Using a Projection Method to Analyze Inflation Bias in a Micro-Founded Model*

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October 23, 2008

Abstract

Since Kydland and Prescott (1977) and Barro and Gordon (1983), a quadratic one-period loss function has been used widely in most of existing literature to analyze inflation bias induced by discretion. In this paper, instead of following the conventional linear-quadratic approach, we use a projection method to investigate the size of inflation bias in a micro-founded non-linear model where firms follow the Calvo pricing-setting rule. The size of inflation bias is between 1% and 4%, depending on parameter values. These results on inflation bias based on a projection method are then compared with those based on the linear-quadratic approximation as in Woodford (2003): The linear-quadratic approximation underestimates the inflation bias by up to about a tenth.

*The authors appreciate comments by Andrew Levin, David Lopez-Salido, Alex Wolman, and participants at 2008 SCE Conference, as well as encouragements from Mike Woodford. The views in this paper are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or any other person associated with the Federal Reserve System.
1 Introduction

Since Kydland and Prescott (1977) initiated the literature of rules versus discretion, improvement upon discretionary equilibria by reducing inflation bias has long been a research theme in policy circles as well as academia, including Barro and Gordon (1983), Clarida, Gali and Gertler (1999), and Woodford (2003). In most of existing papers on the inflation bias, a quadratic one-period loss function is assigned to the central bank:

\[ \pi_t^2 + \lambda(x_t - x^*)^2 \]

where \( \pi \) is inflation rate and \( x \) is output gap. In many applications, the resulting inflation bias is linear in the log deviation of the steady state level of the aggregate output from its efficient level, \( x^* \).

This paper does not follow such a conventional linear-quadratic approach to the inflation bias induced by discretion. Instead, we use a projection method to analyze the inflation bias in a micro-founded non-linear model where firms follow the Calvo pricing-setting rule. To do so, we characterize a set of optimality conditions for the discretionary policy problem without any approximations. We then use Chebyshev polynomials to approximate policy functions that links inflation and output to a set of state variables, thereby converting optimization conditions into a set of non-linear equations for the coefficients of Chebyshev polynomials.

Our choice of a projection method is legitimate in the following sense. It is well known that: Rotemberg and Woodford (1997) and Benigno and Woodford (2006) have provided a micro-foundation for the use of such a loss function by showing that this simple quadratic function is the second-order approximation to the non-linear social welfare function in a Calvo model. But as discussed in Woodford (2003), their derivation does not hold under discretion unless the steady-state level of output under full flexibility is sufficiently close to its efficient level; they approximate the model around the deterministic steady state with zero inflation, but the optimal policy under discretion leads to a unknown positive inflation in a general case. In particular, it is not practically saving time to use a perturbation method in approximating the social welfare if one wants to analyze the optimal policy under discretion. Instead, since we do not know the steady state around which policy functions of the model should be perturbed, it would be reasonable to use a projection method to find the unknown steady state.
Our paper is not the only one to analyze the optimal discretion in a nonlinear Calvo model. Wolman and van Zandweghe (2008) use a fixed point algorithm of dynamic programming to see if multiple Markov Perfect Equilibria can arise in the Calvo model—as compared with the results of King and Wolman (2004) for the Taylor pricing contract. In addition, Adam and Billi (2007) work on the optimal discretion policy in a non-linear setting, and their primary focus is to characterize the optimal policy while allowing for the zero bound for nominal interest rate.

2 Economic Structure and Optimal Policy

2.1 Economic Structure

2.1.1 Households

At period 0, the preference ordering of the representative household is summarized by

\[
E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_{t}^{1-\sigma} - 1}{1 - \sigma} - \frac{\upsilon H_{t}^{1+\chi}}{1 + \chi} \right], \tag{1}
\]

where \( C_t \) denotes consumption, \( H_t \) denotes hours worked. The parameter \( \beta \) denotes discount factor, \( \sigma \) measures the degree of relative risk aversion, \( \chi \) controls the labor supply elasticity, and \( \upsilon \) plays the role of fixing the steady-state level for labor. Households purchase differentiated goods in retail market and combine them into a single composite good using the Dixit-Stiglitz aggregator, and utills of households and government activities depend only upon the amount of the composite good. The demand curve for each good \( z \) can be derived from the following cost-minimization:

\[
\min \int_0^1 P_t(z) C_t(z) \, dz \quad s.t. \quad C_t = \left( \int_0^1 C_t(z) \frac{\epsilon - 1}{\epsilon} \, dz \right)^{\frac{\epsilon}{\epsilon - 1}}, \quad \epsilon > 1, \tag{2}
\]

where \( P_t(z) \) represents the nominal price of good \( z \) and \( C_t(z) \) is its demand. The first-order condition of this cost-minimization problem yields the demand curve for firm \( z \):

\[
C_t(z) = \left( \frac{P_t(z)}{P_t} \right)^{-\epsilon} C_t. \tag{3}
\]
The parameter $\epsilon$ represents the elasticity of demand, and the aggregate price level $P_t$ is
\[
P_t = \left( \int_0^1 P_t^{1-\epsilon}(z) \, dz \right)^{\frac{1}{1-\epsilon}}.
\] (4)

