Discrete hedging approach for pricing warrant bonds in a mixed fractional Brownian environment with Vasicek interest rate model

Zuo-wei You 1,*, Shan-cun Liu 1, Qiang Zhang 2
1. School of Economics and Management, Beihang University, Beijing 100191, China;
2. School of Economics and Management, Beijing University of Chemical Technology, Beijing 100029, China.

Abstract: A pricing approach, based on the discrete hedging principle, for warrant bonds (WBs) with default risk in a mixed fractional Brownian environment is studied. The underlying stock price is described by the mixed fractional version of geometric Brownian motion with Hurst parameter $H$ satisfying $\frac{1}{2}<H<1$, which characterizes the serial autocorrelation in returns. The stochastic interest rate follows the Vasicek process. The warrant bond is viewed as a linear combination of European calls and a bond with default risk, where the default risk is described by the credit spread. The principle of discrete hedging and Taylor’s theorem are employed to derive the pricing formula for call both in the case of no transaction cost and in the case of proportional transaction costs. The results show that the randomness of interest rate leads to a decreasing time interval between rehedges instead of a constant time interval. The value of the warrant bonds decreases as the Hurst parameter $H$ increases, since the “equivalent volatility” of the underlying variable decreases as $H$ increases.

Key words: warrant bond; long memory; mixed fractional Brownian motion; discrete hedging principle; transaction cost; option pricing

1 Introduction

Warrant bonds (WBs) are a very important type of long-term equity-linked debt instrument. Warrant bonds refer to the bonds which proportionally offer the bond subscribers a certain number of stock warrants that can be traded separately. Each warrant provides the holders with an option of buying one share of stock at a predetermined price during a certain period of time before the maturity of the bond. The issuers offer the warrants to reduce the debt financing costs. The warrant contained in the warrant bonds has the early exercise feature of American options while its valuation method is similar to the pricing approach for European call options, which can be traced back to the famous Black-Sholes pricing formula [1].

Since the warrants can be traded separately, the valuation of warrant bonds can be split into the valuation of corporate bonds and that of warrants. The valuation of corporate bonds mainly lies on modeling the interest rate and the credit risk. Vasicek (1977) [2] and Cox-Ingersoll-Ross (CIR, 1985) [3] interest rate models are two of the most popular interest rate models which have mean-reverting property. We can use structural approach or reduced-form approach to deal with the default risk of the bonds. The structural approach was proposed by Merton (1974) [4], where default will happen when the value of the firm’s total assets falls less than the total debts. The reduced-form model uses an exogenous variable ‘hazard rate’ to characterize the instantaneous risk of default as Duffie and Singleton (1999) [5] stated. For an overview of credit risk measurement and bond pricing models, we can refer to the monograph written by Duffie and Singleton (2003) [6].

The key point of warrant valuation is the exact description of the dynamics of the underlying equity value or the stock price. Previous works on the warrant valuation, such as Lauterbach and Schultz (1990) [7], Schulz and Trautmann (1994) [8], Handley (2002) [9], and Bajo and Barbì (2010) [10] used the standard Brownian motion (Bm) to drive the dynamics of the underlying variables. In the field of derivative pricing, unlike other well-studied empirical properties such as volatility clustering, asymmetric and fat-tailed distributed, serial autocorrelation in returns has rarely been studied. Peters (1991) [11] first proposed the Fractal Market Hypothesis (FMH). He used fractals, rescaled range analysis and nonlinear dynamical
models to interpret behavior and understand price movements. Mandelbrot (1997) [12] applied ideas of scaling and self-similarity to analyses of financial data and proposed the substitution of Brownian motion (Bm) with fractional Brownian motion. Based on empirical studies, Shiryaev (1999) [13] pointed out that the process generating prices is a self-similar process of the same type as a fractional Brownian motion or an α-stable Lévy process and that there is a strong aftereffect in prices. Fractional Brownian motion (fBm) is a generalization of standard Brownian motion and has only one more parameter than Bm that is called Hurst parameter denoted by $H$. A fBm with $H \in (1/2, 1)$ is a continuous-time Gaussian process which exhibits ‘self-similarity’, and its increments are long-range correlated, stationary and fat-tailed distributed. The Hurst parameter $H$ not only describes the long-memory property but also the self-similarity. Therefore, replacing Bm with fBm to drive the dynamics of risky asset price makes the model well match the statistical characteristics of the return series.

However, the fBm is neither a semi-martingale nor a Markov process, so the application of fBm to financial modeling is not easy. By means of technique of Malliavin calculus and the concept of Wick product, Duncan et al. (2000) [14], Hu and Øksendal (2003) [15], Elliott and van der Hoek (2003) [16] defined the stochastic calculus w.r.t. fBm as the fractional Wick Itô Skorohod (fWIS) integral, which produces the desirable zero-mean property. They further introduced the notion of Wick value process $S_t \circ \gamma$, which was defined as Wick product of stock price $S_t$ and the number of shares $\gamma_t$, as well as the notion of Wick self-financing strategy. Under such a technique, they proved that the fractional Black-Scholes market with $H \in (1/2, 1)$ is free from strong arbitrage and complete. By using fWIS integral, Necula (2002) [17] defined a type of quasi-martingale based on the quasi-conditional expectation, by which he derived the fractional Black-Scholes formula and concluded that the fractional Black-Scholes formula will degenerate to the Black-Scholes formula when $H \to 1/2$.

From the point of view of pure mathematics, the application of Wick product to the definitions of portfolio value and self-financing strategy is ingenious. However, the Wick portfolio value and the Wick self-financing trading strategy do not have reasonable economic interpretations, and thus the results based on these notions are not economically meaningful, as Björk and Hult (2005) [18] pointed out. Fortunately, Biagini et al. (2008) [19] revised such definitions by introducing market observers denoted by a stochastic test function $\psi$. The process $S_t$ was no longer interpreted as the observed market price but the unobserved fundamental value, an abstract variable. The actual value of the portfolio was the action of the Wick portfolio value $\gamma_t \circ S_t$ to the market observers $\psi$, namely $\langle \gamma_t \circ S_t, \psi \rangle$. Such an interpretation of Wick portfolio value in the sense of generalized function ensures a correct mathematical form and a rational economic meaning of the model. Obviously, this interpretation leads to a low practical value of the model, but it is ingenious in sense of thinking and significant in sense of theory.

Actually, the long-memory of a fBm allows us to have a nontrivial prediction of its development, cf. Gripenberg and Norros (1996) [20]. If we use a fBm as the source of uncertainty of price movement, then the nontrivial predictability of the stock price may be exploited to realize some arbitrage trading strategies as long as the market is frictionless and the traders can act infinitely fast, in other words, the investors can trade continuously without any costs. In order to exclude arbitrage from models based on fBm, additional restrictions have been placed on the market or the behaviors of investors. Cheridito (2003) [21] pointed out that the existence of an arbitrarily small amount of time between two consecutive transactions of an individual investor can exclude any kind of arbitrage opportunities. Rostek and Schöbel (2006, 2009) [22, 23] introduced the preference-based approach to price European options. Xiao et al. (2012) [24] used the same approach to price equity warrants. Guasoni (2006) [25] and Wang (2010) [26] introduced proportional transaction costs to exclude arbitrage from their models.

Another method to exclude arbitrage opportunities is to use a mixed fractional Brownian motion (mfBm) instead of the fBm. A mfBm is a linear combination of a Bm and a fBm, cf. Cheridito(2001) [27]. Bender et al. (2007) [28] had concluded that, for $H \in (1/2, 1)$, the mixed fractional Black-Scholes market, where the stock price dynamics is driven by a mfBm, is arbitrage-free with regular portfolios. He had also pointed out that the regular portfolios constitute an arbitrage-free class of strategies that is sufficiently large to cover hedges for practically relevant options. Therefore, it is also rational to substitute
mfBm for Bm to drive the stock price dynamics when the long-range autocorrelation in returns is considered. Wang et al. (2010) [29] extended the previous model, cf. Wang (2010) [26], to option pricing model under the mixed fractional Brownian environment. Xiao et al. (2012) [30] used the quasi-conditional-expectation based approach to price equity warrants with the total equity value of the issuing firm following the mixed fractional version of geometric Brownian motion.

