eventually offset this negative effect and eventually allow for positive effects of debt-financing government spending on long run growth. To resume, considering productive government spending is the best a priori hypothesis for eventually generate positive long run effects of persistent deficits.

\(^6\) In Barro (1990), productive expenditures have a flow dimension and are not strictly speaking of public investment nature. However, our model can be easily extended to stock productive expenditures (public investment), as we shall see.
In our paper, as in Barro (1990), endogenous growth is achieved by introducing public productive spending in a constant-return-to-scale production function. In contrast with Barro (1990), we allow for debt-financed public spending in the government budget constraint, thus providing a second dynamic relation, besides the Keynes-Ramsey rule. Persistent deficits are introduced via a “deficit rule”, namely a \( m \) deficit-to-output ratio (to illustrate, in Table 1, \( m = 2.5\% \)). Under such an assumption, the productive public expenditures’ path will be endogenously determined in the government budget constraint.\(^7\)

Our results are twofold. First, as regards the steady state balanced growth path, we show that a “Barro-type” balanced budget rule (\( m = 0 \)) generates two long-run steady state equilibria: the “Barro solution”, with zero public debt, and a “Solow solution”, with zero growth. Allowing deficit (\( m > 0 \)) produces two endogenous long-run positive growth paths, a high-growth path and a low-growth path. We show in particular that, similarly to Barro (1990), there is an inverted-U relation between the tax rate and the high-growth path, but the tax rate that maximizes economic growth is higher than Barro’s one, namely the elasticity of output to public expenditures in the production function. As regards the GRPF, we find what we call an “impossibility theorem” in the long run: raising ceteris paribus deficits (with all things being equal, and, in particular, the same tax rate) in order to allow higher government spending, is always long run growth-reducing, which strongly contradicts the GRPF. The reason is straightforward: if Ponzi-finance is forbidden (governments are not allowed to finance the interest payment on debt by issuing new debt – deficit), interest burden is higher than deficit resources in the long run, and government must adjust (diminish) the productive expenditures in the long run, in order to respect the no-Ponzi finance constraint. To see things differently, initially devoted to growth-enhancing public deficits generate less resources then their cost in the long-run, for no-Ponzi finance to hold with other things being equal, the government must diminish productive spending below their initial level, in order to finance the gap between deficit resources and their cost (interest burden) in the long run. Therefore, deficit rules always crowd out productive expenditures in the steady state.

Second, as regards transitional dynamics, the multiplicity of steady states may generate non-linear effects of fiscal impulses. We show that the low steady state is unstable and that the high steady state is a saddle point. Starting from the high-growth steady state, deficit rules like the GRPF may enhance economic growth in the short and medium run. Thus, it looks as if the GRPF were a short run prescription to promote economic growth today, to the detriment of tomorrow, which might somewhat confuse its proponents. As regards welfare, the GRPF is more difficult to assess, but our simulations show that it may increase or decrease households’ utility level, depending on parameter values. Starting from the low-growth steady state, indeterminacy occurs: the economic growth may positively or negatively respond to fiscal deficit impulses. Consequently, the effect of a fiscal deficit impulse on economic growth depends on the level of the initial debt to capital (or income) ratio, which may reproduce stylized facts on non-linearities.

Section one presents the model. Section two deals with steady state issues and shows the multiplicity result. In section three, we discuss the effect of fiscal deficits and taxes in the long run. Section four describes the transitional dynamics of the model, section five depicts the non-linear character of fiscal deficit impulses and section six concludes the paper.

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\(^{7}\) Futagami et al. (1993) introduce public capital as a stock and show that this gives rise to transitional dynamics, in contrast to Barro model with flow expenditures. Introducing public debt also generates transitional dynamics, so we do not need to model public capital (our model is extended to public investment in an appendix).
I. The model

We consider a closed economy with a private sector and a government. The private sector consists on a producer-consumer infinitely-lived representative agent, who maximizes the present value of a discounted sum of instantaneous utility functions based on consumption:

\[ \begin{align*}
U &= \int_0^\infty u(c_t) \exp(-\rho t) dt \\
&= \int_0^\infty (\text{expression for } u(c_t)) \exp(-\rho t) dt
\end{align*} \] (1)

We denote by \( c_t > 0 \) the consumption and \( \rho > 0 \) is the subjective discount rate. To obtain an endogenous growth path, we assume an isoelastic instantaneous utility function:

\[ u(c_t) = \begin{cases} 
\frac{S}{S-1} \left( c_t^{\frac{S}{S-1}} - 1 \right), & \text{for } S \neq 1 \\
\log(c_t), & S = 1
\end{cases} \] (2)

For the intertemporal utility \( U \) to be bounded, we also have to ensure that \( \left( \frac{S-1}{S} \right) \gamma_x < \rho \), with \( \gamma_x \) the growth rate of the variable \( x \).

As a producer, the representative agent generates per capita output \( y_t \) using per capita private capital \( k_t \) and per capita productive public expenditures \( g_t \), with population normalized to unity

\[ y_t = k_t^\alpha g_t^{1-\alpha} \] (3)

with \( 0 < \alpha < 1 \) is the elasticity of output to private capital.

Public expenditures \( g_t \) enter as a flow in the production function and no congestion is present, so that the model is comparable to Barro (1990) (Appendix 5 extends our results to a model with public investment, as in Futagami et al., 1993). The condition \( 0 < \alpha < 1 \) ensures the existence of a competitive equilibrium, since, at the representative agent level, \( g_t \) is exogenous and the production function exhibits decreasing returns to scale. In equilibrium, on the contrary, \( g_t \) is endogenously determined and the production function exhibits constant returns to scale, a necessary condition for a constant growth path to appear in the long run.

The representative agent budget constraint is (a dot over a variable denotes its time derivative):

\[ \dot{k_t} + \dot{b_t} = r b_t + (1 - \tau) y_t - c_t - \delta k_t \] (4)

Households use their income \( (y_t) \) to consume \( (c_t) \), invest \( (\dot{k_t} + \delta k_t) \), with \( \delta \) the rate of private capital depreciation, and to pay taxes. As in Barro (1990), we assume a flat tax rate

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8 This condition corresponds to a no-Ponzi game constraint \( \gamma_x < r \).

9 This form of production function was by Barro (1990). For a general discussion on productive government expenditures, see Barro & Sala-i-Martin (1992) and for empirical evidence, Aschauer (1989) or Munnel (1992).
on output \((\tau)\). Households can buy government bonds \((b_t)\), whose rate of return is the interest rate \(r_t\). In equilibrium, \(r_t\) equals the marginal productivity of private capital.

The government taxes output, makes productive public expenditures and borrows from households whenever current spending exceeds current income, in which case he will have to pay interests on the issued debt:

\[
b_t = r_t b_t + g_t - \tau y_t,
\]

Notice that (5) is an extension of the Barro (1990) government constraint \(g_t = \tau y_t\). In our case, as real figures suggest, public deficits allow (are initially designed for) the government to make productive expenditures eventually higher than fiscal revenues \(\tau y_t\). An interesting feature is then that unproductive public expenditures, in the form of interest payments on debt, may crowd out productive expenditures in the government budget constraint.

The maximization of (1) subject to (2)-(3)-(4), \(k_0\) given and the transversality condition \(\lim_{t \to \infty} \left(\exp\left(-\int_0^t r_s ds\right) (k_t + b_t)\right) = 0\), gives rise to the familiar Keynes-Ramsey relation:

\[
\gamma_c = \frac{\dot{c}}{c} = S \left[\alpha(1-\tau) \left(\frac{g}{k}\right)^{1-\alpha} - \delta - \rho\right],
\]

which describes the rate of growth of consumption (time indexes will be henceforth omitted).

Next, we write (4) and (5) in a more convenient form, showing the rate of growth of private capital (the IS equilibrium) and of public debt:

\[
\gamma_k = \frac{\dot{k}}{k} = \frac{g}{k} \left(\frac{g}{k}\right)^{1-\alpha} - c - \frac{g}{k} - \delta
\]

\[
\gamma_b = \frac{\dot{b}}{b} = \alpha(1-\tau) \left(\frac{g}{k}\right)^{1-\alpha} - \tau \left(\frac{g}{k}\right)^{1-\alpha} + \frac{g}{k} - \left(\frac{b}{k}\right)^{1-\alpha} + \frac{b}{k}
\]

(6)-(7)-(8) form a system with three equations for four variables, thus a free variable remains in the government budget constraint (8). An endogenous growth solution is defined as a constant long-run growth rate of a variable. Looking (for example) at the consumption growth rate, we can see that for a given \(g\), the quotient \(g/k\) is decreasing in \(k\). This means that the growth rate of consumption in (6) comes to zero in the long run – no growth is present. For this scenario not to happen, a continuous growth of public expenditures \(g\) is needed, so that the ratio \(g/k\) remains constant in the long run. If this is the case, consumption grows continuously, at a constant rate. For example, in Barro (1990), the balanced budget hypothesis \((g = \tau y)\) ensures that \(g/k\) is constant in the long run. Nevertheless, as discussed

\(^{10}\) Notice that taxing interest revenues does not affect our model.

\(^{11}\) Of course, Ricardian Equivalence does not hold in our model, since we have productive public spending and distortionary taxes.

\(^{12}\) For this rate to be positive we have to ensure that: \(\alpha(1-\tau) \left(\frac{g}{k}\right)^{1-\alpha} > \delta + \rho\).
in introduction, persistent deficits characterize a large number of countries, and even balanced budget rules are consistent with positive public debt. Therefore, our model allows studying long run effects of persistent deficits, thus generalizing Barro (1990) in a world with positive deficits and/or public debt.

II. The multiplicity of steady state rates of economic growth

In this section we look for the determination of an endogenous \( g/k \) ratio. Indeed, as suggested by Martin & Oxley (1991, p.161), public investment, rather than government consumption and/or the deficit ratio, seems to be the adjusting variable in the government budget constraint: "Most countries have offset such increases by winding back public investment, reflecting the political reality that it is easier to cut-back or postpone investment spending than it is to cut current expenditures". Building on this idea and in order to reproduce persistent fiscal deficits, we specify a deficit rule, namely that the ratio of deficit to output is equal to some value \( m \):

\[
\frac{b}{y} = m \tag{9}
\]

In what follows, we are interested in the effect of deficit and tax rules on economic growth and welfare in a competitive equilibrium, and especially in the long-run, in a restricted class of models with constant deficit and tax rules \( m \) and \( \tau \). Thus, we must wonder about what relevant policy rules we must enforce in a long-run perspective. Clearly, a constant ratio of tax on national income \( \tau \) is a good premise for studying long-run economic growth, as does Barro (1990). What about deficit rules? To our knowledge, only few papers break with the balanced-budget hypothesis. Yet a quick inspection of OECD data described in the introduction shows that positive deficits are the rule, since for three decades the average ratio of fiscal deficit in theses countries has been about 2.5%. Thus, a constant deficit rule such as: \( m \geq 0 \) (for example: \( m = 2.5\% \)), is consistent with OECD data on the long run, while a balanced-budget rule (\( m = 0 \)) hypothesis would not be. In addition, some countries, like EMU’s ones, have passed fiscal rules like (9).

