Tax evasion, public debt, foreign investors and growth

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Abstract

Tax evasion is one of the most studied and the least desired effects of government intervention in the economy. Public expenditure contributes to the economic growth through the provision of infrastructure that increase the productivity of private capital: stagnation and low economic growth may arise as a consequence of tax evasion in a self fuelling mechanism. Most of the model proposed by the literature are set in a static framework, but the impact of tax evasion on growth is clearly a long term problem. In this article we propose an endogenous growth model to analyze the relation between tax evasion and public debt accumulation. We develop a model where debt can be either bought by national or foreign investors. Our paper makes several contributions to the existing literature on tax evasion and debt. In a setting without fiscal illusion evasion may be optimal, provided that the Government sets its level of tax rate higher than the ratio of public expenditure to income. This may happen when Government is trying to reduce pre-existing debt or when, due to some inefficiencies, a part of the tax revenue does not become public expenditure. Although consumers are rational they are not however able to use tax evasion to increase the long run growth of the economy even when public expenditure is not optimal.

Keywords: Dynamic tax evasion, debt, endogenous growth
JEL Classification: G11, H26, H4

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1 Introduction

Tax evasion is one of the most studied and the least desired effects of government intervention in the economy. Since the seminal papers by Allingham and Sandmo (1972) and Yitzhaki (1974), the literature has offered several explanations and possible solutions for this phenomenon. Despite these efforts, tax evasion seems to be increasing.

The most recent estimates (Feige and Cebula, 2011) show that intentional underreporting of income is about 18-19% of the total reported income in the US, leading to a tax gap of about 500 billion dollars. In Europe, the level of tax evasion is about 20% of GDP, accounting for a potential loss of about 1 trillion Euros each year (Buehn and Schneider, 2012; Murphy, 2014). Reducing tax evasion is a priority for most Governments, both in developed and developing countries since the revenue loss is only the tip of the iceberg as concerns the effect of tax evasion (Slemrod 2007; Alm 2012; Dzhumashev and Gahramanov 2011; Markellos et al. 2016). Public expenditure contributes to the economic growth through the provision of infrastructure that increase the productivity of private capital: stagnation and low economic growth may arise as a consequence of tax evasion in a self fuelling mechanism (Dzhumashev et al. (2019).

Most models in this literature are set in a static framework, but the impact of tax evasion on growth is clearly a long term problem. The literature has recently started to study this problem in a dynamic context, but usually using a single country framework where evasion may derive only as a result of fiscal illusion. Furthermore, in a an open economy, debt may be sold to investors abroad, something that in the short run may allow to reduce the domestic budget pressure, but that in the long run it may causes further problems and may spread recession across countries.

In a recent article Marakbi and Villieu (2018) have developed an endogenous growth model to analyze the relation between tax evasion and public debt accumulation. We start from their contribution to develop a model where debt can be either bought by national or foreign investors. Our paper makes several contributions to the existing literature on tax evasion and debt. In a setting without fiscal illusion evasion may be optimal, provided that the Government sets its level of tax rate higher than the ratio of public expenditure to income. This may happen when Government is trying to reduce pre-existing debt or when, due to some inefficiencies, a part of the tax revenue does not become public expenditure. The paper is organised as follows after reviewing the literature, in Section 3 we present the model; in Section 4 and 5 the results of our analysis while in Section 6 we present the implications on growth. Section 7 concludes.

2 Related literature

The literature has long analyzed the optimal taxation policies in economies where tax evasion is widespread, but very little has been produced on the relationship between public debt, tax evasion and long term prospects of the economy. Chen (2003) examines how tax evasion affects the optimal tax rate in a an AK endogenous growth model with productive public expenditures and a balanced-budget rule. Similarly to Barro (1990), his model reproduces the inverted U-shaped curve between optimal taxation and growth and shows that the optimal optimal tax rate is higher as tax evasion becomes more widespread. However, this result holds only when the government has no other instruments to finance public spending. Marakbi and Villieu (2018) introduce debt and study a two step model where at the first stage evasion is an exogenous fraction of the government's revenues. As a second step, they endogenize tax evasion and consider the optimizing behavior of households who make an effort to evade as much taxes as possible. In this setting they show that several equilibria may arise in the steady state; a high-growth and low-public debt solution and a low-growth and high-public debt solution. As concerns the long run consequences of tax evasion on growth and public debt, the initial solution and the productivity of public expenditure are compatible with several solutions. The model proposed is rather peculiar in the assumptions used and does not allow to obtain resuls for more general cases. Our model will instead consider the problem of a consumer that wants to maximise consumptions streams in his/her life, it is fully rational and can anticipate Government choices. Public debt can be owned by residents or non residents and to start with we will consider the case where the quantity domestically owned is chosen by domestic consumers.

