Optimal taxation and policy changes in an endogenous growth model with variable population and public expenditure*

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Abstract
In this work we analyse the issue of optimal taxation and of policy changes in an endogenous growth model driven by public expenditure, in the presence of endogenous fertility and labour supply. Normative analysis confirms the Chamley-Judd result of zero capital income tax and unveils nonzero tax/subsidies on children. Positive analysis reveals the effects of taxes on total employment may be of different sign due to the presence of endogenous fertility.

Keywords: Taxation, endogenous fertility, endogenous growth, population.

J.E.L. Classification: D63, E21, H21, J13, O41.

1. Introduction

An extensive literature on optimal taxation of factor incomes in a general equilibrium-dynamic framework has been flourishing in the last three decades. A well-established finding of such works is that, in the long run, capital income should not be taxed, thus shifting the burden from factor income taxation toward labor (Judd, 1985, Chamley, 1986, Judd, 1999). Although the result is robust with respect to several extensions, some exceptions may arise, such as in the case of borrowing constraints (Aiyagari 1995 and Chamley 2001), market imperfections (Judd 1997), incomplete set of taxes (Correia 1996), overlapping generations (Eros and Gervais 2002), social discounting and disconnected economies (De Bonis and Spataro 2005, 2010), government time-inconsistency and lack of commitment (Reis 2012), externalities from suboptimal policy rules (Turnovsky 1996).

The case of externalities is particularly relevant for endogenous growth models: Romer (1986) introduces externalities deriving from existing capital (spillovers as “learning by doing”); Lucas (1988) shows that decreasing returns to capital could be avoided by adopting a broad view of capital itself that entails human capital as well (externalities from “human capital”); in Barro (1990), spillovers from productive public expenditure avoid diminishing returns to capital and are the engine of sustained long run growth; finally, in a subsequent work, Romer (1990) himself applies the concept of nonrivalty to “ideas” or “discoveries” that can enhance production efficiency and technological progress, and obtained increasing returns in production and thus sustained per-capita income growth.1

In this work we extend the analysis of optimal taxation and public expenditure policies to an endogenous growth setting with productive public expenditure by allowing for endogenous labour supply and endogenous fertility. This has been never done so far.

In fact, several works have analysed the impact of fiscal policies on economic growth, such as Barro (1990), Jones and Manuelli (1990) and Rebelo (1991). As for welfare analysis, Lucas (1990) and Turnovsky (1992) compare the effects of a tax on capital versus a tax on

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1 The literature on endogenous technological change through R&D activities, schumpeterian competition and spillovers has been evolving over the last decade (for a review see Acemoglu 2009 chapters 13 and subsequent ones).
labour and find the former to be inferior to the latter from the viewpoint of economic welfare. Turnovsky (1996) analyses the issue of first-best optimal taxation and expenditure policies in an endogenous growth model with externalities stemming from public goods both in the utility and in the production function and Turnovsky (2000) extends the analysis to the case of endogenous labour supply. In this type of models, direct taxation brings about a natural trade-off: on the one hand, it distorts incentives to save and work, hence reducing growth; on the other, it increases the marginal productivity of private inputs, thus increasing growth and possibly welfare. This is the key contribution of Barro (1990), which was extended in several subsequent studies.

However, in all these works population growth is either absent or exogenous. In fact, the observed large variations in fertility rates both across countries and across times, has led an increasing number of scholars to work on the reformulation of economic theory of endogenous fertility and on the provision of different social criteria for allocation efficiency with variable population. Moreover, most of the endogenous growth models mentioned above suffer from the “scale effect”, meaning that the steady-state growth rate increases with the size (scale) of the economy, as indexed, for example, by population. In order to overcome this problem non-scale models have been provided by Jones (1995) and subsequent works, although still hinging on exogenous population.

In order to breach this gap, in the present work we extend Barro (1990) model, in which the engine of growth is productive public expenditure, by allowing for both endogenous labour supply (as in Turnovsky 2000) and endogenous population (as in Spataro and Renström 2012) and we use this model to analyse fiscal policy in the form of distortionary taxes and government expenditure changes.

We retain Barro (1990) approach since there is consolidated evidence that public expenditure in favour of productive services has a sizable impact on growth (for major insights, see, among others, Turnovsky 1996, García Peñalosa and Turnovsky, 2005). More precisely, we carry out our analysis under two different types of public expenditure: a) optimal amount of public services (as in Barro 1990); b) fixed fraction of GDP (either fixed at the optimal level or not – as in Turnovsky 1996).

We also note that our model allows to avoid two shortcomings of the aforementioned non-scale growth models: first, the direct positive link between economic and demographic growth, which is not supported by post-war data (see Agemoglu 2009, p. 448) and, second, the fact that the long-run equilibrium growth rate is determined by technological parameters and is independent of macro policy instruments.

The assumption of endogenous population, however, may pose major issues related to welfare analysis. In fact, given that under these circumstances welfare evaluations typically imply the comparisons between states of the world in which the size of population is different, the Pareto criterion cannot be used. To overcome this issue, we adopt a Classical Utilitarianism approach, which allows for social orderings that are based on desirable welfarist axioms in presence of variable population and, under our assumptions, can also avoid unpleasant outcomes as for population (for example, the so-called Repugnant Conclusion).

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2 Discussions of the effects of taxation in models of endogenous labour supply are also provided by Rebelo (1991). See also Basu et al. (2004) and Basu and Renström (2007) for indivisible labor economies.


4 See Renström and Spataro (2015) for an endogenous growth model driven by human capital with variable population, where the relationship between population growth and economic growth is not necessarily positive.

5 According to the Repugnant conclusion, any state in which each member of the population enjoys a life above neutrality is declared inferior to a state in which each member of a larger population lives a life with lower utility (see Parfit 1976, 1984, Blackorby et al. 2002). In particular, in a growth model with endogenous population, the RC takes the form of upper-corner solution for the population rate of growth (society reproduces at its physical
In this scenario we show that, while some well-established results on second-best taxation extend also to our model, endogenous population introduces some new results as for both normative and positive analysis.

The work as organized as follows: after laying out the model (section 2), we characterize the decentralized equilibrium (section 3) and the optimal taxation rules and public debt (section 4); in section 5 we present a tax reforms analysis, in order to verify the impact of the latter on economic growth. Section 6 concludes.

2. The model

In this section we lay out the benchmark model. We denote individual quantities by lower case letters, and aggregate quantities by corresponding upper case letters, so that $V = Nv$, with $N$ population size. As anticipated above, we find it instructive to present both the case in which public expenditure is chosen optimally (model 1) and or is fixed as a fraction of GDP (model 2). In the latter case, we will first assume that the share of public expenditure is set arbitrarily while in the second stage it is set optimally. This approach will enhance our understanding of the optimal capital income tax rate which, as it will be shown, depends upon both the socially optimal level of government expenditure and the deviation of actual expenditure from its social optimum.