Using the debt evolution rule (5), we can write (9) in a more convenient way:

\[
\frac{g}{k} = (m + \tau)\left(\frac{g}{k}\right)^{1-\alpha} - \alpha(1-\tau)\left(\frac{g}{k}\right)^{1-\alpha} - \delta \left(\frac{b}{k}\right) \tag{10}
\]

In order to search for endogenous growth solutions we define the three modified variables: \( c_k = c/k \), \( g_k = g/k \) and \( b_k = b/k \). Equations (6)-(7)-(8) and (10) become:

\[\text{An alternative method would be to fix } g/k \text{ and consider an endogenous ratio } m. \text{ However, this would not reflect the stylized fact that the deficit ratio is somehow constant over time and is consistent with the fact that we are not interested in optimal deficit paths – see also footnote 14.}

\[\text{Thus, we do not look for an optimal tax } \{\tau_t\}_{t=0}^\infty \text{ or deficit } \{m_t\}_{t=0}^\infty \text{ paths in a “Ramsey-problem” for a benevolent government, as do, for example, Jones et al. (1993). Notice that none of the optimal plans in Jones et al. (1993) conducts to deficit in the long-run, while we particularly care for this stylized fact.}

\[\text{For example, Greiner & Semsler (2000) and Ghosh & Mourmouras (2004).}

\[\text{In addition, our model will exhibit the balanced-budget rule as a special case.}\]
\[
\left\{ \begin{array}{l}
\frac{\dot{c}_k}{c_k} = \frac{\dot{b}_k}{b_k} - \frac{\dot{g}_k}{g_k} = S \left[ \alpha(1-\tau)(g_k(b_k))^{1-\alpha} - \delta - \rho \right] + g_k(b_k) + c_k + \delta - (g_k(b_k))^{1-\alpha} \\
\frac{\dot{b}_k}{b_k} = \frac{\dot{g}_k}{g_k} = \frac{m(g_k(b_k))^{1-\alpha}}{b_k} + g_k(b_k) + c_k + \delta - (g_k(b_k))^{1-\alpha} \\
g_k = (m+\tau)g_k^{1-\alpha} - rb_k, \text{ with } r = \alpha(1-\tau)g_k^{1-\alpha} - \delta
\end{array} \right. \tag{11}
\]

Notice that \( g_k(b_k) \) is obtained from (11c), which defines an implicit inverse relation between \( g_k \) and \( b_k \), with \( \frac{dg_k}{db_k} = -\frac{g_k[\alpha(1-\tau)g_k^{1-\alpha} - \delta]}{\alpha g_k + (1-\alpha)\delta b_k} < 0 \).

We find the steady state endogenous growth solutions by imposing \( \dot{c}_k = \dot{b}_k = 0 \). Together with the third equation, we obtain constant values of \( c_k, b_k \) and \( g_k \), which means that the four initial variables \( c, k, g \) and \( b \) grow at the same constant rate \( \gamma \). Putting \( \dot{c}_k = \dot{b}_k = 0 \), we eliminate \( c_k \) by extracting (11b) from (11a). Reintroducing the obtained expression of \( b_k \) in (11c) provides an implicit expression in \( g_k \) (note that \( \gamma \) is a (increasing) function of \( g_k \) in (6)), with parameters \( S, \alpha, \delta, \rho, \tau \) and \( m \):

\[
F(g_k) = \gamma(g_k)((m+\tau) - g_k^\alpha) - m\left(\frac{\gamma(g_k)}{S} + \rho\right) = 0 \tag{12}
\]

Since \( \gamma = \gamma_c \) in steady state, we can write (12) as a system:\(^17\):

\[
\left\{ \begin{array}{l}
\gamma^1(g_k) = S \left[ \alpha(1-\tau)g_k^{1-\alpha} - \delta - \rho \right] \\
\gamma^2(g_k) = m\rho \left[ mS^{-1} + \tau - g_k^\alpha \right]^{-1}, \text{ for } g_k^\alpha \neq m\left(\frac{S-1}{S}\right) + \tau
\end{array} \right. \tag{13}
\]

**Proposition 1:** Multiplicity of long-run rates of economic growth

(A) if \( m = 0 \), the system (13) has two solutions. The first is the "Solow solution" corresponding to a zero long run growth rate, while the second is the "Barro solution".

(B) if \( m > 0 \), one can find a set of admissible parameters \( S, \alpha, \delta, \rho, \tau, m \), so that the system (13) has two solutions conducting to positive long run growth rates,

(C) if \( m < 0 \), the system (13) has once again two solutions, but one only leads to a positive long run rate of economic growth, the other being negative. In this case, the positive long run growth rate is associated with a negative stock of public debt.

**Proof:** see Fig.1 for \( S = 1 \) and Appendix 1 for \( S \neq 1 \). Appendix 2 gives approximate analytical values of the two balanced rates of economic growth.

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\(^{17}\) Of course, one might write (13) in a form of a \( g_k^1(\gamma) \) and \( g_k^2(\gamma) \) system, with no changes. What really matters is that \( g_k \) and \( \gamma \) are endogenous and positively related, while \( m \) is exogenous.
Fig. 1 describes the steady-state rates of economic growth for $S = 1$. The $\gamma^2(g_k)$ curve is a hyperbola, with asymptote $g_k = \tau^{1/\alpha}$. For $m > 0$, the $\gamma^2(g_k)$ curve is the continuous line plotted in Fig. 1, while it is the dotted line for $m < 0$. The steady state solutions are defined as the points of intersection between the two curves $\gamma^2(g_k)$ and $\gamma_2^2(g_k)$. First, we can notice that, if $g_k > (m + \tau)^{1/\alpha}$, the steady state stock of public debt is negative in (12). Thus, for $m < 0$, one solution ($\tilde{L}$ point) leads to a negative long run rate of economic growth, which we exclude, and the other ($\tilde{H}$ point) leads to a negative stock of public debt (since $g_k > \tau^{1/\alpha} > (m + \tau)^{1/\alpha}$). Since, as a rule, governments do not have a creditor position in the long run, we restrict our analysis to cases (A) and (B), corresponding to $m \geq 0$.

The (A) case ($m = 0$)

A balanced budget rule means that tax revenues finance productive public expenditures and debt interest payments. Notice that this case does not perfectly correspond to the model of Barro (1990). The difference is that, while in Barro (1990), public debt is always at zero, in our scheme, public debt may be (is) positive. This slight change has important consequences, because two steady state solutions emerge.

Putting $m = 0$ eventually makes the $\gamma^2$ curve (13b) undefined, but we easily find the two solutions from (12). The first solution is the Barro (1990) one, for $g_k^B = \tau^{1/\alpha}$:

$$\gamma^B = S \left[ \alpha(1-\tau)(g_k^B)^{1-\alpha} - \delta - \rho \right] > 0,$$

if

$$g_k^B > \left( \frac{\delta + \rho}{\alpha(1-\tau)} \right)^{1/\alpha}.$$ 

This solution is the $B$ point in Fig. 1 and corresponds to a zero stock of public debt in steady state ($b_k^B = 0$).

The second solution corresponds to a zero growth steady state $\gamma^S = 0$, with

$$g_k^S = \left( \frac{\delta + \rho}{\alpha(1-\tau)} \right)^{1/\alpha}.$$ 

We call this steady state the “Solow solution” ($S$ point of Fig. 1), which is reached for a positive stock of public debt (from (11c)):

$$b_k^S = \left( \frac{g_k^S}{\rho} \right)^{1-\alpha} \left[ \tau - (g_k^S)^{\alpha} \right] > 0.$$ 

The existence of this second solution is due to the fact that, in this situation, the constant level of public debt forces private capital to be constant in the long-run to achieve a constant steady-state $b_k^S$ ratio. Thus, economic growth disappears in the long-run ($\gamma^S = 0$).

The intuition behind this no-growth solution is that, contrary to Barro (1990), not all public expenditures are productive in the steady-state, if public debt is positive: interest payments on debt generate non-productive public expenditures, which produce a crowding-out effect on productive expenditures. Since in steady-state tax revenues must finance

---

18 If $S \neq 1$, the asymptote depends on the $m$ value: $g_k = g_k^m = \left[ m \left( \frac{S-1}{S} \right) + \tau \right]^{1/\alpha}$, making graphical representation less intuitive, but our results are qualitatively unchanged (see Appendix 1).

19 If $m = 0$ and $g_k^a = \tau$, (13b) produces a $0_0$ case of indeterminacy. Thus, for $m = 0$, the $\gamma^2$ curve is either $\gamma = 0$ or $g_k = \tau^{1/\alpha}$, as in Fig. 1.
productive expenditures plus the interest payments on the constant public debt, $b^*_k$ corresponds to such a level of debt that long-run growth disappears, because interest payments absorb a great part of government resources: remaining resources for productive expenditures do not suffice to generate a positive rate of economic growth, which explains why a “Solow solution” emerges in our model.

*Fig. 1 – Steady state rate of economic growth*

The $(B)$ case $(m > 0)$

With persistent deficits, three cases may occur. The most interesting of them, showed in *Fig. 1*, exhibits two points of intersection between $\gamma'$ and $\gamma''$, namely $L$ and $H$, both leading to endogenous positive long run growth rates (further called the low and the high growth rates, $\gamma^L$ and $\gamma^H$). By changing parameters, one can obtain one unique solution for the system (13) or even no solution, when $\gamma'$ passes below $\gamma''$. Approximate analytical values of the two balanced rates of economic growth are given in *Appendix 2*.

Let us try an intuitive interpretation of the $(B)$ case (when multiplicity$^{20}$ occurs), compared to the $(A)$ case. With a deficit rule $(m > 0)$, public debt expands at a positive stationary growth rate ($\dot{b}/b = my/b$), forcing the long run growth rate of the economy to move away from the Solow solution (the low steady state solution moves from $\gamma^L$ to $\gamma^L$). Similarly, introducing deficit generates a positive level of public debt in the long run, which requires interest payments. Therefore the Barro solution can no longer be reached. In this case, the high steady state solution goes from $\gamma^B$ to $\gamma^H$.

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$^{20}$ As usual, by *multiplicity*, we mean the existence of a finite number of solutions and by *indeterminacy*, either the case in which the number of solutions is *infinite* or the case in which we cannot select a determinate solution or trajectory.
III. Fiscal deficits and taxes in the long run

In this section, we study comparative statics in response to a change in the deficit ratio or in the tax rate. Notice that we are not interested in the comparison between taxes and deficit financing, but in deriving the effects of a \textit{ceteris paribus} change in one instrument, with the other (and all other parameters) given. Our results are synthesized in Propositions 2 and 3 below. \textit{A priori} no statement can be made about the effect of a tax rate or a fiscal deficit impulse, because of multiplicity. In fact, as we shall see in Section 4 below, the low solution of system (13) is unstable, therefore no further interest is given to this solution for the moment. Let us turn our attention to the high solution.

