Abstract

This paper examines the response of economies and, in particular, inter-sectoral wage inequality and unemployment to demand shocks. The analysis is performed by using a general equilibrium model with monopolistic competition in hi-tech sectors and perfect competition in a traditional sector. Hi-tech sectors are horizontally differentiated with respect to labor skills required by production technologies. Motivated by higher wages, workers intend to work in hi-tech sectors but face a risk to be unemployed. Unemployment appears as a consequence of job market frictions. The wages of employed workers are agreed through a bargaining mechanism.

Demand shocks are associated with a redistribution of spending from traditional to hi-tech goods. We claim that the whole economy can respond to the shock in two opposite ways increasing a monopolistic or competitive component of the monopolistic competition. In the first case, prices go up, demand for specific varieties decreases, and inter-sectoral wage differential enlarges. All individuals become better off except those who lose their jobs. We predict the reverse shifts in prices, demand and wage differential in the second case. The two responses of an economy are distinguished by consumers’ elasticity of substitution between varieties of hi-tech goods. At last, the unemployment increases in both cases.

Keywords: wage differential; unemployment; monopolistic competition; variable elasticity of substitution; general equilibrium

JEL: L11, D11, E2, J31
Inter-Industry Wage Inequality, Unemployment, and Competitiveness of Monopolistic Competition Driven by Changes in Consumers’ Preferences

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1 Introduction

A remarkable rise of the income dispersion in the US and UK since the 1960s and 1970s respectively and more complex changes of the income inequality in other developed and developing countries have been discussed in many papers; a recent review is presented by Machin (2008). Explaining evolution of income inequalities economists attribute them to the “race” between the relative demand for labor and the relative supply of it, skill-biased technological changes, possibly, international trade, and the role of labor market institutions; see, respectively (Katz and Murphy, 1992; Acemoglu, 2002; Wood, 1995; Lee, 1999) among others. Up to 40% of the income redistribution in favor of high-skilled workers occurs as a result of inter-industry changes (Berman et al., 1998). Explanations of these changes are less conclusive, despite the

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basic arguments remaining the same. In addition, researchers indicate the role of the integral development of industries (Gittleman and Wolff, 1993) and the structure of consumers’ demand (Leonardi, 2003).

We are going to contribute to this analysis theoretically by further exploring the role of consumers’ demand under exogenous variations in the preferences of individuals. We address the question how an economy responds to exogenous shifts in consumers’ preferences for skill-intensive products with general equilibrium modeling. The response of the economy includes changes in the inter-sectoral income inequality, the number of unemployed agents, the redistribution of sector sizes, and the welfare of individuals. We construct a model economy that consists of several hi-tech sectors, in which differentiated goods are produced, and of a homogeneous numéraire sector. Since scale effects are absent in economies characterized by preferences with a Constant Elasticity of Substitution (CES), we use consumers’ preferences with a Variable Elasticity of Substitution (VES), following Zhelobodko et al. (2012). We also assume the existence of frictions on the labor market that prevent rejected job market candidates from finding another job immediately. Frictions are described in details e.g., in a review by Rogerson et al. (2005). We conjecture that under the risk of unemployment, only expected wages are equalized across the sectors. The wages are determined through bargaining between employers and employees a lá Stole and Zwiebel (1996). All three key features of modeling: VES preferences of consumers, labor market frictions, and bargaining are required to assess consequences of demand shocks.

We show that under a shift in consumer preferences for skill-intensive products, the unemployment rate increases whereas an expansion of corresponding hi-tech sectors a priori can occur in two alternative ways. The first way is described by a more competitive monopolistic competition: each firm decreases its prices and increases the output. Then the expansion of sectors occurs at firm as well as at sector levels. This requires an excessive inflow of workers but their wages follow the prices for sectors’ goods since labor is under pressure of unemployed agents. As a result, the inter-sectoral wage inequality goes down. The second possibility strengthens the monopolistic nature of the monopolistic competition. The equilibrium variables mentioned above move in the opposite directions. Our paper suggests that an economy “chooses” its response to a shift in consumer tastes depending on consumers’ elasticity of substitution $\sigma$ between hi-tech goods: if $\sigma$ is a decreasing function of the demand, then monopolistic competition becomes more competitive, and vice versa. The intuition underlying this result is rather simple. Spending additional money for specific goods, consumers exhibit a more elastic
demand. The latter is associated with a decline in demand for specific goods if $\sigma$ is decreasing and with a growth of demand if $\sigma$ is increasing.

According to Zhelobodko et al. (2012), economies characterized by a decreasing $\sigma$ strengthen the competition in hi-tech sectors under exogenous shocks that relax trade barriers. On the contrary, a relax of the competition is associated with an increasing $\sigma$. We argue that economies exhibit the opposite changes in the competition when the expansion of hi-tech sectors is driven by the preferences of consumers.

The case of a decreasing $\sigma$ is supposed to be more relevant (Ottaviano et al., 2002; Bertoletti and Epifani, 2014). We introduce a family of the corresponding utilities and estimate numerically the response of individuals’ welfare. Under changes of preferences in favor of hi-tech goods, all individuals gain except those who lose their jobs.

In our model, changes in the wage inequality are explored as a scale effect, which disappears under preferences with constant elasticity of substitution. On the other hand, the existence of an equilibrium is proved under VES preferences with relatively small deviation from CES. Nevertheless, the response of the wage inequality to a shift in tastes can be distinguished. To our knowledge, this is the first attempt to estimate quantitatively the effect of VES in structural models with monopolistic competition. The scale of the response positively correlates with the efficiency of technologies in the corresponding industries, whose model proxy is the ratio of fixed to variable costs faced by firms.

The novelty of how we address the wage inequality problem is in the introduction of industries with monopolistically competing firms. Namely, such a structure of supply highlights the role of the elasticities of demand explored in this paper. We put conclusions into correspondence with assumptions. As a result of monopolistic competition, the sector size is affected not only explicitly by consumers’ distribution of spending but also implicitly by the elasticity of demand in the sector. Variability of the elasticity of substitution in consumption between varieties of hi-tech goods implies that a demand shift in favor of a single differentiated good affects the production of all sectors. The influence is based on changes in the relative diversity of the differentiated goods, which is the number of the product’s varieties normalized by spending of consumers for this product.

We also contribute to the theory of monopolistic competition when constructing equilibrium in our model with sector-specific wages and VES preferences. We establish that the diversity of individuals’ elasticity of substitution between varieties is limited even under VES, if the economy is large. This limited diversity is required for the existence of the equilibrium.
Our research has links to various papers. Some efforts aim to describe trends in the development of hi-tech sectors and relate one trend to another. As debated in many sources, technological progress shifts the labor demand to favor skilled workers in OECD countries; see, f. e., Machin and Reenen (1998). Less skilled manufacturing workers move to other sectors, mainly to service sectors, Berman et al. (1998). Alongside this trend, relative prices for manufacturing output have been dropping in several developed countries with Australia, US, and Canada as prominent examples (Baldwin and MacDonald, 1998; Ricaurte, 2010; Lowe, 2011). The causality of these trends remains unclear. For example, Loupias and Sevestre (2013) conclude that variations of labor costs affect producer price changes in France. We do not list all possible origins of price changes, however, we mention trade liberalization by Baldwin and MacDonald (1998), exchange rates, business cycles by Messina et al. (2009), inflation by Gagnon (2009), and macro shocks by Clark (2006).

The evolution of hi-tech sectors and changes in inequality of workers’ income are investigated by using dynamic models. Based on classical models of endogenous growth (Judd, 1985; Grossman and Helpman, 1991, 1994), Acemoglu (2012) linked a development of hi-tech sectors to the appearance of new innovative products. Initially, the production of these products requires the labor of skilled workers. Within time innovations are adopted by industries. Eventually, standardized firms competing for production of those adopted products win over hi-tech firms because they face lower labor costs by employing less skill workers. Hi-tech firms, in turn, switch to other innovations. To estimate changes in the gap between wages of skilled and unskilled workers Acemoglu (1997) models the competition between them and finds that this gaps can both increase and decrease depending on frictions within labor markets.

Research methods relate our paper to general equilibrium modeling with monopolistically competing firms. Original Dixit–Stiglitz setting of monopolistic competition involved consumers’ preferences with a Constant Elasticity of Substitution, see Dixit and Stiglitz (1977). As a result, an equilibrium markup does not depend on the size of an economy. Improving this unrealistic model outcome, Ottaviano et al. (2002) constructed a model with a quasi-linear utility and incorporate a pro-competitive response of the equilibrium to an enlargement of the economy: outputs increase and prices decrease. Behrens and Murata (2012) endowed consumers with a sub-utility of the “constant absolute risk aversion” (CARA) type and explored other economies with this pro-competitive response. They decomposed the gain from trade into gains from product diversity and gains from pro-competitive effects. Zhelobodko et al. (2012) describe a class of utilities that lead to this pro-competitive response in terms of the elasticity
of demand between hi-tech goods. An influence of the elasticity of demand on effective tax policies and an income dispersion is derived theoretically by Di Comite et al. (2013) and Os-harin et al. (2014). Melitz (2015) link the Zhelobodko et al. (2012) framework to his trade theory involving heterogeneous firms (Melitz, 2003; Melitz and Redding, 2012) and propose an empirical strategy that separates effects coming from the supply and demand sides to firms’ profit. Empirical evidence that variable elasticity of demand affects profit would overcome the criticism of VES general equilibrium modeling given by Bertoletti and Epifani (2014). Helpman et al. (2008) combine monopolistic competition setting with labor market frictions and a special searching procedures in order to explain the matching between high-skilled workers and efficient firms earlier studied by Amiti and Pissarides (2005). Helpman et al. (2008) succeed with a CES setting but they do not aim to investigate scale effects.

We built our analysis on several recent theoretical contributions that aims to explain links between market size, income inequality, unemployment, and consumer tastes. Helpman et al. (2008) succeed with CES setting but they do not aim to investigate scale effects. Behrens and Robert-Nicoud (2014) proposed a model, in which market size disproportionally favors more skilled agents. They conclude that an enlargement of markets can increase the wage differential through selection effect: the selection process is slower on larger markets and this weakens the competition between workers. Based on a UK data, Leonardi (2003) derives empirically that more educated individuals are hired for high-skilled jobs and consume more skill-intensive goods than less educated individuals. Departing from this finding, he claims that an increase in supply of high-skilled workers (coming from new college graduates) also moves up the demand for them through the shift of the aggregate consumers’ demand. This idea is supported by a simple general equilibrium model. We argue that the structure of consumers’ demand matters by itself. Namely, changes in the demand structure affects both the supply of and the demand for labor, and the integral effect is ambiguous.

The rest of the paper is organized as follows. We formulate the model and prove the existence and uniqueness of equilibrium in section 2. We perform comparative statics with respect to changes in spending and explore how these changes affect unemployment, wage differentials, and competitiveness of monopolistic competition in section 3. In section 3 we also assess the scale of the effect that is revealed by the comparative statics. Section 4 concludes.
2 Model

2.1 Supply Side

We consider an economy that consists of a homogeneous sector (henceforth sector 0) with perfect competition and of \( n \) high-tech sectors with \( N_i \) monopolistically competing single-product firms in each sector, \( i = 1, \ldots, n \). In the homogeneous sector firms price their products at the marginal cost because of the perfect competition. Assigning for simplicity 1 to productivity in this sector, we have that prices \( p_0 \) are equal to wages \( w_0 \).

A firm producing a good \( \xi_i \) in the \( i \)-th hi-tech sector faces sector specific fixed \( c_i^\phi \) costs. Its variable costs are associated with wages of its employees, which work with an inverse productivity \( c_i^v \), homogeneous within sector \( i \). This firm tunes its price \( p(\xi_i) \) to maximize the profit:

\[
\pi(\xi_i) = p(\xi_i)Q(\xi_i) - c_i^v Q(\xi_i)w(\xi_i) - c_i^\phi w(\xi_i) \longrightarrow \max,
\]

where \( Q(\xi_i) \) is the aggregate demand for the good \( \xi_i \) in the \( i \)-th sector that depends on the price \( p(\xi_i) \).

Accepting standard monopolistic competition settings, we expect each particular firm to be so small that the adjustment of its prices does not affect the price index in the \( i \)-th sector. The number (mass) \( N_i \) of goods and firms in the \( i \)-th sector is regulated by the free entry condition

\[
\pi(\xi_i) = 0.
\]

We denote \( \mathcal{L} \) the number of workers in the economy. It consists of employed \( L_i \) and unemployed workers \( L_i^u \), \( i = 1, \ldots, n \), in each hi-tech sector and of \( L_0 \) workers in the homogeneous sector. Symbol \( L_{n+1} \) denotes the total number of unemployed workers: \( L_{n+1} = L_1^u + \ldots + L_n^u \).

2.2 Demand Side

Upper-tier choice. The aggregate demand is based on individual choices of consumers. The income of consumers depends on their jobs; namely on the sector where they work. As a result, there are \( n + 2 \) types of the incomes \( y_0, y_1, \ldots, y_n, y_{n+1} \) in the economy, denoted after the sectors; index \( n + 1 \) describes unemployed agents. A consumer with an income \( y_j \) decides upon her demand in two steps. First, she differentiates between hi-tech goods represented by
consumption indices $H_i$, $i = 1, \ldots, n$, and a homogeneous good $H_0$. Maximizing the upper-tier Cobb-Douglas utility

$$U = H_0^{\beta_0} H_1^{\beta_1} \cdots H_n^{\beta_n} \rightarrow \max,$$

where the exponents $\beta_i \in (0, 1)$, $i = 0, 1, \ldots, n$, are summed up to 1, the consumer allocates her income proportionally to the exponents $\beta_i$ for products of the $i$-th sectors.

**Lower-tier choice.** Second, each consumer “recalls” that her consumption indices consist of individual varieties $\xi_i \in [0, N_i]$, $i = 1, \ldots, n$. Consumers choose the demands $q_j(\xi_i)$, where $j = 0, \ldots, n + 1$ indicates the income $y_j$ of the consumer, for each $i = 1, \ldots, n$ maximizing the consumption index

$$H_i = \int_{N_i} u_i(q_j(\xi_i)) \, d\xi_i \rightarrow \max,$$

with a low-tier four times differentiable utility function $u_i(\chi)$, which reflects preferences for sector $i$’s differentiated good, subject to budget constraint

$$\int_{N_i} p(\xi_i)q_j(\xi_i) \, d\xi_i \leq \beta_i y_j.$$

**Elasticity of substitution between hi-tech varieties.** The following function

$$\sigma_i(\chi) = -\frac{u'_i(\chi)}{u''_i(\chi) \chi},$$

interpreted as the elasticity of the substitution between hi-tech goods, underlies the optimal individual demands $q_j(\xi_i)$. Namely, varying the first order conditions of optimization problem (3), (4) with respect to the price $p(\xi_i)$ for this particular good, the firm finds that the elasticity $E_{p(\xi_i)}q_j(\xi_i)$ of the demand $q_j(\xi_i)$ with respect to its price $p(\xi_i)$ is opposite to the elasticity of substitution:

$$E_{p(\xi_i)}q_j(\xi_i) = -\sigma_i(q_j(\xi_i)).$$

Exploring a multi-sector economy, we introduce the mean value of individuals’ elasticity of substitution

$$\mathcal{S}(\xi_i) = \sum_{j=0}^{n+1} \frac{q_j(\xi_i) L_j}{Q(\xi_i)} \sigma_i(q_j(\xi_i)),$$

between varieties (henceforth, the Mean Elasticity of Substitution MES) where $Q(\xi_i)$ is the aggregated demand $Q(\xi_i) = \sum_{j=0}^{n+1} q_j(\xi_i) L_j$. One can easily check that property (5) is extended to $\mathcal{S}(\xi_i)$:

$$E_{p(\xi_i)}Q(\xi_i) = -\mathcal{S}(\xi_i).$$
If all individuals are identical \((y_i = y_j \text{ for all } i \neq j)\) then the right hand side of equation (7) is simplified to an individual’s elasticity of substitution. Then the first order conditions of consumers’ optimization problem state that the price elasticity of the aggregate demand coincides up to the sign with an individual’s elasticity of substitution between varieties of hi-tech goods. In general case, if the price elasticity of the aggregate demand were opposite to the elasticity of substitution of each individual then Equation (7) would hold. However all these equalities are unlikely to occur, as the market sets a single price for all individuals. The first order conditions can be seen as a market rule that involves an imaginary average consumer with the elasticity of substitution \(S\) between varieties of hi-tech goods. We will use the MES \(S(\xi_i)\) to explore the equilibrium of the model.