2.1. Households

We assume that the representative agent is endowed with a unit of time that can be allocated either to leisure or the work or to child rearing. We also assume, for the sake of simplicity, that each generation lives for one period, and life-time utility is $u(c_t, l_t)$, where $c_t$ is life-time consumption for that individual, $l_t$ is labour supply. We assume that utility is increasing in $c_t$, decreasing in $l_t$ and strictly concave. We also follow the convention that $u(0, .) = 0$ represents neutrality at individual level (i.e. if $u < 0$ the individual prefers not to have been born), and denote the critical level parameter as $\alpha$.

An individual family chooses consumption, labour supply, savings and the number of children (i.e. the change in the cohort size $N$).

We also assume that raising children is costly. We nest the existing approaches in the literature by assuming the cost per family member in the number of children, $\theta(n)$, can either be linear (as in Becker and Barro, 1989, Cremer et al. 2006) or strictly convex (as in Tertilt 2005 and Growiec 2006). Convex cost implies decreasing returns to scale in child rearing.

As for firms, we assume perfectly competitive markets and constant return to scale technology. The consequence of the assumptions on the production side is that we retain the “standard” second-best framework, in the sense that there are no profits and the competitive equilibrium is Pareto efficient in absence of taxation. Otherwise there would be corrective elements of taxation. Finally, we assume the government finances an exogenous stream of per-capita expenditure $x$, that enters as an input in private sector production function, by issuing debt and levying taxes.

To retain the second-best, we levy taxes on the choices made by the families, i.e. savings, labour supply and children. Consequently we introduce the capital-income tax, labour income tax, and a tax/subsidy on the number of children.

2.1.1 Preferences

We focus on a single dynasty (household) or a policymaker choosing consumption and population growth over time, so as to maximize:

$$U = \int_0^\infty e^{-\lambda t} N_t u(c_t, l_t) dt$$ (1)

where $N_t$ is the population (family) size of generation $t$, $u_t = u(c_t, l_t) \geq 0$ is the instantaneous utility function of an individual of generation $t$, such that $u(0, \cdot) = 0$, $u_c > 0$, $u_t < 0$, $u_{cc} < 0$, $u_{lt} > 0$ and $\rho > 0$ the intergenerational discount factor.

More precisely, we will assume the following form of the intratemporal utility function:

$$u(c, l) = c^{\mu(1-\sigma)}(1-\lambda)^{\mu(1-\eta)}$$ (2)

with $1 > \mu(1-\sigma) > 0, 1 > \mu(1-\eta) > 0$, and $1 - \mu(2-\sigma - \eta) > 0$, which are the assumptions to work in presence of sustained long-run per-capita income growth\(^6\).

Moreover, the population dynamics is described by the following law:

$$\frac{d\hat{N}_t}{\hat{N}_t} = n_t$$ (3)

with $n_t \in [\underline{n}, \bar{n}]$.

Hence, the problem of the household is to maximize (3) under the constraint:

$$\dot{A}_t = \ddot{r}_t A_t + \ddot{w}_t (l_t - \ddot{\theta}(n_t))\hat{N}_t - c_t \hat{N}_t$$ (4)

where $A_t$ is household wealth, $\ddot{r}_t \equiv r_t(1 - \ddot{\tau}_t^k)$ is net-of-tax interest rate, $\ddot{w}_t \equiv w_t(1 - \ddot{\tau}_t^k)$ is the net-of-tax wage, $\ddot{\theta}(n_t)$ is the net-of-tax childbearing cost and $\ddot{\tau}_t^k, \ddot{\tau}_t^l$ are the tax rate on capital income, on labour income, respectively. We assume that the gross-of-tax childrearing cost $\dot{\theta}(n_t)$ is a time cost and is specified over the number of children each parent has, so that it is a function of the population growth rate. In fact, in equilibrium each parent has the same number of children, so the per family member population growth rate becomes the economy wide one. More precisely, for the sake of simplicity, we assume that the cost for raising children is increasing in the number of children and linear, so that, $\dot{\theta}(n_t) = (1 + \ddot{\tau}_n^\theta) \cdot \ddot{\theta} \cdot n_t$,\(^7\) where $\ddot{\tau}_n^\theta$ is the tax on the number of children. Notice that, from the budget constraint, the monetary cost for childbearing is $\ddot{w} \cdot \ddot{\theta} \cdot n_t$.

Furthermore, we assume that there are lower and upper bounds on the population growth rate: $n_t \in [\underline{n}, \bar{n}]$. Realistically, there is a physical constraint at each period of time on how many children a parent can have. There is also a constraint on how low the population growth

\(^6\) Notice that the value of utility must be strictly positive, otherwise the value of population would be negative, implying a corner solution for $n$. Hence, the case $\sigma > 1$ (with the general form of utility $u(c, l) = c^{\mu(1-\sigma)}(1-\lambda)^{\mu(1-\eta)}$ discussed by Straub and Werning (2015) as a potentially source of violation of Chamley-Judd zero tax result cannot apply to this model.

\(^7\) Notice that convex childrearing costs, although questionable in terms of realism, are commonly used in population literature (see, among others Tertilt 2005, Growiec 2006), in that convexity is necessary for avoiding a corner solution for $n$. In our work, the time nature of the childbearing costs insures interiority of the solution for $n$ (see Appendix A.1).
can be. The reason for the latter assumption is twofold: first, we do not allow individuals to be eliminated from the population (in that there is no axiomatic foundation for that); moreover, even if nobody wants to reproduce there will always be accidental births.

2.2. Firms

We assume constant-returns-to-scale production technology with labour-augmenting productive public expenditure. More precisely, the production function is:

\[ F_t = F(K_t, x_t, L_t) = F(K_t, x_t, N_t(l_t - \theta(n_t))) = TK_t^{\gamma}(x_t, N_t(l_t - \theta(n_t)))^{1-\gamma} \]  

(5)

with \( T \) the parameter representing total factor productivity, \( K_t \) is capital stock, assumed infinitely durable, \( x_t \) the labour-augmenting flow of services from government spending on the economy’s infrastructure. We also assume that these services are not subject to congestion so that \( x_t \) is a pure public good. \( L_t = (l_t - \theta(n_t))N_t \) is hired labour, with \( l_t - \theta(n_t) \) the fraction of time dedicated to work.