\textit{Effect of an increase in the deficit ratio (m)}

\textbf{Proposition 2:} An “impossibility theorem”

\begin{enumerate}
\item Any \textit{ceteris paribus} (in particular, the same tax rate) increase in the deficit-to-GDP ratio lowers the high-rate of economic growth in the long-run.
\item It is impossible for the long-run growth rate to exceed the Barro solution, obtained with a balanced-budget rule (and no public debt in steady-state), which is the highest long-run rate of economic growth.
\end{enumerate}

\textit{Proof:} From (13b): \[
\frac{d\gamma^h}{dm}\bigg|_{\gamma^h \text{ given}} = \frac{\rho \left( \tau - g^h_k \right)}{\left[ m \left( \frac{S - 1}{S} \right) + \tau - g^\alpha_k \right]^2} \geq 0, \text{ if } m \geq 0. \]

Thus, an increase in the deficit ratio moves the $\gamma^h$ curve towards the left. Since $\gamma$ and $g_k$ are positively linked in the $\gamma^h$ curve, we have: $\gamma^H < \gamma^h$. Notice in particular that: $g_k^H < g_k^H = \tau^{1/\alpha}$.

As we can see in Fig. 2 below, any increase in the deficit ratio raises the (unstable) low rate of economic growth (dotted line in Fig. 2) and reduces the high rate of economic growth (continuous line in Fig. 2). The highest acceptable value for the deficit ratio ($\bar{m}$) is computed in Appendix 2.

\textit{Fig. 2 – The deficit ratio (m) and economic growth (\gamma)}

\begin{itemize}
\item $\gamma^L$
\item $\gamma^H$
\item $\gamma^G$
\item $\gamma^F$
\end{itemize}

---

\textsuperscript{21} Recall that $\gamma$ and $g^h_k$ are endogenous, while $\tau$ and $m$ are exogenous.
Proposition 2 shows that public deficits *ceteris paribus* reduce growth, if we are only concerned with long-run stable solutions. Any deficit rule $m > 0$ never allows the rate of economic growth to exceed the Barro solution ($m = 0$), even if this deficit is *initially fully designed* for productive public expenditures. The explanation of our “impossibility theorem” is due to the transversality condition, which constrains the debt growth rate to be lower than the real interest rate in the long-run. Since, in the long-run, all variables grow at the same rate, the transversality condition $\gamma < r$ means that $b / b < r$, and further that $my < rb$, namely that unproductive public expenditures (interest payments) exceed deficit revenues. Thus, in the government budget constraint, debt service necessarily exceeds the new resources provided by the deficit rule in the steady-state, generating a crowding-out effect on productive public spending (the public investment plays the role of the "adjustment variable", for the government to respect the no-Ponzi constraint): $g / y < \tau$ in equation (5), because $my < rb$.

**Policy implications: revisiting the “Golden Rule of Public Finance”**

Our “impossibility theorem” strongly contradicts the GRPF, which asserts that deficit-financed productive public expenditures may improve economic growth. The advocates of the GRPF defend debt-financed productive public expenditures by considering that the social rate of return of these expenditures (such as expenditures in infrastructures, basic research, new technologies, education or health,…) exceeds the cost of financing (a leverage effect). But this statement is misleading, because government cannot appropriate the social return of productive public expenditures, but only their cost. In the long run, from the government budget constraint’s perspective, the rate of return of fiscal deficit (the public debt growth rate) never can exceed the cost of financing (the real interest rate). Thus, from the point of view of government financial stance, the leverage effect is necessarily negative.

Proposition 2 has some interesting policy implications. For example, concerning EMU, a number of recent papers (Buiter, 2001, Blanchard & Giavazzi, 2004) defend the idea that ruling productive expenditures out of the Stability and Growth Pact (SGP) deficit target may improve long run economic growth, avoiding the risk of a severe contraction of public investment caused by the SGP ceiling. This is an alternative formulation of the GRPF, in which the starting position is not a balanced budget, but a maximum deficit ratio. In our model, interpreting the $m$ ratio as the SGP deficit target, we can examine the effect of a relaxation in the SGP by getting a fraction (say, $\mu$) of productive public expenditures out of the deficit target. In this situation, we have to compare the steady state solution (13) with the solution of a model in which (9) is replaced with:

$$b my g \mu = + \frac{\mu}{1 - \mu}, \quad 0 \leq \mu \leq 1.$$ 

Thus, equation (11c) must be modified: $(1 - \mu)g = (m + \tau)y - rb$, without any other change in the model. With such a rule, the $\gamma^2$ curve (13b) becomes simply:

$$\gamma^2(g, \mu) = m \left( \rho + \mu g^2 \right) \left[ \left( m + \mu g^2 \right) \left( \frac{S - 1}{S} \right) + \tau - g^2 \right]^{-1}$$

(13c)

Remark first that Proposition 2 is unchanged: $\left. \frac{d\gamma^2}{dm} \right|_{g, \mu \text{ given }}> 0$, if $m > 0$, thus, $\gamma'' < \gamma''$ for the transversality condition ($\gamma < r$) to hold. Therefore, the balanced budget rule always dominates any solution with a deficit rule, whether the deficit target includes productive public expenditures or not. Second, remark that, *ceteris paribus*, the larger the share of
productive expenditures ruled out from the deficit target \((\mu)\) is, the smaller productive expenditures and the rate of economic growth will be in the long run\(^{22}\). Effectively, 
\[
\left. \frac{d\gamma^2}{d\mu} \right|_{\text{given}} > 0 \text{ if } m > 0 \text{, thus, the } \gamma^2 \text{ curve moves once again towards the left if } \mu \text{ increases.}
\]

The reason is the same as before: putting \(\mu > 0\) makes more public debt accumulation, so that the crowding out effect on productive public expenditures will be more severe in steady state. Therefore, excluding productive expenditures from a deficit target is not a solution to promote long run economic growth, and, contrary to Blanchard & Giavazzi\(^{23}\) (2004), a positive public debt never encourages public investment in the long run.

Notice that our “impossibility theorem” obviously holds when productive public spending is modeled as a stock (see Appendix 5) and in a model with unproductive primary public expenditures. For example, suppose that a share \(\frac{g_c}{y} = \theta\) of output is allocated to unproductive primary public expenditures (public consumption \(g_c\)). With a balanced-budget rule, the productive expenditures (that we reappoint as \(g_p\) for this topic) to output ratio is: \(\frac{g_p}{y} = \tau - \theta\). With a deficit rule, at each time \(t\), the current stock of public debt must be backed by some future primary surplus: 
\[
b_t = \int_t^\infty \exp\left(-\int_t^u du\right)\left((\tau - \theta) y_s - g_{ps}\right) ds;
\]
consequently, the productive-expenditures-to-output ratio will have to become less than \(\tau - \theta\) in the future. In steady-state, for a positive public debt, we have: \(\frac{g_p}{y} = \tau - \theta - (r - \gamma)b/y < \tau - \theta\), and the long run growth is inferior to the Barro solution.

Of course, if government is able to increase the tax rate or to reduce the share of unproductive public spending, a debt-financed increase in productive expenditures might improve economic growth, because, e.g., the debt burden crowds out unproductive rather productive primary expenditures. Notice that the GRPF allows public deficits devoted to increases in public investment (or productive public spending), but is unclear about the adjustment variable to interest burden\(^{24}\). Our model abstracts from unproductive public expenditures, thus productive public expenditures are the adjustment variable to interest debt burden. Suppose, on the contrary, that public consumption is the adjustment variable, like in a fiscal rule such as\(^{25}\):

\[
\begin{align*}
g_p &= b = my \Rightarrow \frac{g_p}{y} = m, \text{ and:} \\
g_c + rb &= \tau y \Rightarrow \frac{g_c}{y} = \tau - r - \frac{b}{y} \quad (14a)
\end{align*}
\]

\[\text{22} \text{ Of course this effect comes quite mechanically from the fact that we consider an unchanged deficit target, while this target eventually needs a redefinition if some expenditure are ruled out. Nevertheless, the advocates of a “modified” SGP often recommend not changing the 3% ceiling, but only the components of deficit.}\]

\[\text{23} \text{ “there are good arguments for preventing the public debt from disappearing: for instance, public debt […] may be issued to finance public investment projects with a large enough social rate of return” (2004, p.2).}\]

\[\text{24} \text{ For example, Modigliani et al.’s (1998) “Manifesto on unemployment in the European Union” claims that it is necessary that public investment should be financed “neither by cutting other expenditures […] nor by raising taxes”, but by raising public debt. However, it also claims at the same time, that current budget, including the interest cost of public debt, should be balanced. Thus, to service public debt, government will have to cut other expenditures or to raise taxes in the future.}\]

\[\text{25} \text{ The same reasoning applies for fiscal rules used by Ghosh & Mourmouras (2004), such as } g_p = (1 - \varphi) \tau y + b \\
\text{and } g_c + rb = \varphi \tau y, \text{ with } \varphi \in [0,1] \text{ the share of fiscal revenues that is devoted to public consumption.}\]
In system (14) an increase in public borrowing devoted to productive public spending will conduct to a higher productive public spending ratio \( \left( \frac{g_p}{y} \right) \), and thus a higher rate of long-run economic growth, because the debt burden forces public consumption ratio \( \left( \frac{g_c}{y} \right) \) to decrease, for a given tax-rate\(^{26}\). Nevertheless, these growth benefits do not come from deficit-financed productive expenditures \textit{per se}, but \textit{rather from the substitution of unproductive with productive public expenditures}. This is the sense of the “\textit{ceteris paribus}” clause in \textit{Proposition 2a} above: what our proposition says is that, for the same tax-rate and the same ratio of unproductive primary public spending, any increase in debt-financed productive public expenditures lowers the long-run economic growth path.

Remark that the GRPF is precisely advocated on the basis that unproductive primary expenditures (such as wages or transfers) are difficult to bring down, and tax-rates are difficult to increase, so that public borrowing is a solution for public investment not to endure the adjustment to current trends in economic activity\(^{27}\). But deficits generate higher unproductive total expenditures in the future, therefore reducing public investment, unless the government can find tomorrow the formula to do what it cannot do today – to bring down unproductive expenditures or to increase tax-rates. In other words, to assess the benefits of public borrowing for productive expenditures, coherence imposes treating unproductive primary expenditure ratio as given. That is the reason why we take \( g_c = 0 \) in our model.