3 The model

We model a simple endogenous growth model à al Barro Barro (1990) where public expenditure enhances the productivity of private capital. The government produces public expenditure g_t, while agents derive income from the following production technology:

$$y_t = A_t k_t, \tag{1}$$

where

$$A_t = A_0 \left(\frac{g_t}{k_t} \right)^{\psi}, \quad \psi \in (0,1),$$

where k_t is capital in per worker terms, A_t is technological productivity parameter and ψ is a measure of productivity of the public good. Given that there is only a single final good, its price is normalized to one. The production of g_t is financed through a linear income tax at rate τ.

Public expenditure is a constant proportion of the product

$$g_t = \alpha y_t,$$

and so

$$y_t = \left(A_0 \alpha^{\psi} \right)^{\frac{1}{1-\psi}} k_t.$$

Without tax evasion and inefficiencies, Government would set $\alpha = \tau$, the budget would be balanced, and the dynamic equation of capital accumulation would be

$$dk_t = ((1 - \tau) y_t - c_t) dt, \tag{2}$$

where c_t is the instantaneous consumption. Let us now assume that evasion is expedient in this context and that a fraction of income may be concealed from the tax authorities.This process has two effects:

1. it changes the consumers optimization problem because agents have to decide their optimal level of tax evasion;

2. it changes Government decisions since the latter will have to change its decision on how to finance public expenditure.

Both processes change capital accumulation.

3.1 Evasion and capital accumulation

If the agent may hide (evade) a fraction e_t of his income y_t and evasion is detected, a penalty must be paid. If we call $\eta(\tau)$ such a fee (defined as a non-decreasing function of τ), the amount of money that must be paid when caught is

$$\eta(\tau) e_t y_t. \tag{3}$$

Function $\eta(\tau)$ allows us to consider several tax regimes: for $\eta(\tau) = \beta$, the fee is on evaded income as in Allingham and Sandmo (1972); for $\eta(\tau) = \beta\tau$ the fee is on evaded tax as in Yitzhaki (1974); for $\eta(\tau) = \beta + \gamma\tau$ the fee is a mix of the previous cases as in Levaggi and Menoncin (2012, 2013).

Evasion introduces risk in equation (2) since the fee $\eta(\tau)$ may or may not be paid. We model auditing as a Poisson jump process $d\Pi_t$ whose first two moments are:[1]

$$\mathbb{E}_t [d\Pi_t] = \lambda_t dt, \tag{4}$$
$$\mathbb{V}_t [d\Pi_t] = \lambda_t dt, \tag{5}$$

where $\mathbb{E}_t [\cdot]$ and $\mathbb{V}_t [\cdot]$ are the expected value and the variance operators, conditional on the information set at time t, and $\lambda_t \in [0, \infty[$ is the "intensity" of the process and determines the frequency of audits within a time interval. When $\lambda_t = 0$, the probability of being caught is zero, while when λ_t tends towards infinity, the probability of being caught tends towards 1. The stochastic process of capital accumulation can be written as

$$dk_t = ((1 - \tau + \tau e_t) y_t - c_t) dt - \eta(\tau) e_t y_t d\Pi_t, \tag{6}$$