The household’s dynamic budget constraint at period $t$ is given by
\[
C_t + E_t \left[ Q_{t,t+1} \frac{B_{t+1}}{P_{t+1}} \right] = \frac{B_t}{P_t} + (1 - \tau_W) \frac{W_t}{P_t} H_t + \Phi_t - T_t,
\] (5)

where $B_{t+1}$ is the nominal payoff at period $t + 1$ of the bond-portfolio held at period $t$, $W_t$ is nominal wage, and $\Phi_t$ is the real dividend income, $T_t$ is the real lump-sum tax, and $\tau_W$ denotes a constant rate of employment tax (or subsidy when negative) that is proportional to labor income. In addition, $Q_{t,t+1}$ is the stochastic discount factor used for computing the real value at period $t$ of one unit of consumption goods at period $t + 1$. Hence, if $R_t$ represents the risk-free nominal (gross) rate of interest at period $t$, the absence of arbitrage in equilibrium leads to
\[
\frac{1}{R_t P_t} = E_t \left[ \frac{Q_{t,t+1}}{P_{t+1}} \right].
\] (6)

The representative household maximizes (1) subject to the flow budget constraints (5) in each period. The first-order conditions are given by
\[
v C_t^{\sigma} H_t^x = (1 - \tau_W) \frac{W_t}{P_t},
\] (7)
\[
Q_{t,t+1} = \beta \left( \frac{C_t}{C_{t+1}} \right)^{\sigma},
\] (8)

and substitution of (8) into (6) yields the consumption Euler equation:
\[
\beta R_t E_t \left[ \left( \frac{C_t}{C_{t+1}} \right)^{\sigma} \frac{P_t}{P_{t+1}} \right] = 1.
\] (9)

2.1.2 Firms

Each firm produces a differentiated good $z$ using a constant returns to scale production function:
\[
Y_t(z) = A_t H_t(z),
\] (10)
where $Y_t(z)$ denotes the output of firm $z$, and $H_t(z)$ denotes the hours hired by the firm. Firms set prices as in the sticky price model of Calvo (1983). Specifically, each period a fraction of firms $(1 - \alpha)$ are allowed to change prices, whereas the other fraction, $\alpha$, do not change. Let $P^*_t$ be the new price charged by the firms resetting their price. Then, resetting firms choose a new optimal price in order to maximize the following expected discounted sum of profits:

$$
\sum_{k=0}^{\infty} \alpha^k E_t \left[ Q_{t,t+k} \left( (1 - \tau_p) \frac{P^*_t}{P_{t+k}} - \frac{W_{t+k}}{A_{t+k}P_{t+k}} \right) \left( \frac{P^*_t}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} \right],
$$

where $\tau_p$ denotes the amount of proportional revenue tax (or subsidy when negative). Differentiating this expression with respect to $P^*_t$ gives rise to the first-order condition:

$$
\sum_{k=0}^{\infty} \alpha^k E_t \left[ Q_{t,t+k} \left( (1 - \tau_p) \frac{P^*_t}{P_{t+k}} - \frac{\epsilon}{\epsilon - 1} \frac{W_{t+k}}{A_{t+k}P_{t+k}} \right) P^*_{t+k} Y_{t+k} \right] = 0.
$$

Besides, the Calvo type staggering transforms equation (4) into

$$
P_t = \left[ (1 - \alpha) (P^*_t)^{1-\epsilon} + \alpha P_{t-1} \right]^{\frac{1}{1-\epsilon}}. \tag{13}
$$