In addition, substituting public consumption with productive expenditures is probably questionable from a welfare point of view, since public consumption should enter the utility function (except if it is purely wasted expenditure), introducing a wedge between maximizing economic growth and maximizing welfare\(^{28}\). Furthermore, even if we abstract from welfare issues, and suppose that public consumption is purely wasted expenditure, if government can decrease the share of unproductive public spending down to a lower bound \( g_c / y = \bar{\theta} \) (possibly zero, as in our main text), the best solution from the point of view of long-run economic growth, is precisely to adopt this minimum public consumption ratio, by adopting the maximum feasible deficit rule \( (\hat{\bar{m}}) \) in (14b), namely:

\[
\frac{b}{y} = \tau - \bar{\theta} \Rightarrow \frac{\hat{b}}{y} = \tau - \bar{\theta} \Rightarrow \hat{m} = \frac{\gamma (\tau - \bar{\theta})}{r} \text{ in steady-state}^{29}. 
\]

This deficit rule conducts to a lower productive expenditure ratio than the balanced-budget rule: \( g_p / y = \tau - \bar{\theta} \). Effectively, in (14a):

\[
\frac{\hat{g}_p}{y} = \hat{m} = \frac{\gamma (\tau - \bar{\theta})}{r} < \tau - \bar{\theta}, \text{ since } \gamma < r \tag{15}
\]

\(^{26}\) This situation, together with the case of a \textit{net creditor} government in the long run, that we describe in \textit{Fig.1}, (in line with Turnovsky,1995, p.418), are the only cases of positive effects of debt financing on long run growth.

\(^{27}\) While the problem of government consumption is straightforward, analyzing taxes is more complicated. First, with distortionary taxes (considered realistic) the existence of a threshold (see \textit{Proposition 3} below) implies only limited tax resources, from raising the tax rate. Second, growth-enhancing lump-sum taxes are considered highly unrealistic, thus unfeasible, which sheds a further constraint on the government fiscal resources, thus equally strengthening the foundation (importance) of the GRPF and of our analysis.

\(^{28}\) As our model already produces non-trivial welfare effects, we abstract from such issues. Effectively, our model exhibits transitional dynamics, and requires an analysis of welfare, not only in the steady state, but on the entire transition path. By so doing, we obtain interesting intertemporal welfare effects, even without public consumption expenditures in the utility function (see \textit{Section 5} below).

\(^{29}\) If \( m = \hat{m} \), no further substitution of productive for unproductive public expenditures is feasible, as \( g_c = 0 \).
Thus, even if public consumption is the adjustment variable to public debt interest burden, the rate of growth obtained by allowing public borrowing for productive spending will be inferior to the Barro solution, and our “impossibility theorem” of Proposition 2b still holds. In other words, the "growth benefits" obtained from the substitution of unproductive with productive public expenditures, by using the GRPF, come from the fact that long-run growth is very damaged in a steady state, with positive public debt and “wasted” public consumption. Increasing public debt to finance productive spending and reducing public consumption expenditures may improve long-run growth, but if this substitution is seen through to completion, the long-run rate of economic growth cannot exceed the rate obtained with a balanced-budget. Therefore, a fiscal rule such as the GRPF never allows the economy to go beyond the Barro long-run rate of economic growth\textsuperscript{30}. Moreover, since our strategy reunites the best \textit{a priori} conditions to obtaining positive effects of persistent deficits on long run growth, we strongly believe that our "impossibility theorem" not only holds in a model with productive spending, but \textit{in any model with long run growth and any kind of government expenditure}\textsuperscript{31}.

Finally, remark that our “impossibility theorem” is only a steady-state result. Yet, our model exhibits non trivial transitory dynamics. In the following sections we derive the impact of deficit rules along the transition path. As we shall see, deficit-financed increases in productive expenditures may improve economic growth in the short-run and produce non trivial effects on intertemporal welfare. But before analyzing transitional dynamics, let us study the effect of a tax-rate impulse on steady-state.

\textit{Effect of an increase in the tax-rate (\(\tau\))}

The effects of a tax-rate impulse are more delicate to asses, compared to the effects of fiscal deficits, because a higher tax-rate induces a downward movement of the two curves \(\gamma^1\) and \(\gamma^2\). We obtain the following results.

\textbf{Proposition 3: Maximizing-growth tax-rate}

\begin{itemize}
  \item[a)] There is one value of the tax-rate that maximizes the high-rate of economic growth. If \(m = 0\) this value is, as in Barro (1990): \(\tau^0 = 1 - \alpha\). If \(m > 0\), the high-rate of economic growth reaches its maximum at \(\tau^* > \tau^0\).
  \item[b)] The maximizing-growth tax rate \(\tau^*\) is an increasing function of the deficit ratio.
\end{itemize}

\textsuperscript{30}To avoid the crowding out effect of public debt, government has to get some other resources. Issuing money might be a way to exceed the Barro solution, provided that the inflation tax does not penalize economic growth, but the analysis of this case is beyond the scope of this paper.

\textsuperscript{31}One could argue that Proposition 2 only holds for permanent increases of fiscal deficit. But we can slightly amend this proposition for temporary increases. Temporary increases in public borrowing still produce resources for public investment but raise the debt burden in the future. At the time deficit – and the associated revenues – comes to zero, the interest burden on public debt (resulting from the accumulated past deficits) is positive and requires an adjustment of taxes, productive or unproductive spending, to balance government budget. If public investment is the adjustment variable, Proposition 2a has to be modified as follow: “Any \textit{ceteris paribus temporary increase in the deficit-to-GDP ratio, reduces the high-rate of economic growth in the future}” (not in the long-run). Effectively, in the long-run, economic growth is unaffected, because public debt, as a % of GDP, tends to zero in steady-state (since at some future time deficit is zero, so that public debt is constant, while output grows continuously). Thus, temporary fiscal deficits cannot increase economic growth in the long run, and Proposition 2b is unchanged: “\textit{the highest long-run rate of economic growth is (still) […] obtained with a balanced-budget rule}”.

14
Proof: Let us rewrite equation (12) as a function of $\gamma$ and $m$, using the definition of $g_k$ in (13a): 

\[ F(\gamma, m) = \gamma (m + \tau) - \left[ \frac{\gamma + S (\delta + \rho)}{\alpha S (1 - \tau)} \right]^{\alpha - 1} - m \left( \frac{\gamma + \rho}{S + \rho} \right) = 0. \]

The first order condition for $\tau$ to be an optimum is, from the implicit function theorem:

\[ \tau^* = 1 - \alpha + \alpha n \left( \frac{\rho}{\gamma \tau} - \frac{S - 1}{S} \right) \geq 1 - \alpha \]  

(16)

Of course equation (16) gives only an implicit function for $\tau^*$. Nevertheless, for the solvability condition to be enforced, we have: 

\[ \frac{\rho}{\gamma \tau} - \frac{S - 1}{S} > 0, \]

and the maximizing-growth tax rate $\tau^*$ is higher than the Barro solution $\tau^B = 1 - \alpha$, for any positive level of the deficit ratio. Similarly to Barro (1990), flat-rate taxes have a positive growth effect, by providing resources for growth-enhancing public spending, and a negative effect, because they distort of private capital accumulation. Notice that we find the Barro solution as a special case if $m = 0$.

The economic interpretation of Proposition 3 is the following. Let $\tau^*$ be the tax rate that maximizes long-run growth. From (6), the elasticity of $g_y = g / y$ with respect to this tax rate is defined by:

\[ \frac{dg_y}{d\tau \tau} = \frac{\alpha}{1 - \alpha} \left( \frac{\tau^*}{1 - \tau^*} \right) = \varepsilon_r^g(\tau^*) \]  

(17)

In Barro (1990), the balanced-budget rule imposes that: $g_y = \tau$, so that the elasticity of the ratio of public spending to output with respect to the tax-rate is: $\varepsilon_r^g = 1$. Equalizing (17) to this elasticity gives the Barro solution: $\tau^B = 1 - \alpha$. In our model, from the government budget constraint we have: $x g_y = \tau$, where $x = \frac{g_k + (r - \gamma) b_{\gamma}}{g_k}$, so that: $\varepsilon_r^g = 1 - \frac{dx \tau}{d\tau \tau}$. What Proposition 3 shows is that the elasticity of public spending (in % of output) with respect to the tax rate is higher than one$^{33}$: $\varepsilon_r^g(\tau^*) > 1$, which, combined with (17), means that $\tau^* > 1 - \alpha$. In other words, contrary to Barro (1990), one extra percent of tax revenues here yields more than one extra percent of productive government expenditures, because it enables to decrease the (net of deficit resources) burden of public debt service, which explains why the maximizing-growth tax-rate exceeds the Barro one.

In addition, we can remark from relation (16) that the optimal tax-rate is an increasing function of the deficit ratio. Effectively, for small values of $m$,$^34$ we obtain, from (16):
The intuitive explanation of this result is as follow. Suppose that the starting point is the Barro long-run equilibrium with \( \tau^B = 1 - \alpha \) and a zero deficit. If \( m \) jumps to some positive value, long-run economic growth will be lower, since the public-debt burden crowds-out productive spending. To restore (part of) productive public spending, government must increase the tax rate beyond the Barro value. Table 2 reports some simulation results, in which we compute the exact value of \( \tau^* \) and the associated maximized-rate of economic growth.

### Table 2 – The tax rate maximizing the high rate of economic growth

<table>
<thead>
<tr>
<th>( \gamma^H ) (%)</th>
<th>( S = 0.8 )</th>
<th>( S = 1 )</th>
<th>( S = 1.2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 - \alpha = 0.4 )</td>
<td>( m = 1% )</td>
<td>0.4056</td>
<td>0.4032</td>
</tr>
<tr>
<td></td>
<td>( m = 2% )</td>
<td>0.4114</td>
<td>0.4065</td>
</tr>
<tr>
<td></td>
<td>( m = 3% )</td>
<td>0.4176</td>
<td>0.4100</td>
</tr>
<tr>
<td>( 1 - \alpha = 0.3 )</td>
<td>( m = 1% )</td>
<td>0.3047</td>
<td>0.3018</td>
</tr>
<tr>
<td></td>
<td>( m = 2% )</td>
<td>0.3082</td>
<td>0.3037</td>
</tr>
<tr>
<td></td>
<td>( m = 3% )</td>
<td>0.3123</td>
<td>0.3103</td>
</tr>
<tr>
<td>( 1 - \alpha = 0.2 )</td>
<td>( m = 1% )</td>
<td>0.2035</td>
<td>0.2012</td>
</tr>
<tr>
<td></td>
<td>( m = 2% )</td>
<td>0.2071</td>
<td>0.2025</td>
</tr>
<tr>
<td></td>
<td>( m = 3% )</td>
<td>0.2107</td>
<td>0.2037</td>
</tr>
</tbody>
</table>

(* (*)) means that the transversality condition does not hold.