[1] This process can be thought of as the limit of a binomial model whose value is 1 with probability λdt and 0 otherwise. See Levaggi and Menoncin (2012), and Levaggi and Menoncin (2013)

from which,

$$\mathbb{E}_t [dk_t] = ((1 - \tau + (\tau - \eta(\tau) \lambda_t) e_t) y_t - c_t) dt, \tag{7}$$

$$\mathbb{V}_t [dk_t] = \eta(\tau)^2 e_t^2 y_t^2 \lambda dt. \tag{8}$$

3.2 Tax evasion, public debt and capital accumulation

Let us now assume that Government sets $g_t = \alpha y_t$ and that this level of public expenditure will not be decreased because of tax evasion. As an answer to tax evasion by consumers, Government sets an audit process which allows to detect some evaders who pay $\eta(\tau) e_t y_t$ as a fine and evaded taxes. Government is able to actually obtain only a percentage (β) of that amount (because of inefficiencies and costs): $\beta\eta(\tau) e_t y_t$. On average, this is not enough to balance the budget. Let us then assume that the deficit in the presence of tax evasion is financed with the issue of public debt B_t.The public debt B_t:

- increases because of the interest rate that must be paid (r)
- increases because of public expenditure g_t
- decreases because of taxes on the non-evaded income $\tau(1 - e_t) y_t$
- decreases when evaders are caught by the amount $\beta\eta(\tau) e_t y_t$, thus:

$$dB_t = (B_t r - \tau(1 - e_t) y_t + \alpha y_t) dt - \beta\eta(\tau) e_t y_t d\Pi_t$$

As a result the agent's capital k_t:

- increases because of production y_t
- decreases because of consumption c_t
- increases because of coupons on public debt rB_t
- decreases because of taxes paid on the non-evaded product: $-\tau(1 - e_t) y_t$
- decreases by the amount $\eta(\tau) e_t y_t$ when caught evading
- decreases because the agent buy the new public debt dB_t

thus:

$$dk_t = ((1 - \tau + e_t\tau) y_t + B_t r - c_t) dt - dB_t - \eta(\tau) e_t y_t d\Pi_t$$

4 Foreign agent

Let us now assume that the domestic agents buy only a percentage ϕ of the public debt. The other percentage $(1 - \phi)$ is bought by foreign agents. In this case the capital accumulation can be written as follows

$$dk_t = ((1 - \tau + e_t\tau) y_t + \phi B_t r - c_t) dt - \phi dB_t - \eta(\tau) e_t y_t d\Pi_t,$$

and, accordingly, we have

$$dk_t = ((1 - \tau + e_t\tau - \phi\tau e_t + \phi(\tau - \alpha)) y_t - c_t) dt - (1 - \phi\beta) \eta(\tau) e_t y_t d\Pi_t.$$

5 Preliminary results

The representative consumer/investor chooses the optimal level of consumption and tax evasion (c_t^*, e_t^*) in order to maximize the following inter-temporal utility function:

$$\max_{e_t, c_t} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{(c_t - c_m)^{1-\delta}}{1-\delta} dt \right], \tag{9}$$

given the dynamic equation of the state variable k_t:

$$dk_t = ((1 - \tau + e_t \tau) y_t + \phi B_t r - c_t) dt - \phi dB_t - \eta(\tau) e_t y_t d\Pi_t,$$

where ρ is the subjective discount rate.

For $\beta = 1$ (i.e. there are no inefficiencies in Government's ability to collect fees) it is possible to obtain a closed-form solution. In this case the optimal consumption is

$$c_t^* = c_m + \left(k - \frac{c_m}{(1-\alpha) A} \right) \frac{\rho + (\delta - 1)(1 - \alpha) A}{\delta},$$

where $A = \left(\alpha^\psi A_0\right)^{\frac{1}{1-\psi}}$ the optimal percentage of bond is

$$\phi^* = 1,$$

i.e. it is optimal to buy the whole Government's debt, and the optimal evasion is

$$e_t^* = \frac{\tau - \alpha}{\tau - \eta(\tau) \lambda}, \tag{10}$$

which means that when the Government wants to reduce its debt by increasing tax τ over the public expenditure rate α, the it creates positive evasion.