**Examples.** We introduce two examples of VES utilities:

\[ u(\kappa) = 2\sqrt{2\kappa} + 1 + \ln \frac{\sqrt{2\kappa} + 1 - 1}{\sqrt{2\kappa} + 1} \]  
\[ u(\kappa) = \ln \left( \kappa + 1 + \sqrt{\kappa^2 + 2\kappa} \right) \]

They have increasing and, respectively, decreasing elasticity of substitution. These two utilities represent two types of VES families which can be written with hypergeometric functions (see Equations (98) and (99) of Lemma 17 for the definition of the families and, f.i., Whittaker and Watson (1990); Abadir (1999) for description of hypergeometric functions). All utilities from these families are close to the sum of power functions in a positive neighborhood of zero. As we will discuss later in Section 2.5, individual equilibrium demands lie in such neighborhood.

### 2.3 Labor Market

The production side is characterized by technologies that vary from sector to sector but not from firm to firm. Firms face sector-specific productivities \(1/c_i\) of labor. In order to produce an optimal number \(Q(\xi_i)\) of goods firms must hire

\[ l(\xi_i) = c_i^p Q(\xi_i) + c_i^p \]

workers.

Firms and workers agree upon the wages through bargaining, as discussed in details by Stole and Zwiebel (1996). The equal division of surplus between a firm and its all workers underlies bargaining. The firm employs workers one-by-one estimating the gain from hiring an additional
employee. Workers’ surplus is the difference between the wages proposed by the firm and the outer alternative. This alternative is the expected wage in the sector to which the firm belongs.

Workers are motivated to enter hi-tech job markets by larger wages. To be employed in each sector including homogeneous one, individuals have to acquire sector specific skills. One can think that workers choose a desired sector, get required skills, and then enter the job market of the chosen sector. Once the sector is chosen it cannot be altered. The homogeneous sector is able to adopt an arbitrary number of workers. On the contrary, hi-tech sector firms hire the required number of employees and reject the other applications. We assume that the labor market exhibits some frictions. Because of them, rejected candidates cannot find a job in another sector including homogeneous sector because they are not qualified for it. The rejected candidates also fail to find another job in the sector with which their skills are associated: We will show later in Lemma 8 that in the equilibrium, hi-tech sector firms will not hire any unemployed agent (associated with their sector and therefore qualified for the job) for an arbitrary positive wage because because one hired individual cannot alter the demand for the variety of a specific firm.

Homogeneity of technologies inside sectors equalizes intra-sector wages $w_i = w(\xi_i)$. Otherwise, firms have incentives to hire underpaid workers from other firms or unemployed agents: they have enough qualification and agree to get a smaller compensation.

A flat tax with some rate $\alpha \in (0, 1)$ is applied to the wages of all employed workers (including workers employed in the homogeneous sector) and distributed equally between unemployed agents as an unemployment benefit. This sets the (netto) income of employed in the high-tech sectors, employed in the homogeneous sector and unemployed workers to

$$y_i = (1 - \alpha)w_i, \, i = 1, \ldots, n, \quad y_0 = (1 - \alpha)w_0,$$

$$y_{n+1} = \frac{\alpha(L_0w_0 + L_1w_1 + \ldots + L_nw_n)}{L_{n+1}}$$

(11)

respectively.

Workers’ choice of the labor market is assumed to be balanced on average. Given the probabilities $L_i/(L_i + L_i^u)$ and $L_i^u/(L_i + L_i^u)$ to be employed and unemployed respectively in sector $i$, $i = 1, \ldots, n$, agents face identical expected incomes that coincide with the incomes got by workers employed in the homogeneous sector:

$$\frac{y_iL_i}{L_i + L_i^u} + \frac{y_{n+1}L_i^u}{L_i + L_i^u} = y_0.$$

(12)

As discussed above, the expected sector wages are considered as the outer alternative during
bargaining. With the assumption of the labor market balance, this alternative is equal to $w_0$. Following Stole and Zwiebel (1996), one can write out the wages $w(\xi_i) = w_i$ obtained as a result of bargaining:

$$w(\xi_i) = \left( \frac{p(\xi_i)}{c^v_i} + w_0 \right) \frac{l(\xi_i)^2 - (c^v_i)^2}{2l(\xi_i)^2}$$  \hspace{1cm} (13)$$

(see Lemma 4), where $c^v_i$ is the inverse productivity of workers, and the production function is $Q = (1 - c^p)/c^n$ (if the number of the workers exceeds the threshold $c^p$; otherwise the production is zero). Equation (13) indicates that the wages increase with a growth of firms’ revenue $pQ$ and the outer wages $w_0$.

The diversity of sectors, market frictions, and the balance of average incomes stand behind unemployment as a part of general equilibrium. Workers, aiming at larger wages, apply for better compensated jobs but face a risk to remain unemployed. The balance is attained on average.

2.4 Description of Model Variables. Assumptions

The primitives of the economy are 2-tier household preferences with sector specific additive utilities $\{u_i(\cdot)\}_{i=1}^n$, variable $\{c^v_i\}_{i=1}^n$ and fixed $\{c^\phi_i\}_{i=1}^n$ costs, the total number $L$ of workers in the economy, and the exponents $\beta_0$, $\beta_1$, $\ldots$, $\beta_n$ of the Cobb–Douglas upper tier utility. Given those primitives, we describe a general equilibrium in the model.

**Definition.** The set of prices $\{\hat{p}(\xi_i)\}$, individual demands $\{\hat{q}_j(\xi_i)\}$, firms’ outputs $\{\hat{s}(\xi_i)\}$, the number $\hat{N}_i$ of firms, wages $\hat{w}_i$, incomes $\hat{y}_j$, the numbers $\hat{L}_i$ of workers in each sector, and the number $\hat{L}_i^u$ of rejected job market candidates in hi-tech sectors ($i = 1, \ldots, n, j = 0, 1, \ldots, n+1, \xi_i \in [0, \hat{N}_i]$) constitute a general equilibrium, if the following conditions are satisfied:

- For any fixed $j = 0, 1, \ldots, n + 1$, individual demands $\{\hat{q}_j(\xi_i)\}_{\xi_i \in [0,\hat{N}_i], i=1,\ldots,n}$ solve a consumer’s problem (2)–(4) with $p(\xi_i) = \hat{p}(\xi_i)$, $N_i = \hat{N}_i$, $y_j = \hat{y}_j$, $i = 1, \ldots, n$, $\xi_i \in [0, \hat{N}_i]$;

- For any fixed $i = 1, \ldots, n$, and $\xi_i \in [0, \hat{N}_i]$ a firm’s optimization problem with respect to a single price $p(\xi_i)$ is defined by Equation (1), where $w(\xi_i) = \text{const} = \hat{w}$ and $Q(\xi_i) = \sum_{j=0}^{n+1} q_j(\xi_i)\hat{L}_j$ depends implicitly on $p(\xi_i)$. Namely, for any fixed $j = 0, 1, \ldots, n + 1$, we define individual demands $\{q^*_j(\eta_k)\}_{\eta_k \in [0,\hat{N}_k], k=1,\ldots,n}$ as the solutions of a consumer’s problem (2)–(4) with

  - the price for the specific variety $\xi_i$ equalled to $p(\xi_i)$,
the equilibrium prices for the other varieties: \( p(\eta_k) = \hat{p}(\eta_k) \) if \( k \neq i \) or \( k = i \) and \( \eta_i \neq \xi_i \),

and \( N_k = \hat{N}_k, \ k = 1, \ldots, n, \ y_j = \hat{y}_j \).

Then these \( q_i^*(\xi_i) \) are substituted for \( q(\xi) \) into the Equation for \( Q(\xi_i) \). Finally, we require that \( \hat{p}(\xi_i) \) solves a firm’s optimization problem formulated above.

- The market clearance condition is hold:
  \[
  \hat{s}(\xi_i) = \sum_{j=0}^{n+1} \hat{q}_j(\xi_i)\hat{L}_j,
  \]

- Equations (10)–(13) are valid with \( L_j = \hat{L}_j, \ L_i^u = \hat{L}_i^u, \ y_j = \hat{y}_j, \ w_i = \hat{w}_i, \ p(\xi_i) = \hat{p}(\xi_i), \) and \( Q(\xi_i) = \hat{s}(\xi_i), \) where \( i = 1, \ldots, n, \ j = 0, 1, \ldots, n + 1, \ \xi_i \in [0, \hat{N}_i] \). Additionally,
  \[
  \int_{\hat{N}_i} l(\xi_i) \, d\xi_i = \hat{L}_i.
  \]

- Firms are free to enter:
  \[
  \hat{p}(\xi_i)\hat{s}(\xi_i) - c_i^v \hat{s}(\xi_i)\hat{w}(\xi_i) - c_i^\varphi \hat{w}(\xi_i) = 0, \ i = 1, \ldots, n, \ \xi_i \in [0, \hat{N}_i].
  \]

The construction of the equilibrium is performed under the following Assumptions:

**Assumption 1.** We will consider only monotonous differentiable functions \( \sigma_i(\kappa) \) and assume that

\[
\sigma_i(\kappa) > 1, \quad i = 1, \ldots, n,
\]

for any \( \kappa \geq 0 \).

**Assumption 2.** Let \( L_j \) be the equilibrium number of employed workers in sector \( j \). We assume that this number \( L_j \geq 1 \), and

\[
|\sigma'_i(\kappa)| < \frac{L_j}{2C_i} \quad \text{for all } i = 1, \ldots, n, \ j = 0, \ldots, n + 1, \ \kappa \geq 0,
\]

where \( C_i = c_i^\varphi / c_i^v \) is the ratio of the fixed to variable costs.

**Assumption 3.** We also assume that the diversity of the equilibrium individual demands is limited:

\[
\frac{\sigma_i(q_j(\xi_i))}{\sigma_i(q_{j'}(\xi_i))} < 2 \quad \text{for all } i = 1, \ldots, n, \ j, j' = 0, \ldots, n + 1, \ \kappa \geq 0.
\]

If \( \sigma_i \) increases, we assume additionally that there exists some \( \delta_i > 0 \) such that inequalities

\[
\sigma'_i(q_j(\xi_i))q_j(\xi_i) < \delta_i < \sigma_i(q_j(\xi_i)) - 1, \ i = 1, \ldots, n,
\]

is valid for all equilibrium individual demands \( q_{ij}, \ i = 1, \ldots, n, \ j = 0, \ldots, n + 1 \).
Zhelobodko et al. (2012) used an analogue of Assumptions 1 and 2 to prove the existence and uniqueness of the equilibrium in a single-sector economy. The number of workers in their economy (which corresponds to the quantity \( L_j \) in Equation (15)) is a model primitive. In the multi-sector economy explored here, the set of \( L_j \), \( j = 1 \ldots, n \), is endogenous, and Assumption 2 is given in a conditional but straightforward form. It links together three quantities: the equilibrium number of employed workers in each sector, variability of the elasticity of substitution, and the ratio \( C_i \) of fixed to variable costs. Small values of \( C_i \) are consequences of low fixed or/and high variable costs. Both effects can be attributed to less efficient economies since they correspond to smaller deviation from the backyard economy and to less productive labor. Therefore inequality (15) is valid if either sectors are sufficiently large (in terms of \( L_j \)), or preferences weakly deviate from the CES form, or the economy is not developed enough\(^1\).

Assumption 3 is most restrictive. It appears because the economy is multi-sector. Inequality (16) is also written with equilibrium variables. A relevant sufficient condition written in terms of model primitives is, in general, significantly more restrictive and therefore not shown here. We stress that Inequality (16) is valid if \( \sigma \) exhibits a small variability, which is in line with Assumption 2, or the equilibrium individual demands are weakly scattered. In our opinion, Assumption 3 is vital for the construction of the equilibrium with unspecified utility \( u \), but a counterexample is not constructed. Nevertheless, we will see that for economies with large enough number \( \mathcal{L} \) of agents Assumption 3 follows from Assumption 2.

Utilities (98) and (99) including specific cases (8) and (9) satisfy Assumptions 1 and 3 by construction. Assumption 2 is valid, if the number of workers in each sector is large (see Lemmata 18 and 19).

### 2.5 Equilibrium and its Properties

**Proposition 1.** Let Assumptions 1–3 be satisfied. Then a general equilibrium exists, and it is unique. This equilibrium is symmetrical with respect to varieties of the \( i \)-th differentiated product: \( p(\xi_i) \) and \( q_j(\xi_i) \) depends on \( i \) but not on specific varieties. We denote

\[
\begin{align*}
p_i &= p(\xi_i), \quad q_{ij} = q_j(\xi_i), \quad Q_i = Q(\xi_i), \quad \mathcal{G}_i = \mathcal{G}(\xi_i)
\end{align*}
\]

\(^1\)We are able to present a stronger version of Inequality (15), which involves only model primitives:

\[
|\sigma_i'(x)| \leq \min \left\{ \left( \frac{1 - \alpha}{\mu_j} \right) \left( 1 - \frac{1}{\delta_j} \right), \left( 1 - \beta_0 \right) \left( 1 - \frac{1}{\max_k \mu_k} \right) \right\} \frac{\mathcal{L}}{2C_i},
\]

assuming additionally that each \( \sigma_i \) is separated from 1 by some value \( \delta_i \): \( \sigma_i(\kappa) - 1 > \delta_i, \ i = 1, \ldots, n, \ \kappa > 0 \).
the symmetrical equilibrium variables. Then they are given by the following expressions:

\[ Q_i = C_i (\mathcal{G}_i - 1), \quad i = 1, \ldots, n, \]  

(19)

\[ p_i = \frac{\mathcal{G}_i c_i^e w_i}{\mathcal{G}_i - 1}, \quad i = 1, \ldots, n, \]  

(20)

\[ q_{ij} = \frac{(1 - \alpha) \beta_j Q_i}{L_j} = \frac{(\mathcal{G}_j + 1) Q_i}{L \mathcal{G}_j}, \quad j = 1, \ldots, n, \]  

(21)

\[ q_{i0} = \frac{Q_i}{L}, \]  

(22)

\[ q_{i,n+1} = \frac{\alpha Q_i}{L_{n+1}}. \]  

(23)

The following Equations determine the equilibrium wages, number of employed and unemployed workers, and number of firms:

\[ w_i = \frac{\mathcal{G}_i + 1}{\mathcal{G}_i} w_0, \quad i = 1, \ldots, n \]  

(24)

\[ L_i = (1 - \alpha) \beta_i \frac{\mathcal{G}_i}{\mathcal{G}_i + 1}, \quad i = 1, \ldots, n, \quad L_0 = (1 - \alpha) \beta_0 L, \]  

(25)

\[ L_{n+1} = L \left( \alpha + (1 - \alpha) \sum_{j=1}^{n} \frac{\beta_j}{\mathcal{G}_j + 1} \right), \]  

(26)

\[ L^u_i = \frac{\beta_i / (\mathcal{G}_i + 1)}{\sum_{j=1}^{n} \beta_j / (\mathcal{G}_j + 1)} L_{n+1}, \quad i = 1, \ldots, n, \]  

(27)

\[ N_i = \frac{(1 - \alpha) \beta_i L}{c_i^e (\mathcal{G}_i + 1)}, \quad i = 1, \ldots, n. \]  

(28)

The economic interpretation of Proposition 1 is in line with previous studies on the subject; see, f. i., Helpman and Krugman (1985). We discuss here several issues. Equilibrium variables depend on the MES \( \mathcal{G}_i \). When \( \mathcal{G}_i \) is large, the demand is basically determined by the most preferred representative of the differentiated product. Then the diversity of the product is subdued and the output of specific goods enlarges as stated in Equations (28) and (19); the \( i \)-th hi-tech sector becomes similar to the homogeneous one, and the wage differential between them disappears, Equation (24). The number \( L_i \) of employed workers follows the sector size measured by \( \beta_i L \). The linear relationship (25) between \( L_i \) and \( \beta_i \) is a consequence of the Cobb–Douglas setting (2).

On the contrary, small values of \( \mathcal{G}_i \) enlarge the diversity of the differentiated product. Therefore, the quantity \( 1/\mathcal{G}_i \) is interpreted as the love for variety in various papers (see Equation (28), where \( N_i \) is proportional to \( 1/ (\mathcal{G}_i + 1) \)). An expanding diversity of the differentiated product affects the total number of employed workers in a sector in the two opposite directions: the number of employees in a particular firm decreases but the number of firms increases. According to (25), the first effect is stronger. The wage bargaining balances the pressure from
unemployed agents and the competition of firms for skilled workers. The second force dominates, and wages follow the size of diversity, Equation (24). The primary role of diversity has been already mentioned by Vives (2001): in the first best outcome of a multi-sector economy individuals face a wider diversity of goods than in a competitive equilibrium.