**Proposition 3** shows that, with a deficit rule \( (m > 0) \), the optimal size of the public sector is larger than with the balanced budget rule of Barro (1990). This result is not incompatible with **Proposition 2** statement, that the rate of economic growth is lower than Barro’s one, as we can see in Fig.3:

***Fig.3–The tax-rate (\( \tau \)) and the high rate of economic growth (\( \gamma^H \))***

If \( \tau > \tau^* \), any increase in the tax rate or in the deficit ratio lowers long-run economic growth. In this case, the two policy instruments are substitutes: one can reach the same rate of economic growth by increasing (respectively decreasing) \( \tau \) and decreasing (respectively raising) \( m \). If \( \tau < \tau^* \), on the contrary, the two instruments are complementary: to reach an unchanged rate of economic growth, one have to move the tax rate and the deficit ratio in the same direction, since long-run economic growth is positively affected by \( \tau \) and negatively affected by \( m \). In particular, the optimal \((\tau, m)\) couple, from the point of view of long-run economic growth is: \( m = 0 \) and \( \tau = 1 - \alpha \).
IV. Transitional dynamics

In order to look for the stability of the two equilibria, we examine the transitional dynamics of \( (c_k, b_k) \) in system (11). Note that \( b_k \) is the only predetermined (no-jump) variable in (11). To study the transitional dynamics of the model we proceed as follows. First, we derive some analytical results from the case \( m = 0 \), and then we offer a graphical representation for the case \( m > 0 \). Our analysis is supplemented with numerical simulations.

A balanced budget rule (\( m = 0 \))

For \( m = 0 \) (no deficit), we find the two steady states expounded in the previous section. First, the Barro steady state is reached for: \( b_k = b_k^B = 0 \), so that \( \dot{b}_k = 0 \) in (11b). When public debt is zero, productive public expenditures are: \( g_k = g_k^B = \tau^{1/\alpha} \), and the associated steady consumption-to-capital ratio is: \( c_k = c_k^B = \left( g_k^B \right)^{1-\alpha} - \gamma^B - g_k^B - \delta \), which makes \( \dot{c}_k = 0 \) in (11a). Second, we find the Solow steady state for \( g_k = g_k^S \), i.e. \( b_k = b_k^S \), both values defined above, and \( c_k = c_k^S = \left( g_k^S \right)^{1-\alpha} - g_k^S - \delta \), so that \( \dot{c}_k = \dot{b}_k = 0 \) with no long run growth.

Graphically, the stability locus of \( c_k \) (\( \dot{c}_k = 0 \)) has an inverted-U shape with a maximum at \( \hat{b}_k \), as represented in Fig.4\textsuperscript{35}. In the same chart we plot the locus of stability points for \( b_k \) (\( \dot{b}_k = 0 \)), which is made of two curves: the vertical curve \( b_k = b_k^B = 0 \) and an inverted-U shaped curve in \( c_k \): \( c_k = \left( g_k \left( b_k \right) \right)^{1-\alpha} - g_k \left( b_k \right) - \delta \), with a maximum at \( \tilde{b}_k \), \( (\tilde{b}_k < \hat{b}_k \text{ except for very high values of } S) \textsuperscript{36} \).

On Fig.4, the B point is the “Barro solution” and the S point is the “Solow solution” (no growth). On the right hand of \( b_k^S \), the rate of economic growth is negative, while it is positive on the left hand side. In (11), it can be seen that the \( \dot{b}_k = 0 \) curve is always above the \( \dot{c}_k = 0 \) curve, except when the long run rate of growth is zero (point S) or negative (on the right hand of \( b_k^S \)). Thus, the dynamics clearly show that B is a saddle point, while S is unstable.

\textsuperscript{35} This is another motivation for our choice to consider \( g \) as a flow. Indeed, with \( g \) modeled as stock, the reduced form is in three accumulation equations (see Appendix 5), with little possibilities for a (comprehensive) graphical representation.

\textsuperscript{36} \( \hat{b}_k \) is implicitly defined by: \( g \left( \hat{b}_k \right) = \left( (1-\alpha)(\alpha(1-\tau)S-1) \right)^{1/\alpha} \) and \( \tilde{b}_k \) by: \( g \left( \tilde{b}_k \right) = (1-\alpha)^{1/\alpha} > g \left( \hat{b}_k \right) \), if \( S < 2/\alpha(1-\tau) \). Whether \( b_k^S \) is greater or smaller than \( \tilde{b}_k \) or \( \hat{b}_k \) depends on parameters. Notice that \( \tilde{b}_k < 0 \) if \( \tau < \tau^B = 1-\alpha \) and \( \hat{b}_k > 0 \) if \( \tau > \tau^B = 1-\alpha \). In Fig.4, we consider, without generality loss, a particular representation in which \( \tilde{b}_k = 0 \) (this is obtained for \( \tau = \tau^B = 1-\alpha \)), so that \( \tilde{b}_k < b_k^S < \hat{b}_k \).
Analytical results confirm this diagnostic. In Appendix 4, looking at the local stability of the two steady states, we observe that the determinant of the Jacobian matrix of system (11) is negative in the neighborhood of the Barro steady state, while positive in the neighborhood of the Solow steady state, with a positive trace.

A deficit rule \((m > 0)\)

When the government runs into debt \((m > 0)\), analytical results are delicate to obtain. We can find a set of admissible parameters, so that the phase diagram changes slightly in comparison with the case \(m = 0\)\(^{37}\). The major change is that the \(\dot{b}_k = 0\) locus now has an asymptote at \(b_k = b^s_k\), so that the \(B\) steady state can no longer be reached. Moreover, we know from the previous section that the two steady state solutions \(L\) and \(H\) are respectively on the left and on the right of \(S\) and \(B\).

We construct the steady state curve for different values of the deficit ratio by extracting \(m\) from the \(\dot{c}_k = 0\) relation and substituting it into the \(\dot{b}_k = 0\) relation. Numerical simulations show that, for admissible values of parameters, this steady state locus takes the form of an increasing curve, represented by the \(SS\) curve in Fig.5 (for \(m \geq 0\))\(^{38}\). The two extreme points of this curve are the Barro steady state (\(B\) point, for \(b_k = 0\)), and the Solow steady state (\(S\) point) associated to the highest public-debt-to-capital ratio (with no growth). These two extrema correspond to a balanced budget rule \((m = 0)\). Steady-state points are located inside the segment \(BS\), like points \(L\) and \(H\), correspond to a deficit rule \((m > 0)\).

\(^{37}\) Note that \(m\) is small, ordinary less than 5%. Appendix 4 offers some simulation results on local stability.

\(^{38}\) For some parameters values, numerical simulations show that the \(SS\) line may become curved when public debt is high. The analysis below is unchanged, except that the interpretation of the sustainable level (see footnote 41 below) of public debt must be slightly modified.
Of course, the multiplicity of steady states implies that each value of \( m \) is associated with two points on the \( SS \) curve.

*Fig. 5 – Phase diagram in the case of a deficit rule (\( m > 0 \))*

The phase diagram (Fig. 5) points out that, similarly to the \( m = 0 \) case, the high steady state \( (H) \) is a saddle point and the low steady state \( (L) \) is unstable, knowing that \( b_h \) is predetermined and \( c_k \) can jump. In Fig. 5 we also remark that the paths of the variables turn around \( L \), following an unstable spiral. Thus, an economy starting with a very high debt stock \( b_h > \bar{b}_h \) cannot reach any steady state\(^{39}\). In this situation, there is no initial jump of \( c_k \) that allows the economy to follow a saddle path. In order to avoid such unstable paths, in which public debt-to-capital (or output) ratio explodes, we impose a superior bound for \( b_h \) \( (b_h \leq b_h^{\text{max}}) \), which, in turn, defines a certain maximum “sustainable” level of public debt\(^{40}\) that is the highest debt ratio that the society is prepared to finance.

\(^{39}\) This circular movement occurs when \( \partial c_k / \partial b_h \leq 0 \) in the neighborhood of \( L \) point. For some parameter values, \( \partial c_k / \partial b_h > 0 \), which strengthen this result: an economy whose initial debt-to-capital ratio is higher than \( b_h^{L} \) cannot reach any steady state.

\(^{40}\) The *sustainability* condition points out the social and political bound on public borrowing, and it has to be distinguished from the *solvency* condition \( \gamma < r \). Such an upper bound is often assumed when the public debt path is unstable (see, e.g. Sargent & Wallace, 1981).
This means that, at some date $T$, to be endogenously determined, the debt-to-capital ratio will reach $b_k^{\text{max}}$, and, at this point of time, the government must adopt the deficit rule $m$ which makes $T$ a steady state. In what follows, we define $b_k^{\text{max}}$ as the highest debt-to-capital ratio that is consistent with the existence of a steady state. In Fig.5, this ratio corresponds to the Solow solution ($b_k^{\text{max}} = b_k^S$). Thus, the steady deficit rule that must be adopted at $T$ is $m = 0$. In this case public debt is seen as “sustainable” as long as $\gamma \geq 0^{41}$.

This set-up produces indeterminacy. In our model, indeterminacy has two meanings. First, it takes the form of multiple (two) balanced long run growth paths: an endogenous growth path ($H$ solution) and a “poverty trap” without growth ($S$ solution), obtained when the public debt sustainability condition is reached$^{42}$. Second, indeterminacy means that one cannot choose between the multiple (two) transition paths to steady state equilibrium. Fig.5 exhibits three kinds of solutions, depending on the initial public-debt-to-capital ratio. Ceteris paribus, an economy endowed with a low initial public debt ratio ($b_{k0} < b_k^S$) converges to the high balanced growth path ($H$), while an economy endowed with a high initial public debt ratio ($b_{k0} > b_k^S$) converges to the “poverty trap” solution ($S$)$^{43}$. But for economies endowed with an intermediate initial public debt ratio ($b_k < b_{k0} < b_k^S$), the transition path is radically indeterminate. These economies may reach either the convergent path to the high equilibrium $H$ (the consumption-to-capital ratio jumps down), or the path leading to the “no growth trap” $S$ (if the consumption-to-capital ratio jumps up). This situation reminds of the “history versus anticipations” scenario of Krugman (1991). For countries with an initial high or low public debt, “history”, in the form of the initial public debt ratio, determines the equilibrium path, while for countries with an intermediate initial public debt ratio, the transition path is subjected to “expectations”, in the form of “optimistic” or “pessimistic” views on debt sustainability.

V. Non-linear effects of fiscal deficit on welfare and economic growth

In order to illustrate some preliminary intuitions, let us suppose an upward jump of the deficit-to-output ratio, from an initial steady state position $m$ to the new steady state described by the deficit ratio $m' (m' > m)$. Thus, the two steady states ($L$ and $H$) shift respectively to $L'$ and $H'$ on the $SS$ curve, as in Fig.7 below. If the economy departs from the $H$ steady state, for a predetermined public-debt-to-capital ratio ($b_{k0} = b_k^H$), $g_{k0}$ jumps up in (11c). Consequently, the rate of economic growth increases initially. Thus, unlike our steady state result, debt-financed productive expenditures may, in the short run, lead to a higher rate of economic growth than tax-financed$^{44}$ ones. Thus, quite paradoxically, it looks

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$^{41}$ If the $SS$ curve bends, $b_k^{\text{max}}$ should be distinguished from $b_k^S$, since the highest debt-to-capital ratio that is consistent with the existence of a steady state is no longer $b_k^S$. It follows that the sustainability level of public debt is reached for a positive rate of economic growth.