Next, we will show that the profit maximization condition (12) can be rewritten in a recursive way. In order to do this, note that substituting (8) into (12) and then rearranging, we have

$$
(1 - \tau_p) \sum_{k=0}^{\infty} (\alpha \beta)^k E_t \left[ \left( \frac{Y_{t+k}}{C_{t+k}^\alpha} \right) \left( \frac{P_{t+k}}{P_t} \right)^{\epsilon-1} \right] \frac{P^*_t}{P_t} \tag{14}
$$

$$
= \frac{\epsilon}{\epsilon - 1} \sum_{k=0}^{\infty} (\alpha \beta)^k E_t \left[ \left( \frac{W_{t+k} Y_{t+k}}{A_{t+k} P_{t+k} C_{t+k}^\alpha} \right) \left( \frac{P_{t+k}}{P_t} \right) \right].
$$

It is now useful to define two variables, $F_t$ and $V_t$, as follows.

$$
F_t = (1 - \tau_p) \sum_{k=0}^{\infty} (\alpha \beta)^k E_t \left[ \left( \frac{Y_{t+k}}{C_{t+k}^\alpha} \right) \left( \frac{P_{t+k}}{P_t} \right)^{\epsilon-1} \right], \tag{15}
$$

$$
S_t = \frac{\epsilon}{\epsilon - 1} \sum_{k=0}^{\infty} (\alpha \beta)^k E_t \left[ \left( \frac{W_{t+k} Y_{t+k}}{A_{t+k} P_{t+k} C_{t+k}^\alpha} \right) \left( \frac{P_{t+k}}{P_t} \right) \right].
$$
We then have the following recursive representations of the two variables \( F_t \) and \( S_t \):

\[
F_t = (1 - \tau_p) \frac{Y_t}{C_t} + \alpha \beta E_t \left[ \Pi_{t+1}^{\epsilon-1} F_{t+1} \right],
\]

(16)

\[
S_t = \frac{\epsilon}{\epsilon - 1} \left( \frac{W_t}{P_t A_t} \right) \left( \frac{Y_t}{C_t} \right) + \alpha \beta E_t \left[ \Pi_{t+1}^{\epsilon} S_{t+1} \right],
\]

(17)

with two terminal conditions,

\[
\lim_{T \to \infty} (\alpha \beta)^T E_t \left[ \prod_{k=1}^{T} \Pi_{t+k-1} \right] F_{t+T} = 0,
\]

\[
\lim_{T \to \infty} (\alpha \beta)^T E_t \left[ \prod_{k=1}^{T} \Pi_{t+k}^{\epsilon} \right] S_{t+T} = 0,
\]

where \( \Pi_t = P_t/P_{t-1} \). We now substitute the definitions of \( F_t \) and \( S_t \) specified in (15) into the profit maximization condition (14) to yield

\[
\frac{P^*_t}{P_t} = \frac{S_t}{F_t}.
\]

(18)

In addition, substituting equation (18) into (13) leads to

\[
1 = (1 - \alpha) \left( \frac{S_t}{F_t} \right)^{1-\epsilon} + \alpha \Pi_t^{\epsilon-1}.
\]

(19)

As a result, we have expressed the profit maximization condition (14) and the price level definition (13) in terms of \( F_t \) and \( S_t \) with their intertemporal evolution equations (16) and (17).

### 2.1.3 Social Resource Constraint

In any sticky price model with a staggered price setting, relative prices can differ across firms. Besides, if firms have different relative prices, there are distortions that create a wedge between the aggregate output measured in terms of production factor inputs and the aggregate demand measured in terms of the composite goods. In order to see such relative price distortions, individual outputs are linearly aggregated:

\[
A_t H_t = Y_t \int_0^1 \left( \frac{P_t(z)}{P_t} \right)^{-\epsilon} dz,
\]
where $H_t = \int_0^1 H_t(z)dz$. By defining a measure of relative price distortions as

$$\Delta_t = \int_0^1 \left( \frac{P_t(z)}{P_t} \right)^{-\epsilon}dz,$$  

(20)

the aggregate production function can be written as follows:

$$Y_t = \frac{A_t}{\Delta_t}H_t.$$  

(21)

In order to obtain a law of motion for the measure of relative price distortion described above, note that the Calvo type staggering allows one to rewrite the measure of relative price distortions specified in equation (20) as

$$\Delta_t = (1 - \alpha) \left( \frac{P_t^*}{P_t} \right)^{-\epsilon} + \alpha \Pi_t \Delta_{t-1}.$$  

(22)

Hence, substituting (13) into (22), one can derive an expression of how the measure of relative price distortions evolves over time:

$$\Delta_t = (1 - \alpha) \left( \frac{1 - \alpha \Pi_t^{-1}}{1 - \alpha} \right)^{\frac{-\epsilon}{1+\epsilon}} + \alpha \Pi_t \Delta_{t-1}. $$  

(23)