The purpose of this article is to build a rational pricing model for warrant bonds in the fractal market where the stock transaction costs are proportional to the value of the stocks traded. Since warrant bonds have long lifespans, we substitute the Vasicek interest rate model for the constant interest rate. Then we extend the option pricing model in Wang et al. (2010) [29] to that in the case where the interest rate is not constant but stochastic. In section 2, our framework for warrant bond pricing is presented and the pricing environment is described in detail. The dynamics of the underlying stock price is described by a SDE driven by a mfBm with $1/2 < H < 1$. The interest rate follows the Vasicek process. The existence of a small amount of time between two consecutive transactions is assumed to exclude arbitrage. Inspired by the idea from Tsiveriotis and Fernandes (1998) [31], the default risk is described by the credit spread and then the task of pricing the WB turns into task of pricing the call option. In Section 3, we first get the delta hedging strategy, according to the discrete hedging principle, in the assumption of no transaction cost. After we get the explicit pricing formula for call option we derive Lemma A1 as shown in Appendix A. In section 4, we continue to price the call option in the assumption of proportional transaction costs. We resort to the delta hedging strategy and Lemma A1 which we have gotten in section 3 to derive the semi-explicit pricing formula for call option. In Section 5, we show the semi-explicit pricing formula for the WB with default risk described in Section 2. We further picture the numerical results to show the impact of randomness of interest rate on the time interval of rehedges and the impact of Hurst parameter on the value of the WB. All we have utilized is Taylor’s theorem and the analyses based on small changes. We do not use Wick portfolio value or Wick self-financing strategy which will lead to meaningless economic interpretations. Our results are not against economic intuitions.

2 the framework of pricing for warrant bonds

We are now in a position to establish a reasonable pricing model for warrant bonds. To begin with, we assume that a listed firm, which has already issued $m$ shares of stock in the financial markets, issues and sells $n$ WBs at time $t = 0$. Each WB contract consists of two parts: first, a corporate bond with a face value of $F$ and a maturity of $T_b$; second, $\alpha$ warrants which will expire at time $T$, where $T < T_b$. Each warrant gives its holder a right to buy one share of the stock at $K = \beta S_0$ before $T$, where $\beta > 1$ and $S_0 > 0$ is the stock price at time $t = 0$. We denote by $(S_t)_{t \geq 0}$ the underlying stock price, by $(V_t)_{t \geq 0}$ the value of the WB. The task is to show the value $(V_t^f)_{t \geq 0}$ of the WB.

Let $\hat{B} = (\hat{B}_t)_{t \geq 0}$ and $\bar{B} = (\bar{B}_t)_{t \geq 0}$ be two standard Brownian motions and let $B^\alpha = (B^\alpha_t)_{t \geq 0}$ be a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. These three processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and pairwise independent, where $\mathbb{P}$ is the real world probability measure, $\mathcal{F}_t = \sigma(\hat{B}_t, \bar{B}_t, B^\alpha_t; 0 \leq s \leq t)$, $\mathcal{F}_0$ is trivial, $\mathcal{F} = \mathcal{F}_T$.

Since warrant bonds have long lifespans, the supposition of constant interest rates may be not valid. We need to consider the randomness of the interest rate. The interest rate is assumed to follow the Vasicek process:

$$dr_t = k(\theta - r_t)dt + \sigma d\hat{B}_t, \quad t \in [0, T_b],$$

where $k > 0$, $\theta > 0$, and $\sigma > 0$ are constants. By solving this SDE, $\forall s > t > 0$, $r_s$ can be shown as

$$r_s = \theta + (r_t - \theta)e^{-k(s-t)} + \sigma \int_t^s e^{k(u-t)}d\hat{B}_u, \quad \forall s > t > 0.$$
Let $D = (D_t)_{t \geq 0}$ denote the value of riskless zero-coupon bonds with $1$ par value and maturity $T$, then
\begin{equation}
D_t := D(t,r_t; T) = E[e^{-\int_0^t r_s ds} | \mathcal{F}_t] = e^{-G_t+\lambda t}, \quad t \in [0,T].
\end{equation}
where $G_t := G(t; T) = \frac{1}{k} [1 - e^{-k(T-t)}]$, $A_t := A(t; T) = \frac{4\theta^2 - 3\theta^2}{4k} - \frac{\sigma^2 - 2\theta^2}{2k^2} (T-t) + \frac{\sigma^2 - \theta^2}{k^2} e^{-k(T-t)} - \frac{\sigma^2}{4k} e^{-2k(T-t)}$.

The stock price of the underlying company is assumed to satisfy the following process:
\begin{equation}
S_t = S_0 \exp \left( \int_0^t \mu_s ds + \varepsilon (\rho B_t + \sqrt{1 - \rho^2} \tilde{B}_t) + \sigma_H B^H_t \right), \quad t \in [0,T], \quad H \in (1/2,1),
\end{equation}
where $\varepsilon > 0$, $\sigma_H > 0$, $\rho \in (-1,1)$ are constants, and $\mu_s$ is adapted to $\mathcal{F}_t$. Let $B_t := \rho B_t + \sqrt{1 - \rho^2} \tilde{B}_t$, then $B_t$ is a BM.

Our model is based on the following assumptions for simplification.

**S1.** There is no restriction on shorting and the biggest amount of shares one can trade and hold.

**S2.** During the survival time of the WBs, the firm will neither issue new bonds and shares of stock nor pay any interests and dividends.

**S3.** Essentially, the warrant is an American call option. Since the value of an American call option is equal to the value of a European call option in the case of no dividend in the framework of martingale pricing theory, we can further assume that the warrants only be exercised at the maturity date.

If the investors choose to exercise the warrant at time $T$, the company will have to issue new shares of stock. At the moment when the exercise of all the warrants is finished, denoted by $T := T + 0^+$, the stock price will jump to $S_{r_t} = (mS_t + n\alpha K)/(m + \alpha n)$. This means that the terminal condition for value of the warrants of each WB contact is
\begin{equation}
W_T := \max \{0, S_T - K\} = \alpha \left( \frac{mS_T + \alpha nK}{m + \alpha n} - K \right) \mathbf{1}_{[S_T > K]} + \frac{\alpha m}{m + \alpha n} (S_T - K) \mathbf{1}_{[S_T > K]} = \frac{\alpha m}{m + \alpha n} (S_T - K).
\end{equation}

The above equation implies the split of the warrant bond into $\frac{\alpha m}{m + \alpha n}$ calls and a bond. Let the credit spread of the bond be a constant $\lambda > 0$, then the value of zero-coupon bond with par value $1$ and maturity $T$ is
\begin{equation}
E[e^{-\lambda T} | \mathcal{F}_t] = e^{-\lambda T} E[t, r_T; T]. \quad \text{Let } C = (C_t)_{t \geq 0} \text{ denote the value of a call with exercise price } K \text{ and maturity } T, \text{ then}
\end{equation}
the value of the warrant bond is
\[ V_t = \frac{\alpha_m}{m + \alpha_m} C_t + F_t e^{-\delta(t-T)} D(t, r_T; T_0), \quad 0 < t \leq T. \] (8)

Thus we convert the valuation of the warrant bond into the valuation of the European call.

S4. Prices are monitored continuously but hedging takes place discretely. The portfolio is revised every \( \delta t \), where \( \delta t > 0 \) is a small timestep and \( \delta t \in \mathcal{F}_t \). That is to say, if hedging takes place at \( t \), then the next hedging will occur at \( t + \delta t \), where \( \delta t \) is predetermined at \( t \). This assumption excludes arbitrage from our model due to Cheridito (2003) [21].

We take \( s = t + \delta t \) in Eqs. (2) and (6), and define \( \delta r_t := r_{t+\delta t} - r_t \) and \( \delta S_t := S_{t+\delta t} - S_t \) as the change of interest rate and change of stock price from \( t \) to \( t + \delta t \) respectively, then we have
\[ \delta r_t \equiv r_{t+\delta t} - r_t = (\theta - r_t) (1 - e^{-\delta t}) + \sigma e^{-\delta t} \int_t^{t+\delta t} e^\delta dB_h, \] (9)

\[ \delta S_t := S_{t+\delta t} - S_t = S_t \left\{ \exp \left[ \bar{\mu} \delta t + \mathcal{N}(B_{t+\delta t} - B_t) + \mathcal{N}(B_{t+\delta t} - B_t) \right] - 1 \right\}. \] (10)

S5. There is no cost for trading riskless bonds. Trading each share of stock costs an amount \( \frac{\kappa}{2} S_t \) for both buying and selling, where \( \kappa > 0 \) is a constant. That is to say the cost is proportional to the value traded.