Assuming perfect competition, firms hire capital, \( K \), and labour services, \( L \), on the spot market and remunerate them according to their marginal productivity, such that

\[ F_{K_t} = r_t \]  

(6)

\[ F_{L_t} = w_t \]  

(7)

Moreover, the economy resource constraint is:

\[ \dot{K}_t = F(K_t, x_t, L_t) - c_tN_t - x_tN_t. \]  

(8)

Defining \( k \equiv \frac{K}{N} \) as the capital intensity and \( f \equiv \frac{F}{N} \) as per capita output the latter expressions read as:

\[ f = \delta T^{\gamma}(l - \theta(n))^{1-\gamma} k \]  

(5’)

\[ r = F_{K_t} = \gamma L \]  

(6’)

\[ w = F_{L_t} = \frac{F_{L_t}}{N} = (1 - \gamma) \frac{f}{l - \theta(n)} = (1 - \gamma) \delta T^{\gamma}(l - \theta(n))^{1-2\gamma} k \]  

(7’)

\[ \dot{k}_t = f_t - n_t k_t - c_t - x_t \]  

(8’)

Notice that monotonicity and concavity of the production function implies \( \frac{1}{2} < \gamma < 1 \).

2.3. The government

We allow the government to finance an exogenous stream of public expenditure \( x_t \) by levying taxes, both on capital and labour income and issuing debt, \( B \), whose law of motion is:
\[ \dot{B}_t = r_t B_t - \tau^k_t r_t A_t - \tau^l_t w_t l_t N_t - w_t \theta n_t N_t [(1 - \tau^l_t) \hat{\theta} - 1] + x_t N_t \quad (9) \]

The expenditure flow model that we present in this section is assumed to take the following form:

\[ xN = \partial F \quad (10) \]

In words, public expenditure is a fixed fraction of total output. Hence, we can summarize the economy's resource constraint as follows:

\[ \dot{K}_t = \bar{F}_t - c_t N_t, \quad (11) \]

where \( \bar{F} = F - xN = (1 - \delta) \tilde{\delta} \gamma T^\gamma (l - \theta(n)) \tilde{\gamma} K \).

### 3. The decentralized equilibrium

We now characterize the decentralized equilibrium of the economy. The problem of the individual (household) is to maximize (1) subject to (4), taking \( A_0 \) and \( N_0 \) as given. The current value Hamiltonian is:

\[ H_t = N_t u_t + q_t [\bar{F}_t A_t + \bar{w}_t (l_t - \bar{\theta} n_t) N_t - c_t N_t] + \lambda_t n_t N_t \quad (13) \]

with \( q_t \) and \( \lambda_t \) the shadow price of wealth and of population, respectively.

The first-order conditions are the following:

\[ \frac{\partial H}{\partial A} = \rho q - \dot{q} \Rightarrow \dot{q} = (\rho - \bar{\gamma}) q \quad (14) \]

\[ \frac{\partial H}{\partial c} = 0 \Rightarrow u_c = q \quad (15) \]

\[ \frac{\partial H}{\partial \lambda} = 0 \Rightarrow -u_t = q \bar{w} \quad (16) \]

\[ \frac{\partial H}{\partial N} = \rho \lambda - \dot{\lambda} \Rightarrow \dot{\lambda} = (\rho - n) \lambda - u - q [\bar{w}(l - \bar{\theta} n) - c] \quad (17) \]

\[ \frac{\partial H}{\partial n} = 0 \Rightarrow \lambda = q \bar{w} \hat{\theta} \quad (18) \]

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8 In fact, by using \( xN = \partial F = \delta \tilde{\delta} K^\gamma (xN(l - \theta(n))) \tilde{\gamma} \) we get that \( xN = (\tilde{\delta} \gamma T^\gamma (l - \theta(n))) \tilde{\gamma} K \), such that \( F = \delta \tilde{\gamma} T^\gamma (l - \theta(n)) \tilde{\gamma} K \) and \( \bar{F} = F - xN = (1 - \delta) \tilde{\delta} \gamma T^\gamma (l - \theta(n)) \tilde{\gamma} K \).

9 We focus on interior solutions for \( n \), so that the potential constraint \( n_t \in [\underline{n}, \bar{n}] \) is not binding. See Appendix A.1 for details.

10 We omit the subscript referring to time when it causes no ambiguity to the reader.
and the transversality conditions are\textsuperscript{11}
\[
\lim_{t \to \infty} e^{-\rho t} q_t A_t = 0, \quad \lim_{t \to \infty} e^{-\rho t} \lambda_t N_t = 0.
\] (19)

The last condition for the competitive equilibrium is capital market clearing condition:
\[
A_t = K_t + B_t,
\] (20)
which, in per capita terms, is:
\[
a_t = k_t + b_t.
\] (21)

Note that, given the policy under investigation, equilibrium market price (interest rate) and equilibrium labour price are equal to (private) marginal product of capital and labour, respectively (see eqs. 6’ and 7’), and the latter can be different from social marginal product of capital and labour:
\[
r^* = \tilde{F}_k = (1 - \delta)\frac{1}{\gamma} T^{\frac{1}{\gamma}} \frac{1}{l - \theta(n)}
\] (22a)
\[
w^* = \tilde{F}_l = \frac{1 - \gamma}{\gamma} (1 - \delta)\frac{1}{\gamma} T^{\frac{1}{\gamma}} \frac{1}{l - \theta(n)} \frac{1 - \sigma}{\gamma} k
\] (22b)

This difference is due to the presence of externality brought about by public expenditure. More precisely, we get that
\[
r > r^* \Leftrightarrow w > w^* \Leftrightarrow \frac{\gamma}{(1 - \delta)} < \frac{\gamma}{(1 - \delta)}
\] (23)

In case the policymaker aims to correct for this externality, it can either choose $\delta$ optimally (i.e. equal to the production elasticity of public expenditure, $1 - \gamma$, as in Turnovsky 1996, for example), or raise corrective taxes.

**Balanced growth path**

Finally, we characterize the balanced growth path (BGP), along which all per-capita variables grow at the same rate.

By using eq. (14), doing time derivative of (15) and recognizing that, along the BGP $n, l$ and $c/k$ are constant, we get the usual expressions for the per-capita consumption, capital intensity and wage rate growth rates (which, along the BGP are equal):
\[
\frac{\dot{c}}{c} = \frac{\bar{r} - \rho}{1 - \sigma}
\] (24a)

\textsuperscript{11} It can proved that transversality conditions are satisfied along the BGP.
\[
\begin{align*}
\frac{\dot{w}}{w} &= \bar{r} - \frac{1-\mu(1-\sigma)}{\theta\mu(1-\eta)} + l \frac{1-\mu(2-\sigma-\eta)}{\theta\mu(1-\eta)} 
\quad \text{(24b)} \\
\frac{k}{k} &= Gf - n 
\quad \text{(24c)}
\end{align*}
\]

with \( G \equiv (1 - \delta) - \frac{(1-\sigma)(1-\gamma)(1-\tau)}{(l-\theta)(1-\eta)} (1 - l) > 0. \)

These equations provide the implicit solutions for \((l, n, c/k)\) and of the per-capital growth rate of the economy along the BGP. We shall assume that the latter growth rate is positive.