$^{42}$ Of course, if the sustainability condition is $b_k^{\text{max}} \neq b_k^S$, the “poverty trap” means a slow growth balanced path.

$^{43}$ In these two cases, the saddle path is reached by an initial jump of $c_{k0}$, which rules out any divergent path.

$^{44}$ This is the case if the economy initially departs from the $B$ steady state. In other words, in the short run, the rate of economic growth may exceed the Barro solution.
as if the GRPF were a short run prescription to increase economic growth, to the detriment of long run growth potential.

For example, in our simulations (see Table 3A), an increase in the deficit ratio from \( m = 1\% \) to \( m = 2\% \) initially raises the rate of economic growth from \( \gamma^H = 8.66\% \) to \( \gamma^H = 9.95\% \), but lowers this rate to \( \gamma^H = 8.13\% \), in the long run. On Fig. 8 below, the hatched area depicts the increase of economic growth following the jump of the deficit ratio, and the grey area depicts growth losses.

Table 3A. Simulation results for an increase in the deficit-to-GDP ratio from \( m = 1\% \) to \( m = 2\% \) — Starting from \( H \) Steady State

<table>
<thead>
<tr>
<th>(%)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Steady State</td>
<td>6.20</td>
<td>18.91</td>
<td>21.13</td>
<td>( \gamma^H = 8.66 )</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial jump of variables</td>
<td>6.20</td>
<td>18.60</td>
<td>22.02</td>
<td>9.95</td>
<td>8.97</td>
<td>9.31</td>
<td>8.64</td>
<td>2.45</td>
</tr>
<tr>
<td>Final Steady State</td>
<td>13.03</td>
<td>19.42</td>
<td>20.41</td>
<td>( \gamma^H = 8.13 )</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For simulation values\(^45\): \( \alpha = 0.6\), \( \rho = 0.1\), \( S = 2\), \( \tau = 0.4\), \( \delta = 0.05\)

Deficit rules and intertemporal welfare

From a welfare point of view, the fact that the growth rate can increase in the short (medium) run, while (always) diminish in the long run, raises a serious difficulty in estimating the effect of deficit rules, because we need to compute the present value of the net intertemporal welfare loss associated to the two paths (with and without change in the deficit ratio), in terms of agents' utility level.

In Barro (1990), maximizing economic growth is equivalent to maximizing welfare. This is no longer the case in our model with persistent deficits and public debt. Let us compute the value of discounted intertemporal welfare (1) for different deficit rules, compared with the Barro steady state growth rate. With a balanced budget rule \( (m = 0)\), transitional dynamics vanish, and, similar to Barro (1990), the consumption path \( \{c^B_t\}_0^\infty \) is a steady-growth path \( c^B_t = k_0 c^B_{t=0} \exp\left(\gamma^B t\right) \), where \( c^B_{t=0} = (1 - \tau) f - \delta - \gamma^B \) comes from the IS relation (7) and \( k_0 \) is predetermined. If deficit jumps to some value \( m > 0\), starting from the Barro steady state, the initial consumption-to-capital ratio jumps to a value \( c^m_{t=0} \) (to be computed), and we define the consumption path associated with a \( m \) deficit rule as \( \{c^m_t\}_0^\infty \), with \( c^m_t = k_0 c^m_{t=0} \exp\left(\int_0^t \gamma^m ds\right) \) and \( \gamma^m \) is computed from the Keynes-Ramsey relation (6). In the long-run, the rate of growth of consumption will be lower than initially, but it may become higher for some time on the transition path. We are interested in computing the net welfare loss between the two paths (with or without change in the deficit ratio), in terms of

\(^{45}\) \( \alpha = 0.6 \) is consistent with Aschauer (1989) findings. The tax rate takes the value which maximizes growth, with no deficit. Other parameters are chosen so that the phase diagram is meaningful (of course, the transversality condition holds). MatLab\textsuperscript{®} codes are available upon request.
households’ utility level. Let $\Delta_t(m)$ be the present value of the net loss of welfare between the two paths at date $t$, namely:

$$\Delta_t^m = \left( u \left( c^m_t \right) - u \left( c^B_t \right) \right) \exp(-\rho t)$$  \hspace{1cm} (15)$$

so that the total discounted net loss of welfare associated to a $m$ deficit rule can be easily defined (see (1)) as:

$$\Delta U(m) \equiv U^m - U^B = \int_0^\infty \Delta_t^m \, dt$$  \hspace{1cm} (16)$$

$U^B \equiv \int_0^\infty u \left( c^B_t \right) \exp(-\rho t) \, dt$ is the intertemporal welfare associated with an economy which stays on the $B$ point of Fig.5 from $t = 0$ to infinity, while $U^m = \int_0^\infty u \left( c^m_t \right) \exp(-\rho t) \, dt$ is the intertemporal welfare of an economy which, starting from $B$, with the same initial predetermined values for $b_0$ and $k_0$, jumps to $c^m_0$ and then moves to $H$ in Fig.5. To compute the adjustment path, we linearize system (11) in the neighbourhood of the high steady-state (see Appendix 4). By so doing, we obtain the adjustment path for $c^m_0$ and $b_0$ (and notably the initial jump of $c^m_0$ or equivalently $c^m_0$, since $c^m_{k0} \equiv c^m_0 / k_0$ and $k_0$ is predetermined), and thus for all dependent variables, among which the rate of growth of consumption $\gamma_t^m$, that we use to compute $c^m_t$. Fig.6 depicts the path of the discounted net gain of welfare $\Delta_t^m$ for several values of the deficit rule ($m = 1\%, m = 2\%, m = 3\%$), and different values of the consumption elasticity of substitution ($S = 0.5\%, S = 1\%, S = 1.5\%$). Fig.6 shows that a fiscal deficit rule may improve or weaken welfare, depending on parameters. If the consumption elasticity of substitution is “high” ($S \geq 1$ in Fig.5), the total value of the discounted net gain of welfare $\Delta U(m)$ (the integral of $\Delta_t^m$) is negative. On the contrary, if the consumption elasticity of substitution is “low” ($S = 0.5\%$, for example), the total value of the discounted net gain of welfare $\Delta U(m)$ becomes positive. These implications are mainly driven by the fact that, for the net intertemporal welfare to be positive, the short run gain must be higher than the (always) long run loss. With a "high" ("low") elasticity of substitution $S = 1.5\%$ ($S = 0.5\%$), the agents easily accept to substitute present to future consumption, thus a negative (positive) jump in initial consumption $c^m_0 < c^B_0$ ($c^m_0 > c^B_0$) and a lower (higher) short run welfare, compared to the long run value, thus net intertemporal welfare loss (gain).
Fig.6: Assessing the net welfare gain of deficit rules

For: $\rho = 0.075$, $\alpha = 0.6$, $\tau = 0.4$, $\delta = 0.025$, $k_0 = 1$. 
Thus, the welfare effect of a jump of the deficit ratio depends on parameter values, and especially on the shape of the utility function. Unfortunately, empirical evidence does not allow us to solve this question, because most of empirical investigations find a value of $S$ near or smaller than one, which, in our model, is consistent with either an improvement or a deterioration of intertemporal welfare. To have a more precise idea about welfare implications, Table 4 computes the net gain of intertemporal welfare (in %) for different deficit ratios, drawing on Fig.6. In accordance with Fig.6, deficit rules $(m > 0)$ dominate balanced-budget rule $(m = 0)$ if $S$ is low, and inversely if $S$ is high. We can notice that the welfare effect of different deficit rules is quite large, compared to the balanced-budget rule, notably for high deficit rules. Remark, in particular, that jumping from a balanced-budget rule to a $m = 3\%$ deficit rule gives rise to a near $12\%$ intertemporal welfare gain, if $S = 0.5$, but to a more than $15\%$ welfare loss, if $S = 1.5$.

Table 4: Net gain of welfare (in %) with a deficit rule of $m > 0$

<table>
<thead>
<tr>
<th>$\Delta U (m)/U^B$</th>
<th>$m = 0.01$</th>
<th>$m = 0.02$</th>
<th>$m = 0.03$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S = 0.5$</td>
<td>-0.02</td>
<td>2.49</td>
<td>11.89</td>
</tr>
<tr>
<td>$S = 1$</td>
<td>-4.41</td>
<td>-9.16</td>
<td>-14.13</td>
</tr>
<tr>
<td>$S = 1.5$</td>
<td>-4.44</td>
<td>-9.69</td>
<td>-15.74</td>
</tr>
</tbody>
</table>

For: $\rho = 0.075$, $\alpha = 0.6$, $\tau = 0.4$, $\delta = 0.025$, $k_0 = 1$

Table 4 and Fig. 6 show that the optimal deficit rule that a benevolent government, which maximizes household intertemporal welfare, should implement, is not necessarily the rule that maximizes long-run economic growth\(^{46}\) $(m = 0)$. Computing the first order condition for maximizing welfare\(^{47}\) in the lines of Judd (1982) or Futagami et al. (1993) would be inconclusive, since we do not have an explicit definition for the rate of economic growth. But simulations (see Table 4) clearly show that the deficit rule that maximizes intertemporal welfare is $m^* = 0$, if $S$ is large enough, and $m^* = \bar{m}$, for “low” values of $S$, where $\bar{m}$ is the maximum deficit rule consistent with the existence of a positive long-run growth rate (defined in Fig.2 above and whose value is computed in Appendix 2). In the latter case, the deficit rule that maximizes welfare conducts to a much damaged economic growth in long-run.

Non-linear effects of fiscal deficits on growth

To assess the welfare implications of a positive impulse in the deficit ratio, we supposed that the initial steady state is the Barro growth rate and the economy moves to a "high" growth rate. This experiment showed that, depending on parameters and particularly on the elasticity of substitution $S$, there may be conflicting effects between short run and long run growth. While these effects are important, notice that they represent only a (small) part of the non-linearities between deficits and growth, as emphasized by the rich dynamics in Fig.5. To fix ideas, we describe in Fig.7 initial adjustments and transitional dynamics, following a positive impulse on the deficit ratio, with initial steady states $L$ or $H$, in the spirit of Fig.5.

\(^{46}\)Futagami et al. (1993) already show that the tax rate that maximizes welfare is different from the tax rate that maximizes long-run economic growth, because of transitional dynamics, in their model with public capital. In our model, the same result appears with transitional dynamics coming from the fiscal rule.