Equation (19) also relates the output of varieties to the efficiency of the economy: More efficient economies produce more amount of specific varieties. The drop in variable costs, which is equivalent to a growth in labor productivity in the model, strengthens monopolistic competition between firms and makes them reduce the prices. This negative relation between prices and variable costs are given by Equation (20).

The number of unemployed workers expectedly increases with the taxation rate $\alpha$: a growth of the unemployment benefit subdues the risk of unemployment, Equation (26). Tastes affect unemployment primary with development of sectors reflected by $\beta_i L$ and secondary with the elasticity of demand. Development of sectors requires additional labor force. However, not only the demand for labor but also the supply of it changes. Complex relations between sectoral unemployment, total unemployment, tastes, and the number of individuals in the economy are covered by Equations (26) and (27).

The size of each sector can be measured in terms of the number of firms operated in the sector or the number of workers employed by those firms. As in other general equilibrium models, these two quantities are proportional Equations (25) and (28). The distribution of markets’ size ($L$ or $N$) follows the distribution of consumers’ tastes $\beta$ between different differentiated products.

Wages $w_0$ in the homogeneous sector cannot be found in the model. One could consider the homogeneous product as a *numeraire* and assign 1 to $w_0$ to simplify the notation.

In our model, full employment is unattainable under monopolistic competition. If high-tech sectors are similar to the standard sector, i.e., values of $S_i$ are large for all $i$, and unemployment benefits measured by $\alpha$ are small, then the unemployment almost disappears, Equation (26). Once the high-tech sector deviates from the standard one, the balance of expected wages induces unemployment. The introduction of risk aversion could alter this conclusion.

Equations (21)–(23) are technical but lead us to a quantitative description of the aggregate and individual demands. Initially, we note that under variable elasticities of substitution preferences, Equation (19) determines the equilibrium aggregate demand implicitly. The MES $S_i$, by Proposition 1, a function of $i$ and not of $\xi$, depends on the aggregate $Q_i$ and individual $q_{ij}$ demands (Equation (6)). All the demands are affected by prices $p_i$. Therefore, each firm faces
Equation (19) with its own price as a single unknown, as expected.

Equation (21), in particular, links individual demands to the size of the economy measured by \( L \). Under CES preferences, individuals consume less amount of each variety enjoying an increasing diversity of differentiated goods, when the economy enlarges. This consumption of each variety scales as \( 1/L \). Then the aggregate demand for a variety is expanded into the sum of individual demands each of which is close to zero. We argue that a departure from CES preferences conserves this small contribution of each individual demand into the aggregate demand. Indeed, under VES preferences, the size \( L \) of the economy also affects the outputs \( Q_i \) and the average elasticities of substitution \( G_i \), but this influence is weak\(^2\). Therefore the integral impact of the size \( L \) of the economy, the aggregate demand, and the diversity of differentiated goods described by Equation (21) on the individual demands is primarily driven by \( L \), if it is large enough. With a growth of the economy, outputs become more and more independent on \( L \), the diversity of goods unrestrictedly increases, and individual demands for specific varieties tend to zero\(^3\). In other words, properties of the equilibrium are affected by the behaviour of the preferences only in a positive neighborhood of zero. Therefore, a restrictive part of Assumption 3, Inequality (16), is not restrictive anymore and stays in line with Assumption 2 claiming that the variability of \( \sigma \) is small with respect to each \( L_j \)\(^4\).

We note that our families of hypergeometric functions (98) and (99), which give an example of preferences satisfying to Assumptions 1–3, are simplified to a sum of two power functions up to negligible terms:

\[
u(\kappa) = \frac{2^{-1/A}A}{A-1} \left( \kappa^{1 - \frac{1}{A}} - \frac{A-1}{2A(2A-1)} \kappa^{2 - \frac{1}{A}} \right) + O(\kappa^{3 - \frac{1}{A}}), \quad A > 1, \quad \kappa \ll 1,
\]

and, respectively,

\[
u(\kappa) = \frac{A}{A-1} \left( \kappa^{1 - \frac{1}{A}} + \frac{A-1}{A(2A-1)} \kappa^{2 - \frac{1}{A}} \right) - O(\kappa^{3 - \frac{1}{A}}), \quad A > 1, \quad \kappa \ll 1,
\]

where minus is used ahead of big-O just to stress that the next term of the series is negative. Computations are moved into the Appendix.

\(^2\)see (Zhelobodko et al., 2012) that deals with an analogue of Equation (19).

\(^3\)A rigorous proof of this statement follows from Equations (21)–(23) and (19), where the series expansion at 0 is substituted for \( G \) given by Equation (6).

\(^4\)Indeed, the ratio (16) scales as

\[
\frac{\sigma_i(0) + |\sigma_i'(0)|K_i/L \sigma_i(0)}{\sigma_i(0)} = 1 + \frac{|\sigma_i'(0)|K_i}{\sigma_i(0)L},
\]

where \( K_i > 0 \) is an appropriate constant, and Assumption 3 requires that \( |\sigma_i'(0)| \) is small in terms of the size \( L \) of the economy. Therefore ratio (16) is indeed less than 2.
An unspecified separable utility at the lower tier is used to exhibit the role of the demand side. A general form of the upper tier utility seems not to be so crucial. Therefore we make it as simple as possible when choosing the Cobb–Douglas form. The lost of generality here is compensated by significant simplification of mathematics.

We sketch here the idea of the Proof of Proposition 1. Under Assumptions 1 and 2, which involve only properties of the function $\sigma_i$, we derive Equation (19) from the first order condition and prove the existence and uniqueness of its solution (Lemma 2). It limits the set of admissible equilibrium prices $p(\xi_i)$ to a single value. This value of the price determines the optimal aggregate demand $Q_i$ and elasticity of substitution $\Theta_i$ and other variables through Equations (25)–(28). Then using the other Assumptions, we check that those variables indeed constitute the equilibrium. Namely this step relies on the bounded diversity of the equilibrium demand formulated by Assumption 3 (see details of the Proof of Proposition 1 in the Appendix).

3 Changes in Spending: Comparative Statics

3.1 Rigorous results

Now that the general equilibrium is completely defined, we assume that tastes of consumers slowly vary. Let consumers be ready to (slightly) reduce the consumption of the homogeneous product to buy more of the $i$-th differentiated product: $\beta_i$ increases, $\beta_0$ decreases by the same amount, and the other $\beta_j$, $j \neq 0$, $j \neq i$, do not vary.

Describing the response of the economy to those changes, we introduce a new Assumption, which is a modified version of Assumption 2.

Assumption 2’. Let

$$B = \max_{\kappa, \kappa' \in \left[0, \frac{\min \ q_k, \max \ q_k}{\min \ q_k, \max \ q_k}\right]} \frac{\left|\sigma'_k(\kappa)\right|}{\left|\sigma'_k(\kappa')\right|}. \quad (29)$$

We assume that

$$\max_{\kappa \in \left[0, \frac{\min \ q_k, \max \ q_k}{\min \ q_k, \max \ q_k}\right]} |\sigma'_k(\kappa)| < \frac{L_k}{4C_k(2Bn + 1)} \quad k = 1, \ldots, n. \quad (30)$$

This assumption includes a moderate deviation of the utilities from CES functions measured through the ratio of fixed to variable costs, which is in line with Assumption 2. Assumption 2’ also requires relatively small diversity of the equilibrium demands. This requirement agrees with Assumption 3.
We denote $S^*$ the maximal equilibrium mean elasticity of substitution:

$$S^* = \max_{k=1,...,n} \mathcal{S}_k.$$ (31)

**Proposition 2.** Let Assumptions 1–3 and 2' be satisfied. We also assume that all functions $\sigma_i(\sigma), i' = 1,\ldots,n,$ decreasing or all these functions increasing and

$$\alpha < \frac{1 - \beta_0}{2B S^* - S^* - \beta_0}.$$ $\quad$  
Then qualitative response to an increase of $\beta_i, i = 1,\ldots,n$ is given by Table 1. The variables move into the opposite directions, if $\beta_i$ decreases.

**Comment 1.** If $\sigma_i$ is an increasing function, then our prediction regarding the response of the number of unemployed agents is valid under the following additional assumption: the equilibrium MES is uniformly bounded from above:

$$\mathcal{S}_k < \frac{16}{1 - \beta_0} - 1, \quad k = 1,\ldots,n.$$ $\quad$  
Proof of proposition 2 can be found in Lemmata 9-15.

We link an increase in spending for differentiated product $i$ to a higher significance of product specifications for the consumer side. As soon as the elasticity of substitution $\sigma_i(q_{ij})$ between varieties of the $i$th good is variable, such attention of consumers to products’ details makes representatives of the differentiated product more distant substitutes. As a result, the elasticity $\sigma_i(q_{ij})$ increases. Then two possibilities naturally arise. If $\sigma_i$ is an increasing function then the individual demands $q_{ij}$ go up, and vise versa. The aggregate demand $Q_i$ for the $i$th differentiated product will follow the individual demands.

Each mean elasticity of substitution $\mathcal{S}_k$ is obtained as a weighted average

$$\mathcal{S}_k = (1 - \alpha) \sum_{j=0}^n k \sigma_k(q_{kj}) + \alpha \sigma_k(q_{k,n+1})$$

of sector specific values $\sigma_k(q_{kj})$, see Equations (6) and (22). Under the demand shock $(\beta_i \rightarrow \beta_i + \Delta \beta, \beta_0 \rightarrow \beta_0 - \Delta \beta)$ in favor of sector $i$, main changes of $\mathcal{S}_k$ are given by the term

$\Delta \beta(\sigma_k(q_{ki}) - \sigma_k(q_{k0}))$. Its sign is determined by the sign of the derivative $\sigma'_k$.

The integral effect of demand shocks is decomposed into two unequal parts called here primary and secondary. The primary effect is based on the redistribution of spending in favor of the $i$th differentiated good. This redistribution of spending drives an expansion of sector $i$, increasing the number of firms/the diversity of the varieties $N_i$ and the number $L_i$ of employees
Table 1: Comparative statics with respect to $\beta_i$ shows the response of sector $k$’s number of firms $N_k$, the number of employed workers $L_k$, output $Q_k$, prices $p_k$, relative diversity (RD) $N_k/(\beta_k L)$, relative wages $w_k/w_0$; $k = 1 \ldots n$.

<table>
<thead>
<tr>
<th>Response</th>
<th>$\beta_i \uparrow$</th>
<th>$\sigma' &gt; 0$</th>
<th>$\sigma' &lt; 0$</th>
<th>$\sigma' = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td># {firms}</td>
<td>$N_i \uparrow$</td>
<td>$N_k \downarrow$</td>
<td>$N_i \uparrow$</td>
<td>$N_k \uparrow$</td>
</tr>
<tr>
<td>RD</td>
<td>$N_i/\beta_i E \downarrow$</td>
<td>$N_k/\beta_k E \downarrow$</td>
<td>$N_i/\beta_i E \uparrow$</td>
<td>$N_k/\beta_k E \uparrow$</td>
</tr>
<tr>
<td>Outputs</td>
<td>$Q_i \uparrow$</td>
<td>$Q_k \uparrow$</td>
<td>$Q_i \downarrow$</td>
<td>$Q_k \downarrow$</td>
</tr>
<tr>
<td>Sector output</td>
<td>$N_i Q_i \uparrow$</td>
<td>$N_k Q_k \uparrow$</td>
<td>$N_i Q_i \downarrow$</td>
<td>$N_k Q_k \downarrow$</td>
</tr>
<tr>
<td># {employed}</td>
<td>$L_i \uparrow$</td>
<td>$L_k \uparrow$</td>
<td>$L_i \uparrow$</td>
<td>$L_k \downarrow$</td>
</tr>
<tr>
<td>Prices</td>
<td>$p_i \downarrow$</td>
<td>$p_k \downarrow$</td>
<td>$p_i \uparrow$</td>
<td>$p_k \uparrow$</td>
</tr>
<tr>
<td>Relative wages</td>
<td>$w_i/w_0 \downarrow$</td>
<td>$w_k/w_0 \downarrow$</td>
<td>$w_i/w_0 \uparrow$</td>
<td>$w_k/w_0 \uparrow$</td>
</tr>
<tr>
<td># {unemployed}</td>
<td>$L_{n+1} \uparrow$</td>
<td>$L_{n+1} \uparrow$</td>
<td>$L_{n+1} \uparrow$</td>
<td>$L_{n+1} \uparrow$</td>
</tr>
</tbody>
</table>
in this sector. This effect, based on the love for variety, is observed in the economy with CES preferences, \( n = 1 \), and without unemployment.

Changes in the elasticity of demand measured with the aggregate elasticities of substitution \( \Theta_k, k = 1, \ldots, n \), underlie secondary effects, which are observed in response of other equilibrium variables to demand shocks. We note that the love for variety determines the relative diversity \( N_k/(\beta_k L) \) of differentiated goods itself, which exhibits the diversity of the \( k \)th differentiated good normalized by the sector size (see Equation (28), where this quantity is proportional to \( 1/(\Theta_k + 1) \)). The secondary effects are described with changes in the relative diversity of differentiated goods. Under increasing elasticity of substitution, the relative diversity of differentiated goods decreases. Under decreasing elasticity of substitution, the relative diversity of differentiated goods increases. We stress that secondary effects are revealed only with models that involve VES preferences.

A growth of unemployment as the response to a demand shock for a differentiated good is also explained by the love for variety (i.e., relative diversity itself). This explanation involves moving of workers from sector to sector. Such a transition serves for illustrative purposes only, since our model is static. After a demand shock, the economy achieves a new equilibrium following the scheme described in Section 2. Then old and new equilibria are compared. Nevertheless, for the sake of simplicity one can argue that initially employed workers are still employed in the expanding sector whereas other individuals attracted by sector expansion enter its job market. These entrants can be among currently unemployed individuals earlier qualified for the job in the expanding sector or among other individuals who have just trained for this job. The number of accepted candidates is regulated by firms’ equilibrium output so that only a part of them is accepted. If \( n = 1 \) and preferences have a CES form, the fraction of unemployment agents is proportional to the sector size \( \beta_i L \). In a multi-industry case with VES preferences, the effect can be either amplified or subdued, depending on the relative diversity of varieties produced in the other sectors. Finally, if the expanding sector is too close to the homogeneous one (\( \Theta_i \) is large enough) and \( \sigma' > 0 \), then the additional demand for labor from the other hi-tech sectors prevails over the rejection rate of the expanding sector. This case is exceptional because of similarity of the expanding and homogeneous sectors.

3.2 Competitive and monopolistic responses

Zhelobodko et al. (2012) claimed that hi-tech sectors response pro-competitively to inflow of
workers, increasing output and decreasing prices, only if the elasticity of substitution $\sigma(\cdot)$ is a decreasing function. A redistribution of tastes launches opposite processes, Table 1. There are no contradictions: the two models differ not only formally but also conceptually: Zhelobodko et al. (2012) described an effect of an exogenous increase of the whole economy whereas we investigate an intra-economy size effect. The general equilibrium in this paper is “more general”, since the number of workers in each sector is endogenous. Its changes are induced by consumers’ tastes.

Let $\beta$ (slightly) increase. According to Table 1, the sector output $Q_i N_i$ and the number of firms $N_i$ follow the tastes and also increase, as expected. This is the primary effect. At the firm level, an increasing demand for the varieties “competes” with an increasing diversity of the differentiated product. The “race” outcome is determined by the relative diversity of varieties. If the relative diversity impact is negative, then the diversity “wins the race”, and firms’ outcome $Q_i$ increases. As stated in section 3.1, it occurs, if the preferences have an increasing elasticity of substitution (in the equilibrium).

It is worth repeating that industries react pro-competitively on changes in $\beta$. Only at the firm level, the response is ambiguous.

### 3.3 Response of the whole economy

We show that a demand shock in preferences for a differentiated product $i$ affects the production of the other differentiated products. An explicit (primary) influence of spending disappears ($\beta_i$ is absent in the equations describing the equilibrium variables in sector $j$; see Proposition 1). Nevertheless, an implicit (secondary) effect linked to the relative diversity of differentiated products is still expected.