We can notice that, as usual, the economy growth rate is proportional to the net-of-tax interest rate. This, in turn, depends on the whole set of the endogenous variables, namely, the population growth rate, labour supply, capital intensity and per-capita consumption, and on the deep parameters of the economy, comprising taxes. We will address the effects of the latter on the economy growth rate in section 4, after characterizing the optimal taxation rules.

4. The Ramsey problem

We now solve the optimal tax problem (Ramsey problem). In doing so, we adopt the primal approach, consisting of the maximization of a direct social welfare function through the choice of quantities (i.e. allocations; see Atkinson and Stiglitz 1972)\(^{13}\). For this purpose we must restrict the set of allocations among which the government can choose to those that can be decentralized as a competitive equilibrium. We now provide the constraints that must be imposed on the government’s problem in order to comply with this requirement.

In our framework there is an implementability constraint associated with the individual family’s intertemporal consumption choice. More precisely this constraint is the individual budget constraint with prices substituted for by using the consumption Euler equation, which yields (see Appendix A.2):

\[
A_0u_c = \int_0^\infty e^{-\rho t}u_tN_tdt - N_0 \int_0^\infty e^{-\rho t}[u_t - u_c(t)(c_t + T) - u_l l_t]dt 
\quad \text{(25)}
\]

Finally there three feasibility constraints, one which requires that private and public consumption plus investment be equal to aggregate output (eq. 9); the other is given by eq. (17) (again, where we make use of individuals’ FOCs (15) and (16)), the last one is eq. (4).

Hence, supposing that the policy is introduced in period 0, the problem of the policymaker is to maximize (1) subject to eq. (25), and, \( \forall t \geq 0, \) eqs. (9’), (17) and (4). Hence, the current value Hamiltonian is:

\[
H_t = N_t \left[ u_t + \Omega \left[ u_t - \frac{N_0}{N_t} u_t + \frac{N_0}{N_t} u_l l_t + \frac{N_0}{N_t} u_c(t)(c_t + T) \right] \right] + \omega_t (\bar{F}_t - c_t N_t) + \varphi_t n_t N_t 
\quad \text{(26)}
\]

where \( \Omega, \omega_t \) and \( \varphi_t \) are the multipliers associated with the constraints.

First order condition with respect to consumption (omitting time subscripts) and labour imply:

\[\text{Note that we have made use of the relationship } \frac{c}{k} = (1 - l) \frac{(1-\sigma)}{(1-\eta)} k \text{ and eq. (7’).}\]

\[\text{On the contrary, the dual approach takes prices and tax rates as control variables. For a survey see Renström (1999).}\]
\[
\frac{\partial H}{\partial c} = 0 \Rightarrow (1 + \Omega) u_c - \Omega \frac{N_0}{N_t} \Delta_c u_c = \omega \tag{27}
\]
\[
\frac{\partial H}{\partial l} = 0 \Rightarrow (1 + \Omega) u_l - \Omega \frac{N_0}{N_t} \Delta_l u_l = -\omega \frac{\bar{\ell}_1}{N} \tag{28}
\]

with \( \Delta_c \equiv \frac{u_c c}{u_c} - \frac{u_c l}{u_c} > 0 \) and \( \Delta_l \equiv \frac{u_l l}{u_l} - \frac{u_c c}{u_l} < 0 \) usually referred to as the “general equilibrium elasticity” of consumption and of labour, respectively.

By using eq. (15)-(16), (27) and (28) can be written as:

\[
(1 + \Omega) = \Omega \frac{N_0}{N_t} \Delta_c + \frac{\omega}{u_c} \tag{29}
\]
\[
(1 + \Omega) = \Omega \frac{N_0}{N_t} \Delta_l - \frac{\omega w^*}{u_c \bar{\omega}} \tag{30}
\]

with \( w^* \equiv \bar{\ell}_1 \). Finally, we get:

\[
\frac{\partial H}{\partial k} = \omega r^* = \rho \omega - \dot{\omega} \tag{31}
\]

with \( r^* \equiv \bar{F}_K \).

\[
\frac{\partial H}{\partial n} = (1 + \Omega) u - \omega c + \varphi n = \rho \varphi - \dot{\phi} \tag{32}
\]
\[
\frac{\partial H}{\partial n} = 0 \Rightarrow \omega \bar{F}_l \theta = \varphi N \tag{33}
\]

From the expressions above it emerges that there are two alternative candidate equilibria: the former implies a positive value of \( n \) and transitional dynamics, the latter entails \( n=0 \). In the next Proposition we characterize the BGP:

**Proposition 1.** Along the optimal BGP population is constant.

**Proof.** See Appendix B.1. \( \square \)

We can provide the following Proposition concerning second best taxation for the case when \( x \) is chosen optimally:

**Proposition 2.** In model 1, along the optimal BGP, second best optimal taxation implies:

\[
\tau^* = 0, \quad (1 - \tau^*) = \frac{1 + \Omega - \Omega \frac{N_0}{N} \Delta_c}{1 + \Omega - \Omega \frac{N_0}{N} \Delta_l} \in (0,1) \quad \text{and} \quad \frac{(1-\tau^*) \beta}{\theta} < 1. \]

**Proof.** See Appendix B.2. \( \square \)
The analysis of second best optimal taxation carried out so far shows that the Chamley-Judd result holds also in our scenario, provided that labour income taxation can be conditioned on the number of children that are present in the household: in fact, along the BGP, capital income tax should be zero and effective labour income tax should be positive. Moreover, the implicit tax on children \((\frac{1-\tau_l}{\theta})\) (it is implicit given that at the optimum \(n=0\)), can be either a tax or subsidy.

Moreover, as in Turnovsky (1996), nonzero capital income tax arises, although in a second best context, for correcting suboptimal public expenditure. In fact, when the fraction of public expenditure is above (below) the social second-best optimum, the social return to capital is less than its private marginal physical product. Consequently, capital income should be taxed to obtain the social optimum.

Finally, notice that the government finds it (second best) optimal to have constant population. Intuitively, this is due to the fact that the tax instrument that it used \((\tilde{\theta})\) entails a tax break at 1 child per adult, (i.e. \(n=0\)). When the number of children increases, although the second best tax structure would imply zero taxes on both capital income and labour income (see Proposition 2 for \(N\to\infty\)), the distortion brought about by a nonzero \(\tau^n\) would be too high and, consequently, the associated allocation of resources would be suboptimal.