\(^{47}\)Formally:

$$
\frac{dU^*}{dm} = k \int_0^T \left[ \frac{dc^*}{dm} + c^*_s \left( \int \frac{d\gamma^*}{dm} \right) \right] \exp \left( \int_{\gamma^*_s}^{\gamma^*} \frac{d\gamma'}{dm} \right) u^* \left( c^*_s \right) \exp \left(-\rho t\right) dt
$$
If the economy is initially in \( H \), we can either have a negative (as in Fig.7) or a positive jump in initial consumption, with a lower growth rate in the steady state \( H' \). However, if the economy initially departs from the low steady state \( L \), indeterminacy occurs. It should be kept in mind that \( L' \) is unstable, so that it cannot be reached. For a predetermined public-debt-to-capital ratio \( b_{k_0} = b_k^L \), we can either have an upward jump of the initial consumption-to-capital ratio (to \( c_{k_0}^H \)), if the agents anticipate that public debt will become unsustainable at some future date \( T \), or a downward jump (to \( c_{k_0}^L \)), if they are “optimistic” about public debt sustainability. In both cases, economic growth increases initially, but, because of the collapse of the consumption-to-capital ratio, it increases much more in the second case. In Table 3B, the rate of economic growth jumps from \( \gamma^H = 0.8\% \) to \( \gamma^L = 1.81\% \) in the second case, but only to \( \gamma^{L_0} = 0.84\% \) in the first one, with capital accumulation going through similar changes.

**Table 3B. Simulation results for an increase in the deficit-to-GDP ratio\(^{\dagger} \)**

<table>
<thead>
<tr>
<th>(%)</th>
<th>( b_k )</th>
<th>( c_k )</th>
<th>( g_k )</th>
<th>( \gamma_y )</th>
<th>( \gamma_k )</th>
<th>( \gamma_c )</th>
<th>( \gamma_b )</th>
<th>( \gamma_{bs} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Steady State</td>
<td>53.57</td>
<td>25.01</td>
<td>11.97</td>
<td>( \gamma^H = 0.80 )</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial jump of variables(^{\dagger}) (case 2, case 3)</td>
<td>53.57</td>
<td>24.28</td>
<td>12.60</td>
<td>1.81</td>
<td>1.78</td>
<td>1.44</td>
<td>-0.15</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>25.12</td>
<td>12.60</td>
<td>0.84</td>
<td>0.95</td>
<td>1.44</td>
<td>0.68</td>
<td>-0.27</td>
<td></td>
</tr>
<tr>
<td>Final Steady State(^{\dagger}) (( H',S ))</td>
<td>13.35</td>
<td>19.44</td>
<td>21.52</td>
<td>( \gamma^H = 8.13 )</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>54.60</td>
<td>25.46</td>
<td>11.21</td>
<td>( \gamma^S = 0 )</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For simulation values: \( \alpha = 0.6 \), \( \rho = 0.1 \), \( S = 2 \), \( \tau = 0.4 \), \( \delta = 0.05 \)

These initial jumps in consumption and in the growth rate of output or capital accumulation determine the transition path that will be reached by the economy. With a
downward jump of consumption, the economy is able to produce a sufficiently high rate of capital accumulation to reduce the debt-to-capital ratio. Consequently, all variables converge to the high steady state $H'$, with declining public debt ratio and self-enforcing economic growth. With an upward jump in consumption, on the contrary, the initial jump of economic growth does not allow public debt to decline. Then, the economy moves to the “poverty trap” (the $S$ point), which it reached at time $T$. In the simulation of Table 3B, $T = 2.35$.

Fig. 8: Non-linear effects of fiscal deficit on economic growth

Our model reveals the non-linear characteristic of a fiscal deficit impulse in the medium and long run. In all three cases of Fig. 8, public deficit is positively linked to economic growth in the short run. On the contrary, in the medium or long run, fiscal deficit may be positively or negatively related to economic growth, with an unchanged set of parameters, according to the initial public-debt-to-capital ratio. Both variables are negatively related in the neighborhood of the high steady state, where the public-debt-to-capital ratio is small (case 1). In the neighborhood of the low steady state, with a high public-debt-to-capital ratio, the relation between economic growth and fiscal deficit depends on expectations. If the fiscal policy stance is seen to become unsustainable at some future date, the economy converges to the no-growth solution: once more, economic growth and fiscal deficits are negatively related (case 3). If agents are “optimistic”, however, the initial cut in consumption is enough to generate a permanent positive growth effect and the economy converges to the high steady state. In this case, the link between public deficit and economic growth is positive (case 2). Thus, our model can reproduce the so-called “Keynesian” or “anti-Keynesian” behavior of consumption and economic growth, following a raise in the deficit to GDP ratio. When the public debt ratio is high, an expectation switch about its sustainability may reverse the effect of deficits on economic growth, as in Feldstein (1982) or Sutherland (1997), but in a very different set up without uncertainty.\(^{48}\)

In the spirit of the empirical work described in introduction, what produces this non-linearity is an expectation effect on the credibility of fiscal policy, when the initial debt-to-capital ratio is very high (Perotti, 1999). If households judge public debt as sustainable, the rise of public deficit is credible and the economy grows to the high solution. On the contrary, if the deficit rise is seen to turn unsustainable, the economy is condemned to a “poverty trap”.

\(^{48}\) Our model produces another kind on non-linearity, namely the asymmetric effects of a cut in deficit. Even if we abstract indeterminacy issues, a deficit cut will have a much bigger impact on the transitory growth path if the economy starts from the $L$ steady state (high initial debt), than if it starts from point $H$ (low initial debt).
VI. Conclusion

Does higher public deficit, increasing the burden of public debt, generate a crowding out effect on economic growth or does it ensure the development of productive infrastructures which promote long run growth-enhancing? In a very stylized model of endogenous growth, we show that the two effects may occur, depending fundamentally on i) the initial debt-to-capital (or output) ratio and ii) expectations about government debt sustainability. Although analytically simple, our model exhibits some complex dynamics, allowing for non-linear effects of fiscal policy, which may reproduce a number of empirical results.

Our model does not contradict Keynesian views in the short and medium run (a fiscal deficit impulse may increase economic growth), but it does in the long run. The same is true for the GRPF, and our model shows that one must be careful in assessing the benefits of fiscal deficits in terms of growth and/or welfare. Since Ricardian Equivalence does not hold, a debt financed increase in productive expenditures boosts economic growth in the short run, but the burden of repayment will weigh the balanced growth path down in the long run. The GRPF may be a good prescription in the short run, to reduce unemployment for instance, (this is the goal of Modigliani et al.’s (1998) Manifesto), but it can never promote long-run economic growth. Thus, despite the assertions of the GRPF proponents, allowing for debt-financed productive public expenditures does not ensure to fill a shortage of public investment in the long run. On the contrary, the shortage of public investment may be the consequence of excessive government borrowing, either for productive or unproductive expenditures.

Moreover, if the initial fiscal stance is close to equilibrium, allowing for debt-financed productive public expenditures does not ensure to fill a shortage of public investment in the long-run. On the contrary, the shortage of public investment may be the consequence of excessive government borrowing, either for productive or unproductive expenditures. However, if the initial fiscal stance is much damaged, our model produces another effect: a deficit-financed impulse in productive expenditures may (or not) generate self-fulfilling optimistic expectations, which allow the economy to leave the low-growth balanced path and to catch up with the high-growth balanced path. Unfortunately, the alternative between optimistic or pessimistic views on the future is not a matter of choice for the government, and the risk of such a debt policy is to condemn economy to a no growth trap.

Nevertheless, this high-growth balanced path will remain inferior to the path that may be obtained with a balanced-budget rule. In this perspective, one might ask why do governments borrow, if this implies a lower balanced growth path? Probably because governments are “short-termists”\textsuperscript{49}. Note that governments’ “short-termism” may come up to households’ expectations, since, in some cases, our model shows that borrowing for productive expenditures is welfare enhancing. However, the GRPF makes the burden of public debt be postponed to future generations, thus its intergenerational fairness might be questioned. Making our children pay for infrastructures that they will use in the future is one way that we really borrow from our children – and never pay them back\textsuperscript{50}. Besides, bringing investment projects forward in the name of our children is probably questionable, because they might want to do other choices tomorrow. Even if our current productive expenditures are designed for our children, these might criticize us tomorrow for making it impossible for them, because of our borrowing, to do for their own children what we have done for them.

\textsuperscript{49} Public debt may also produce extra resources for the government, like an “inflation tax” on the nominal issued debt or a mark-up between the market rate of interest and the public debt cost of finance, but these extra resources are unlikely to persist in the long run.

\textsuperscript{50} See, for example, Eisner (1996, p.89): “It is absurd to try to balance a budget that makes no distinction between government expenditures for current consumption and government expenditures of an investment nature […] balancing the budget at the expense of our public investment in the future is one way that we really borrow from our children – and never pay them back”.

27
Appendix 1: The steady state rates of economic growth with $S \neq 1$

If $S < 1$, notice first that the asymptote $\hat{g}_k = \left[ m\left(\frac{S-1}{S}\right) + \tau \right]^{1/a}$ is located on the left of $g_k = \tau^{1/a}$, so Proposition 2 trivially holds (this situation is depicted by the dotted lines, and the two steady state are $\hat{L}$ and $\hat{H}$). If $S > 1$, we can see that the $\gamma^2(g_k)$ curve intersects the $g_k = \tau^{1/a}$ line for the higher acceptable rate of economic growth $\left(\gamma^\text{max} = \frac{Sp}{S-1}\right)$, subject to the transversality condition. This implies that $\gamma^1(g_k)$ and $\gamma^2(g_k)$ intersect on the left-hand side of $g_k = \tau^{1/a}$, i.e. that the high growth solution $\gamma^H < \gamma^B$. This situation is depicted by the continuous curves, and the two steady states are $\hat{L}$ and $\hat{H}$.

Appendix 2: Analytical value of the steady state growth rates

Equation (12) can be rewritten as:

\[
\frac{\gamma}{S} + \rho + \delta = f(\tau) \left[ 1 - \frac{m}{\tau} \left( \frac{(1-S)\gamma + S\rho}{S\gamma} \right) \right]^{1-a}
\]

or in logarithm

\[
\log \left( \frac{\frac{\gamma}{S} + \rho + \delta}{f(\tau)} \right) = \left( \frac{1-a}{a} \right) \log \left( 1 - \frac{m}{\tau} \left( \frac{(1-S)\gamma + S\rho}{S\gamma} \right) \right),
\]

where

\[f(\tau) = \alpha(1-\tau)(\tau)^{\frac{1-a}{a}}.\]

For $m \to 0$, we can write:

\[
\frac{\gamma}{S} + \rho + \delta = \left[ 1 - \frac{(1-\alpha)m}{\alpha \tau} \left( \frac{(1-S)\gamma + S\rho}{S\gamma} \right) \right] f(\tau)
\]

remark that $\frac{\gamma}{S} + \rho + \delta \to f(\tau)$ in (A1), which is a second-degree equation in $\gamma$: $\gamma^2 + A\gamma + B = 0$, where
where: \( A \equiv S \left\{ \rho + \delta - f(\tau) \left[ 1 - \frac{m(1-\alpha)}{\alpha \tau S} \right] \right\} \) and \( B \equiv \frac{Sm \rho (1-\alpha) f(\tau)}{\alpha \tau} \). As the
determinant of this second degree equation in \( \tau \) is positive, it has two real (positive)
solutions: \( \gamma_+ = \frac{1}{2} \left[ -A + \sqrt{\Delta} \right] \) and \( \gamma_- = \frac{1}{2} \left[ -A - \sqrt{\Delta} \right] \), where: \( \Delta = A^2 - 4B \). For \( \Delta = 0 \), we
find the value of the deficit ratio \( m = \bar{m} \) (depicted in Fig.2) that equalizes the two rates of
economic growth (there are two solutions, but only one value verifies the transversality
condition).