The actual tax rate paid by the consumer is equal to:

$$\tilde{\tau}_t \equiv \tau - (\tau - \eta(\tau) \lambda) e_t$$

Substituting the optimal leve of tax evasion, from equation 10, we can write

$$\tilde{\tau}_t \equiv \alpha \tag{11}$$

which implies that the in equilibrium the effective tax rate that consumer are willing to pay allows to raise enough resources to finance productive public expenditure. Any level of expenditure below this level cannot be financed through taxes if evasion is expedient. This implies that once a debt exists, the only way to reduce it is to combat tax evasion first.

In this case the optimal capital accumulation is given by

$$dk_t = \left(k_t - \frac{c_m}{(1-\alpha) A} \right) \frac{(1-\alpha) A - \rho}{\delta} dt.$$

We can conclude that the economy grows at the rate

$$\frac{(1-\alpha)\,A-\rho}{\delta}$$

which is equivalent to what obtained by Barro (1990) as one might expect. The absence of fiscal illusion means that the long run growth is not influenced by the presence of the debt nor by tax evasion. It is interesting to note that this result is true because in this context there is no administrative costs arising from tax evasion, i.e. audits are run at no costs. In this case the consumer tries to reduce its fiscal burden to the level at which it maximises long run growth, given the constraint that public expenditure is set by the Government and consumers choices as concerns tax evasion do not influence this level of expenditure. Public debt will be used to balance any difference.

The optimal debt is given by

$$dB_t = \left(B_t r + \eta\frac{\tau-\alpha}{\tau-\eta\lambda}\lambda A k_t\right)dt - \eta\frac{\tau-\alpha}{\tau-\eta\lambda}A k_t d\Pi_t,$$

and its expected value is .

$$\mathbb{E}_t\left[\frac{dB_t}{B_t}\right] = r dt,$$

which means that the debt grows at the same rate as the interest rate. Again, given equation 11 once for some reason the budget is not balanced and Government issue some debt, it is not possible to reduce it unless the audit intesity and the penalty are increased to the point where evasion is no longer convenient.

Extensive audit is however incompatible with the assumption of zero costs and this may be the reason why most real world health care systems are reducing the frequency of audit while increasing the penalties Slemrod (2018, 2007).

6 , Tax evasion, public debt and growth

Let us now examine what are the implications of tax evasion on growth. Following Barro (1990), we can determine which is the tax rate that maximize growth. The long-run growth rate can be written as

$$\gamma = \frac{(1-\alpha)\left(\alpha^{\psi}A_0\right)^{\frac{1}{1-\psi}}-\rho}{\delta}.$$

and the optimal (growth maximizing) tax rate is maximise when

$$\alpha = \psi.$$

which implies that the optimal level of public expediture should be

$$g_t = \psi y_t$$

Any other level of expenditure is not optimal. In particular, if $\alpha > \psi$ evasion could maximise the long-run growth because in this case, even if all the tax revenue is used to increase public expenditure. However, in this model,

7 Futher research and extensions

The study of the relationship between tax evasion, public debt and growth will have to be furtherly extended. For the time being we have run some simulations showing that the effects may be either positive or negative. The next step is to find either an analytical solution for the effect of the debt hold by non resident investors or some simulations using real data. Another avenue we would like to explore is to find from CG point of view the optimal level of debt to be sold abroad. Finally in this model we have considered a fixed interest rate. Marakbi and Villieu (2018) uses a risk adjusted interest rate which will certainly be more realistic.

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A First demonstration

If we call $J(t,k,B)$ the value function of the optimization problem, then $J(t,k,B)$ must solve the following (Hamilton-Jacobi-Bellman) partial differential equation:

$$\begin{aligned} 0 = {} & \frac{\partial J}{\partial t} - (\rho + \lambda) J + \frac{(c_t - c_m)^{1-\delta}}{1-\delta} + \frac{\partial J}{\partial k}\left(\left(1 - (1-\phi)\alpha - \tau\phi + \tau\phi e_t\right) y_t + \phi B_t r - c_t\right) \\ & + \frac{\partial J}{\partial B}\left(B_t r - \tau(1-e_t) y_t + \alpha y_t\right) + \lambda J\left(t, k - (1-(1-\phi)\beta)\eta(\tau) e_t y_t, B - \beta\eta(\tau) e_t y_t\right) \end{aligned}$$