Finally, the Proposition that follows provides the result concerning the sign of the optimal level of debt:

**Proposition 3.** Under model 1 and 2 optimal debt is negative.

**Proof.** See Appendix B.3. \(\square\)

This result states that along the second-best optimal BGP public expenditure should entirely be financed by labour income taxes, by taxes on children and by the returns of public assets or by also capital income taxes, if \(\gamma > (1-\delta)\).

### 5. Tax reforms

We now analyse the effects of policy changes on the equilibrium levels of the main variables (i.e. labour supply \(1-\theta(n)\), population growth rate \(n\) and the growth rate of the economy, \(g\), which is proportional to \(\dot{r}\), in that \(dg = \frac{d\dot{r}}{1-\mu(1-\sigma)}\), along the BGP.

In this section, for the sake of simplicity, we will assume that:

\[
H1: \left(\frac{1-\delta}{\gamma}\right) \frac{1-\mu(2-\sigma)}{1-\mu(1-\eta)} > 1.
\]

which allows unambiguous effects of most tax-reforms (see Appendix C for details)

From decentralized equilibrium, using eqs. (24a)-(24c) and market equilibrium condition (6’)? we get (see Appendix C.1):

\[
\tilde{\rho} \left[\frac{1-\delta}{1-\mu(1-\sigma)} - \frac{g}{(1-\tau^k)\gamma}\right] = \frac{\rho}{1-\mu(1-\sigma)} - n > 0
\]

\[
\tilde{\rho} = (1 - \tau^k)F_K = (1 - \tau^k)F_K \gamma \delta \frac{1}{T\gamma(l - \theta n)} \frac{1}{\gamma}
\]
\[
\tilde{\theta} r \left[ 1 - \frac{g}{(1-\tau^k)\gamma} \right] = \frac{1-\mu(1-\sigma)}{\mu(1-\eta)} - \frac{1-\mu(2-\sigma-\eta)}{\mu(1-\eta)} - \tilde{\theta} n > 0
\]  

(36)

With the economy always being on a balanced growth path, the effects of government policy on the equilibrium are obtained by taking the differentials of (37)-(39). Routine calculations yield the qualitative responses with respect to \( \tau^k \), \( \tau^l \) and \( \tilde{\theta} \) (i.e. \( d\tau^n = d\tilde{\theta} \)) which we can summarize in the following proposition:

**Proposition 3.** Along the BGP, the effects of policy changes are the following:

\[
\frac{\partial g}{\partial \tau^l} < 0, \quad \frac{\partial (l - \theta n)}{\partial \tau^l} < 0
\]

\[
\frac{\partial g}{\partial \tau^k} < 0, \quad \frac{\partial (l - \theta n)}{\partial \tau^k} < 0,
\]

\[
\frac{\partial g}{\partial \tilde{\theta}} < 0, \quad \frac{\partial (l - \theta n)}{\partial \tilde{\theta}} > 0, \quad \frac{\partial n}{\partial \tilde{\theta}} < 0
\]

Moreover, both \( \frac{\partial n}{\partial \tau^l} \) and \( \frac{\partial n}{\partial \tau^k} \) are ambiguous.

Proof: See Appendix C.

The results on economic growth are somehow intuitive: higher taxes produce lower growth. However, differently from previous literature on endogenous growth models and endogenous labour supply, the effects of taxes on employment may be different. In fact, as in Turnovsky 2000, the signs of the effect of either taxes on labour supply are quite the same and negative. However, the difference in our results arises as for the effect of taxes/subsidies on children. In fact, while, on the one hand, an increase of the tax on children decreases the economy per capita growth rate and, on the other hand, it increases total employment. The reason is that an increase of this tax reduces the number of children per household and this effect more than offsets the (positive) effect that the same tax exerts on leisure (in fact, higher taxes on children reduce the net amount of resources that workers obtain from work).

As for the effect of taxes on the population growth rate, although the sign of \( \frac{\partial n}{\partial \tau^l} \) is ambiguous, it can be shown that it is positive up to a certain level of \( \tau^l \). Whether such a level of \( \tau^l \) is binding (lower than 1) depends on the set of parameters. Moreover, around \( \tau^k = \tau^l \) and for \( n \) sufficiently close to 0, \( \frac{\partial n}{\partial \tau^l} \) is positive.

The sign of \( \frac{\partial n}{\partial \tau^k} \) is ambiguous as well and depends, among other things, on the level of \( \tau^l \).

For illustrative purposes, in Figure 1 we depict this non-monotonic relationship between the population growth rate (n) and taxes on capital income (\( \tau^k \)) and on labour income \( \tau^l \) along the BGP. In particular, it emerges that, for plausible values of the parameters, an increase of the tax on labour income increases the number of children. On the contrary, the effect of \( \tau^k \) can be negative (positive) for low (high) values of \( \tau^l \).

Finally, as expected, \( \frac{\partial n}{\partial \tilde{\theta}} < 0 \). Note that the full monetary cost for raising children is \( \tilde{w} \tilde{\theta} \) and that, given that \( \frac{\partial (\tilde{w})}{\partial \tilde{\theta}} = \frac{\partial (\tilde{w})}{\partial (1-\theta n)} \frac{\partial \theta (1-\theta n)}{\theta} < 0 \), it follows that, in our model, the direct effect of the increase of \( \tilde{\theta} \) prevails on the indirect effect on the wage rate.
**Figure 1:** The relationship between population growth rate \((n)\) and taxes on capital income \((\tau^k)\) and on labour income \(\tau^l\) along the BGP.

Parameters: \(\eta = 0.3, \sigma = 0.7, \theta = \bar{\theta} = 2, \mu = 0.4, \gamma = 0.6, \delta = 0.4, T = 5, \rho = 0.6.\)

### 6. Conclusions

In the present work we have carried out an analysis of optimal taxation and policy changes in an endogenous growth model in presence of endogenous fertility and labour supply. As far as the normative analysis is concerned, we show that, at the steady state the second-best policy entails zero capital income tax, positive labour income tax, nonzero tax/subsidy on children and negative debt. Optimal nonzero taxes on capital and labour income result as a corrective device only in the case of suboptimal public spending, as in Turnovsky (1996), although in a second best analysis.

From a positive standpoint we show that a rise of taxes (either on labour income or on capital income), depresses both per-capita growth and total employment. However, while an increase of fiscal pressure on children reduces economic growth, it increases labour supply, thus enriching the effects of taxes unveiled by the existing literature (see for example Turnovsky 2000), due to the presence of endogenous fertility.