We construct a hedged portfolio with value \( \Pi_t = \Delta_t D_t + \Delta S_t S_t - C_t \). Its value at time \( (t + \delta t) + \delta t \), based on the assumption S5, is
\[ \Pi_{t+\delta t} = \Delta_t D_{t+\delta t} + \Delta S_t S_{t+\delta t} - C_{t+\delta t} - \frac{\kappa}{2} S_{t+\delta t} \left| u_{t+\delta t} \right|, \] (11)

where \( u_{t+\delta t} \) represents the number of shares of stock that changes at time \( t + \delta t \). The change of value of the portfolio is
\[ \delta \Pi_t := \Pi_{t+\delta t} - \Pi_t = \Delta_t (\delta D_t) + \Delta S_t (\delta S_t) - \frac{\kappa}{2} S_{t+\delta t} \left| u_{t+\delta t} \right|, \] (12)

where \( \delta D_t := D(t + \delta t, t + \delta r_t) - D(t, r_t) \) and \( \delta C_t := C(t + \delta t, t + \delta r_t, S_t, S_t) - C(t, r_t, S_t) \).

The hedging strategy and valuation policy are as follows as Wilmott [32] stated:

( i ) Choose \( \Delta = (\Delta_1, \Delta_2) \) to minimize the variance of \( \delta \Pi_t \).

( ii ) Value the call by setting the expected return on \( \Pi_t \) equal to the risk-free rate.

We continue to analyze \( \delta \Pi_t \) by \( \delta D_t, \delta S_t, \delta C_t \), which requires Taylor’s theorem and the following lemma (Wang (2010) [26]) that is known as Lévy’s modulus of continuity theorem.

**Lemma 1.** If \( B^{\mathbb{H}} = (B^{\mathbb{H}}_t)_{t \geq 0} \) is a fractional Brownian motion with Hurst \( \mathbb{H} \in (0,1) \), then we have
\[ \lim_{\delta t \to 0} \sup_{\delta > 0} \frac{|B^{\mathbb{H}}_{t+\delta t} - B^{\mathbb{H}}_{t}|}{\mathbb{H} \sqrt{2 \log(\delta T)}} = 1 \quad \text{a.s.}. \]

Thus we have \( \delta B^{\mathbb{H}} = O(\delta t^{\mathbb{H}} \sqrt{\log(\delta t)^{-1}}) \). (13)

where \( O(\cdot) \) is explained as follows. A random function \( h(x) \) is said to be \( O(g(x)) \), if there exist a constant \( M > 0 \) such that \( \frac{h(x)}{g(x)} \leq M \) a.s. as \( x \to 0 \). It is not difficult to find that
\[ O(\delta B_t + \delta B^{\mathbb{H}}_t) = O(\delta B_0), \mathbb{H} \in (1/2,1). \] (14)
Note 2. The valuation based on the discrete hedging principle in a fractional Brownian motion environment has two advantages. First, the market is free from arbitrage. Second, the concepts of Wick portfolio value and Wick self-financing strategy are abandoned.

3 European call pricing with no transaction cost

In this section we take $\kappa = 0$ and let $\delta t$ be a fixed small time step. We derive the explicit pricing formula for call in this case before we eventually arrive at the explicit pricing formula for call with proportional transaction costs in Section 4.

In this case, the change of value of the hedged portfolio denoted by Eq. (12) turns to be

$$\delta \Pi_0 = \Delta_d (\delta D_t) + \Delta_s (\delta S_t) - \delta C_t.$$  (15)

Since the time step and the price changes are small we can use Taylor’s theorem to analyze the behaviors of $\delta D_t, \delta S_t, \delta C_t$ respectively based on Eqs. (9-10). We can get the following theorem.

Theorem 1 Based on the discrete hedging principle, the value of riskless zero-coupon bonds $D = (D_t)_{t=0}$ in a market with no transaction cost satisfies the following equation

$$\frac{\partial D}{\partial t} \delta t + k(\theta - r) \frac{\partial D}{\partial r} \delta r + \frac{1}{2} \frac{\partial^2 D}{\partial r^2} \delta r^2 = O((\delta t)^{\frac{3}{2}} (\log(\delta t)^{-\frac{1}{2}}), \quad t \in [0, T].$$  (16)

Proof. According to the independence of increments of Brownian motion as well as Taylor’s theorem, we have

$$\delta r_t = (\theta - r_t)(1-e^{k(\delta t)}) + \sigma \int_0^{\delta t} e^{k(\delta t-\tau)} dB_u = k(\delta t)(\theta - r_t) + \sigma \int_0^{\delta t} e^{k(\delta t-\tau)} dB_u + O((\delta t)^2).$$  \hspace{1cm} (17)

$$\delta r_t = \sigma \int_0^{\delta t} e^{k(\delta t-\tau)} dB_u + O((\delta t)^2).$$  \hspace{1cm} (18)

Now we employ the discrete hedging principle to derive the PDE of $D$. We assume there are two riskless zero-coupon bonds $D_1, D_2$ with maturities $T_1, T_2$ respectively. We construct a hedged portfolio with value of $\Pi_0 = D_1(t, r_t; T_1) - \Delta D_2(t, r_t; T_2)$. Due to Taylor’s theorem, we have

$$\delta D(t, r_t) = D(t + \delta t, r_t + \delta r_t) - D(t, r_t) = \frac{\partial D}{\partial t} \delta t + \frac{\partial D}{\partial r} \delta r_t.$$

Therefore, we have

$$\delta \Pi_0 = \delta D_1(t, r_t; T_1) - \Delta \delta D_2(t, r_t; T_2) = \frac{\partial D_1}{\partial t} \delta t + \frac{\partial D_1}{\partial r} \delta r_t + \frac{1}{2} \frac{\partial^2 D_1}{\partial r^2} \delta r_t^2 + O((\delta t)^3 (\log(\delta t)^{-\frac{1}{2}}).$$

Since we need choose $\Delta$ to minimize the variance of $\delta \Pi_0$, we can get $\Delta = \frac{\partial D_1}{\partial r} / \frac{\partial D_2}{\partial r}$. Then due to $E[\delta \Pi_0] = \Pi_0 r_0(\delta t)$, we have

$$\delta \Pi_0 = \frac{\partial D_1}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 D_1}{\partial r^2} \delta r_t^2 + O((\delta t)^2 (\log(\delta t)^{-\frac{1}{2}}).$$

Rearranging the terms in the above equation, we have

$$\frac{\partial D_1}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 D_1}{\partial r^2} \delta r_t^2 + \frac{1}{2} \frac{\partial^2 D_1}{\partial r^2} \delta r_t^2 + O((\delta t)^2 (\log(\delta t)^{-\frac{1}{2}}) = \left[D_1 - \frac{\partial D_1}{\partial r} \frac{\partial D_1}{\partial r} \right] r_0(\delta t).$$
\[
\left[ \frac{\partial D_i}{\partial t} + \frac{1}{2} \frac{\partial^2 D_i}{\partial r^2} \sigma_i \sigma_i - D_i r_i (\Delta t) \right] \frac{\partial D_i}{\partial r} + \left[ \frac{\partial D_i}{\partial t} + \frac{1}{2} \frac{\partial^2 D_i}{\partial r^2} \sigma_i \sigma_i - D_i r_i (\Delta t) \right] \frac{\partial D_i}{\partial r} = O((\Delta t)^{3/2} (\log(\Delta t)^{-1/2})^2).
\]

Let \( \lambda(t) \) be the market price of risk of interest rate, we can get

\[
\left[ \frac{\partial D}{\partial t} + \frac{1}{2} \frac{\partial^2 D}{\partial r^2} \sigma_i \sigma_i - D_i r_i (\Delta t) \right] \frac{\partial D}{\partial r} + O((\Delta t)^{3/2} (\log(\Delta t)^{-1/2})^2) = -k(\theta - r) \Delta t + \lambda(t) \Delta t.
\]

Without loss of generality, we assume \( \lambda(t) = 0 \) for ease of calculation. Thus we can get Eq. (16) and complete the proof. \( \blacksquare \)

According to Theorem 1, we have

\[
\delta D_i = -k(\theta - r) \frac{\partial D}{\partial t} + r_i D_i \Delta t + O((\Delta t)^{3/2} (\log(\Delta t)^{-1/2})^2).
\]

**Theorem 2** Based on the discrete hedging principle, the value of European call \( C = (C_t)_{t \in [0, T]} \) in a market with no transaction cost satisfies the following equation

\[
\frac{\partial C}{\partial t} + r S \frac{\partial C}{\partial S} + k(\theta - r) \frac{\partial C}{\partial r} \left[ \frac{1}{2} \sigma^2 (\Delta t)^{1/2} \right] S^2 \frac{\partial^2 C}{\partial S^2} + \rho \sigma \frac{\partial C}{\partial S} \frac{\partial C}{\partial r} = O((\Delta t)^{3/2} (\log(\Delta t)^{-1/2})) , \quad t \in [0, T].
\]