FOC c

$$c_t^* = c_m + \left(\frac{\partial J}{\partial k}\right)^{-\frac{1}{\delta}}$$

FOC e

$$\begin{aligned} 0 = {} & \frac{\partial J}{\partial k}\tau\phi y_t + \frac{\partial J}{\partial B}\tau y_t \\ & + \lambda\left(\frac{\partial J}{\partial\left(k - (1-(1-\phi)\beta)\eta(\tau) e_t y_t\right)}\left(-(1-(1-\phi)\beta)\eta(\tau) y_t\right) + \frac{\partial J}{\partial\left(B - \beta\eta(\tau) e_t y_t\right)}\left(-\beta\eta(\tau) y_t\right)\right) \end{aligned}$$

Guess

$$J = F^{\delta}\frac{(k+hB-H)^{1-\delta}}{1-\delta}$$

$$0 = \delta F^{\delta-1}\frac{(k+hB-H)^{1-\delta}}{1-\delta}\frac{\partial F}{\partial t} - F^{\delta}(k+hB-H)^{-\delta}\frac{\partial H}{\partial t} - (\rho+\lambda)F^{\delta}\frac{(k+hB-H)^{1-\delta}}{1-\delta}$$
$$+\frac{(c_t-c_m)^{1-\delta}}{1-\delta} + F^{\delta}(k+hB-H)^{-\delta}\left((1-(1-\phi)\alpha-\tau\phi+\tau\phi e_t)y_t+\phi B_t r - c_t\right)$$
$$+hF^{\delta}(k+hB-H)^{-\delta}(B_t r-\tau(1-e_t)y_t+\alpha y_t)+\lambda F^{\delta}\frac{(k+hB-H-(1-(1-\phi)\beta+h\beta)\eta(\tau)e_t y_t)^{1-\delta}}{1-\delta}$$

FOC c

$$c_t^* = c_m + \frac{k+hB-H}{F}$$

FOC e

$$e_t^* = \frac{k+hB-H}{(1-(1-\phi)\beta+h\beta)\eta(\tau)y_t}\left(1-\left(\frac{(1-(1-\phi)\beta+h\beta)\eta(\tau)\lambda}{(\phi+h)\tau}\right)^{\frac{1}{\delta}}\right)$$

HJB

$$0 = \delta F^{\delta-1}\frac{(k+hB-H)^{1-\delta}}{1-\delta}\frac{\partial F}{\partial t} - F^{\delta}(k+hB-H)^{-\delta}\frac{\partial H}{\partial t} - (\rho+\lambda)F^{\delta}\frac{(k+hB-H)^{1-\delta}}{1-\delta}$$
$$+\frac{\delta}{1-\delta}F^{\delta-1}(k+hB-H)^{1-\delta}$$
$$+F^{\delta}(k+hB-H)^{-\delta}(1-(1-\phi)\alpha-\tau\phi-\tau h+\alpha h)A_{\psi}\left(k_t+\frac{\phi r+hr}{(1-(1-\phi)\alpha-\tau\phi-\tau h+\alpha h)A_{\psi}}B_t-H\right)$$
$$+F^{\delta}(k+hB-H)^{-\delta}(1-(1-\phi)\alpha-\tau\phi-\tau h+\alpha h)A_{\psi}H$$
$$-F^{\delta}(k+hB-H)^{-\delta}c_m$$
$$+F^{\delta}(k+hB-H)^{1-\delta}\frac{\tau\phi+h\tau}{(1-(1-\phi)\beta+h\beta)\eta(\tau)}\left(1-\left(\frac{(1-(1-\phi)\beta+h\beta)\eta(\tau)\lambda}{(\phi+h)\tau}\right)^{\frac{1}{\delta}}\right)$$
$$+\lambda F^{\delta}(k+hB-H)^{1-\delta}\frac{\left(\frac{(1-(1-\phi)\beta+h\beta)\eta(\tau)\lambda}{(\phi+h)\tau}\right)^{\frac{1-\delta}{\delta}}}{1-\delta}$$