In this paper we have treated public expenditure as a flow variable (services from current expenditure). A natural extension of our study is to analyse the case of public expenditure as financing a stock of public goods (infrastructure): this case is left for future research.

### References


Princeton and Oxford;


Appendix

Appendix A.1. The value Function

In this Appendix we show that the problem is concave. The Bellman equation of the problem can be written as (we omit time subscripts):

\[ \rho \hat{V}(K, N) = \max\{Nu + \hat{V}_K K + \hat{V}_N N\} \]  
\[ \text{(A.1)} \]

with \( \dot{K} = fK - cN \)

with \( f \equiv \frac{F}{N} \)

FOCs w.r.t. K and N give:

\[ \rho \hat{V}_K = \hat{V}_{KK} \dot{K} + \hat{V}_{NK} \dot{N} + \hat{V}_K f \]  
\[ \text{(A.2)} \]

\[ \rho \hat{V}_N = \hat{V}_{KN} \dot{K} + \hat{V}_{NN} \dot{N} + \hat{V}_N n - \hat{V}_K c - u \]  
\[ \text{(A.3)} \]
by combining (A.2) and (A.3) and (A.1) if follows given that \((\tilde{V}_K + \tilde{V}_N) = \tilde{V}(K, N)\), so that the value function turns out to be homogeneous of degree 1. Hence, by defining \(v \equiv K/N\) and \(v(k, 1) \equiv \tilde{V}(K, N)/N\), so that \(\tilde{V}_K = v'(k)\) and \(\tilde{V}_N = v(k) - v'(k)k\), eq. (C.1) can be written as

\[\rho v(k) = \max\{u + v'(k)\dot{k} + nv(k)\}\]  \(\text{(A.4)}\)

Next, by defining \(\tilde{c} \equiv c/k\) and exploiting eq. (2) it follows that

\[\rho v(k)k^{-\mu(1-\sigma)} = \max\left\{u(\tilde{c}, l) + v'(k)\frac{k}{k}k^{1-\mu(1-\sigma)} + nv(k)k^{-\mu(1-\sigma)}\right\}\]  \(\text{(A.5)}\)

which shows that \(v(k)\) is homogenous of degree \(\mu(1-\sigma)\). Hence, by using the property of homogeneous functions of degree \(s\): \(v'(k) = (1/s)\lambda^{1-s}v'({\lambda}k)\), with \(s = \mu(1-\sigma)\) and \(\lambda = k^{-1}\), eq. (A.5) can be written as:

\[\rho v(1) = \max\left\{u(\tilde{c}, l) + v'(1)\frac{k}{k} + nv(1)\right\}\]  \(\text{(A.6)}\)

Next, FOCS yield:

\[u_{\tilde{c}} - v'(1) = 0\]

\[u_l + (1 - \delta)\dot{v}'(1) = 0\]

\[v(1) + v'(1)[-(1 - \delta)f'\theta - 1] = 0\]

where \(\frac{k}{k} = \hat{f} - n - \tilde{c}\) and \(\hat{f} \equiv \frac{f}{k} = \hat{f}(l - \theta n)\). Finally, the Hessian matrix reads as

\[
\begin{bmatrix}
u_{\tilde{c}\tilde{c}} & u_{\tilde{c}t} & 0 \\ u_{\tilde{c}t} & u_{tt} & -(1 - \delta)\hat{f}'\theta \\ 0 & 0 & v'(1)(1 - \delta)f''\theta^2 \\
\end{bmatrix}
\]

Given that \(u_{\tilde{c}\tilde{c}} < 0\), \((u_{\tilde{c}\tilde{c}}u_{tt} - u_{\tilde{c}t}^2) > 0\) and the Determinant is \(u_{\tilde{c}\tilde{c}}v'(1)(1 - \delta)f''\theta^2(u_{\tilde{c}\tilde{c}}u_{tt} - u_{\tilde{c}t}^2) < 0\), the value function is globally concave, so that the solutions for all variables will be interior.

**Appendix A.2. Implementability constraint**

First, by integrating eq. (17) one gets:

\[\lambda_0 = \int_0^\infty e^{-\rho t}[u_t - u_{ct}c_t - u_{lt}l_t]dt\]  \(\text{(A.7)}\)

Second, let us take the following time derivative:

\[
\frac{d(A_tq_t)}{dt} = A_tq_t + A_tq_t
\]

which, exploiting eqs. (4) and (14) can be written as
\[
\frac{d(A_t q_t)}{dt} - \rho q_t A_t = \rho q_t A_t + q_t N_t \left[ \bar{w}_t (l_t - \bar{n}_t) - c_t \right] = q_t N_t \left[ \bar{w}_t l_t - c_t \right] - \lambda_t \dot{N}_t
\]

Hence, pre-multiplying by \( e^{-\rho t} \) and integrating both sides, making use of transversality conditions and of eqs. (15)-(16), we obtain

\[
-A_0 q_0 = \int_0^\infty e^{-\rho t} [u_{ct}(c_t) - u_{lt} l_t] dt - \int_0^\infty e^{-\rho t} \lambda_t \dot{N}_t dt
\]

Integrating by parts the last integral and using (17) we get:

\[
A_0 q_0 + \lambda_0 N_0 = \int_0^\infty e^{-\rho t} u_t dt
\]

Finally, using (A.7) eq. (25) in the text follows:

\[
A_0 u_{c_0} = \int_0^\infty e^{-\rho t} u_t N_t dt - N_0 \int_0^\infty e^{-\rho t} [u_t - u_{ct} c_t - u_{lt} l_t] dt
\]

Appendix B.1. Proof of Proposition 1

[TB TO BE CONTINUED]

Appendix B.2. Proof of Proposition 2

As for the capital income tax, from (32) we get:

\[
\dot{\omega} = (\rho - r^*) \omega
\]

and given that \( \frac{\dot{\omega}}{q} \) must be constant along the BGP (by eq. 22), the rate of growth of \( \omega \) and \( q \) must be equal, so that \( r = r^* \).

If \( x \) is productively efficient (i.e. \( \delta = 1-\gamma \), \( r = r^* \), then \( r^k = 0 \).