New opportunities in the expanding sector $i$ attract entrepreneurs from other fields. However changes in the demand for other differentiated products are ambiguous; they are opposite to differences of the relative diversity of differentiated products. If $\sigma' = 0$, and the relative diversity of differentiated products is unchanged, then the $i$th sector expands at the expense of the homogeneous one; workers precisely follow tastes. Let now $\sigma'_k < 0$, $k = 1 \ldots, n$. Then the relative diversity of all differentiated products increases. In particular, it enlarges the diversity of the $k$th product, $k \neq i$ but the demand for specific varieties of the differentiated product goes down and prices go up. The whole sector incurs a negative scale effect: the redistribution of the output to larger amount of firms with a fall of individual outputs reduces the sector output.
The demand for labor follows the sector output, and the number $L_k$, $k \neq i$, of employees decreases. The same is true at the firm level. Hiring less workers, firms value each of them more (since the linear production function has a fixed input cost) and agree to increase their wages through the bargaining mechanism. Applying the same arguments, we end up with the opposite conclusion regarding the direction of changes in the equilibrium outputs $Q_k$, prices $p_k$, the number of employees $L_k$, and the ratio $w_k/w_0$ of the relative wages, $k \neq i$, under a decrease of the relative diversity of differentiated products (occurred if $\sigma' > 0$).

As stated above, the size of secondary effects in the $k$-th hi-tech sector depends on changes in the relative diversity of the $k$-th differentiated product. Then ranking sectors with respect to these changes $\sim \partial (1/S_k)/\partial \beta_i$, $k = 1, \ldots, n$ and $k \neq i$, we exhibit the elasticity of sectors’ response to demand shocks. The following Proposition quantifies changes in the wage inequality.

**Proposition 3.** We assume that the equilibrium is given by Equations (19)–(28). Let also Assumption (2′) be valid. Then the relative change of wages in hi-tech and homogeneous sectors is defined by the following formulae

$$\left(\frac{w_j}{w_0} - 1\right)^{-1} \frac{\partial}{\partial \beta_i} \left(\frac{w_j}{w_0} - 1\right) = -\frac{K_{ji}\sigma_j'(\kappa)C_j(S_j - 1)}{\mathcal{L}_i\mathcal{S}_j}$$

where $j = 1, \ldots, n$, $i \in \{1, \ldots, n\}$, $K_{ji} \in (1/3, 2)$, $\kappa \in (q_{ji}, q_{ji})$.

Proof of Proposition 3 is given in the Appendix, Lemma 16. Proposition 3 indicates, first, that the percentage change in the wage differential is small for preferences with a weak variability in the elasticity of substitution. This observation is expected, since the wage differential does not vary under CES. Second, the wage differential is sensitive to the value of $C_j$, which is the ratio of fixed to variable costs. Large values of $C_j$ are associated with developed industries (economies) so that less developed industries are more likely to suppress perturbations initiated by demand shocks. We claim that competition provides a mechanism through which shocks affect the wage differential. Third, the wage differential $w_j/w_0 - 1$ is most sensitive to changes in spending $\beta_i$ when sector $j$ corresponding to the wages is characterized by subdued monopolistic competition ($\mathcal{S}_j$ is large) but sector $i$ is, on the contrary, rather competitive. Interestingly, if $i = j$ then the both effects in the competition can dominate such that the maximal sensitivity with respect to the diversity is attained at intermediate $\mathcal{S}_i = \mathcal{S}_j = 2$.

New we uncover quantitative results from technical Equation (32). Propositions 2 and 3 are valid under relatively small variability of the elasticities of substitution $\sigma_k$, $k = 1, \ldots, n$, described in Assumptions 2 and 2’. However the size of the effect is proportional to this relative
variability of $\sigma_j$, see the right hand side in Equation (32). We note that, given the equilibrium, Proposition 3 requires Assumption $2'$ but not Assumption 2. Substituting for $\sigma_j'$ its maximal feasible value given by Assumption $2'$ (the second argument of the maximum in Definition (61) is assumed to be larger than the first argument), we derived that

$$\left(\frac{w_i}{w_0} - 1\right)^{-1} \frac{\partial}{\partial \beta_i} \left(\frac{w_i}{w_0} - 1\right) < \frac{2(1 + \mathcal{S}_j)}{\mathcal{S}_j \mathcal{S}_i} \frac{K_{ii}}{(2Bn + 1)}.$$ 

Substitution $\mathcal{S}_j = \mathcal{S}_i = 3$ serving only as an illustration fixes the first multiplier to 8/9. The estimate of the second multiplier is unclear because Assumption $2'$ is rough. For example, if $n = 1$, then the second multiplier is equal to 1/2 (the Proof of this statement follows from the Proof of Proposition 3), and a percentage change in the wage differential follows up to 0.44 part of percentage changes in spending measured through the exponent $\beta_i$.

### 3.4 Welfare

Individual welfare is understood as the indirect utility function. The latter is obtained by substitution of equilibrium variables into utility (2). There are $n + 2$ types of individuals: workers of hi-tech sectors ($n$ types), workers of the homogeneous sector, and unemployed agents. They differ by their incomes. Each type is represented by its own welfare denoted by $W_0, W_1, \ldots, W_n, W_{n+1}$.

We are going to estimate qualitatively the response of welfare to a growth of spending for products of sector 1 ($i = 1$ in Table 1). The expressions for the individual demands appear to be too complicated to be tackled explicitly. Therefore we use the specific family of utilities given by Equation (98) with $\sigma' < 0$ (in particular, by the simplest utility (9)) and analyze changes in the welfare numerically. For the sake of simplicity we consider only two hi-tech sectors, i.e., $n = 2$. Changes in the re-distribution of spending affect the welfare explicitly, Equation (2). Namely, individuals decide to consume more products from sector 1 at expense of the homogeneous product. This decision influences the welfare positively. The welfare of hi-tech workers and unemployed agents is also affected implicitly through changes in incomes and equilibrium individual demands. Thus welfare appears to follow the direction of changes in the income. According to our computer simulation, as a rule, the explicit effect dominates.

Let the welfare change from $W_i$ to $\tilde{W}_i$, $i = 0, \ldots, n + 1$, as a result of a growth of the Cobb–Douglas exponent from $\beta_1$ to $\tilde{\beta}_1$. The welfare of homogeneous sector workers are affected only by the explicit effect since $w_0$ does not changes. Therefore, $\tilde{W}_0 > W_0$. We proved (Proposition 2 and Table 1 for $\sigma' < 0$) that the income of workers employed in all hi-tech sectors increases but
the growth of the wages in sector 1 is the largest. As a composition of two positive effects, the welfare of hi-tech workers goes up and

$$\frac{\tilde{W}_1}{W_1} > \frac{\tilde{W}_2}{W_2} > 1.$$  

We find that, as a rule,

$$\frac{\tilde{W}_0}{W_0} > \frac{\tilde{W}_1}{W_1},$$

despite the wages in the homogeneous sector remain the same. The following quantitative arguments underlie the last inequality. The welfare is represented by the product of the consumption indices $H_i$ to the power of the corresponding $\beta_i$, $i = 0, \ldots, n$, Equation (2). When $\beta_1$ increases and $\beta_0$ decreases, the ratios of the new to old welfares are primary affected by shifts in $H_1^{\beta_1}$ and $H_0^{\beta_0}$: they both follow the change in their $\beta$. Substituting equilibrium variables into $H_0$ and $H_1$, we get

$$\frac{\tilde{W}_1}{W_1} \cdot \frac{\tilde{W}_0}{W_0} \approx \left( \frac{u(q_{11})}{u(q_{10})} \frac{\mathcal{S}_1}{\mathcal{S}_1 + 1} \right)^{\beta_1 - \beta_1},$$

where the first and the second fractions correspond to the shifts in $H_1^{\beta_1}$ and $H_0^{\beta_0}$ respectively. If $L$ is large, then, according to Proposition 1, the individual demands $q_{11}$ and $q_{10}$ are close to zero, and the ratio $u(q_{11})/u(q_{10})$ of the utilities in the last formula is close to 1. In our numerical example, $\mathcal{S}_1$ is found to lie in a neighborhood of 2, and the fraction $\mathcal{S}_1/(\mathcal{S}_1 + 1)$ is much less than 1. These arguments are extended to economies that are characterized by moderate values of $\mathcal{S}_1$ and small individual demands. The latter is a consequence of large values of $L$ (see the discussion after Proposition 1).

The welfare of unemployed agents, as the welfare of the other individuals, undergoes an explicit positive effect from the re-distribution of spending. However, the number of unemployed agents increases (Table 1) and their income goes down (Equation (49)). This leads to a negative implicit effect. The negative effect is smaller, in general, so that

$$\frac{\tilde{W}_{n+1}}{W_{n+1}} > 1.$$  

Our simulation supports this observation.

The model cannot predict the changes in the employment of each specific individual. Simplifying interpretation, we assume that expanding sectors only hire workers, and vice versa. Then sector 1 attracts new workers primary from the homogeneous sector and, in exceptional cases, from the other hi-tech sector, Figure 1. A part of them is employed, whereas the rest are rejected. These rejected job market candidates become worse off, where as all the other individuals gain from the shock in demand.
Figure 1: The difference $\Delta L_i = \tilde{L}_i - L_i$, $i = 0, 1, 2, 3$, between new and old numbers of individuals associated with sector $i$; the horizontal dashed line indicates 0.

Figure 1 serves only as an illustration of differences $\tilde{L}_i - L_i$, $i = 0, 1, 2, 3$, where $\tilde{L}_i$ corresponds to $\tilde{\beta}_i$. Relative changes in these differences are small for a range of parameters. In our simulation, we use $\alpha \in [0.01, 0.15]$, $\mathcal{L} \in [100, 10000]$, $C_i \in [1, 1000]$, $c_i^\varphi \in [0.1, 10]$, $A \in [2, 4]$, $\beta_j \in [0.1, 0.5]$, $j = 1, 2$, with $\beta_1 + \beta_2 < 0.9$.

4 Conclusion

In this paper, we describe the role of the elasticity of demand in the response of hi-tech sectors to demand shocks. The modeling is based on the Zhelobodko et al. (2012) description of the firms’ decision making under monopolistic competition and additive unspecified preferences of consumers.

This description is extended to the case of heterogeneous consumers. In contrast to various studies, we do not limited ourselves to the description of symmetrical equilibriums but prove the existence of a unique equilibrium in production, if consumers are not diversified too much. The latter condition seems to be essential such that an excessive diversity can ruin the equilibrium. This unique equilibrium is predictably symmetrical, since intra-sector characteristics are identical.

In our model, firms select an optimal amount of workers among job market candidates. Job market frictions do not allow rejected agents to find another job immediately. They are limited to receiving an unemployment benefit. Choosing a job market, workers move between sectors until the expected wages are equalized. Accepted candidates agree to their wages with firms through the bargaining mechanism described in (Stole and Zwiebel, 1996). This bargaining allows the construction of a general equilibrium with the number of workers in each sector balanced endogenously (unlike numerous general equilibrium models with monopolistic competition and a preset number of workers qualified for each available job). Such a setting allows to making a theoretical prediction regarding wage inequalities and the unemployment
rate. In particular, we argue that the capacity of expanding sectors is insufficient to fully absorb the inflow of workers attracted by the sector expansion and its increasing demand for labor.

As far as we know, researchers debating the need to switch from CES to VES preferences in order to describe certain phenomena of economic development do not estimate the size of these phenomena in their theoretical models. This gap is filled in this study; we argue that the influence of demand shocks on the wage differential deviates from zero in the framework of the approach and determine the deviation.

Further development of this research could involve workers with different productivities and consumers with various differing utilities. Our construction of the general equilibrium is likely to survive under those extensions. A new model could also contribute to problems of sorting and matching.

A Proof of Main Results

A.1 Proof of Proposition 1

At the first stage, the existence and uniqueness of the solution of Equation (19) is established. Since \( S \) depends on \( Q(\xi_i) \), this Equation defines \( Q(\xi_i) \) only implicitly. It is worth noting that firms tune their prices, and the demand is formed as a response to these prices. Therefore, merely the price \( p(\xi_i) \) is a single but “hidden” unknown in Equation (19). Dropping product name \( \xi_i \) in the notation, we put

\[
h(p) = Q(p) - C(S(p) - 1).
\]

Lemma 1 states that the function \( h(p) \) monotonously decreases. Then the equation \( h(p) = 0 \) has a unique solution. Indeed, monotonically decreasing function \( h(p) \) is positive, when \( p \) is close to 0, negative, when \( p \) is large, and therefore crosses zero at some unique \( p \). Since the functions \( h(p) \) are identical within sector \( i \), the unique \( p = p_i \) depends on \( i \) but not on specification \( \xi_i \) of the product. The aggregate demand \( Q_i = Q(\xi_i) \) found with this \( p_i \) equalizes the both hand sides in Equation (6). According to Lemma 2, it is well defined and symmetrical with respect to \( \xi_i \). The other variables are also symmetrical with respect to \( \xi_i \) (Lemma 2). Therefore notation (18) neglecting the product specification is justified. Equations (19)–(28) are proved in Lemmata 2–6.

Lemma 7 verifies the second order conditions for optimization problem (1). Namely, it
proves that \( \partial^2 \pi_i/\partial p_i^2 < 0 \) at the point \( p_i \) with \( \partial \pi_i/\partial p_i = 0 \). Then the profit \( \pi_i \) attains its maximum at \( p_i \).

Here we place five Lemmata that are used in the proof of Proposition 1.

**Lemma 1.** Let Assumption 2 be valid in the following form

\[
|\sigma_i'(\pi)| < \frac{L_i}{2nC_i} \quad \text{for all } i = 1, \ldots, n, \ j = 0, \ldots, n + 1, \text{ and } \pi \geq 0.
\]  

(33)

Then the function \( h(p) \) monotonically decreases and has a unique root.

**Proof.** The product specification \( \xi_i \) and the sector specification \( i \) are dropped in the formulation of the Lemma and in the Proof; in particular, \( \sigma_i(q_{ij}) \) is reduced to \( \sigma(q_j) \). Evaluating \( \partial h/\partial p \) with Equations (7) and using the formula

\[
\frac{\partial \mathcal{S}}{\partial p} = -\frac{1}{pQ^2} \sum_{j=0}^{n+1} \sum_{k=j+1}^{n+1} q_j q_k (\sigma(q_j) - \sigma(q_k))^2 L_j L_k - \frac{1}{pQ} \sum_{j=0}^{n+1} \sigma'(q_j) \sigma(q_j) q_j^2 L_j,
\]

(34)

proved in lemma 20, one can see that the condition \( \partial h/\partial p < 0 \) is equivalent to

\[
\frac{1}{p} \sum_{j=0}^{n+1} q_j \sigma(q_j) \left( 1 - \frac{C \sigma'(q_j) q_j}{Q} \right) L_j > \frac{C}{pQ^2} \sum_{j=0}^{n+1} \sum_{k=j+1}^{n+1} q_j q_k (\sigma(q_j) - \sigma(q_k))^2 L_j L_k.
\]

(35)

It is enough to prove

\[
q_j \sigma(q_j) \left( 1 - \frac{C \sigma'(q_j) q_j}{Q} \right) L_j > \frac{C}{Q^2} \sum_{k=j}^{n+1} q_k (\sigma(q_j) - \sigma(q_k))^2 L_j L_k \quad \text{for all } j = 0, \ldots, n + 1.
\]

or

\[
\sigma(q_j) \left( 1 - \frac{C \sigma'(q_j) q_j}{Q} \right) > \frac{C}{Q^2} \sum_{k=j}^{n+1} q_k (\sigma(q_j) - \sigma(q_k))^2 L_k \quad \text{for all } j = 0, \ldots, n + 1.
\]

(36)

Among all the terms at the right hand side of (36) which are nonnegative let us choose the one delivering maximum. Let it be \( q_k (\sigma(q_j) - \sigma(q_k))^2 L_k \) for some \( k \). So, it is enough to prove that

\[
\sigma(q_j) \left( 1 - \frac{C \sigma'(q_j) q_j}{Q} \right) > \frac{C n}{Q^2} q_k L_k (\sigma(q_j) - \sigma(q_k))^2 \quad \text{for all } j = 0, \ldots, n + 1.
\]

(37)

We consider the cases of decreasing and increasing functions \( \sigma(\cdot) \) separately. Without loss of generality we can assume that the individual demands \( q_j, j = 1, \ldots, n \) are sorted in descending or ascending order. Otherwise, they can be properly reordered. Let us first assume that \( \sigma(\cdot) \) decreases and \( q_j < q_k \) for \( k > j \). Then \( \sigma(q_j) > \sigma(q_k) \). As for \( \sigma'(\cdot) < 0 \) the expression

\[
- \frac{C \sigma'(q_j) q_j}{Q}
\]
is positive, Inequality (37) is valid, if for some \( \kappa \in [q_k, q_j] \) the following inequality is valid:

\[
\sigma(q_j) > \frac{Cnq_kL_k}{Q^2}(-\sigma'(\kappa))(q_k - q_j)(\sigma(q_j) - \sigma(q_k)).
\] (38)

This inequality follows from

\[
\sigma(q_j) > \frac{Cnq_kL_k}{Q^2}(-\sigma'(\kappa))q_k\sigma(q_j),
\]

or

\[
1 > \frac{Cn(q_jL_j)(q_kL_k)}{L_jQ^2}(-\sigma'(\kappa)).
\]

Using the evident inequality \( q_jL_j/Q < 1 \), \( j = 0, \ldots, n + 1 \), we claim that the last condition is valid if

\[
1 > \frac{Cn}{L_j}(-\sigma'(\kappa)),
\]

following from (33).