Finally, the aggregate market clearing condition is given by

$$Y_t = C_t,$$  

(24)

so the social resource constraint in period $t$ is therefore given by

$$\frac{A_t}{\Delta_t}H_t = C_t.$$  

(25)

2.2 Optimal Policy under Discretion

Following Woodford (2003), we define optimal policy under discretion as “a procedure under which at each time that an action is to be taken, the central bank evaluates the economy’s current state and hence its possible future paths from now on, and chooses the optimal current actions in the light of this analysis, with no advance commitment about future actions, except that they will similarly be the ones that seem best in whatever state may be reached in the future.” Before proceeding, it is worthwhile to discuss
implementation conditions of the optimal policy problem, which restrict the feasible allocations of the social planner. First, the household budget constraint is not included as a constraint for the optimal policy problem because of the lump-sum tax. Second, the size of employment subsidy rate determines whether the profit maximization condition is binding or not as an implementation condition in the optimal policy problem.

In order to gain some insights about the role of employment subsidy, we describe equilibrium condition of the flexible price model and then compare them with those of the first-best equilibrium. Since \( \alpha = 0 \) corresponds to the flexible price model, it follows from (12) that the profit maximization condition for the flexible price model turns out to be

\[
\frac{\bar{W}_t}{\bar{P}_t} = (1 - \tau_P) \left(1 - \epsilon^{-1}\right) A_t.
\]

(26)

where \( \bar{W}_t \) and \( \bar{P}_t \) are the nominal wage rate and the price level in the flexible price model. Combining (7) with (26), we can see that the relationship between MRS and MPL in the flexible price model is given by

\[
v \frac{\bar{C}_t}{\bar{H}_t} = (1 - \tau_P) \left(1 - \tau_W\right) \left(1 - \epsilon^{-1}\right) A_t,
\]

(27)

where \( \bar{C}_t \) and \( \bar{H}_t \) denote consumption and labor in the flexible price model. We can see from (27) that when we set \((1 - \tau_P) (1 - \tau_W) (1 - \epsilon^{-1}) = 1\), the flexible price model can achieve the efficient level of output—which would be attained at the perfectly competitive equilibrium.

We now characterize the optimal policy problem under discretion. The government at period 0 chooses a set of decision rules for \{ \( C_t, H_t, F_t, S_t, \Pi_t, \Delta_t \} \}_{t=0}^{\infty} \) in order to maximize

\[
V(\Delta_{t-1}) = \max \left\{ \frac{C_t^{1-\sigma} - 1}{1 - \sigma} - \frac{vH_t^{1+\chi}}{1 + \chi} + \beta E_t [V(\Delta_t)] \right\},
\]

(28)

subject to the following equilibrium conditions in each period \( t = 0, 1, \ldots, \infty \):

\[
C_t = \frac{A_t}{\Delta_t} H_t,
\]

(29)

\[
F_t = (1 - \tau_P) \frac{A_t H_t}{\Delta_t C_t^{\sigma}} + \alpha \beta E_t \left[ \Pi_{t+1}^{-1} F_t \right],
\]

(30)

\[
S_t = \frac{vH_t^{1+\chi}}{(1 - \tau_W) (1 - \epsilon^{-1}) \Delta_t} + \alpha \beta E_t \left[ \Pi_{t+1}^{-1} S_t \right],
\]

(31)
\[ \Delta_t = (1 - \alpha) \left( \frac{1 - \alpha \Pi_t^{t-1}}{1 - \alpha} \right)^{\frac{1}{1 - \epsilon}} + \alpha \Pi_t \Delta_{t-1}, \quad (32) \]

\[ S_t = F_t \left( \frac{1 - \alpha \Pi_t^{t-1}}{1 - \alpha} \right)^{\frac{1}{1 - \epsilon}}. \quad (33) \]

The overall distortion in the steady-state output level as a result of taxes/subsidies and market power is summarized by

\[ \Phi = 1 - (1 - \tau_P) (1 - \tau_W) \left( 1 - \epsilon^{-1} \right). \]

3 Projection Methods and Numerical Results

This section describes our use of a projection method to obtain solutions for the model under discretion. Our nonlinear system of equations has features that complicate the solution process, and we describe the strategies that we have adopted to overcome these complications. In particular, we found that in the course of obtaining valid solutions over the relevant range of state variables, it was important to have flexibility in setting and readjusting the range of these variables. We will describe a homotopy procedure—our iterative scheme for reliably improving on our initial guesses for the nonlinear solution. This section also present the numerical results regarding the size of optimal inflation under discretion and compares these results from a nonlinear numerical analysis with those from a linear-quadratic approximation analysis (e.g. Woodford, 2003).