As for the labour income tax, from (29) and (31)

\[
\bar{w} = \frac{1 + \Omega - \frac{N_0}{N} \Delta_i}{1 + \Omega - \frac{N_0}{N} \Delta_i} \quad \Delta_e
\]

(B.17)

Given that if \( \delta = 1-\gamma \), \( w = w^* \), then
\[
(1 - \tau') = \frac{1 + \Omega - \Omega \frac{N_0 \Delta_c}{N}}{1 + \Omega - \Omega \frac{N_0 \Delta_j}{N}} < 1
\]

(since \( \Delta_c = 1 - \mu(2 - \sigma - \eta) + \frac{\mu(1-\eta)}{1-l} > 0 \) and \( \Delta_j = \frac{1-\mu(2-\sigma-\eta)}{1-l} < 0 \)

As for \( \bar{\theta} \), along the steady state growth path, after tax wage and consumption grow at the same rate. Combining (24b) and (B.3), with labour constant, we have

\[
\bar{r} = \frac{1 - l - \mu(1-\sigma)}{\mu(1-\sigma)} - \frac{l}{\bar{\theta}} = \frac{\bar{r} - \rho}{1 - \mu(1 - \sigma)}
\]  

(B.18)

Equation B.7 gives \( c/k \) along the growth path, then combined with (B.18) and using \( n=0 \), \( \bar{r} = \bar{F}_k \), we have

\[
\frac{c}{k} = \frac{1 - l - \mu(1-\sigma)}{\mu(1-\sigma)} + \frac{l}{\bar{\theta}}
\]  

(B.19)

(B.2) and (B.7) give

\[
\frac{c}{k} = \frac{1 - \sigma}{1 - \eta} \left( \frac{1 + \Omega}{\mu(1 - \sigma)} - \left[ \frac{1 + \Omega - \Omega \frac{N_0 \Delta_c}{N}}{1 + \Omega - \Omega \frac{N_0 \Delta_j}{N}} \right] \right)
\]  

(B.20)

Hence, B.19 and B.20 provide the following:

\[
\frac{\bar{w} \bar{\theta}}{w \bar{\theta}} = \frac{\bar{w}}{w} \frac{\bar{w}}{\bar{w}} = \frac{w^*}{w} \left( (1 - l) \frac{1 - \mu(1-\sigma)}{\mu(1-\sigma)} + l \right) \left( \frac{1 + \Omega - \Omega \frac{N_0 \Delta_c}{N}}{1 + \Omega - \Omega \frac{N_0 \Delta_j}{N}} \right)
\]

Finally by using (27) and (28), as well as the individual’s FOCs (15) and (16), gives

\[
\frac{\bar{w} \bar{\theta}}{w \theta} = \frac{w^*}{w} \left( \frac{1 - \mu(1-\sigma)}{\mu(1-\sigma)} + l \right) \left( \frac{1 + \Omega - \Omega \frac{N_0 \Delta_c}{N}}{1 + \Omega - \Omega \frac{N_0 \Delta_j}{N}} \right)
\]  

(B.21)

[TO BE CONTINUED]

Appendix B.3. Proof of Proposition 3

Let us start from inputs remuneration. Eqs. (22a) and (22b) give
Next, eq. (8') yields:

\[ \dot{k} = \frac{\tilde{F}}{k} \left( c - \frac{c}{k} - n = r^* - \frac{c}{k} - n \right), \]  

so that

\[ \frac{c}{k} = r^* - n - \frac{k}{\dot{k}}. \]

Given that along the BGP \( \frac{k}{c} = \frac{\dot{c}}{c} = \frac{r^* - \rho}{1 - \mu(1 - \sigma)} \), we get that:

\[ \frac{c}{k} = r^* - n - \frac{r^* - \rho}{1 - \mu(1 - \sigma)} \equiv z \]

Next, let us exploit the individuals' budget constraint,

\[ \dot{a} = (\bar{r} - n) a + \bar{w} (l - \bar{\theta} n) - c \]

(B.25)

and given that

\[ \frac{\dot{a}}{a} = \frac{\dot{c}}{c} = \frac{r^* - \rho}{1 - \mu(1 - \sigma)} \]

(B.26)

it follows:

\[ \frac{c}{a} = (\bar{r} - n) - \frac{r^* - \rho}{1 - \mu(1 - \sigma)} + \frac{\bar{w}}{a} (l - \bar{\theta} n) \]

(B.27)

Moreover, exploiting (B.24)

\[ \frac{\bar{w}}{a} (l - \bar{\theta} n) = \frac{\bar{w}}{w^* a} \frac{k w^* (l - \bar{\theta} n)}{r^* k} r^* = \frac{\bar{w}}{w^* a} \frac{k 1 - \gamma}{\gamma} r^* \]

we can rewrite eq. (B.27) as:

\[ \frac{c}{a} = z + \frac{\bar{w}}{w^* a} \frac{k 1 - \gamma}{\gamma} r^* . \]

(B.28)

Finally,

\[ \frac{a}{k} = \frac{\frac{c}{k}}{\frac{c}{a}} = \frac{z}{z + \frac{w k 1 - \gamma}{w^* a} r^*} = \frac{\frac{a}{c}}{\frac{a}{k} + \frac{w 1 - \gamma}{w^* r^*}} \]
and collecting terms we get:

\[ z \left( \frac{a}{k} - 1 \right) + \frac{\ddot{w}}{w} \frac{1 - \gamma}{\gamma} \frac{r^*}{r} = 0 \]

and, by recalling that \( \left( \frac{a}{k} - 1 \right) = \frac{b}{k} \) we get:

\[ \frac{b}{k} = \frac{\ddot{w}}{w} \frac{1 - \gamma}{\gamma} \frac{r^*}{r - \eta} < 0 \]

(B.29)

**Appendix C. Proof of Proposition 4**

Total differentiation of (34)-(36) yields:

\[ Bd\ddot{r} + [1 + \Pi \ddot{r}(1 - l)]dn - \ddot{r}\Pi(1 - \theta n)dl = \frac{G\ddot{r}}{(1 - \tau^k)^\gamma} d\tau^k + \frac{G\ddot{r}}{(1 - \tau^k)^\gamma} Q d\tau^l \]  

(C.1)

\[ d\ddot{r} + \frac{1 - \gamma}{\gamma} \frac{\ddot{r}\theta}{l - \theta n} dn - \frac{1 - \gamma}{\gamma} \frac{\ddot{r}}{l - \theta n} dl = - \frac{\dddot{r}}{(1 - \tau^k)\gamma} d\tau^k \]  

(C.2)

\[ H\dddot{d}\ddot{r} + \ddot{\theta}[1 + \Pi \ddot{r}(1 - l)]dn - \left[ (\dddot{r}\Pi(1 - \theta n) - \frac{1 - \mu(2 - \sigma - \eta)}{\mu(1 - \eta)} \right] dl = \frac{G\ddot{r}\ddot{\theta}}{(1 - \tau^k)\gamma} d\tau^k + \frac{G\ddot{r}\ddot{\theta}}{(1 - \tau^k)\gamma} Q d\tau^l - (H\dddot{\theta} + n) d\ddot{\theta} \]  

(C.3)

with \( B \equiv \frac{1}{1 - \mu(1 - \sigma)} - \frac{G}{(1 - \tau^k)^\gamma} > 0 \), \( \Pi \equiv \frac{(1 - \sigma)(1 - \gamma)(1 - l)}{\gamma(1 - \theta n)(1 - \eta)(1 - \tau^k)^\gamma} = \frac{(1 - \delta - G)}{\gamma(1 - \theta n)(1 - \eta)(1 - l)} > 0 \), \( Q \equiv \frac{\partial q}{\partial \tau^l} = \frac{(1 - \sigma)(1 - \gamma)(1 - l)}{(1 - \theta n)(1 - \eta)} > 0 \), \( H \equiv B - \frac{\mu(1 - \sigma)}{1 - \mu(1 - \sigma)} = 1 - \frac{G}{(1 - \tau^k)^\gamma} > 0 \).