Condition: there exists h^* such that

$$\frac{\phi r+h^* r}{(1-(1-\phi)\alpha-\tau\phi-\tau h^*+\alpha h^*)A_{\psi}} = h^*,$$

and so

$$
\begin{aligned}
0 =& \delta F^{\delta-1} \frac{(k+hB-H)^{1-\delta}}{1-\delta} \frac{\partial F}{\partial t} - (\rho+\lambda) F^{\delta} \frac{(k+hB-H)^{1-\delta}}{1-\delta} + \frac{\delta}{1-\delta} F^{\delta-1} (k+hB-H)^{1-\delta} \\
&+ F^{\delta} (k+hB-H)^{1-\delta} (1-(1-\phi)\alpha - \tau\phi - \tau h + \alpha h) A_{\psi} \\
&+ F^{\delta} (k+hB-H)^{1-\delta} \frac{\tau\phi + h\tau}{(1-(1-\phi)\beta + h\beta)\eta(\tau)} \left(1 - \left(\frac{(1-(1-\phi)\beta + h\beta)\eta(\tau)\lambda}{(\phi+h)\tau}\right)^{\frac{1}{\delta}}\right) \\
&+ F^{\delta} (k+hB-H)^{1-\delta} \frac{\lambda}{1-\delta} \left(\frac{(1-(1-\phi)\beta + h\beta)\eta(\tau)\lambda}{(\phi+h)\tau}\right)^{\frac{1-\delta}{\delta}} \\
0 =& -F^{\delta} (k+hB-H)^{-\delta} \frac{\partial H}{\partial t} + F^{\delta} (k+hB-H)^{-\delta} (1-(1-\phi)\alpha - \tau\phi - \tau h + \alpha h) A_{\psi} H - F^{\delta} (k+hB-H)
\end{aligned}
$$

$$
\begin{aligned}
0 =& \frac{\partial F}{\partial t} - (\rho+\lambda) F \frac{1}{\delta} + 1 \\
&+ \frac{1-\delta}{\delta} F (1-(1-\phi)\alpha - \tau\phi - \tau h + \alpha h) A_{\psi} \\
&+ \frac{1-\delta}{\delta} F \frac{\tau\phi + h\tau}{(1-(1-\phi)\beta + h\beta)\eta(\tau)} \left(1 - \left(\frac{(1-(1-\phi)\beta + h\beta)\eta(\tau)\lambda}{(\phi+h)\tau}\right)^{\frac{1}{\delta}}\right) \\
&+ \frac{\lambda}{\delta} F \left(\frac{(1-(1-\phi)\beta + h\beta)\eta(\tau)\lambda}{(\phi+h)\tau}\right)^{\frac{1-\delta}{\delta}} \\
0 =& \frac{\partial H}{\partial t} - (1-(1-\phi)\alpha - \tau\phi - \tau h + \alpha h) A_{\psi} H + c_m
\end{aligned}
$$

$$
H = \frac{c_m}{(1-(1-\phi)\alpha - \tau\phi - \tau h + \alpha h) A_{\psi}}
$$

$$
\begin{aligned}
0 =& \frac{\partial F}{\partial t} + 1 \\
&- F \left(\frac{\rho+\lambda}{\delta} + \frac{\delta-1}{\delta} (1-(1-\phi)\alpha - \tau\phi - \tau h + \alpha h) A_{\psi} + \frac{(\phi+h)\tau}{(1-(1-\phi)\beta + h\beta)\eta(\tau)} \left(\frac{\delta-1}{\delta} - \left(\frac{(}{(1-(1-}\right.\right.\right.
\end{aligned}
$$

$$
F = \frac{1}{\frac{\rho+\lambda}{\delta} + \frac{\delta-1}{\delta} (1-(1-\phi)\alpha - \tau\phi - \tau h + \alpha h) A_{\psi} + \frac{(\phi+h)\tau}{(1-(1-\phi)\beta + h\beta)\eta(\tau)} \left(\frac{\delta-1}{\delta} - \left(\frac{(\phi+h)\tau}{(1-(1-\phi)\beta + h\beta)\eta(\tau)\lambda}\right)^{-\frac{1}{\delta}}\right)}
$$