Let now \( \sigma'(\cdot) > 0 \). Then, by (33),

\[
0 < \sigma'(q_j)\frac{Cq_j}{Q} < \frac{L_j}{2C} \cdot \frac{Cq_j}{Q} < \frac{q_jL_j}{2Q} < \frac{1}{2},
\]

The expressions in the brackets in the left hand side of Inequality (37) are between 1/2 and 1. Then Inequality (37) is valid, if

\[
\sigma(q_j) > \frac{2Cn}{Q^2}q_kL_k(\sigma(q_j) - \sigma(q_k))^2 \quad \text{for all } j = 0, \ldots, n + 1.
\] (39)

Deriving Inequality (39) we apply the same arguments as in the proof of Inequality (38). Indeed, the inequality \( q_j > q_k \) is valid up to re-ordering of \( q_j \). Then \( \sigma(q_j) > \sigma(q_k) \). Inequality (39) is derived, if for some \( \kappa \in [q_k, q_j] \) the following inequality

\[
\sigma(q_j) > \frac{2Cnq_kL_k}{Q^2}\sigma'(\kappa)(q_j - q_k)(\sigma(q_j) - \sigma(q_k)),
\]

is valid. It is weaker than inequality

\[
\sigma(q_j) > \frac{Cnq_kL_k}{Q^2}(-\sigma'(\kappa))q_j\sigma(q_j),
\]

which is, in turn, weaker than

\[
1 > \frac{Cn(q_jL_j)(q_kL_k)}{L_jQ^2}(-\sigma'(\kappa)).
\]

The last inequality follows from (33).

Finally, the equation \( h(p) = 0 \) has a unique solution. Indeed, monotonically decreasing function \( h(p) \) is positive, when \( p \) is close to 0, negative, when \( p \) is large, and therefore crosses zero at some unique \( p \).
Lemma 2. If the first order condition of profit maximization problem \(1\) determines a unique optimal price \(p_i\) in sector \(i\), then this price \(p_i\) is given by Equation \(20\). The price \(p_i\) and individual demands \(q_{ij}\) for goods in sector \(i\) of workers employed in sector \(j\) are independent of the product specification \(\xi_i\); therefore the product specification is dropped. Additionally, Equation \(19\) is valid, and the aggregate demand \(Q_i = Q(\xi_i)\) is also independent of \(\xi_i\).

Proof. We apply a standard method to prove Lemma 2. From the first order condition \((\partial\pi(\xi_i)/\partial p(\xi_i) = 0)\) we immediately obtain Equation \(20\). Combining it with the free entry condition \((\pi = 0)\) we find that the optimal supply satisfies Equation \(19\). The latter equation is equivalent to the equation \(h(p(\xi_i)) = 0\). According to Lemma 1, the function \(h\) has a unique root. Therefore \(p(\xi_i) = p_i\) depends on \(i\) but not on \(\xi_i\). Equations \(20\) and \(19\) relate \(Q(\xi_i)\) to \(p_i\). Then the equilibrium aggregate demand (equalled to the firms’ supply) also does not depend on the product specification \(\xi_i\), \(Q(\xi_i) = Q_i\). From the first order conditions \(u'(q_j(\xi_i)) = \lambda_i p_i\) of consumers’ optimization problem and the monotonicity of \(u'\) it follows that the individual demands \(q_j(\xi_i)\) are also symmetrical \(q_j(\xi_i) = q_j(\xi_i') = q_{ij}\). \(\square\)

Lemma 3. Let the first order conditions of firms’ optimization problem be satisfied. Then Equations \(21\)–\(23\), \(28\) and the following relationships are valid in the equilibrium:

\[
L_i = (1 - \alpha) L \beta_i \frac{y_0}{y_i}, \quad i = 0, \ldots, n
\]
\[
L_{n+1} = L \left( 1 - (1 - \alpha) \sum_{j=0}^{n} \frac{\beta_j y_0}{y_j} \right)
\]
\[
L_i^u = L \beta_i \left( 1 - \frac{y_0}{y_i} \right) \frac{1 - (1 - \alpha) \sum_{j=0}^{n} \frac{\beta_j y_0}{y_j}}{1 - \sum_{j=0}^{n} \frac{\beta_j y_0}{y_j}}
\]

Proof. The revenue of firms is transmitted to their workers:

\[
p_i Q_i = w_i l_i = w_i \frac{L_i}{N_i}, \quad i = 1, \ldots, n.
\]

The income of workers is

\[
y_i = (1 - \alpha) w_i, \quad i = 0, \ldots, n.
\]

The number of all unemployed agents is

\[
L_{n+1} = \sum_{i=1}^{n} L_i^u.
\]

The income of the unemployed agents is

\[
y_{n+1} = \frac{\alpha}{L_{n+1}} \sum_{i=0}^{n} w_i L_i = \frac{\alpha}{(1 - \alpha) L_{n+1}} \sum_{i=0}^{n} y_i L_i.
\]
The balance of the incomes is re-written in the form:

\[ L_i y_i + L_i^u y_{n+1} = L_i y_0 + L_i^u y_0. \]  

(46)

Summing the balance of the incomes (46) up from 1 to \( n \), we get

\[ \sum_{i=1}^{n} L_i y_i + L_{n+1} y_{n+1} = y_0 \sum_{i=1}^{n} L_i + L_{n+1} y_0 = y_0 (\mathcal{L} - L_0). \]

Simplifying,

\[ \sum_{i=0}^{n+1} L_i y_i = \mathcal{L} y_0. \]  

(47)

Multiplying individual budgets (4) by \( L_j \) and summing them up from \( j = 0 \) to \( n + 1 \), we have

\[ p_i Q_i N_i = \beta_i \sum_{j=0}^{n+1} y_j L_j. \]

(48)

With Equations (43) and (47), the last Equation is transformed into (40) for \( i = 1, \ldots, n \).

From (45) it follows that

\[ (1 - \alpha) y_{n+1} L_{n+1} = \alpha \sum_{i=0}^{n} y_i L_i \]

or

\[ y_{n+1} L_{n+1} = \alpha \sum_{i=0}^{n+1} y_i L_i. \]

Applying (47) to this equation, we get

\[ y_{n+1} L_{n+1} = \alpha \mathcal{L} y_0. \]

(49)

The following step uncovers the value of \( L_0 \). Combining Equations (45) and (40), we have

\[ (1 - \alpha) \alpha \mathcal{L} y_0 = \alpha \left( y_0 L_0 + \sum_{i=1}^{n} (1 - \alpha) \beta_i \mathcal{L} y_0 \right). \]

Using \( \beta_0 = 1 - \sum_{i=1}^{n} \beta_i \), we end up with

\[ y_0 L_0 = (1 - \alpha) \left( \mathcal{L} y_0 - \sum_{i=1}^{n} \beta_i \mathcal{L} y_0 \right) = (1 - \alpha) \mathcal{L} y_0 \beta_0. \]

This equation is equivalent to (40), in which \( i = 0 \) is substituted.

The total number of unemployed agents complements the number of the all employed workers to \( \mathcal{L} \):

\[ L_{n+1} = \mathcal{L} - \sum_{i=0}^{n} L_i. \]
Substitution of Equations (40) into this equation leads to Equation (41).

We use Equation (46) to find $L^u_i$:

$$L^u_i = \frac{(y_i - y_0)L_i}{y_0 - y_{n+1}}.$$  

Substituting Equations (43) and (41) into the last equation, we get

$$L^u_i = \frac{(y_i - y_0)(1 - \alpha)\beta_i L y_0}{y_0(1 - \frac{\alpha}{1 - \alpha y_0 \sum_{j=0}^{m}(\beta_j / y_j)}) y_i}.$$  

This equation is transformed into (42).

Establishing (21)–(23), we solve budget constraint (4) with respect to $q_{ij}$:

$$q_{ij} = \frac{\beta_i y_j}{p_i N_i}.$$  

The combination of this equation with (43) and (44) results in

$$q_{ij} = \frac{\beta_i y_j Q_i}{w_i L_i} = \frac{(1 - \alpha)\beta_i y_j Q_i}{y_i L_i}.$$  

We substitute expression (50) into the definition of the aggregate demand:

$$Q_i = \sum_{j=0}^{n+1} q_{ij} L_j = (1 - \alpha)\frac{\beta_i Q_i}{y_i L_i} \sum_{j=0}^{n+1} y_j L_j.$$  

With Equation (47) it turns out to

$$Q_i = (1 - \alpha)\frac{\beta_i Q_i}{y_i L_i} L y_0.$$  

Returning to Equation (50), we obtain Equations (21)-(23). Equation (28) follows from Equations (48), (47), and (44).

Lemma 4. Let the output $Q$ relate to the number $l$ of workers by the production function $Q = \theta(l - l_*)$, where $l_*>0$ indicates a minimal labor requirement to run a firm. We assume that the firm sells its production at the price $p$ per unit and get the profit $\pi = pQ - lw$, where $w$ is the wage of each worker. Then the bargain between the firm and its workers, described in (Stole and Zwiebel, 1996), leads to the wages

$$w = \theta p \frac{l^2 - l_*^2}{2l^2} + \frac{l^2 - l_*^2}{2l^2} w_0,$$  

if the both sides of the bargain estimate alternative wages in the labor market in $w_0$. 
Proof. Let \( F(l) = \theta p(l - l_*) \) if \( l > l_* \) and \( F(l) = 0 \) if \( l \leq l_* \) be the revenue of the firm as a function of labor. Then hiring \( l_* + 1 \) workers instead of \( l_* \), the firm obtains the revenue \( \Delta F(l_* + 1) \). Each of \( l_* \) workers benefits from this deal since their wages increase from 0 to \( w(l_* + 1) \). Since the alternative of workers is to be paid \( w_0 \), their surplus is \( w(l_* + 1) - w_0 \). The equal division of the surplus between the firm and each of its workers leads to the equation

\[
\Delta F(l_* + 1) - (l_* + 1)w(l_* + 1) = w(l_* + 1) - w_0.
\]

With a hire of the \((l_* + 2)\)-nd worker, the equal division arguments result in

\[
\Delta F(l_* + 2) - w(l_* + 2) + (l_* + 1)(w(l_* + 1) - w(l_* + 2)) = w(l_* + 2) - w_0.
\]

Solving this Equation with respect to \( w(l_* + 2) \), we get

\[
w(l_* + 2) = \frac{\Delta F(l_* + 2) + (l + 1)w(l_* + 1) + w_0}{l_* + 3}.
\]

In general,

\[
w(l) = \frac{\Delta F(l) + (l - 1)w(l - 1) + w_0}{l_* + 1},
\]
given \( l > l_* \). The differential equation

\[
w'(l) = -\frac{2w(l)}{l} + \frac{F'(l) + w_0}{l}
\]
corresponds to the obtained difference equation. The solution of the differential equation is

\[
w = \frac{1}{l^2} \int_{l_*}^{l} \left( \theta p + w_0 \right) x \, dx = \theta p \frac{l^2 - l_*^2}{2l^2} + \frac{l^2 - l_*^2}{2l^2} w_0.
\]

\[\square\]

**Lemma 5.** Let the first order conditions of firms’ optimization problem be satisfied. Then the wages of workers employed in the \( i \)th hi-tech sector are given by Equation (24).

Proof. The index \( i \) indicating the sector is dropped. Substituting Equation (20) into Equation (51) we get

\[
\left( 1 - \frac{\mathcal{G}}{\mathcal{G} - 1} \frac{l^2 - l_*^2}{2l^2} \right) w = \frac{l^2 - l_*^2}{2l^2} w_0.
\]

The zero profit condition leads to

\[
pQ = lw.
\]

With Equation (20) it turns to

\[
\frac{\mathcal{G}}{\mathcal{G} - 1} (l - c^e) = l.
\]
Then the optimal number of workers is

\[ l = c^\phi \mathcal{S}. \]

Combining this Equation with (52), we obtain

\[ \left(1 - \frac{\mathcal{S}}{\mathcal{S} - 1} \frac{l^2 - (c^\phi)^2}{2l^2} \right) w = \frac{l^2 - (c^\phi)^2}{2l^2} w_0. \]

This equation is re-written as

\[ \left(1 - \frac{\mathcal{S} + 1}{2\mathcal{S}} \right) w = \frac{\mathcal{S}^2 - 1}{2\mathcal{S}^2} w_0 \]

Eventually,

\[ w = \frac{\mathcal{S} + 1}{\mathcal{S}} w_0. \]

\[ \square \]

**Lemma 6.** Let the first order conditions of firms’ optimization problem be satisfied. Then the number and income of unemployed workers are given by Equations (26) and

\[ y_{n+1} = \alpha y_0 \left(1 + \frac{\alpha y_0}{\alpha + (1 - \alpha) \sum_{j=1}^{n} \beta_j / (\mathcal{S}_j + 1)} \right) \]

respectively. If \( \alpha \leq 1 \) then the income of unemployed workers is less than or equal to the income of workers employed in the homogeneous sector: \( y_{n+1} \leq y_0 \). The equality is attained if \( \alpha = 1 \).

**Proof.** Equation (26) follows from Equation (41), Lemma 3 and Equation (24), Lemma 5. Then, substituting Equation (26), into Equation (49), we obtain Equation (53). The inequality \( y_{n+1} \leq y_0 \) is trivial.

\[ \square \]

**Lemma 7.** Let Assumption 1 and Condition (16) be valid. Then \( \partial^2 \pi / \partial p^2 < 0 \) at the point \( p \) defined by the first order condition.

**Proof.** Computing the derivative of profit (1) and taking into account the derivative \( \partial Q / \partial p \) given by (7), we get

\[ \frac{\partial \pi}{\partial p} = Q \left(1 - \left(1 - \frac{c^\nu}{p}\right) \mathcal{S}\right). \]

Equalizing this derivative to zero, we obtain

\[ \frac{p - c^\nu}{p} \mathcal{S} = 1 \]

Computing the second derivative, we get

\[ \frac{\partial^2 \pi}{\partial p^2} = \frac{\partial Q}{\partial p} \left(1 - \left(1 - \frac{c^\nu}{p}\right) \sigma\right) + Q \left(- \frac{c^\nu}{p^2} - \left(1 - \frac{c^\nu}{p}\right) \frac{\partial \mathcal{S}}{\partial p}\right). \]
Substituting the equilibrium point \( p = p_{\text{opt}} \), i.e., using that the first derivative is zero, we obtain

\[
\frac{\partial^2 \pi}{\partial p^2} \bigg|_{p=p_{\text{opt}}} = -\frac{\mathcal{S} - 1}{p} Q - \frac{Q}{\mathcal{S}} \frac{\partial \mathcal{S}}{\partial p}
\]

From lemma 20, equation (118) the previous formula leads to:

\[
\frac{\partial^2 \pi}{\partial p^2} \bigg|_{p=p_{\text{opt}}} p = (-2\mathcal{S} + 1)Q + \frac{1}{\mathcal{S}} \sum_{j=0}^{n+1} q_j \sigma(q_j)(\sigma(q_j) + \sigma'(q_j)q_j) L_j.
\]

We plan to show that the obtained expression is negative:

\[
\frac{\partial^2 \pi}{\partial p^2} \bigg|_{p=p_{\text{opt}}} < 0.
\]

It is equivalent to

\[
\mathcal{S} Q \cdot Q + QL \sum_{j=0}^{n+1} q_j \sigma(q_j)(\sigma(q_j) + \sigma'(q_j)q_j) L_j - 2\mathcal{S}^2 Q^2 < 0. \quad (55)
\]

Writing \( \mathcal{S} \) and \( Q \) as series we get

\[
\sum_{j=0}^{n+1} \sum_{j'=0}^{n+1} q_j q_{j'} \sigma(q_j)(\sigma(q_j) + \sigma'(q_j)q_j + 1) L_j L_{j'} - 2 \sum_{j=0}^{n+1} \sum_{j'=0}^{n+1} q_j q_{j'} \sigma(q_j)\sigma(q_j) L_j L_{j'} < 0.
\]

By \( T_{jj'} \) we denote the sum of all terms corresponding to the indices \( j \) and \( j' \):

\[
T_{jj'} = q_j q_{j'} (\sigma^2(q_j) + \sigma^2(q_{j'}) + \sigma(q_j)(\sigma'(q_j)q_j + 1) + \sigma(q_{j'})(\sigma'(q_{j'})q_{j'} + 1) - 4\sigma(q_j)\sigma(q_{j'}) \gamma_j \gamma_{j'}). \quad (56)
\]

From Assumption 1 it follows that \( \sigma(q_j)/\sigma(q_{j'}) < 2\).