The determinant of the associated matrix of such a system reads as (for the sake of simplicity we compute it assuming \( \dddot{\theta} = \theta \)):

\[
\text{Det} = -\dddot{r}^2 \Pi \theta \left( \frac{(1 - \gamma)(1 - \sigma)\mu}{\gamma(1 - \mu(1 - \sigma))} - \frac{1 - \mu(2 - \sigma - \eta)}{\mu(1 - \eta)} + \frac{\theta \dddot{r} \mu(1 - \sigma)(1 - \gamma)}{\gamma(l - \theta n)(1 - \eta)\mu(1 - \sigma)} \right) \\
+ B \dddot{r} \left( \frac{(1 - \gamma)(1 - 2\sigma - \eta)}{\mu(1 - \eta)(1 - \theta n)} - \frac{(1 - \mu(2 - \sigma - \eta))}{\mu(1 - \eta)} \dddot{r} \Pi(1 - l) \right)
\]

By exploiting the definitions of \( B, Q \) and \( \Pi \) it follows:

\[
\text{Det} = -\dddot{r}^2 \Pi \theta^2 \left( \frac{(1 - \gamma)(1 - \sigma)\mu}{\gamma(1 - \mu(1 - \sigma))} - \frac{1 - \mu(2 - \sigma - \eta)}{\mu(1 - \eta)} \\
- [(1 - \delta - G) \frac{\theta \dddot{r} (2\gamma - 1)(1 - \mu(2 - \sigma - \eta))}{\gamma(1 - \tau^k)(l - \theta n)\mu(1 - \eta)}} \right) \\
- \frac{\theta \dddot{r} (1 - \gamma)}{\gamma(l - \theta n)\mu(1 - \eta)} \left[ \frac{1 - \delta}{\gamma} \frac{(1 - \mu(2 - \sigma - \eta))}{(1 - \tau^k)} - (1 - \mu(1 - \eta)) \right]
\]

19
Notice that the first three addends are negative. Sufficient condition for the last term to be negative is $\tau^k \geq 1 - \left(1 - \frac{\delta}{\gamma} \right) \frac{1 - \mu(2 - \sigma - \eta)}{1 - \mu(1 - \eta)}$, which holds, under H1 in the text, for any $\tau^k \geq 0$.

Moreover, using Cramer’s rule, one gets:

$$ Det \frac{\partial (l - \theta n)}{\partial \tau^l} = [2 \Pi (1 - \theta n) \theta \bar{r} + \frac{1 - \mu(2 - \sigma - \eta)}{\mu(1 - \eta)}] \frac{Q \theta \bar{r}}{\gamma(1 - \tau^k)} > 0 \quad (C.5) $$

and

$$ Det \frac{\partial g}{\partial \tau^l} = Det \frac{\partial \bar{r}}{\partial \tau^l} = \left[2 \Pi (1 - \theta n) \theta^2 \bar{r} + \frac{1 - \mu(2 - \sigma - \eta)}{\mu(1 - \eta)} \right] \frac{Q \theta \bar{r}^2 (1 - \gamma) \Gamma(1 - \tau^k)(l - \theta n) > 0}{\gamma^2 (1 - \tau^k) \gamma(1 - \tau^k)} \quad (C.6) $$

where the first equality follows from eq. (24a).

As for the effect of the tax on children, we get:

$$ Det \frac{\partial (l - \theta n)}{\partial \tau^k} = \bar{r}^2 \theta^2 \Pi \left[\frac{2(1 - \theta n) - \mu(l - \theta n)(1 - \sigma)}{(1 - \tau^k)(1 - \mu(1 - \sigma))} + \frac{\theta \bar{r}(1 - \mu(1 - \eta))}{(1 - \eta)(1 - \tau^k)} \right] > 0 \quad (C.7) $$

$$ Det \frac{\partial g}{\partial \tau^k} = \frac{1 - \mu(2 - \sigma - \eta)}{\mu(1 - \eta)} \frac{\bar{r}}{\gamma(1 - \tau^k)} \left[\frac{G \theta \bar{r}(1 - \gamma)}{(l - \theta n)(1 - \tau^k)^2} + 1 + \Pi \theta (1 - l) \right] > 0 \quad (C.8) $$

As for the effect of the tax on children, we get:

$$ Det \frac{\partial (l - \theta n)}{\partial \theta} = \left[1 - \frac{\bar{r}}{\gamma(1 - \tau^k)} \right] [-1 + \bar{r} \Pi \theta (l - \theta n)] < 0 \quad (C.9) $$

The first term in brackets is positive. By exploiting the definition of $\Pi = \frac{(1 - \sigma)(1 - \gamma)(1 - \tau^l)}{\gamma(1 - \theta n)(1 - \eta)(1 - \tau^k)^2}$ and the equilibrium relationship $1 - l = \frac{c}{k} \frac{1 - \eta}{k W 1 - \sigma}$ it follows that $[-1 + \Pi \theta (l - \theta n)] = -1 + \frac{c}{k}$. Given that $\frac{c}{k} = r^* - n - \frac{c}{k} > 0$, for plausible values of parameters $\frac{c}{k} < 1$, so that the inequality of the above equation follows.

As for $\frac{\partial g}{\partial \theta}$, we get:

$$ Det \frac{\partial g}{\partial \theta} = - \frac{(1 - \gamma)}{\gamma} \frac{\bar{r}}{l - \theta n} \left[1 - \frac{\bar{r}}{\gamma(1 - \tau^k)} \right] [-1 + \bar{r} \Pi \theta (l - \theta n)] = - \frac{(1 - \gamma)}{\gamma} \frac{\bar{r}}{l - \theta n} Det \frac{\partial (l - \theta n)}{\partial \theta} > 0 \quad (C.10) $$

[TO BE CONTINUED]