**Proof.** Due to Taylor’s formula of \( e^x = 1 + \frac{1}{2} x^2 + \frac{1}{6} e^{\phi x} x^3, \phi \in (0, 1) \), we have

\[
\exp \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right] = 1 + \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right] + \frac{1}{2} \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right]^2 + \frac{1}{6} e^{\frac{1}{6} \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right]} + O((\Delta t)^{3/2} (\log(\Delta t)^{-1/2}),
\]

\[
\left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right]^3 = \left[ \frac{\mu_i (\Delta t)}{\sigma_i (\Delta t)^{1/2}} \right]^3 + 3 \left[ \frac{\mu_i (\Delta t)}{\sigma_i (\Delta t)^{1/2}} \right] \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right] + 3 \left[ \frac{\mu_i (\Delta t)}{\sigma_i (\Delta t)^{1/2}} \right]^2 \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right] + O((\Delta t)^{3/2} (\log(\Delta t)^{-1/2})) = O((\Delta t)^{3/2} (\log(\Delta t)^{-1/2})),
\]

and

\[
e^{\frac{1}{6} \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right]} = O((\Delta t)^{3/2} (\log(\Delta t)^{-1/2})),
\]

we have

\[
\exp \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right] = 1 + \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right] + \frac{1}{2} \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right]^2 + O((\Delta t)^{3/2} (\log(\Delta t)^{-1/2})).
\]

Substituting the above Eq. into Eq. (10) leads to the following equations.

\[
\delta S_i = S_i \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right]^2 + \frac{1}{2} \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right]^2 + O((\Delta t)^{3/2} (\log(\Delta t)^{-1/2})^2).
\]

(21)

\[
(\delta S_i)^2 = S_i^2 \left[ \frac{\mu_i (\Delta t) + \sigma_i (\Delta t)^{1/2}}{\sigma_i (\Delta t)^{1/2}} \right]^2 + O((\Delta t)^{3/2} (\log(\Delta t)^{-1/2})^2).
\]

(22)
\[(\delta r_i)S_i = \left[k(\delta t)(\theta - r_i) + \sigma \int_0^{\delta t} e^{-\delta t - \delta t u} dB_u + O((\delta t)^3)\right] \\
\cdot S_i \left[\{\mu_r(\delta t) + \epsilon_r(\delta B_r^\nu) + \sigma_r^\mu(\delta B_r^\nu)\} + \frac{1}{2} \sigma_r^\mu(\delta B_r^\nu) \right]^2 + O((\delta t)^3 (\log(\delta t)^{-1})^3) \]

\[= \sigma \int_0^{\delta t} e^{-\delta t - \delta t u} dB_u \cdot S_i \left[\epsilon_r(\delta B_r^\nu) + \sigma_r^\mu(\delta B_r^\nu)\right] + O((\delta t)^3 (\log(\delta t)^{-1})^3) \]

\[= \sigma(\delta B_r^\nu) S_i \left[\epsilon_r(\delta B_r^\nu) + \sigma_r^\mu(\delta B_r^\nu)\right] + O((\delta t)^3 (\log(\delta t)^{-1})^3), \]

where we have applied \(e^{-\delta t - \delta t u} = 1 - k(\delta t - u) + \frac{1}{2} k^2(\delta t - u)^2 + O((\delta t - u)^3)\). Due to Taylor’s formula, we have

\[
\delta C(t_r, r_s) = C(t + \delta t, r_r, r_s, \delta S_r) - C(t, r_r, S_r)
\]

\[
= \frac{\partial C}{\partial t} \delta t + \frac{\partial C}{\partial r} \delta r + \frac{\partial C}{\partial S} \delta S + \frac{1}{2} \frac{\partial^2 C}{\partial r^2} (\delta r)^2 + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} (\delta S)^2 + \frac{1}{2} \frac{\partial^2 C}{\partial r \partial S} (\delta r)(\delta S) + O((\delta t)^3 (\log(\delta t)^{-1})^3)
\]

Due to Eqs. (17-19) and (21-24), the change in value of the hedged portfolio is

\[
\delta \Pi_1 = (\delta r_i \frac{\partial D}{\partial r} - \frac{\partial C}{\partial r}) \delta r_i + \Delta_r \left[-k(\theta - r_i) \frac{\partial D}{\partial t} + r_i \frac{\partial D}{\partial t}\right] + (\Delta_r - \frac{\partial C}{\partial S}) \delta S, \]

\[= \frac{\partial C}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 C}{\partial r^2} (\delta r)^2 + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} (\delta S)^2 + \frac{1}{2} \frac{\partial^2 C}{\partial r \partial S} (\delta r)(\delta S) + O((\delta t)^3 (\log(\delta t)^{-1})^3)
\]

\[= (\Delta_{r_i} \frac{\partial D}{\partial r} - \frac{\partial C}{\partial S}) \delta S
\]

\[+ \Delta_r \left[-k(\theta - r_i) \frac{\partial D}{\partial t} + r_i \frac{\partial D}{\partial t}\right] - \frac{\partial C}{\partial t} \delta t
\]

\[- \frac{1}{2} \frac{\partial^2 C}{\partial r^2} \left[\sigma^2 \int_0^{\delta t} e^{-\delta t - \delta t u} dB_u^2 + 2k(\delta t)(\theta - r_i) \sigma \int_0^{\delta t} e^{-\delta t - \delta t u} dB_u\right] - \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \left[\sigma^2 \int_0^{\delta t} e^{-\delta t - \delta t u} dB_u^2\right] + \frac{1}{2} \frac{\partial^2 C}{\partial r \partial S} \left[\sigma^2 \int_0^{\delta t} e^{-\delta t - \delta t u} dB_u\right]
\]

\[- \frac{1}{2} \frac{\partial^2 C}{\partial r \partial S} \left[\sigma^2 \int_0^{\delta t} e^{-\delta t - \delta t u} dB_u\right] + O((\delta t)^3 (\log(\delta t)^{-1})^3).
\]

Since \(E[\delta B^\nu_t] = 0\) and \(E[\delta B^\nu_t]^2 = (\delta t)^{2\mu}\), we have

\[E\left[\int_0^{\delta t} e^{-\delta t - \delta t u} dB_u\right] = E\left[\int_0^{\delta t} e^{-2k(\delta t)} du\right] = \frac{1-e^{-2k(\delta t)}}{2k} = \frac{2k(\delta t) + O((\delta t)^2)}{2k} = (\delta t) + O((\delta t)^2).
\]

\[E\left[\epsilon_r(\delta B_r^\nu) + \sigma_r^\mu(\delta B_r^\nu)\right] = \epsilon_r(\delta t) + \sigma_r^\mu(\delta t) = \epsilon_r(\delta t) + \sigma_r^\mu(\delta t)^{2\mu}.
\]

\[E\left[\sigma(\delta B^\nu) S_i \left[\epsilon_r(\delta B_r^\nu) + \sigma_r^\mu(\delta B_r^\nu)\right]\right] = \rho \sigma \varepsilon S_i(\delta t).
\]

So

\[E[\delta \Pi_1] = (\delta r_i \frac{\partial D}{\partial r} - \frac{\partial C}{\partial S}) \left[k(\delta t)(\theta - r_i) + \frac{1}{2} \sigma_r^\mu(\delta B_r^\nu)^2 + \frac{1}{2} \sigma_r^\mu(\delta B_r^\nu)^{2\mu}\right] + \Delta_r \left[-k(\theta - r_i) \frac{\partial D}{\partial t} + r_i \frac{\partial D}{\partial t}\right]
\]

\[= \frac{\partial C}{\partial r} \delta r_i - \frac{1}{2} \sigma_r^2 \frac{\partial^2 C}{\partial r \partial r} \delta r_i - \frac{1}{2} \sigma_r^2 \frac{\partial^2 C}{\partial r \partial S} \delta S_i + \frac{1}{2} \sigma_r^2 \sigma_r^\mu(\delta B_r^\nu)^{2\mu} - \frac{\partial C}{\partial S} \delta S_i \sigma(\delta B^\nu) + O((\delta t)^3 (\log(\delta t)^{-1})^3).
\]