3.1 Projection Methods

Yun (2005) provides an analytic solution for optimal policy—for both discretion and commitment—in the case of no monopolistic distortion. We have not been able to obtain an analytic solution for the model under distortion and must apply some numerical approximation technique. Since computation of the optimal policy under discretion involves how future variables are determined as a function of current variables, we cannot apply a perturbation method. Instead, we apply a projection method.

Judd (1998) has provided a generic description of how to apply projection methods which we will use in the following pages to organize the description
of our particular implementation. We solve the operator equation

$$\mathcal{N}(f) = 0$$

where $\mathcal{N} : B_1 \to B_2$ and $f : D \subset \mathbb{R}^m \to \mathbb{R}^n$

as follows:

1. Choose a basis and a norm over $B_1$, and a basis and an inner product over $B_2$.
2. Choose a degree of approximation, $n$, to define $\hat{f} \equiv \sum_{i=1}^{n} a_i \varphi_i(x)$.
3. Construct approximation $\hat{N}$ to define $R(x; a) \equiv \mathcal{N}(\hat{f}(\cdot; a))(x)$.
4. Compute $\| R(\cdot; a) \|$ or choose $p_i$ and compute $P_i(a) \equiv < R(\cdot; a), p_i(\cdot) >$.
5. Find $a \ni P(a) = 0$.
6. Verify the quality of the solution.

**Step 1: Choose basis, norms and inner products**

Since, for the economic models we plan to address, we are only interested in bounded values of the state variables, we choose $B_1$ to be a tensor product of closed finite one dimensional intervals $B_1 = [l_{x_1}, u_{x_1}] \otimes \ldots \otimes [l_{x_n}, u_{x_n}]$. For the experiments with the model at hand, there is only one state variable $\Delta_{t-1}$, so $B_1$ will just be a one dimensional interval containing the values of $\Delta_{t-1}$.

The economic models we will entertain will have both state and non-state variables. The models will have variables dated time $t$ as well as future values of the variables, dated time $t+1$. These models will generally have a number of equations, $m$, $m = m_n + m_s$, relating these variables where $m_s$ is the number of state and $m_n$ is the number of non state variables. As a result, $B_2$ will be contained in $\mathbb{R}^m$.

We have chosen Chebyshev polynomials of the first kind for our basis functions. The Chebyshev polynomials are a sequence of orthogonal polynomials which are important in approximation theory because the resulting interpolation polynomial provides an approximation that is close to the polynomial of best approximation to a continuous function under the maximum norm. However, since Chebyshev polynomials are only defined on the interval $[-1, 1]$, we have linearly interpolated each
finite range to that interval. For $B_1$, we choose the norm induced by the inner product:\footnote{In the future we intend to apply a Galerkin Method, in which case we will use an analogous inner product for for $B_2$. Alternatively, one can solve an orthogonal collocation problem. The roots of the Chebyshev polynomials of the first kind, which are also called Chebyshev nodes, are useful as nodes in polynomial interpolation since, their concentration of nodes near the border of $B_1$ reduces the magnitude of polynomial oscillation there and gives the polynomial approximation its maximum norm optimality property. Since, so far, we have only applied the collocation method. The choice of inner product, $B_2$, has played no role.}

$$< f, g > = \int_{B_1} f(x)g(x) \left[ \prod_{i=1}^{n} (1 + x_i^2)^{-\frac{1}{2}} \right] dx$$

so that $\| f \| = < f, f >$.

**Step 2: Choose a degree of approximation**

We make a distinction between state and non state variables. We construct a tensor product of Chebyshev polynomials for the state variables alone. For each state variable, we specify a polynomial degree. Although the code is more general, the results presented here are for a univariate polynomial. We evaluate each basis element at each of the Chebyshev nodes. The state and non-state variables at time $t$ are a weighted sum of the basis functions. We employ function composition of the polynomial functions in order to compute the time $t + 1$ values. We take care to ensure that the values of the variables will stay within the range defined for the Chebyshev polynomial since this is not be guaranteed for arbitrary settings of the polynomial weights.

**Step 3: Construct approximation**

We use a simplified version of the nonlinear equation to construct an initial guess for the zeroth order polynomial for the endogenous variables. We then increase the degree using the $k$th order solution as an initial guess for the $(k + 1)$-order solution.