**Appendix 3: Proof of Proposition 3**

From (12), extracting \( g_k \) from (13a) and using the implicit function theorem, we compute:
\[
\frac{d\gamma}{d\tau} = -\frac{\partial F}{\partial \tau} \left/ \frac{\partial F}{\partial \gamma} \right., \quad \text{with:} \quad \frac{\partial F}{\partial \tau} = \gamma' \left( (1-\tau)(1-\alpha) - \alpha g_k^a \right) (1-\alpha).
\]
Thus, the tax rate \( \tau^* \) that maximizes
economic growth is defined by: \( g_k \left( \tau^* \right) = \left( (1-\tau^*)(1-\alpha) / \alpha \right)^{1/\alpha} \). Introducing this value in
Eq.(13a), we obtain the maximized growth rate, as a function of \( \tau^* \):\[
\left[ \frac{\gamma^*}{S} + \delta + \rho \right]^a = \left[ \alpha \left( 1 - \tau^* \right) \right]^a \left[ g_k \left( \tau^* \right) \right]^{-a} = \left( 1 - \tau^* \right) Z, \quad \text{where:} \quad Z \equiv \alpha^{2a-1} (1-\alpha)^{-a}, \text{and}
\]
from (12): \( \left( 1 - \tau^* \right) \frac{\alpha}{m} = \left[ 1 + m \left( \frac{S-1}{S} \right) - \frac{m \rho}{\gamma} \right] \), thus:
\[
\left[ \frac{\gamma^*}{S} + \delta + \rho \right] = \left[ \frac{\gamma^*}{S} \left( \alpha Z \right)^{1/\alpha} - 1 \right] = \left[ 1 + m \left( \frac{S-1}{S} \right) - \frac{m \rho}{\gamma} \right].
\]
For
\[
m \to 0, \text{ we can write:} \quad \alpha \left( \frac{\gamma^*}{S} + \delta + \rho \right) = \frac{m \left( S-1 \right)}{S} - \frac{m \rho}{\gamma}, \text{ which is a second-degree}
equation in \( \gamma^* \): \( \left( \gamma^* \right)^2 + Ay^* + B = 0 \), where: \( A \equiv S (\rho + \delta) - S (\alpha Z)^{1/\alpha} - m (S-1)(\alpha Z)^{1/\alpha} / \alpha \)
and \( B \equiv Sm \rho (\alpha Z)^{1/\alpha} / \alpha \), with solutions \( \gamma_1^* \) and \( \gamma_2^* \). The maximum value of economic
growth is: \( \gamma_1^* = \frac{1}{2} \left[ -A + \sqrt{A^2 - 4B} \right] \). To compute \( \tau^* \), note that in (13b):
\[
\tau^* = 1 - \alpha + \alpha m \left( \frac{\rho}{\gamma} - \frac{S-1}{S} \right) > 1 - \alpha \quad \text{for the transversality condition to be respected, which}
proves Proposition 3.

**Appendix 4: The local stability of steady states**

In the neighborhood of the Barro steady state, we have, with
\[
\psi(b_k) = g_k \left( b_k \right) + \delta - \left( g_k \left( b_k \right) \right)^{1-a}:
\]
\[
J^B = \left[ \begin{array}{c}
c_k^a \\
g_k^a \\
\alpha S (1-\tau)(1-\alpha) \left( \frac{dg_k}{db_k} \right) (g_k^a)^{-a} + \frac{d\psi}{db_k} \end{array} \right].
\]
Clearly \( Det \left( J^B \right) = -c_k^a \gamma^a < 0 \), so that the Barro steady state is a saddle path.

In the neighborhood of Solow steady state, we have: 29
 Clearly  \( \text{Det}(J^s) = -\alpha S(1-\tau)(1-\alpha) \left( \frac{dg}{db} \right) S_i \left[ (1-a_i) - (g_i)^a \right] \)  and  \( c_i^s b_i^s (g_i^s)^a > 0 \). To show that:

\[
\begin{align*}
\text{Tr}(J^s) &= c_i^s + b_i^s \left( \frac{dy}{db} \right) S_i \left[ (1-a_i) - (g_i)^a \right] > 0, \\
&\text{note that: } b_i \left( \frac{dy}{db} \right) S_i \left[ (1-a_i) - (g_i)^a \right] = \frac{rb_i (g_i)^{1-a} - (g_i)^a}{\alpha g_k + (1-\alpha) \delta b_k}.
\end{align*}
\]

\( c_k = (g_k)^{1-a} - g_k - \delta = (1-\tau)(g_k)^{1-a} - \delta + rb_k = r(1-\tau)(g_k)^{1-a} + rb_k, \) so:

\[
\begin{align*}
c_k + b_k \left( \frac{dy}{db} \right) S_i &= r(1-\tau)(g_k)^{1-a} + \frac{(1-\alpha) rb_k ((g_k)^{1-a} + \delta b_k - g_k)}{\alpha g_k + (1-\alpha) \delta b_k} > 0
\end{align*}
\]

**Local stability of steady states if \( m > 0 \) (\( \lambda_i \) are the eigenvalues of \( J^s \))**

<table>
<thead>
<tr>
<th>(%)</th>
<th>( \text{Det}(J^H) )</th>
<th>( \text{Det}(J^L) )</th>
<th>( \text{Tr}(J^H) )</th>
<th>( \text{Tr}(J^L) )</th>
<th>( \lambda_1^H, \lambda_2^H )</th>
<th>( \lambda_1^L, \lambda_2^L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 1 )</td>
<td>-1.48</td>
<td>1.83</td>
<td>10.03</td>
<td>27.47</td>
<td>18.17</td>
<td>16.18</td>
</tr>
<tr>
<td>( m = 2 )</td>
<td>-1.24</td>
<td>1.47</td>
<td>10.92</td>
<td>25.39</td>
<td>17.85</td>
<td>16.41</td>
</tr>
<tr>
<td>( m = 3 )</td>
<td>-0.91</td>
<td>1.04</td>
<td>12.29</td>
<td>22.83</td>
<td>17.51</td>
<td>16.56</td>
</tr>
</tbody>
</table>

For simulation values: \( \alpha = 0.6, \rho = 0.1, S = 2, \tau = 0.4, \delta = 0.05 \)

**Appendix 5: The model with public expenditure as a stock variable**

Following the analysis of Barro & Sala-i-Martin (1992) and Futagami et al. (1993), we extend our model to public expenditure modeled as a stock variable. Replacing \( g \) with \( g + \delta^s \) (\( \delta^s \) is the public capital depreciation rate), the IS equilibrium (7) becomes:

\[
\gamma_k = (g_k)^{1-a} - c_k - \delta - (\gamma_k + \delta^s) g_k.
\]

The public debt evolution in (8) now is:

\[
\gamma_b = \alpha(1-\tau)(g_k)^{1-a} - \delta + \frac{\tau (g_k)^{1-a} - (\gamma_s + \delta^s) g_k}{(b_k)},
\]

with (6) unchanged. In steady state, public capital grows at the same rate as the other variables: \( \gamma^s = \gamma \). Thus, (13) becomes:

\[
H(g_k; \tau, m) = \gamma(g_k) \left[ (m+\tau) g_k^{1-a} - g_k (\gamma(g_k) + \delta^s) \right] - mr(g_k) g_k^{1-a} = 0
\]

with \( r(g_k) = \alpha(1-\tau)(g_k)^{1-a} - \delta \) and \( \gamma(g_k) = S [ r(g_k) - \rho ] \). We can easily remark that our steady state results are qualitatively unchanged (in particular Propositions 2 and 3). The two figures below represent respectively \( F(g_k; \tau, m) = 0 \) and \( H(g_k; \tau, m) = 0 \).
For simulation values: $\alpha = 0.6$, $\rho = 0.05$, $S = 1$, $\tau = 0.4$, $\delta = 0.05$

Note that $\gamma^2$ function becomes (we extend the more general expression (13c), and we express the inverse $g^a_k(\gamma^2)$ function for simplicity): $g^a_k(\gamma^2) = \frac{\gamma^2 \left[m \left(\frac{S-1}{S}\right) + \tau\right] - \rho m}{\gamma^2 \left[1 - \mu \left(\frac{S-1}{S}\right)\right] + \rho \mu \gamma^2 + \delta^\prime}$. This function describes a growing relation between $\gamma^2$ and $g_k$ for $\gamma^2 < \gamma$, and a decreasing curve for $\gamma^2 > \gamma$. $\gamma$ is the only positive rate of growth that maximizes $g^a_k(\gamma^2)$. For instance, if $\mu = \delta^\prime = 0$, $\gamma = 2\rho m \left[m \left(\frac{1-S}{S}\right) + \tau\right]^{-1}$. The $\gamma$ curve (13a) is unchanged. It is easy to compute: $\frac{dg_k}{dm}_{\gamma \text{ given}} < 0$ and $\frac{dg_k}{d\mu}_{\gamma \text{ given}} < 0$, which extends Proposition 2 to a model with public investment expenditures (the $\gamma^2$ curve moves towards the left).

To extend Proposition 3, note that $\frac{d\gamma}{d\tau} = -\frac{\partial H}{\partial \tau}/\frac{\partial H}{\partial \gamma} = 0$ if $(\gamma + \delta^\prime) \left(g_k(\tau^*)\right)^\alpha = \frac{(1-\tau^*) (1-\alpha)}{\alpha}$. Now, in the government budget constraint:

$(\gamma + \delta^\prime) g^a_k = \left(\tau - m \left(\frac{r - \gamma}{\gamma}\right)\right)$. Thus, the tax rate $\tau^*$ that maximizes economic growth is still:

$\tau^* = 1 - \alpha + \alpha m \left(\frac{\rho}{\gamma} - \frac{S-1}{S}\right) > 1 - \alpha$.

As regards transitional dynamics, in an appendix available on request, we show that the two steady states have the same local stability properties. Note that, with public investment, system (11) becomes a three variables system, with an additional accumulation (predetermined) variable $g$, which does not allow for two-dimensional representations. Using simulations, we show that, except for a scale effect, none of our results are affected: the low equilibrium is unstable, with a negative and two positive eigenvalues, while the high equilibrium is a saddle path, with a positive and two negative eigenvalues.
References