Therefore, \(\delta \Pi_1 = X + Y + Z\), where
\[ X_t = (\Delta_{t^0} \frac{\partial D}{\partial r} \Delta_{t^0} \frac{\partial C}{\partial r}) \left[ \sigma^2 \int_0^t e^{2(\beta - r) \alpha} d\hat{B}_t \right]. \]

\[ Y_t = (\Delta_{t^0} \frac{\partial C}{\partial S}) \left[ \left( \sigma^2 \int_0^t e^{2(\beta - r) \alpha} d\hat{B}_t \right)^2 \sigma^2 \right] + \frac{1}{2} \left[ \sigma^2 \sigma^2 \int_0^t e^{2(\beta - r) \alpha} d\hat{B}_t \right]^2 - \frac{1}{2} \sigma^2 \sigma^2 \int_0^t e^{2(\beta - r) \alpha} d\hat{B}_t \right]^2. \]

\[ Z_t = \frac{1}{2} \sigma^2 \int_0^t e^{2(\beta - r) \alpha} d\hat{B}_t \right]^2 - \frac{1}{2} \sigma^2 \sigma^2 \int_0^t e^{2(\beta - r) \alpha} d\hat{B}_t \right]^2. \]

In order to minimize \( \text{Var}(\delta \Pi), \) it should be that \( X_t = 0 \) and \( Y_t = 0, \) namely, \( \Delta_{t^0} \frac{\partial D}{\partial r} \Delta_{t^0} \frac{\partial C}{\partial r} \) and \( \Delta_{t^0} \frac{\partial C}{\partial S}. \) Thus,

\[ \text{Var}(\delta \Pi) = E[(\delta \Pi - E[\delta \Pi])^2] = E[X_t + Y_t + Z_t]^2 = E[X_t^2 + Y_t^2 + Z_t^2 + 2X_tY_t + 2X_tZ_t + 2Y_tZ_t]. \]

Then according to \( E[\delta \Pi] = r_t \Pi_t \sigma_t, \) we can obtain

\[ E[\delta \Pi] = \left( \frac{\partial C}{\partial r} \Delta_{t^0} \frac{\partial D}{\partial r} \Delta_{t^0} \frac{\partial C}{\partial S} \right) \left[ -k(\theta - r) + D_t \Delta_{t^0} \frac{\partial C}{\partial r} \Delta_{t^0} \frac{\partial D}{\partial r} \right] = \Pi_t \sigma_t. \]

By rearranging the terms in the above equation, we can obtain Eq. (20), which completes the proof.

**Theorem 3** Based on the discrete hedging principle, the value of European call \( C = (C_t)_{t=0}^T \) in a market with no transaction cost can be presented explicitly as

\[ C(t, r, S_t) = S_t N(d_1) - KD_t N(d_2), \]

where \( K \) is the exercise price, and \( d_2 = \frac{\ln S_t - r_t}{\sqrt{2r_t}}, \)

\[ d_1 = d_2 + \sqrt{2r_t}, \]

\[ \tau_t := \tau(t) = \frac{1}{2} \left[ \sigma^2 \right] (s) ds, \quad \sigma^2(t) = \sigma^2 G_t^2 + 2\rho \sigma G_t + \varepsilon^2 + \sigma^2 (\delta t)^{\frac{3}{2}}. \]

By integration, \( \tau_t \) becomes

\[ \tau(t) = \frac{1}{2} \left[ \sigma^2 \right] (T - t) - 2G_t + \frac{1}{2} \left[ 1 - e^{2(\beta - r) \alpha} \right] \right] + \frac{2 \rho \sigma G_t}{k} \left[ (T - t) - G_t \right] + \left[ \varepsilon^2 + \sigma^2 (\delta t)^{\frac{3}{2}} \right](T - t). \]

The proof is shown in Appendix B.

**Note 3.** Comparing **Theorem 3** and the BS option pricing formula, we can conclude that \( \sigma(t) \) in Eq. (27) corresponds to the volatility in BS formula. We call \( \sigma(t) \) the “equivalent volatility” for ease of expression.

**4 European call pricing with proportional transaction costs**

It is time to resolve the pricing task in section 2. Due to Eqs. (12) and **Theorem 2**, the value of European call in the market with transaction costs as described in Section 2 satisfies the following equation
\[
\left[ \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial S^2} + \rho \sigma \xi S \frac{\partial C}{\partial r \partial S} + \frac{\partial C}{\partial r} \right] \cdot \left[ \varepsilon^2 + \sigma^2 \left( \delta t \right)^{2H-1} \right] + r S_r \frac{\partial C}{\partial S} - r C_r + k(\theta - r) \frac{\partial C}{\partial r} \delta t + E\left[ \frac{K}{2} S_{t+\delta t} \right] |_{t+\delta t} = O(\left( \delta t \right)^{3} (\log(\delta t))^{-1 \frac{3}{2}}) .
\]

We will follow the same hedging strategy as before, namely, the delta hedging strategy\(^1\). The number of shares of stock traded at time \(t + \delta t\) is
\[
\nu_{t+\delta t} = \frac{\partial C}{\partial S} (t + \delta t, r_r + \delta r, S_r + \delta S, S_r) - \frac{\partial C}{\partial S} (t, r, S_r) = \delta S_r \frac{\partial^2 C}{\partial S^2} (t, r, S_r) + \delta t \frac{\partial^2 C}{\partial S \partial r} (t, r, S_r) + O((\delta S_r)^2) .
\]

To leading order the number of stock bought or sold at time \(t + \delta t\) is
\[
\nu_{t+\delta t} = \frac{\partial^2 C}{\partial S^2} S \left[ (\rho \delta \hat{B}_t + \sqrt{1 - \rho^2} \delta \tilde{B}_t) + \sigma^2 \delta^2 \hat{B}_t \right] + \frac{\partial^2 C}{\partial S \partial r} \sigma \delta \hat{B}_t + O((\delta S_r)^2)
\]
\[
= \frac{\partial^2 C}{\partial S^2} S \left[ (\rho \delta \hat{B}_t + \sqrt{1 - \rho^2} \delta \tilde{B}_t) + \sigma^2 \delta^2 \hat{B}_t \right] + \frac{\partial^2 C}{\partial S \partial r} \sigma \delta \hat{B}_t + O((\delta S_r)^2)
\]
\[
= \frac{\partial^2 C}{\partial S^2} S \left[ (\sigma^2 + \rho \sigma \xi) \delta \hat{B}_t + \sqrt{1 - \rho^2} (\sigma \delta \tilde{B}_t) \right] + O((\delta S_r)^2)
\]

where we have applied \(\frac{\partial^2 C}{\partial S^2} / S \frac{\partial^2 C}{\partial S \partial r} = G_r\), cf. Lemma A1. The expectation of trading cost is
\[
E\left[ \frac{K}{2} S_{t+\delta t} \right] = E\left[ \frac{K}{2} S_t \right] \left| \delta S \right| = E\left[ \frac{K}{2} S_t \right] \left| \sigma^2 + \rho \sigma \xi \right| \delta \hat{B}_t + E\left[ \frac{K}{2} S_t \right] \sigma \delta \hat{B}_t
\]
\[
= E\left[ \frac{K}{2} S_t \right] \left| \sigma^2 + \rho \sigma \xi \right| \delta \hat{B}_t + E\left[ \frac{K}{2} S_t \right] \sigma \delta \hat{B}_t
\]
\[
= E\left[ \frac{K}{2} S_t \right] \left[ (\sigma^2 + \rho \sigma \xi) \delta \hat{B}_t + \sqrt{1 - \rho^2} (\sigma \delta \tilde{B}_t) \right] + O(\delta S_r^2)
\]

where we have used \(\frac{\partial^2 C}{\partial S^2} > 0\). Since \(|\delta S| \ll S_t\), we can ignore the term \(Cost_1\) and only consider the dominated term \(Cost_2\).