**Step 4: Compute** $P_i(a) \equiv < R(\cdot ; a), p_i(\cdot ) >$

We evaluate the system of equations using each basis element at each of the Chebyshev nodes. The code provides a vector of values and a matrix of derivatives for use in Newton’s method for solving the
nonlinear system of equations. We use Mathematica to generate Java Code that implements the derivatives.\footnote{We have implemented univariate Gauss Hermite integration but do not use it in the present exercise.}

**Step 5: Solve Collocation Equation**

Given an initial guess for polynomial weights, we use Newton’s method with analytic derivatives to solve the system of collocation equations.

**Step 6: Verify the Quality of Solution**

We have compared the analytic solution with our polynomial solution for the case when \( \Phi = 0 \), and the solution is the same as the analytic solution available in Yun (2005); the 10th degree polynomial agrees to very high precision.

### 3.2 Some Implementation Issues

Initially, we investigated adapting existing code for solving the problem. We have located freely available FORTRAN code from Judd and MATLAB code from Gapen and Cosimano. We found that the code was very useful for benchmarking and validation but difficult to modify to solve our particular problem.

We undertook developing generic Java code with the expectation that it would be faster than MATLAB, comparable to “C”/Fortran in performance. In addition our implementation uses Mathematica and we anticipate developing a convenient interface to MATLAB and DYNARE. Further practical aspects of the choice include the benefit that Java code will run on Linux and Windows machines. In addition, there are many free development tools for debugging and documenting the code. We have since discovered that JBENGE, a tool for estimating and simulating DSGE models has also adopted Java as their implementation language.

Solving the nonlinear system in Step 4 is by far the most difficult task to accomplish reliably. There is no guarantee that the current approximation will generate values for the time-\( t \) state variables that lie in \( B_1 \) consequently, the Chebyshev polynomials variables may be applied to values outside their range of definition. This may not prove to be a problem so long as the Newton step converges to a solution where the values are within the range...
of definition, but the intermediate polynomial functions used in conjunction with the model equations will produce other numerical errors (e.g. the log of a negative number).

For the problem at hand, we have found that—to obtain convergence for a given degree of approximation—it is important to start with a small range of values of $\Delta_{t-1}$ in the definition of the Chebyshev polynomials and then to gradually extend the range. We have written the code to make it easy to adjust the variable interpolation range. For a given set of parameters and a given degree of approximation, we systematically adjust the range from small to large.

There are a number of desirable features of the code that could be addressed in the future. We envision developing a generic open source tool, but currently the code depends on Mathematica; we would like to develop a Dynare interface. The program uses a simple operator overloading while it would be preferable to use more efficient automatic differentiation techniques.

### 3.3 Numerical Results

To investigate the size of optimal inflation under discretion, we have to give numerical values to the parameters. Although we experiment with many different values, the benchmark parameter values are taken from Yun (2005). For example, we assume that utility is logarithmic in consumption ($\sigma = 1$) and quadratic in labor ($\chi = 1$). We also set $\epsilon = 11$, $\alpha = 0.75$, and $\beta = 0.99$. The only difference from his paper is that, in our paper, there is no subsidy to nullify the monopolistic distortion so the degree of monopolistic distortion is kept at $\Phi = \epsilon^{-1}$.

At this benchmark specification, we solve the model using a projection method where the range of the state vector is specified as $(1, 3)$. Figure 1 illustrates the solution of this discretionary equilibrium. The solid line represents the values of $\Delta_t$ as a function of $\Delta_{t-1}$, shown for the range of $(1, 1.02)$. This line crosses the 45-degree line (dotted) at around 1.0016—that is the steady-state value for the dispersion measure. At this steady state, the value of $\bar{\Pi}$ is about 1.0044 (dashed line). In terms of the annualized rate for net inflation, this steady state corresponds to 1.8% inflation.

It has been widely known that the size of inflation bias depends on the degree of monopolistic competition, since imperfect competition induces the difference between the outcome of the market economy and that of the social planner. In our benchmark specification used in Figure 1, the elasticity of
substitution ($\epsilon$) determines how monopolistically competitive the economy is, and the size of distortion ($\Phi$) is equal to its inverse. Figure 2 shows how the discretionary equilibrium behaves as we increase the value of $\Phi$. In terms of the model parameters, we are increasing the value of $\tau_P$ or $\tau_W$ from the efficient level of subsidy as specified in Yun (2005). Note that the range of $\Phi$ in this figure covers our benchmark parametrization of $\Phi = 0.09$. The solid line represent the size of inflation bias, and the dashed line corresponds to the steady-state value for the price dispersion ($\Delta_t$). The size of inflation bias increases slightly faster than linearly with respect to the size of monopolistic distortion.