We first consider the case \( \sigma' < 0 \). We apply Lemma 21 with \( \delta = 0 \) (\( B \) is changed to 2, according to the Comment to Lemma 21) and obtain that

\[
T_{jj'} < 0. \quad (57)
\]

Let \( \sigma' > 0 \). Now we apply Lemma 21 with \( \delta \) given by (17) and obtain

\[
\sigma(q_j)^2 - 4\sigma(q_j)\sigma(q_{j'}) + \sigma^2(q_{j'}) + (1 + \delta)\sigma(q_j) + (1 + \delta)\sigma(q_{j'}) < 0.
\]

Estimating \( \sigma'(q_j)q_j \) and \( \sigma'(q_{j'})q_{j'} \) by Inequality (17) we conclude that the brackets in (56) and \( T_{jj'} \) are negative.

Now we derive that a single firm becomes worse off, if it hires any unemployed agent (qualified for the job) and pays her even slightly more than the unemployment benefit. Let \( l' \) be an additional labor paid at \( w' \), and \( \Delta Q \) be the surplus of the output (index \( i \) indicating the sector
is dropped to simplify the notation). Then a considered firm has to decrease its price to some $p - \Delta p$ to shift the aggregate demand up to the level of $Q + \Delta Q$. The structure of the firm’s expenses changes. The firm pays $w$ to its first $c^v Q + c^\varphi$ workers and $w' < w$ to new $l'$ workers that produce $Q'$. The profit becomes

$$
\pi' = (p - \Delta p)(Q + \Delta Q) - c^v Q w - c^\varphi Q' w' - c^\varphi w.
$$

We are going to uncover changes in the demand share for the output of a single firm. This quantity is ill-defined because the set of firms is represented by a continuous (not discrete) set. Therefore we assume that a (small) part of firms, whose joint mass is $\varepsilon$, hires current unemployed agents and behaves identically.

**Lemma 8.** If in the equilibrium, a small mass $\varepsilon$ of firms in a single sector hires the same additional small number of unemployed agents proposing them identical compensations, then each of these firms becomes worse off in terms of their profit $\pi'$ given by (58).

**Proof.** Budget constraint (4) turns to

$$
pq_j(N - \varepsilon) + (p - \Delta p)(q_j + \Delta q_j) = \beta y_j,
$$

where index $j$ is related to consumers. Index $i$ indicating the sector is dropped to simplify the notation. In general, Equation (59) is valid only approximately. Indeed, new workers get a larger salary (their $y_j$ is enlarged), and the prices for the other products also vary. However, new workers accept an arbitrary salary that is larger than the unemployment benefit. Therefore, changes in the income $y_j$ are negligible. Changes in the other prices and corresponding outputs are also negligible, since the deviation of a small mass of firms is considered. Summing Equation (59) over all consumers, we get

$$
pQ(N - \varepsilon) + (p - \Delta p)(Q + \Delta Q) = \beta Y,
$$

where $Y$ is the total amount of money in the economy. Taking into account the balance $pQ N = \beta Y$, we conclude that

$$
(p\Delta Q + Q\Delta p - \Delta p\Delta Q)\varepsilon = 0.
$$

Then the revenue $(p - \Delta p)(Q + \Delta Q)$ remains $pQ$, and the profit

$$
\pi' = pQ - c^v Q w - c^\varphi Q' w' - c^\varphi w = -c^\varphi Q' w' < 0
$$

is negative.
A.2 Proof of Proposition 2

We put

\[ M_k = \frac{\mathcal{S}_k}{\mathcal{G}_k - 1} \max \left\{ \beta_k \frac{\mathcal{S}_k}{4(1 + \mathcal{G}_k)} \right\}, \quad k = 1, \ldots, n. \]  \hspace{1cm} (61)

and

\[ \gamma_k = \max_{j=1, \ldots, n+1} \left| \sigma_k'(q_{kj}) \right| C_k M_k, \]  \hspace{1cm} (62)

**Lemma 9.** Let monotonous \( \sigma_k' \) have an identical sign for all values of \( k = 1, \ldots, n \). We assume that for all \( k = 1, \ldots, n \)

\[ \gamma_k \leq \frac{1}{4(2Bn + 1)} \quad \text{if} \quad \sigma_k'(q_{kj}) > 0, \quad j = 1, \ldots, n + 1, \]  \hspace{1cm} (63)

\[ \gamma_k \leq \frac{1}{2(2Bn + 1)} \quad \text{if} \quad \sigma_k'(q_{kj}) < 0, \quad j = 1, \ldots, n + 1. \]  \hspace{1cm} (64)

If \( \sigma_k'(q_{kj}) > 0, \quad j = 1, \ldots, n + 1 \), we also assume that Assumption 2 is satisfied. Then Equation

\[ \frac{\partial Q_j}{\partial \beta_i} = C_j \frac{\partial \mathcal{S}_j}{\partial \beta_i} = K_{ji} C_j \left( \sigma_j(q_{ji}) - \sigma_j(q_{j0}) \right), \quad \text{where} \quad K_{ji} \in \left( \frac{1}{3}, 2 \right). \]  \hspace{1cm} (65)

is valid.

**Proof.** At the first step we write out a system of linear equation which the derivatives \( \frac{\partial Q_k}{\partial \beta_i} \), \( i, k = 1, \ldots, n \), satisfy to. Definition (6) of the aggregate elasticity of demand is evaluated with Lemma 2:

\[ \mathcal{S}_k = (1 - \alpha) \sum_{j=0}^{n} \beta_j \sigma_k(q_{kj}) + \alpha \sigma_k(q_{k,n+1}), \quad k = 1, \ldots, n. \]

Then the derivative of \( \mathcal{S}_i \) with respect to \( \beta_i \) is given by the following formula:

\[ \frac{\partial \mathcal{S}_k}{\partial \beta_i} = (1 - \alpha) \left( \sigma_k(q_{ki}) - \sigma_k(q_{k0}) \right) + (1 - \alpha) \sum_{j=1}^{n} \beta_j \sigma_k'(q_{kj}) \frac{\partial q_{kj}}{\partial \beta_i} \]

\[ + (1 - \alpha) \left( 1 - \sum_{j=1}^{n} \beta_j \right) \sigma_k'(q_{k0}) \frac{\partial q_{k0}}{\partial \beta_i} + \alpha \sigma_k'(q_{k,n+1}) \frac{\partial q_{k,n+1}}{\partial \beta_i}, \quad i, k = 1, \ldots, n. \]  \hspace{1cm} (66)

Using (19), which relates \( \mathcal{S}_k \) to \( Q_k \), we get

\[ \frac{\partial Q_k}{\partial \beta_i} = C_k \left( (1 - \alpha) \left( \sigma_k(q_{ki}) - \sigma_k(q_{k0}) \right) + (1 - \alpha) \sum_{j=1}^{n} \beta_j \sigma_k'(q_{kj}) \frac{\partial q_{kj}}{\partial \beta_i} \right. \]

\[ + (1 - \alpha) \beta_0 \sigma_k'(q_{k0}) \frac{\partial q_{k0}}{\partial \beta_i} + \alpha \sigma_k'(q_{k,n+1}) \frac{\partial q_{k,n+1}}{\partial \beta_i} \left( \right), \quad i, k = 1, \ldots, n. \]  \hspace{1cm} (67)

We are going to substitute the aggregate for individual demands in Equation (67). According to (22),

\[ \frac{\partial q_{k0}}{\partial \beta_i} = \frac{1}{\mathcal{L}} \frac{\partial Q_k}{\partial \beta_i}. \]  \hspace{1cm} (68)
Similarly, from (21) we obtain that
\[
\frac{\partial q_{kj}}{\partial \beta_i} = \frac{1}{\mathcal{L}} \left( \frac{\bar{S}_j + 1}{\bar{S}_j} \frac{\partial Q_k}{\partial \beta_i} - \frac{Q_k}{\bar{S}_j} \frac{\partial \bar{S}_j}{\partial \beta_i} \right) = \frac{1}{\mathcal{L}} \left( \frac{\bar{S}_j + 1}{\bar{S}_j} \frac{\partial Q_k}{\partial \beta_i} - \frac{Q_k}{C_j \bar{S}_j^2} \frac{\partial Q_j}{\partial \beta_i} \right). \quad (69)
\]

According to (23),
\[
\frac{\partial q_{k,n+1}}{\partial \beta_i} = \frac{\alpha}{L_{n+1}} \frac{\partial Q_k}{\partial \beta_i} - \frac{\alpha Q_k}{L_{n+1}^2} \frac{\partial L_{n+1}}{\partial \beta_i}.
\]

The derivative of \(L_{n+1}\) is computed with (26), Proposition 1
\[
\frac{\partial L_{n+1}}{\partial \beta_i} = \mathcal{L} (1 - \alpha) \left( \frac{1}{\bar{S}_i + 1} - \sum_{j=1}^{n} \frac{\beta_j}{(\bar{S}_j + 1)^2} \frac{\partial \bar{S}_j}{\partial \beta_i} \right) = \mathcal{L} (1 - \alpha) \left( \frac{1}{\bar{S}_i + 1} - \sum_{j=1}^{n} \frac{\beta_j}{C_j (\bar{S}_j + 1)^2} \frac{\partial \bar{S}_j}{\partial \beta_i} \right). \quad (71)
\]

Combining the last equation and Equation (70) we get:
\[
\frac{\partial q_{k,n+1}}{\partial \beta_i} = \frac{\alpha}{L_{n+1}} \frac{\partial Q_k}{\partial \beta_i} - \frac{\alpha(1 - \alpha)Q_k}{L_{n+1}^2} \mathcal{L} \left( \frac{1}{\bar{S}_i + 1} - \sum_{j=1}^{n} \frac{\beta_j}{C_j (\bar{S}_j + 1)^2} \frac{\partial \bar{S}_j}{\partial \beta_i} \right). \quad (72)
\]

Substituting Equations (68), (69) and (72) into (67), we have
\[
C_k^{-1} \frac{\partial Q_k}{\partial \beta_i} = (1 - \alpha)(\sigma_k(q_{ki}) - \sigma_k(q_{k0})) + \frac{1 - \alpha}{\mathcal{L}} \sum_{j=1}^{n} \frac{\beta_j}{\bar{S}_j} \frac{\partial \bar{S}_j}{\partial \beta_i} \left( \frac{\bar{S}_j + 1}{\bar{S}_j} \frac{\partial Q_k}{\partial \beta_i} - \frac{Q_k}{C_j \bar{S}_j^2} \frac{\partial Q_j}{\partial \beta_i} \right) + \frac{1 - \alpha}{\mathcal{L}} \frac{\alpha}{\beta_0} \frac{\partial Q_k}{\partial \beta_i} + \alpha \sigma'_k(q_{k,n+1}) \left( \frac{\alpha}{L_{n+1}} \frac{\partial Q_k}{\partial \beta_i} - \frac{\alpha(1 - \alpha)Q_k}{L_{n+1}^2} \mathcal{L} \left( \frac{1}{\bar{S}_i + 1} - \sum_{j=1}^{n} \frac{\beta_j}{C_j (\bar{S}_j + 1)^2} \frac{\partial \bar{S}_j}{\partial \beta_i} \right) \right). \quad (73)
\]

Extracting the terms containing \(\frac{\partial q_k}{\partial \beta_i}\) we have
\[
C_k^{-1} \frac{\partial Q_k}{\partial \beta_i} = (1 - \alpha)(\sigma_k(q_{ki}) - \sigma_k(q_{k0})) + \frac{(1 - \alpha)\beta_k}{\mathcal{L}} \sigma'_k(q_{kk}) \frac{\bar{S}_k + 1}{\bar{S}_k^2} \frac{\partial Q_k}{\partial \beta_i} + \frac{1 - \alpha}{\mathcal{L}} \sum_{j=1}^{n} \frac{\beta_j}{\bar{S}_j} \frac{\partial \bar{S}_j}{\partial \beta_i} \left( \frac{\bar{S}_j + 1}{\bar{S}_j} \frac{\partial Q_k}{\partial \beta_i} - \frac{Q_k}{C_j \bar{S}_j^2} \frac{\partial Q_j}{\partial \beta_i} \right) + \frac{1 - \alpha}{\mathcal{L}} \frac{\alpha}{\beta_0} \sigma'_k(q_{k0}) \frac{\partial Q_k}{\partial \beta_i} + \frac{\alpha^2}{L_{n+1}} \sigma'_k(q_{k,n+1}) \frac{\partial Q_k}{\partial \beta_i} + \frac{\alpha^2(1 - \alpha)Q_k \mathcal{L}}{L_{n+1}^2} \sigma'_k(q_{k,n+1}) \left( \frac{\bar{S}_i + 1}{\bar{S}_i} \frac{\partial Q_k}{\partial \beta_i} - \frac{Q_k}{C_j \bar{S}_j^2} \frac{\partial Q_j}{\partial \beta_i} \right) + \frac{\alpha^2(1 - \alpha)Q_k \mathcal{L}}{L_{n+1}^2} \sigma'_k(q_{k,n+1}) \sum_{j=1}^{n} \frac{\beta_j}{C_j (\bar{S}_j + 1)^2} \frac{\partial Q_j}{\partial \beta_i}. \quad (74)
\]
Grouping terms in this equation, we get

\[
\begin{pmatrix}
C_k^{-1} - \frac{1 - \alpha}{\mathcal{L}} \left( \sum_{j=1, j \neq k}^{n} \beta_j \frac{\sigma'_k(q_{kj})(\mathcal{G}_j + 1)}{\mathcal{G}_j} + \beta_k \sigma'_k(q_{kk}) \frac{(\mathcal{G}_k + 1)^2}{\mathcal{G}_k^2} + \beta_0 \sigma'_k(q_{k0}) \right)
\end{pmatrix}
\]

\[
\quad
- \frac{\alpha^2 \sigma'_k(q_{k, n+1})}{L_{n+1}} \left( 1 + \frac{(1 - \alpha) \mathcal{L} Q_k}{L_{n+1} C_k (\mathcal{G}_k + 1)^2} \right) \frac{\partial Q_k}{\partial \beta_i}
\]

\[
+ \sum_{j=1, j \neq k}^{n} \left( (1 - \alpha) \beta_j \sigma'_k(q_{kj}) \frac{Q_k}{\mathcal{G}_j \mathcal{G}_j} - \frac{\alpha^2 (1 - \alpha) \mathcal{L} \sigma'_k(q_{k, n+1}) Q_k}{L_{n+1}^2 C_j (1 + \mathcal{G}_j)^2} \right) \frac{\partial Q_j}{\partial \beta_i}
\]

\[
= (1 - \alpha) \left( \sigma_k(q_{ki}) - \sigma_k(q_{k0}) - \frac{\alpha^2 \mathcal{L} \sigma'_k(q_{k, n+1}) Q_k}{L_{n+1}^2 (1 + \mathcal{G}_i)} \right), \quad k = 1, \ldots, n. \quad (75)
\]

Introducing \(a_{kk}, a_{kj},\) and \(b_k,\) as shown in the last formula, we get the system of the linear equations

\[
A \begin{pmatrix}
\frac{\partial Q_1}{\partial \beta_i} \\
\vdots \\
\frac{\partial Q_n}{\partial \beta_i}
\end{pmatrix} = b, \quad (76)
\]

where \(A = (a_{kj})_{k,j=1}^{n}, b = (b_1, \ldots, b_n)^T.\) Thus, System (75) of linear equations with respect to \(\partial Q_k/\partial \beta_i\) is found.