Then the expected transaction cost over the timestep \(\delta t\) is simplified as
\[
E\left[ \frac{K}{2} S_{t+\delta t} \right] = Cost_2 \approx E\left[ \frac{K}{2} S_t \right] \left| \sigma^2 + \rho \sigma \xi \right| \delta \hat{B}_t + \sigma \delta \hat{B}_t
\]
\[
= E\left[ \frac{K}{2} S_t \right] \left| \sigma^2 + \rho \sigma \xi \right| \delta \hat{B}_t + \sigma \delta \hat{B}_t
\]
\[
\approx E\left[ \frac{K}{2} S_t \right] \left| \sigma^2 + \rho \sigma \xi \right| \sqrt{\frac{2}{\pi}} \sqrt{\frac{\delta \hat{B}_t}{\delta \hat{B}_t}} + \sigma \delta \hat{B}_t
\]
\[
= E\left[ \frac{K}{2} S_t \right] \left| \sigma^2 + \rho \sigma \xi \right| \sigma \delta \hat{B}_t + \sigma \delta \hat{B}_t
\]
\[
= E\left[ \frac{K}{2} S_t \right] \left| \sigma^2 + \rho \sigma \xi \right| \sigma \delta \hat{B}_t + \sigma \delta \hat{B}_t
\]
\[
\approx E\left[ \frac{K}{2} S_t \right] \left| \sigma^2 + \rho \sigma \xi \right| \delta \hat{B}_t + \sigma \delta \hat{B}_t
\]

Substituting Eq. (31) into Eq. (28) yields the following PDE of the call value
\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial S^2} + \rho \sigma \xi \frac{\partial C}{\partial r \partial S} + r S_r \frac{\partial C}{\partial S} - r C_r + k(\theta - r) \frac{\partial C}{\partial r} \delta t + E\left[ \frac{K}{2} S_{t+\delta t} \right] |_{t+\delta t} = O((\delta t)^2 (\log(\delta t))^{-1 \frac{3}{2}})
\]

Solving this PDE for \(C\) is similar to solving the Eq. (B1) in Appendix B. The similar tedious manipulation yields the following theorem.

**Theorem 4** Based on the discrete hedging principle, the value of European call \(C\) in a market with proportional transaction costs can be presented as
\[\footnote{When we consider trading costs, we take the delta hedging strategy, which has been derived in Section 3, as an exogenous strategy in this section.}
\[ C_i = S_i N(d_i) - KD_i N(d_2), \]

where \( d_2 = \frac{\ln \frac{S_i}{K} - \bar{\gamma} \tau}{\sqrt{2 \bar{\gamma} \tau}}, \quad d_3 = d_2 + \sqrt{2 \bar{\gamma} \tau}, \]

\[ \tilde{\tau}(t) = \frac{1}{\tau} \int_0^\tau \tilde{\sigma}^2 ds, \quad \tilde{\sigma}^2 = \sigma^2 G_t^2 + 2 \rho \sigma \bar{G}_t + \varepsilon^2 + \sigma_{\mu}^2 (\tilde{\tau})^{3/2} + \kappa \sqrt{2 \pi} (\sigma G_t)^2 + 2 \rho \sigma \bar{G}_t + \varepsilon^2 + \sigma_{\mu}^2 (\tilde{\tau})^{3/2} \tilde{\tau}^{1/4}, \]

where \( \tilde{\tau} \) is the solution of the following Eq. (36).

This formula is a semi-explicit pricing formula. Comparing Eq. (33) and BS option pricing formula, we know \( \tilde{\sigma} \) is the corresponding equivalent volatility. Now we show that there exist \( \tilde{\tau}' \in (0, 1) \) such that \( \tilde{\sigma}_{\min}^2 = \min_{\tilde{\tau} \in (0, 1)} \tilde{\sigma}^2 = \tilde{\sigma}^2(\tilde{\tau}') \) holds. Substituting \( \tilde{\sigma}_{\min}^2 \) into \( \tilde{\sigma}^2 \) in Eq. (35) we obtain the actual call value.

Let \( g(\tilde{\tau}) = \sigma_{\mu}^2 (\tilde{\tau})^{3/2} + \kappa \sqrt{2 \pi} (\sigma G_t)^2 + 2 \rho \sigma \bar{G}_t + \varepsilon^2 + \sigma_{\mu}^2 (\tilde{\tau})^{3/2} \tilde{\tau}^{1/4} \), then \( \tilde{\tau}' \) is the solution of equation \( g'(\tilde{\tau}) = 0 \). From \( g'(\tilde{\tau}) = 0 \) we can get

\[ \frac{1}{\kappa} \sqrt{\frac{\pi}{2}} \tilde{\tau}^{1/2} + \frac{1}{2} \left( \frac{(\sigma G_t)^2 + 2 \rho \sigma \bar{G}_t + \varepsilon^2}{(\tilde{\tau})^{3/2}} + \sigma_{\mu}^2 \right) = \frac{1}{2(2H-1)\sigma_{\mu}^2} \left( \frac{(\sigma G_t)^2 + 2 \rho \sigma \bar{G}_t + \varepsilon^2}{(\tilde{\tau})^{3/2}} + \sigma_{\mu}^2 \right), \]  

(36)

Let \( g_1(\tilde{\tau}) = \frac{1}{\kappa} \sqrt{\frac{\pi}{2}} (\tilde{\tau})^{1/2} + \frac{1}{2} \left( \frac{(\sigma G_t)^2 + 2 \rho \sigma \bar{G}_t + \varepsilon^2}{(\tilde{\tau})^{3/2}} + \sigma_{\mu}^2 \right) \) and \( g_2(\tilde{\tau}) = \frac{1}{2(2H-1)\sigma_{\mu}^2} \left( \frac{(\sigma G_t)^2 + 2 \rho \sigma \bar{G}_t + \varepsilon^2}{(\tilde{\tau})^{3/2}} + \sigma_{\mu}^2 \right). \)

Obviously, the function \( g_1(\tilde{\tau}) > 0 \) is strict monotonic increasing w.r.t. \( \tilde{\tau} \), while the function \( g_2(\tilde{\tau}) > 0 \) is strict monotonic decreasing w.r.t. \( \tilde{\tau} \). Since \( \lim_{\tilde{\tau} \to -\infty} g_1(\tilde{\tau}) = 0 \) as well as \( \lim_{\tilde{\tau} \to +\infty} g_2(\tilde{\tau}) = +\infty \), there exists a unique solution of \( \tilde{\tau} > 0 \) for equation \( g_1(\tilde{\tau}) = g_2(\tilde{\tau}) \). Let \( g_{1/2}(\tilde{\tau}) = g_1(\tilde{\tau}) - g_2(\tilde{\tau}) \), then we have \( \lim_{\tilde{\tau} \to 0+} g_{1/2}(\tilde{\tau}) = -\infty < 0 \) and \( g_{1/2}(1) = \frac{1}{\kappa} \sqrt{\frac{\pi}{2}} + \frac{1}{2} \left( \frac{(\sigma G_1)^2 + 2 \rho \sigma \bar{G}_1 + \varepsilon^2}{(1)^{3/2}} + \sigma_{\mu}^2 \right) > 0 \). Therefore, the equation (36) has a unique solution \( \tilde{\tau}' = f(T-t, H) \in (0, 1) \). In other words, the time between rehedges is not fixed but a function of time to maturity. Thus \( \tilde{\sigma}_t^2 \) in Eq. (35) turns to be

\[ \tilde{\sigma}_t^2 = \tilde{\sigma}_{\min}^2 = \sigma^2 G_t^2 + 2 \rho \sigma \bar{G}_t + \varepsilon^2 + g(\tilde{\tau}') \]  

(37)

**Note 4** If we let \( \kappa = 0 \) and \( \tilde{\tau} \to 0 \), which means that the market is frictionless and the investors can trade continuously, then we have \( \tilde{\sigma}_t^2 = \sigma^2 G_t^2 + 2 \rho \sigma \bar{G}_t + \varepsilon^2 \). This means that the uncertainty resulting from fBm has no contribution to the call price. Or, the randomness of the call resulting from fBm can be eliminated by a continuous adapted hedging strategy. The reason may be that the quadratic variation of Bm is the length of the time interval but the quadratic variation of fBm is zero.
Note 5 $\delta t^*$ is a function of $T - t$ since $G_t$ is a function of $T - t$, thus the call value is a function of $T - t$. Recalling that the option price, in the case where Wick self-financing strategy is used, is a function of $T^{2H} - t^{2H}$, we can claim the rationality of our model.

5 Results and Discussions

Incorporating Section 2 with section 4, we have proved the following theorem, which is the pricing formula for the warrant bonds with default risk in our model.

**Theorem 5.** The value of the warrant bonds with the assumptions from S1 to S5 has the following explicit presentation

$$V_t = \frac{m}{m + \alpha n} C_t + F \cdot e^{-\delta(t-t^*)}D(t, r; T), \quad 0 < t \leq T,$$

(38)

where $C_t$ is shown by **Theorem 4**, and $D(t, r; T)$ is shown as Eq. (3) with maturity $T$.