To better understand how other parameters affect the size of inflation bias, we do some comparative statics of the model. First, the smaller the curvature parameters ($\sigma$ and $\chi$), the bigger the inflation bias. When the utility function moves closer to linearity with respect to consumption and labor, the size of inflation bias becomes very high. Second, the higher the elasticity parameter ($\epsilon$), the smaller the inflation bias. Third, a higher value of $\alpha$ increase the size of inflation bias only slightly.

3.4 Comparison with the Linear-Quadratic Approximations

We now compare our results with those from the conventional linear-quadratic approach (e.g. Woodford, 2003). The size of inflation bias under this approach would be

$$\bar{\pi} = \left( \frac{\kappa \lambda}{(1 - \beta) \lambda + \kappa^2} \right) \frac{\Phi}{\sigma + \chi},$$

where $\kappa = (1 - \alpha)(1 - \alpha \beta)(\sigma + \chi)/\alpha$ and $\lambda = \kappa/\epsilon$. Under our benchmark calibration, this formula produces an inflation bias of 1.6% per annum. The size of inflation bias based on the projection method is 1.8% in annual rate. These two values indicate that the linear-quadratic approach underestimate the inflation bias by about a tenth.

Figure 3 depicts how much the linear-approximation underestimates the inflation bias depending on the value of $\Phi$. The solid line is inherited from the previous figure; the dashed line represent the level of inflation bias under the linear-quadratic approximation: $(1 + \bar{\pi})$. The dashed line is linear and

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3These numerical results are presented in the Appendix. These results are all based on $\beta = 0.99$ and $\Phi = 0.11$. 

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tangential to the solid line near $\Phi = 0$. The difference between the two lines becomes larger as monopolistic distortion moves the economy farther away from the efficient outcome. Finally, the table in the Appendix shows that as the two preference parameters ($\sigma$ and $\chi$) get smaller, the difference between the two approaches becomes larger. In particular, the linear-quadratic approximation can underestimate the size of inflation bias by about thirteen percent when $\chi = 0$ and $\sigma = 1$.

Even though the linear-quadratic approach underestimate the size of inflation bias, the simplicity and transparency of this approach would be very helpful in relating the size of inflation bias to the values for the parameters. Since the discount factor is close to unity, we can approximate $\bar{\pi}$ by $\Phi/\epsilon(\sigma + \chi)$. This expression clearly tells us that the inflation bias is (approximately) proportional to the size of monopolistic distortion and that it is inversely related to $\epsilon$, $\sigma$ and $\chi$. It should, of course, be noted that such a result from the linear-quadratic approximation could potentially be misleading. For example, when the discount factor is less than unity, the expression for $\bar{\pi}$ implies that an increase in $\alpha$ would lead to an increase in $\bar{\pi}$. However, according to the results based on the projection method, an increase in $\alpha$ decreases the size of inflation bias.

4 Conclusion

We have demonstrated how a projection method can be used to compute the inflation bias in a full nonlinear version of the Calvo model. The annual inflation rate is between 1% and 4% under plausible parameter values. We also point out that our analysis can be extended to solve the optimal discretionary policy problem in models with random shocks. Once we find a deterministic steady state using the projection method, it will facilitate the analysis of random shocks based on a perturbation method.

Furthermore, it would be interesting to find additional mechanisms that can affect the size of inflation bias. For example, Debortoli and Nunes (2007) have modelled an imperfect commitment setting in which there is a continuum of loose commitment possibilities ranging from full commitment to full discretion. Specifically, we expect that the larger the degree of commitment, the smaller the size of inflation bias. In addition, we note that it would be possible to use the same projection method to analyze the effects of loose commitment on the inflation bias.
Appendices

This appendix includes a full description of the discretionary equilibrium and a table including projection results.

A Lagrangian

In the presence of technology shocks and government spending, the Lagrangian of this problem is

\[
\mathcal{L} = \frac{C_{t}^{1-\sigma} - 1}{1 - \sigma} - \frac{vH_{t}^{1+\chi}}{1 + \chi} + \beta E_{t}[V(\Delta_{t})]
\]

\[
+ \phi_{1t} \left[ \frac{A_{t}H_{t} - C_{t} - G_{t}}{\Delta_{t}} \right]
\]

\[
- [(1 - \tau_{p}) \phi_{2t}] \left[ \frac{A_{t}H_{t}}{\Delta_{t}C_{t}^{\sigma}} + \alpha \beta E_{t} \left[ \frac{L(\Delta_{t})}{1 - \tau_{p}} \right] - \left( \frac{F_{t}}{1 - \tau_{p}} \right) \right]
\]