At the next step we are going to apply Lemma 23. First, we estimate the diagonal matrix elements \(a_{kk}\)

\[
a_{kk} = C_k^{-1} - (1 - \alpha) \left( \sum_{j=1, j \neq k}^{n} \beta_j \frac{\sigma'_k(q_{kj})(\mathcal{G}_j + 1)}{\mathcal{G}_j} + \beta_k \sigma'_k(q_{kk}) \frac{(\mathcal{G}_k + 1)^2}{\mathcal{G}_k^2} + \beta_0 \sigma'_k(q_{k0}) \right)
\]

\[
- \frac{\alpha^2 \sigma'_k(q_{k, n+1})}{L_{n+1}} \left( 1 + \frac{(1 - \alpha) \mathcal{L} Q_k}{L_{n+1} C_k (\mathcal{G}_k + 1)^2} \right) \quad (77)
\]

in Equations (76). We note that all the terms subtracted from \(C_k^{-1}\) in \(a_{kk}\) follow the sign of \(\sigma'_k.\) If \(\sigma'_k \leq 0\) for all \(k = 0, \ldots, n,\) then

\[
a_{kk} \geq C_k^{-1}. \quad (78)
\]

We claim that if \(\sigma'_k > 0,\) then the term \(C_k^{-1}\) constitutes the main part of \(a_{kk}.\) To be precise, we prove that \(a_{kk} > (2C_k)^{-1}.\) To establish this, we evaluate the subtracting terms one by one. According to Assumption 2 and (25), that is, using \(|\sigma'_k(x)| < L_j/(2C_k)\) and

\[
\mathcal{L} = \frac{L_j (\mathcal{G}_j + 1)}{(1 - \alpha) \beta j \mathcal{G}_j}, \quad (79)
\]
we have
\[
\frac{\beta_j \sigma'_k(q_{kj})(\mathcal{G}_j + 1)}{L \mathcal{G}_j} < \frac{\beta_j \sigma'_k(q_{kj})(\mathcal{G}_j + 1)(1 - \alpha)\beta_j \mathcal{G}_j}{L_j (\mathcal{G}_j + 1) \mathcal{G}_j} < \frac{(1 - \alpha)\beta_j^2 \sigma'_k(q_{kj})}{L_j} < \frac{\beta_j}{2C_k} \quad \text{for } j = 1, \ldots, n \text{ and } j \neq k. \tag{80}
\]

Further, using (79) again, we obtain:
\[
\beta_k \sigma'_k(q_{kk}) \frac{(\mathcal{G}_k + 1)^2}{L \mathcal{G}_k^2} < \frac{\beta_k \sigma'_k(q_{kk})}{L_k} \left(1 + \frac{1}{\mathcal{G}_k}\right) < \frac{\beta_k}{2C_k}. \tag{81}
\]

Since \( \mathcal{G}_k > 1 \) it follows that
\[
1 + \frac{1}{\mathcal{G}_k} < 2.
\]

Applying assumption (30) in the form \( |\sigma'_k(\mathcal{G})| < L_k/(4C_k) \), we get
\[
\beta_k \sigma'_k(q_{kk}) \frac{(\mathcal{G}_k + 1)^2}{L \mathcal{G}_k^2} < \frac{\beta_k \sigma'_k(q_{kk})}{L_k} \left(1 + \frac{1}{\mathcal{G}_k}\right) < \frac{\beta_k}{2C_k}. \tag{81}
\]

From Assumption 2 it follows that
\[
\frac{\beta_0 \sigma'_k(q_{ko})}{L} < \frac{\beta_0 \sigma'_k(q_{ko})}{L_k} < \frac{\beta_0}{2C_k}. \tag{82}
\]

Combining (80), (81) and (82), we have that
\[
(1 - \alpha) \sum_{j=1\atop j \neq k}^n \frac{\beta_j \sigma'_k(q_{kj})(\mathcal{G}_j + 1)}{L \mathcal{G}_j} + \beta_k \sigma'_k(q_{kk}) \frac{(\mathcal{G}_k + 1)^2}{L \mathcal{G}_k^2} + \beta_0 \sigma'_k(q_{ko}) \frac{1}{L} < \frac{1}{2C_k} \sum_{j=0}^n \beta_j < \frac{1 - \alpha}{2C_k}. \tag{83}
\]

Now we are going to prove that
\[
\frac{\alpha^2 \sigma'_k(q_{kn+1})}{L_{n+1}} \left(1 + \frac{(1 - \alpha)\mathcal{L}Q_k \beta_k}{L_{n+1}C_k(\mathcal{G}_k + 1)^2}\right) < \frac{\alpha}{2C_k}. \tag{84}
\]

Taking into account Assumption 2 and Equation (19) we see that the last inequality follows from
\[
\frac{\alpha L_k}{2C_k L_{n+1}} \left(1 + \frac{(1 - \alpha)\mathcal{L}(\mathcal{G}_k - 1)\beta_k}{L_{n+1}(\mathcal{G}_k + 1)^2}\right) < \frac{1}{2C_k}
\]
or from
\[
L_k \left(\alpha + \frac{(1 - \alpha)\mathcal{L} \beta_k}{L_{n+1}(\mathcal{G}_k + 1)}\right) < L_{n+1}.
\]

According to (26),
\[
L_{n+1} > \alpha \mathcal{L}. \tag{85}
\]

Substituting (26) to the right-hand side of the last inequality we get the evident estimate
\[
L_k \left(\alpha + (1 - \alpha) \frac{\beta_k}{\mathcal{G}_k + 1}\right) < \mathcal{L} \left(\alpha + (1 - \alpha) \sum_{j=1}^n \frac{\beta_j}{\mathcal{G}_j + 1}\right)
\]
that proves (84). Now the element \( a_{kk} \) given by (77) is estimated with Inequalities (83) and (84):

\[
a_{kk} > \frac{1}{C_k} - \frac{1}{2C_k} - \frac{\alpha}{2C_k} = \frac{1}{2C_k}.
\]

Second, we estimate the non-diagonal elements \( a_{kj} \) by the diagonal element \( a_{jj} \). Namely, we will prove that

\[
|a_{kj}| < \frac{1}{2} a_{jj} \gamma \frac{\varsigma_k}{\varsigma_j},
\]

with an appropriate choice of positive \( \gamma, \varsigma_k, \) and \( \varsigma_j \). We are going to use \( \gamma \) given by (62) and put

\[
\varsigma_j = \frac{\varsigma_j - 1}{M_j}.
\]

By (19) and (62),

\[
\left| (1 - \alpha) \beta_j \sigma_k'(q_{kj}) \frac{Q_k}{C_j L \varsigma_j^2} \right| = (1 - \alpha) \frac{|\sigma_k'(q_{kj})| C_k (\varsigma_j - 1) \beta_j (\varsigma_j - 1)}{L (\varsigma_j - 1)^2} \frac{1}{C_j} \\
< (1 - \alpha) \frac{|\sigma_k'(q_{kj})| C_k M_k (\varsigma_j - 1) M_j \frac{1}{C_j}}{(\varsigma_j - 1) M_k C_j} = (1 - \alpha) \gamma \frac{\varsigma_k}{\varsigma_j} \frac{1}{C_j}.
\]

Dealing with the second term in \( a_{kj} \), we estimate

\[
\left( \frac{L_{n+1}}{L} \right)^2 \geq \left( \alpha + (1 - \alpha) \frac{\beta_j}{\varsigma_j + 1} \right)^2 \geq 4 \alpha (1 - \alpha) \frac{\beta_j}{\varsigma_j + 1}
\]

with Equation (26) and the elementary inequality \((a + b)^2 \geq 4ab\). Given \( \varsigma_k \) \((k = 1, \ldots, n, \) see (88)), \( \gamma, \) and the last inequality, the second term in \( a_{kj} \) is estimated with Equation (19) in the following way:

\[
\frac{\alpha^2 (1 - \alpha) L \sigma_k'(q_{k,n+1}) Q_k}{L_{n+1}^2} \frac{\beta_j}{C_j(1 + \varsigma_j)^2} \leq \frac{|\sigma_k'(q_{k,n+1})| C_k M_k (\varsigma_k - 1) M_j \frac{1}{C_j}}{(\varsigma_k - 1) M_k C_j} \frac{\beta_j}{4(1 + \varsigma_j) C_j} \leq \alpha \frac{|\sigma_k'(q_{k,n+1})| C_k M_k (\varsigma_k - 1) M_j \frac{1}{C_j}}{(\varsigma_k - 1) M_k C_j} \frac{1}{C_j} \frac{\beta_j}{\varsigma_j} \frac{1}{C_j}.
\]

Now Inequality (87) is a consequence of Inequalities (89), (90), (78), and (86).

We turn to the right hand side of Equation (75). By Lemma 22,

\[
|\sigma_k(q_{ki}) - \sigma_k(q_{k0}) - \alpha^2 L \sigma_k'(q_{k,n+1}) Q_k| \geq \frac{1}{2} |(\sigma_k(q_{ki}) - \sigma_k(q_{k0}))|.
\]

Equations (21)–(22) imply that for some \( \zeta, \zeta' \in (0, 1) \)

\[
\frac{\sigma_k(q_{ki}) - \sigma_k(q_{k0})}{\sigma_{k'}(q_{k'i}) - \sigma_{k'}(q_{k'0})} = \frac{\sigma_k'(\zeta q_{ki} + (1 - \zeta) q_{k0}) \left( \frac{(\varsigma_{k,i} + 1) Q_{k}}{C_{k,i}} - \frac{Q_k}{C_k} \right)}{\sigma_{k'}'(\zeta' q_{k'i} + (1 - \zeta') q_{k'0}) \left( \frac{(\varsigma_{k',i} + 1) Q_{k'}}{C_{k',i}} - \frac{Q_k}{C_k'} \right)}
\]

\[
= \frac{\sigma_k'(\zeta q_{ki} + (1 - \zeta) q_{k0}) Q_k}{\sigma_{k'}'(\zeta' q_{k'i} + (1 - \zeta') q_{k'0}) Q_k}.
\]
Then
\[ \left| \frac{b_k}{b_{k'}} \right| < \frac{2\sigma'_k(\zeta q_{k_i} + (1 - \zeta)q_{k_0})Q_k}{\sigma'_{k'}(\zeta' q_{k'i} + (1 - \zeta')q_{k'0})Q_{k'}}. \]

If \( \sigma'_k > 0 \) for all \( k \), we continue in the following way:
\[ \frac{b_k a_{k'j}}{b_{k'} a_{jj}} < \frac{2\sigma'_k(\zeta q_{k_i} + (1 - \zeta)q_{k_0})Q_k}{\sigma'_{k'}(\zeta' q_{k'i} + (1 - \zeta')q_{k'0})Q_{k'}} \frac{\sigma'_{k'}(q_{k'j})Q_{k'}}{L_j \Theta_j Q_j a_{jj}} \]

(91)

By the definition of the constant \( B \),
\[ \left| \frac{\sigma'_{k'}(q_{k'j})}{\sigma'_{k'}(\zeta' q_{k'i} + (1 - \zeta')q_{k'0})} \right| < B. \]

Using definitions (62) and (88) of \( \gamma \) and \( \kappa \), we claim that
\[ \left| \frac{2\sigma'_k(\zeta q_{k_i} + (1 - \zeta)q_{k_0})}{L_j} \right| < \gamma \frac{c_k^v \kappa}{2c_k^v \kappa_j} \]

By (86) and the last two inequalities, Inequality (91) is transformed into
\[ \frac{b_k a_{k'j}}{b_{k'} a_{jj}} < \frac{2B\gamma \kappa}{\Theta_j \kappa_j} < 2B\gamma \frac{\kappa}{\kappa_j}. \]

(92)

Now let \( \sigma'_k < 0 \) for all \( k = 1 \ldots, n \). Then we use \( \gamma \) defined in Equation (62) and also end up with Inequality (92). As above, the estimates involve (64) but not (63).

Applying Lemma 23 with
\[ \frac{1}{2} \left| \sigma_k(q_{ki}) - \sigma_k(q_{k0}) \right| < |b_{kk}| < |\sigma_k(q_{ki}) - \sigma_k(q_{k0})| \]

and
\[ \frac{1}{2C_k^v} < a_{kk} < \frac{3}{2C_k^v}, \]

we conclude that the derivatives \( \partial Q_k/\partial \beta_i \) (\( k = 1 \ldots, n \), \( i \) is fixed) follow the sign of the differences \( \sigma_k(q_{ki}) - \sigma_k(q_{k0}) \) and satisfy Equation (65).

Lemma 10.
\[ \frac{\partial N_i}{\partial \beta_i} = (1 - \alpha) \frac{L}{c_i^v (\Sigma_i + 1)} \left( 1 - \frac{\beta_i K_{ij}}{\Sigma_i + 1} (\sigma_i(q_{ii}) - \sigma_i(q_{i0})) \right), \]

and for \( k \neq i \)
\[ \frac{\partial N_k}{\partial \beta_i} = -\frac{(1 - \alpha) L \beta_k}{c_i^v (\Sigma_i + 1)^2} K_{ki} (\sigma_k(q_{ki}) - \sigma_k(q_{k0})), \]

In particular, \( \frac{\partial N_i}{\partial \beta_i} > 0 \), and the sign of \( \frac{\partial N_k}{\partial \beta_i} \) for \( k \neq i \) is opposite to that of \( \sigma'_k \).

Proof. From (28) it follows that
\[ \frac{\partial N_i}{\partial \beta_i} = (1 - \alpha) \frac{L}{c_i^v (\Sigma_i + 1)} \left( 1 + \frac{\beta_i}{\Sigma_i + 1} \frac{\partial \Sigma_i}{\partial \beta_i} \right), \]

and for \( k \neq i \)
\[ \frac{\partial N_k}{\partial \beta_i} = (1 - \alpha) L \beta_k \frac{\partial \Sigma_k}{\partial \beta_i}. \]

The proof follows from (65) of Lemma 9 and Assumption 2.
Next, from (28) it follows that
\[
\frac{N_k}{\beta_k L} = \frac{(1 - \alpha)}{c_k^v (\mathfrak{S}_k + 1)},
\]
and we get

**Lemma 11.**
\[
\frac{\partial}{\partial \beta_i} \left( \frac{N_k}{\beta_k L} \right) = -\frac{(1 - \alpha)}{c_k^v (\mathfrak{S}_k + 1)} K_{ki} (\sigma_k(q_i) - \sigma_k(q_{i0})).
\]
So, \( \frac{N_k}{\beta_k L} \) decreases if \( \sigma_k \) is increasing function and \( \frac{N_k}{\beta_k L} \) increases if \( \sigma_k \) is decreasing function.