We now present a numerical example using representative values of the parameters. Let $S_0 = 10$ (dollars), $m = 10^5$, $n = 2 \times 10^5$, $T = 5$ (years), $F = 100$ (dollars), $K = 12$ (dollars), $T = 4.5$ (years), and $\alpha = 10$. The values of parameters in the Vasicek interest rate model are: $k = 1$, $\theta = 3\%$, $\sigma = 0.15$, and $r_0 = 3\%$. The parameters in the SDE of stock price are: $\rho = -0.1$, $\varepsilon = 0.2$, $\sigma_H = 0.2$, $H = 0.6$. The credit spread is chosen as $\lambda = 0.05$. We keep the values of above parameters unchanged and let $t$ vary from 0.01 to 4.49 in three cases where $\kappa = 0.1\%, 0.15\%, 0.02\%$. We calculate the term $\delta t^*$ and show it as a function of $t$ in figure 1. Figure 1 shows that the portfolio is not revised every fixed timestep. The time between two consecutive rehedges shrinks very slowly as time increases to maturity. Moreover, when the transaction cost increases, the time between two consecutive rehedges increases.

![Figure 1](image1.png)

Figure 1 $\delta t^*$ with varying times that increase to maturity

Now we choose $H \in (0.52, 0.98)$, $T - t = 2$, and $S_t = 12$, we draw the picture of $V(t)$ and the picture of $\sigma_{\text{min}}^{2}$ with varying H, which are shown in Figure 2 and Figure 3 respectively. From Figure2, the value of warrant bond decreases as the Hurst parameter H increases from 0.52 to 0.9, just as we expected. The reason can be obtained by figure3. As H increases $\sigma_{\text{min}}^{2} = \sigma^{2}G_t^{2} + 2\rho\sigma\varepsilon G_t + \varepsilon^{2} + g(\delta t^*)$ shrinks. Based on the section 3 and the section 4, we have the following causalities, which shows how the transaction costs, time to maturity and Hurst parameter affect the value of the warrant bonds.

$$T - t \rightarrow G_t \quad H \rightarrow \delta t^* \rightarrow \sigma_{\text{min}}^{2} = \sigma^{2}G^2 + 2\rho\sigma\varepsilon G_t + \varepsilon^{2} + g(\delta t^*) \rightarrow C_t \rightarrow V_t$$

Here $A \rightarrow B$ means that the value of B is affected by the value of A.
6 Conclusions

Previous studies on warrant bond valuation models always used semi-martingales to describe the values of underlying assets. It has rarely been studied to consider the long-range serial autocorrelation in returns of the underlying assets when pricing warrant bonds. Even if such a property was taken into account, the existing models are still imperfect allowing for improvements.

Our valuation of the warrant bonds takes the stock price as underlying variable. Our goal is to establish a rational model to value the warrant bonds in the framework of fractal market hypothesis, and the focus is on the impact of the serial autocorrelation in returns, which is implicated by the Hurst parameter, on the value of the warrant bonds. We do not use concepts of Wick portfolio value or Wick self-financing strategy. So our model is not lack of economic meanings and practical values.

Our model can be viewed as an extension of the model by Wang et al. (2010)[29] which assumed a constant interest rate. Since warrant bonds have long lifespans, the assumption of constant interest rates may be inapposite. We assume that the interest rate follows the Vasicek process instead of keeping unchanged. The numerical results show that the randomness of the interest rates leads to a decreasing timestep of revising the hedged portfolio instead of a constant timestep. Roughly speaking, the revision timestep goes down with an increasing speed.

The warrant bond is split into two components: options and a bond with default risk. The default risk is described by the credit spread and hence the pricing for WB then turns into pricing for option. We obtain the pricing formula for WBs as well as for options. By comparing the option pricing formula in our model with the BS option pricing formula, we get the term called “equivalent volatility”. So we can analyze how the factors influence the option price by analyzing how the factors affect the “equivalent volatility”, which makes us understand the causality clear.

The influence of Hurst parameter on the WB value is based on its influence on the value of the embedded call options. The value of the warrant bond decreases as the Hurst parameter increases. The serial autocorrelation becomes stronger when the Hurst parameter $H$ increases from 0.5 to 1, namely, the uncertainty goes weaker. This gives rise to the diminution of the “equivalent volatility” of the underlying variable, and thus the diminution of the value of WBs. The result is not against economic intuitions.

That we draw a large length of wring to price option with no trading cost in Section 3 is not needless. We need the delta hedging strategy that obtained in Section 3 as exogenous strategy in Section 4 and need the Lemma A1 to calculate the trading costs before arriving at the option pricing formula with transaction costs.

Appendix A

Lemma A1. Assume the financial market has no trading cost, the option value represented by Eq. (25) in Theorem 3
satisfies the following relationship:
\[
\frac{\partial^2 C}{\partial S \partial r} = S \frac{\partial^2 C}{\partial S^2} = G_t. \tag{A1}
\]

**Proof.** Substituting \( D_t = e^{-G_{t}C(T \in A, r; T)} = e^{-G_{t}+A} \) into Eq. (26), we have
\[
d_z = \frac{\ln S_t - \ln D_t - \ln K - r}{\sqrt{2\tau_t}} = \frac{\ln S_t + G_t r - A_t - \ln K - r}{\sqrt{2\tau_t}}.
\]

Noticing that \( d_z = d_z + \sqrt{2\tau_t} \), we have \( \frac{\partial d_z}{\partial S} = \frac{\partial d_z}{\partial S} = \frac{1}{S_t \sqrt{2\tau_t}} \) and \( \frac{\partial d_z}{\partial r} = \frac{\partial d_z}{\partial r} = \frac{G_t}{\sqrt{2\tau_t}} \). \tag{A2}

Differentiating both sides of Eq. (25), namely, \( C(t, r, S_t) = S_t N(d_z) - KD_t N(d_z) \), we can get
\[
\frac{\partial C}{\partial S} = N(d_z) + S_t \phi(d_z) \frac{\partial d_z}{\partial S} - KD_t \phi(d_z) \frac{\partial d_z}{\partial S} - N(d_z) + \phi(d_z) \frac{1}{S_t \sqrt{2\tau_t}} - KD_t \phi(d_z) \frac{1}{S_t \sqrt{2\tau_t}}.
\]
\[
\frac{\partial^2 C}{\partial S^2} = \phi(d_z) \frac{\partial d_z}{\partial S} + \phi'(d_z) \frac{\partial d_z}{\partial S} \frac{1}{\sqrt{2\tau_t}} - KD_t \phi'(d_z) \frac{\partial d_z}{\partial S} \frac{1}{S_t \sqrt{2\tau_t}} + KD_t \phi'(d_z) \frac{1}{S_t \sqrt{2\tau_t}}
\]
\[
= \phi(d_z) + \phi'(d_z) \frac{1}{S_t \sqrt{2\tau_t}} - KD_t \phi'(d_z) \frac{1}{S_t \sqrt{2\tau_t}} + KD_t \phi'(d_z) \frac{1}{S_t \sqrt{2\tau_t}}.
\]
\[
\frac{\partial^2 C}{\partial S \partial r} = \phi(d_z) \frac{\partial d_z}{\partial r} + \phi'(d_z) \frac{\partial d_z}{\partial r} \frac{1}{\sqrt{2\tau_t}} - KD_t \phi'(d_z) \frac{\partial d_z}{\partial r} \frac{1}{S_t \sqrt{2\tau_t}} - K \frac{\partial D_t}{\partial r} - \phi(d_z) \frac{1}{S_t \sqrt{2\tau_t}}
\]
\[
= \phi(d_z) \frac{G_t}{\sqrt{2\tau_t}} + \phi'(d_z) \frac{G_t}{\sqrt{2\tau_t}} \frac{1}{\sqrt{2\tau_t}} - KD_t \phi'(d_z) \frac{G_t}{\sqrt{2\tau_t}} \frac{1}{S_t \sqrt{2\tau_t}} + KD_t \phi'(d_z) \frac{G_t}{\sqrt{2\tau_t}} \frac{1}{S_t \sqrt{2\tau_t}}.
\]

where \( \phi(\cdot) \) represents the probability density function of the standard normal distribution, and \( \phi'(\cdot) \) is its derivative. Comparing Eq.(L4) and Eq. (L5), we can get Eq. (L1), which completes the proof. \( \blacksquare \)

**Appendix B**

**Proof of Theorem 3.** Ignoring the higher order terms and omitting the subscript \( t \) in Eq. (20), we have that \( C \) satisfies the following PDE with boundary condition
\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial r^2} + \rho \sigma S \frac{\partial^2 C}{\partial r \partial S} + \frac{1}{2} \left[ e^{-2} + \sigma^2 (\delta t)^{2\mu-1} \right] S \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C + k(\theta - r) \frac{\partial C}{\partial r} = 0, \quad t \in [0, T].
\]
\[
C_T = (S_T - K)^+. \tag{B1}
\]

In order to solve this boundary value problem, we need change variables three times.