\[
- [(1 - \tau_{p}) \phi_{3t}] \left[ \frac{vH_{t}^{1+\chi}}{(1 - \tau_{p})(1 - \tau_{w})(1 - \epsilon^{-1})} + \alpha \beta E_{t} \left[ \frac{M(\Delta_{t})}{1 - \tau_{p}} \right] - \left( \frac{S_{t}}{1 - \tau_{p}} \right) \right]
\]

\[
+ \phi_{4t} \left[ (1 - \alpha) \left( \frac{1 - \alpha \Pi_{t}^{-1}}{1 - \alpha} \right) \right]
\]

\[
- [(1 - \tau_{p}) \phi_{5t}] \left[ \left( \frac{F_{t}}{1 - \tau_{p}} \right) \left( \frac{1 - \alpha \Pi_{t}^{-1}}{1 - \alpha} \right) \right]
\]

where auxiliary functions \( L(\Delta_{t}) \) and \( M(\Delta_{t}) \) are defined as

\[
L(\Delta_{t}) = \Pi_{t+1}^{\epsilon^{-1}} F_{t+1}, \quad (34)
\]

\[
M(\Delta_{t}) = \Pi_{t+1}^{\epsilon^{-1}} S_{t+1}. \quad (35)
\]

Having described the optimal policy problem under discretion, the first-order conditions can be summarized as follows:

\[
1 + \sigma \frac{A_{t}H_{t}}{\Delta_{t}C_{t}^{\sigma}} \phi_{2t} = \phi_{1t}C_{t}^{\sigma}, \quad (36)
\]

\[
v \Delta_{t}C_{t}^{\sigma} \chi H_{t}^{\chi} + A_{t} \phi_{2t} + \frac{v(1 + \chi)}{(1 - \tau_{w})(1 - \epsilon^{-1})} \phi_{3t}C_{t}^{\sigma} H_{t}^{\chi} = \phi_{1t}A_{t}C_{t}^{\sigma}. \quad (37)
\]
\[ \phi_{2t} = \phi_{5t} \left( \frac{1 - \alpha \Pi_t^{\epsilon-1}}{1 - \alpha} \right)^{\frac{1}{1-\epsilon}} , \]  
(38)

\[ \phi_{3t} = -\phi_{5t} , \]  
(39)

\[ \epsilon \left( \left( \frac{1 - \alpha \Pi_t^{\epsilon-1}}{1 - \alpha} \right)^{\frac{1}{1-\epsilon}} - \Pi_t \Delta t^{-1} \right) \phi_{4t} = -\frac{1}{1 - \alpha} \left( \frac{1 - \alpha \Pi_t^{\epsilon-1}}{1 - \alpha} \right)^{\frac{1-\epsilon}{\epsilon-1}} F_t \phi_{5t} , \]  
(40)

\[ \frac{A_t H_t}{\Delta_t^2 \sigma_t} \phi_{2t} + \phi_{3t} \frac{\nu H_t^{1+\chi}}{(1 - \tau_W) (1 - \epsilon^{-1}) \Delta_t^2} - \phi_{4t} + \beta E_t \left[ V' (\Delta_t) \right] = \phi_{1t} \frac{A_t H_t}{\Delta_t^2} + \alpha \beta E_t \left[ \phi_{2t} L' (\Delta_t) + \phi_{3t} M' (\Delta_t) \right] , \]  
(41)

where \( \phi_{1t} , \phi_{2t} , \phi_{3t} , \phi_{4t} , \) and \( \phi_{5t} \) are Lagrange multipliers for (29), (30), (31), (32), and (33) respectively, and \( \tau_p \) is assumed to be zero for simplicity. In this general specification, the measure for the overall distortion in the steady-state output level as a result of taxes/subsidies and market power is

\[ 1 - \Phi = (1 - \tau_p) (1 - \tau_W) (1 - \epsilon^{-1}) . \]
### B Sensitivity Analysis

The following table presents the results of the projection analysis. All these results are based on $\beta = 0.99$ and $\Phi = 0.11$.

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REFERENCES

Figure 1: $\Delta_t(\Delta_{t-1})$ and $\Pi_t(\Delta_{t-1})$
Figure 2: \( \bar{\Pi} \) and \( \bar{\Delta} \) as a function of \( \Phi \)
Figure 3: $\bar{\Pi}$ and $(1 + \bar{\pi})$ as a function of $\Phi$