First, let \( y = \frac{S}{D(t,r)} \), \( \hat{C}(t, y) = \frac{C(t, r, S)}{D(t, r)} \). We need to note that the complete form of these variable substitutions is
\[
y = \frac{S}{D(t, r; T)} \hat{C}(t, y) = \frac{C(t, r; S)}{D(t, r; T)}. \tag{B2}
\]

According to the chain rule of derivation, we can obtain
\[
\begin{align*}
\frac{\partial \hat{C}}{\partial t} &= \frac{\partial C}{\partial t} + D \frac{\partial \hat{C}}{\partial r} + \frac{\partial}{\partial y} \left( \frac{\partial \hat{C}}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial C}{\partial y} \right), \\
\frac{\partial \hat{C}}{\partial r} &= \frac{\partial C}{\partial r} + \frac{\partial}{\partial r} \left( \frac{\partial \hat{C}}{\partial r} \right) - \frac{\partial}{\partial r} \left( \frac{\partial C}{\partial r} \right), \\
\frac{\partial^2 \hat{C}}{\partial y^2} &= \frac{\partial^2 C}{\partial y^2} - \frac{\partial^2}{\partial y^2} \left( \frac{\partial C}{\partial y} \right), \\
\frac{\partial \hat{C}}{\partial S} &= \frac{\partial C}{\partial S} - \frac{\partial}{\partial S} \left( \frac{\partial C}{\partial S} \right), \\
\frac{\partial^2 \hat{C}}{\partial \eta \partial \tau} &= \frac{\partial^2 C}{\partial \eta \partial \tau} - \frac{\partial^2}{\partial \eta \partial \tau} \left( \frac{\partial C}{\partial \eta} \right). 
\end{align*}
\] (B3)

Substituting the above equations into Eq. (B1) and rearranging the terms, we obtain

\[
\frac{\partial \hat{C}}{\partial t} + \frac{1}{2} \sigma^2(t) \frac{\partial^2 \hat{C}}{\partial y^2} = 0. 
\] (B4)

Second, let \( x = \ln y \), \( y = e^x \) in order to eliminate term \( y^2 \) in Eq. (B4), then we have

\[
\begin{align*}
\frac{\partial \hat{C}}{\partial y} &= \frac{\partial \hat{C}}{\partial x} \frac{\partial x}{\partial y} = 1 \frac{\partial \hat{C}}{\partial x}, \\
\frac{\partial^2 \hat{C}}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{\partial \hat{C}}{\partial y} \right) = 1 \frac{\partial^2 \hat{C}}{\partial x^2} \frac{\partial x}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\partial \hat{C}}{\partial y} \right) = \frac{\partial^2 \hat{C}}{\partial x^2} - \frac{\partial}{\partial x} \left( \frac{\partial \hat{C}}{\partial x} \right), \\
\frac{\partial \hat{C}}{\partial S} &= \frac{\partial C}{\partial S} - \frac{\partial}{\partial \eta} \left( \frac{\partial \hat{C}}{\partial \eta} \right). 
\end{align*}
\]

Thus Eq. (B4) turns to be

\[
\frac{\partial \hat{C}}{\partial t} + \frac{1}{2} \sigma^2(t) \frac{\partial^2 \hat{C}}{\partial x^2} = 0. 
\] (B5)

Third, let \( \hat{C}(t,y) = u(\tau, \eta) \), where \( \tau = \tau(t) \) satisfying \( \tau(T) = 0, \) then

\[
\begin{align*}
\frac{\partial \hat{C}}{\partial \tau} &= \frac{\partial u}{\partial \tau} \frac{\partial \tau}{\partial \eta} + \frac{\partial u}{\partial \eta} \frac{\partial \tau}{\partial \tau}, \\
\frac{\partial^2 \hat{C}}{\partial \eta \partial \tau} &= \frac{\partial}{\partial \eta} \left( \frac{\partial \hat{C}}{\partial \tau} \right) = \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \tau}{\partial \eta} + \frac{\partial u}{\partial \eta} \frac{\partial \tau}{\partial \tau} \frac{\partial \tau}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \tau}{\partial \eta} \frac{\partial \tau}{\partial \tau} - \frac{\partial^2 u}{\partial \eta \partial \tau} \frac{\partial \tau}{\partial \eta} \frac{\partial \tau}{\partial \tau}, \\
\frac{\partial \hat{C}}{\partial \tau} &= \frac{\partial u}{\partial \eta} \frac{\partial^2 \tau}{\partial \eta^2} \left( \frac{\partial \tau}{\partial \eta} \right) + \frac{\partial u}{\partial \eta} \theta'(t) = \frac{\partial u}{\partial \eta} \tau'(t) + \frac{\partial u}{\partial \eta} \eta'(t). 
\end{align*}
\]

Substituting the above equations into Eq. (B5) and rearranging the terms, we obtain

\[
\frac{\partial u}{\partial \tau} \tau'(t) + \frac{1}{2} \sigma^2(t) \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \tau}{\partial \eta} - \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \tau}{\partial \eta} \frac{\partial \tau}{\partial \eta} = 0. 
\]

Let \( \tau'(t) = -\frac{1}{2} \sigma^2(t) \) and \( \theta'(t) = -\frac{1}{2} \sigma^2(t), \) then the above Eq. becomes

\[
\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \eta^2}. 
\] (B6)

Noticing the conditions of terminal value \( \tau(T) = 0 \) \ and \( \theta(T) = 0, \) we can get

\[
\tau(t) = \frac{1}{2} \int_t^T \sigma^2(s) ds, \quad \theta(t) = -\frac{1}{2} \int_t^T \sigma^2(s) ds = -\tau(t). 
\]
The terminal condition \( \dot{C}(t, y) = (y - K)' \) then turns to be \( u(0, \eta) = (e^\theta - K)' \).  \( (B7) \)

Due to Poisson formula for solving heat conduction equation, we have

\[
u(t, \eta) = \frac{1}{\sqrt{2\pi \cdot 2^{\tau}}} \int_{-\infty}^{\infty} (e^\xi - K) e^{-\frac{(q+1)^2}{2\cdot 2^{\tau}}} d\xi, \quad (0 < \tau < \tau(0))
\]

\[
= \frac{1}{\sqrt{2\pi \cdot 2^{\tau}}} \int_{-\infty}^{\infty} e^{(q+1)^2} d\xi - K \frac{1}{\sqrt{2\pi \cdot 2^{\tau}}} \int_{-\infty}^{\infty} e^{\frac{\xi^2}{2\cdot 2^{\tau}}} d\xi,
\]

\[
= \frac{1}{\sqrt{2\pi \cdot 2^{\tau}}} \frac{1}{2^{\tau}} \int_{-\infty}^{\infty} e^{\frac{\xi^2}{2\cdot 2^{\tau}}} d\xi - K \frac{1}{\sqrt{2\pi \cdot 2^{\tau}}} \int_{-\infty}^{\infty} e^{\frac{\xi^2}{2\cdot 2^{\tau}}} d\xi,
\]

\[
y \frac{1}{\sqrt{2\pi \cdot 2^{\tau}}} \frac{1}{2^{\tau}} \int_{-\infty}^{\infty} e^{\frac{\xi^2}{2\cdot 2^{\tau}}} d\xi - K \frac{1}{\sqrt{2\pi \cdot 2^{\tau}}} \int_{-\infty}^{\infty} e^{\frac{\xi^2}{2\cdot 2^{\tau}}} d\xi
\]

\[
= y \frac{1}{\sqrt{2\pi \cdot 2^{\tau}}} \frac{1}{2^{\tau}} \int_{-\infty}^{\infty} e^{\frac{\xi^2}{2\cdot 2^{\tau}}} d\xi - K \frac{1}{\sqrt{2\pi \cdot 2^{\tau}}} \int_{-\infty}^{\infty} e^{\frac{\xi^2}{2\cdot 2^{\tau}}} d\xi
\]

Therefore, we have \( \dot{C}(t, y) = u(t, \eta) = yN(d_1) - KN(d_2) \). Due to Eq. (B1) and noticing that

\[
\eta(t) = x(t) + \theta(t) = \ln y(t) + \theta(t) = \ln \frac{S_t}{D_t} + \theta(t) = \ln \frac{S_t}{D_t} - \tau(t),
\]

we can get Eq. (25). ■

References