To describe the change of sector output \( Q_k N_k \) let us first notice that
\[
Q_k N_k = \frac{(1 - \alpha) L \beta_k \mathfrak{S}_k - 1}{c_k^v \mathfrak{S}_k + 1} = \frac{(1 - \alpha) L \beta_k}{c_k^v (\mathfrak{S}_k + 1)^2} \left( 1 - \frac{2}{\mathfrak{S}_k + 1} \right).
\]

**Lemma 12.** The change of sector output is described by the following formulae
\[
\frac{\partial (Q_k N_k)}{\partial \beta_i} = 2(1 - \alpha) L \beta_k \frac{\partial \mathfrak{S}_i}{\partial \beta_i} + 2(1 - \alpha) L \beta_i \frac{\partial \mathfrak{S}_k}{\partial \beta_i}.
\]

for \( k \neq i \) and
\[
\frac{\partial (Q_i N_i)}{\partial \beta_i} = \frac{Q_i N_i}{\beta_i} \left( 1 + \frac{\beta_i C_i K_{ii}}{L \mathfrak{S}_i (\mathfrak{S}_i + 1)} \sigma'_i(\mathfrak{x}) \right).
\]

**Proof.** The first formula of the lemma is evident. To prove the second one we can find
\[
\frac{\partial (Q_i N_i)}{\partial \beta_i} = \frac{(1 - \alpha) L \mathfrak{S}_i - 1}{c_i^v \mathfrak{S}_i + 1} + \frac{2(1 - \alpha) L \beta_i \mathfrak{S}_i}{c_i^v (\mathfrak{S}_i + 1)^2} \frac{\partial \mathfrak{S}_i}{\partial \beta_i}.
\]

As for some \( \mathfrak{x} \in (q_{i0}, q_{ii}) \)
\[
\frac{\partial \mathfrak{S}_i}{\partial \beta_i} = K_{ii} (\sigma_i(q_{ii}) - \sigma_i(q_{i0})) = K_{ii} \sigma'_i(\mathfrak{x}) (q_{ii} - q_{i0}) = K_{ii} \sigma'_i(\mathfrak{x}) \frac{Q_i}{L \mathfrak{S}_i} = C_i K_{ii} \sigma'_i(\mathfrak{x}) \frac{\mathfrak{S}_i - 1}{L \mathfrak{S}_i}
\]
\[
\frac{\partial (Q_i N_i)}{\partial \beta_i} = \frac{(1 - \alpha) L (\mathfrak{S}_i - 1)}{c_i^v (\mathfrak{S}_i + 1)} \left( 1 + \frac{\beta_i C_i K_{ii}}{L \mathfrak{S}_i (\mathfrak{S}_i + 1)} \sigma'_i(\mathfrak{x}) \right).
\]

Let us rewrite Equation (25) for \( L_k \) in the form
\[
L_k = (1 - \alpha) \beta_k L \left( 1 - \frac{1}{\mathfrak{S}_i + 1} \right).
\]

**Lemma 13.** For \( k \neq i \)
\[
\frac{\partial L_k}{\partial \beta_i} = \frac{(1 - \alpha) L \beta_k \partial \mathfrak{S}_k}{(\mathfrak{S}_k + 1)^2} \frac{\partial \mathfrak{S}_i}{\partial \beta_i} \tag{93}
\]
and
\[
\frac{\partial L_i}{\partial \beta_i} = \frac{L_i}{\beta_i} \left( 1 + \frac{\beta_i K_{ii} C_i (\mathfrak{S}_i - 1)}{L \mathfrak{S}_i (\mathfrak{S}_i + 1) \mathfrak{S}_i^2} \sigma'_i(\mathfrak{x}) \right), \tag{94}
\]
for some \( \mathfrak{x} \in (q_{i0}, q_{ii}) \). So, \( \frac{\partial L_k}{\partial \beta_i} \) follows the sign of \( \frac{\partial Q_k}{\partial \beta_i} \) and \( \frac{\partial L_i}{\partial \beta_i} > 0 \).
Proof. The formula (93) is straightforward while the formula (94) can be obtained by the same reasoning as in lemma 12. Moreover, according to assumption 2
\[
\frac{\beta_i K_i C_i (\xi_i - 1) |\sigma_i'(\lambda)|}{\mathcal{L}(\xi_i + 1) \xi_i^2} < \frac{K_i C_i L_i}{\mathcal{L} 2C_i} < 1.
\]
The Lemma is proved. \(\square\)

According to (20) and (24)
\[
p_i = c_i^\nu w_0 \frac{\xi_i + 1}{\xi_i - 1} = p_i = c_i^\nu w_0 \left(1 + \frac{2}{\xi_i - 1}\right)
\]
Lemma 14.
\[
\frac{\partial p_k}{\partial \beta_i} = -\frac{2c_i^\nu w_0}{(\xi_k - 1)^2} \frac{\partial \xi_k}{\partial \beta_i}
\]
From (71) we can get the following
Lemma 15. If \(\sigma_i\) is a decreasing function then
\[
\frac{\partial L_n+1}{\partial \beta_i} > 0.
\]
\(\text{If } \sigma_i \text{ is an increasing function, we assume additionally that the quantity } \xi^* \text{ defined in (31) satisfies the following condition}
\[
\xi^* < \frac{16}{1 - \beta_0} - 1.
\]
Then the inequality (95) is valid.
Proof. From (71) if follows that
\[
\frac{\partial L_n+1}{\partial \beta_i} = \mathcal{L}(1 - \alpha) \left(\frac{1}{\xi_i + 1} - \sum_{j=1}^{n} \frac{\beta_j}{(\xi_j + 1)^2} \frac{\partial \xi_j}{\partial \beta_i}\right).
\]
So, the derivative \(\frac{\partial L_n+1}{\partial \beta_i} > 0\), if
\[
\sum_{j=1}^{n} \frac{\beta_j}{(\xi_j + 1)^2} \frac{\partial \xi_j}{\partial \beta_i} < \frac{1}{\xi_i + 1},
\]
or
\[
\sum_{j=1}^{n} \frac{\beta_j}{(\xi_j + 1)^2} K_{ji} (\sigma_j(q_{ji}) - \sigma_j(q_{j0})) < \frac{1}{\xi_i + 1};
\]
\[
\sum_{j=1}^{n} \frac{\beta_j}{(\xi_j + 1)^2} K_{ji} \sigma_j'(\lambda) \frac{C_j(\xi_j - 1)}{L_i \xi_j} < \frac{1}{\xi_i + 1}.
\]
According to Assumption 2, it is enough to prove that
\[
\sum_{j=1}^{n} \frac{\beta_j}{(\xi_j + 1)^2} K_{ji} \frac{L_i C_j(\xi_j - 1)}{2C_j L_i \xi_j} < \frac{1}{\xi_i + 1}.
\]
or,
\[
\sum_{j=1}^{n} \beta_j (\mathcal{S}_j - 1) \frac{1}{(\mathcal{S}_j + 1)^2 \mathcal{S}_j} < \frac{1}{\mathcal{S}_i + 1}.
\]  (96)

One can check that
\[
\frac{\mathcal{S}_j - 1}{(\mathcal{S}_j + 1)^2 \mathcal{S}_j} < \frac{1}{16},
\]
for \(\mathcal{S}_j > 1\) so that (96) is valid if
\[
\frac{1}{16} \sum_{j=1}^{n} \beta_j < \frac{1}{\mathcal{S}_i + 1},
\]
or,
\[
\frac{1 - \beta_0}{16} < \frac{1}{\mathcal{S}_i + 1}
\]
and, finally,
\[
\mathcal{S}_i < \frac{16}{1 - \beta_0} - 1,
\]
which proves the Lemma.

\[\square\]

### A.3 Proof of Proposition 3

\textbf{Lemma 16.}

\[
\left(\frac{w_j}{w_0} - 1\right)^{-1} \frac{\partial}{\partial \beta_i} \left(\frac{w_j}{w_0} - 1\right) = -C_j K_{ji} \sigma_j^{\prime}(\varkappa) (\mathcal{S}_j - 1) \frac{\mathcal{L} \mathcal{S}_i \mathcal{S}_j}{\mathcal{S}_j^2 \mathcal{S}_i} \]  (97)

where \(K_{ji} \in (1/3, 2)\) is given in Lemma 9.

\textbf{Proof.} According to (24),
\[
\frac{\partial}{\partial \beta_i} \left(\frac{w_j}{w_0} - 1\right) = \frac{\partial}{\partial \beta_i} \left(1 + \frac{1}{\mathcal{S}_j}\right) = -\frac{1}{\mathcal{S}_j^2} \frac{\partial \mathcal{S}_j}{\partial \beta_i}.
\]

Using Lemma 9, we get
\[
\frac{\partial}{\partial \beta_i} \left(\frac{w_j}{w_0}\right) = -\frac{K_{ji}}{\mathcal{S}_j^2} \sigma_j^{\prime}(\varkappa) \left(\sigma_j(q_ji) - \sigma_j(q_j0)\right), \quad i = 1, \ldots, n.
\]

With the Lagrange difference formula, we continue:
\[
\frac{\partial}{\partial \beta_i} \left(\frac{w_j}{w_0} - 1\right) = -\frac{K_{ji}}{\mathcal{S}_j^2} \sigma_j^{\prime}(\varkappa)(q_ji - q_j0),
\]
for some \(\varkappa \in (q_ji, q_j0)\). Applying Equations (21)-(22) for the individual demands, we get:
\[
\frac{\partial}{\partial \beta_i} \left(\frac{w_j}{w_0} - 1\right) = -\frac{K_{ji}}{\mathcal{S}_j} \frac{\sigma_j^{\prime}(\varkappa) Q_j}{\mathcal{L} \mathcal{S}_j} \frac{1}{\mathcal{S}_j} - C_j K_{ji} \sigma_j^{\prime}(\varkappa) (\mathcal{S}_j - 1) \frac{\mathcal{L} \mathcal{S}_i \mathcal{S}_j}{\mathcal{S}_j^2 \mathcal{S}_i} \left(\frac{w_j}{w_0} - 1\right).
\]

We get Equation (97). \[\square\]
A.4 Justification of examples

Lemma 17. Let the family of utility functions be given by equation

\[
 u(\kappa) = \begin{cases} 
 \frac{A}{2(A-1)}(\kappa(\kappa + 2))^{\frac{A-1}{A}}_2F_1 \left(1, 2 - \frac{1}{A} ; 2 - \frac{1}{A} ; -\frac{\kappa}{2} \right), & \text{if } A > 1, \\
 \ln \left( \kappa + 1 + \sqrt{\kappa^2 + 2}\kappa \right), & \text{if } A = 1,
\end{cases}
\]  

(98)

where \( A \geq 1 \) is a parameter, \( _2F_1(a, b; c; z) \) is a standard notation for hypergeometric functions (see Whittaker and Watson (1990); Abadir (1999) for details). Then the elasticity of substitution is

\[
 \sigma(\kappa) = A \left( 1 + \frac{1}{\kappa + 1} \right), \quad A \geq 1.
\]

Let the family of utility functions be given by equation

\[
 u(\kappa) = \begin{cases} 
 \frac{A}{A-1} \kappa^{1-\frac{1}{A}} \left( 2\kappa + 1 \right)^{1+\frac{A}{A}}_2F_1 \left(1, 2 - \frac{1}{2A} ; 2 - \frac{1}{A} ; -2\kappa \right), & \text{if } A > 1, \\
 2\sqrt{2\kappa + 1} + \ln \frac{\sqrt{2\kappa + 1} - 1}{\sqrt{2\kappa + 1} + 1} & \text{if } A = 1,
\end{cases}
\]  

(99)

where \( A \geq 1 \) is a parameter. Then the elasticity of substitution is

\[
 \sigma(\kappa) = A \left( 2 - \frac{1}{\kappa + 1} \right).
\]

The Proof of the Lemma consists of elementary computations.

Lemma 18. Let

\[
 u(x) = \int \left( \frac{\sqrt{2x + 1}}{x} \right)^{1/A} \, dx, \quad A > 1
\]  

(100)

and

\[
 \mathcal{L} > \max \left\{ \frac{(A + 1)(A(2 - \bar{\delta}) - 1)C_i\bar{\delta}}{A(1 - \bar{\delta})}, \frac{4(2n + \bar{\delta}^2)AC_iM_i}{\bar{\delta}^2} \right\} \quad \text{for any } i, j = 1, \ldots, n.
\]

(101)

We also assume that Assumption 1 is satisfied. Then Propositions 1 and 2 are valid, and

\[
 q_{ij} < \frac{1}{n}.
\]

Comment 2. If \( A > 1 \) then the utility (100) can be expressed in term of hypergeometric functions (99). If \( A = 1 \) then the corresponding utility is given by utility (8).

Proof. A direct computation justifies that

\[
 \sigma(x) = -\frac{u'(x)}{u''(x)x} = A \left( 2 - \frac{1}{x + 1} \right).
\]

The main equation is

\[
 Q_i = C_i(\mathcal{G}_i - 1)
\]
We evaluate the right hand side for the utility in question:

\[ S_i = \sum_{j=0}^{n+1} q_{ij} L_j \sigma_i(q_{ij}) = A \sum_{j=0}^{n+1} \frac{q_{ij} L_j}{Q_i} \left( 2 - \frac{1}{1 + q_{ij}} \right) = A \sum_{j=0}^{n+1} \frac{L_j}{Q_i} \left( 2q_{ij} - 1 + \frac{1}{1 + q_{ij}} \right). \]

We evaluate the obtained sum taking into account that \(1/(1 + \varepsilon) \sim 1 - \varepsilon\). More precisely, given arbitrary \(\delta < 1\), if \(q_{ij} < 1 - \delta\), then

\[ 1 - q_{ij} < \frac{1}{1 + q_{ij}} < 1 - \delta q_{ij}. \] (102)

Returning to \(S_i\), we get

\[ A \sum_{j=0}^{n+1} \frac{L_j q_{ij}}{Q_i} < S_i < A \sum_{j=0}^{n+1} \frac{L_j q_{ij} (2 - \delta)}{Q_i}. \]

Using \(Q_i = \sum_{j=0}^{n+1} L_j q_{ij}\), we continue

\[ A < S_i < A (2 - \delta). \]

In particular

\[ \frac{S_i + 1}{S_i} < \frac{A + 1}{A} \]

as a decreasing function with respect to \(S_i\). According to the main equation \(Q_i = C_i(S_i - 1)\), the output \(Q_i\) satisfies the following double inequality

\[ C_i(A - 1) < Q_i < C_i(A(2 - \delta) - 1). \]

Since

\[ q_{ij} = \frac{Q_i(S_j + 1)}{C_i \mathcal{L} S_j} \quad i, j = 1, \ldots, n \]

and the right hand side of this Equation increases with \(Q_i\), it follows that

\[ q_{ij} < \frac{C_i(A + 1)(A(2 - \delta) - 1)}{A \mathcal{L}}. \] (103)

Then inequalities

\[ \mathcal{L} > \frac{(A + 1)(A(2 - \delta) - 1) C_i \delta}{A(1 - \delta)} \quad \forall i, j = 1, \ldots, n \] (104)

implies that \(q_{ij} < (1 - \delta)/\delta\). It justifies double Inequality (102). Now we are going to check inequalities (63) and (64). They both follow from inequality

\[ \max_{i=1, \ldots, n+1} \max_{x \in [\min q_{ij}, \max q_{ij}]} \max_{\sigma_i} \frac{4|\sigma_i(x)\sigma_i M_i}{\mathcal{L}} \quad \forall i, j = 1, \ldots, n \]

\[ \left( 2n \sigma'(x') \sigma'(x') + 1 \right) < 1. \] (105)
We have already proved that $q_{ij} < (1 - \delta) / \delta$ for $i, j = 1, \ldots, n$. Then

$$\sigma'(x) = \frac{A}{(x + 1)^2} > A\delta^2, \quad \sigma'(x) < A$$

and

$$\frac{\sigma'(x)}{\sigma'(x')} < \frac{1}{\delta^2}.$$  

Therefore Inequality (105) is weaker than

$$\max_{i=1,\ldots,n} 4AC_iM_i \left( \frac{2n}{\delta^2} + 1 \right) < 1.$$  

It is can be re-written as

$$\mathcal{L} > \frac{4(2n + \delta^2)AC_iM_i}{\delta^2} \quad i = 1 \ldots, n.$$  

We choose $\delta = 1/(n + 1)$. When $\delta$ is changed to its largest value value $1/(1 + 1) = 1/2$, the last inequality becomes stronger and turns to:

$$\mathcal{L} > 32(n + 1)AC_iM_i \quad i = 1 \ldots, n \quad (106)$$

With $\delta = 1/(n + 1)$ Inequality (104) is weaker than

$$\mathcal{L} > 2AC_iM_iC_i(n + 1). \quad (107)$$

Inequalities (106) and (107) follow from (101). Thus, Inequalities (63) and (63) are justified.

Finally, Assumption 1 is also valid. Let $x_m$ be the minimal equilibrium individual demand:

$$x_m = \min_{j=0,\ldots,n+1; i=1,\ldots,n} q_{ij}.$$  

It is positive. Put,

$$\delta = \frac{1}{2} \left( \frac{x_m}{(x + 1)^2} + \frac{x_m}{x + 1} \right).$$

This $\delta$ agrees with Assumption 1. \qed

Figure 2 illustrates the behavior of three utilities (with $A$ equalled to 1.5, 2, and 3).

Let us now consider another family of utility examples satisfying main Assumptions

**Lemma 19.** Let

$$u(x) = \int \left( \frac{1}{x(x + 2)} \right)^{1/A} dx, \quad A > 1 \quad (108)$$

and

$$\mathcal{L} > \max \left\{ \left( \frac{2A - 1}{2} \right) C_i \left( A(1 + \delta) + 1 + \delta \right), 4AC_iM_i \left( \frac{2n}{\delta^2} + 1 \right) \right\} \quad \forall i, j = 1, \ldots, n.$$  

We also assume that Assumption 1 is satisfied. Then Propositions 1 and 2 are valid, and

$$q_{ij} < \frac{1 - \delta}{\delta}.$$
Figure 2: The utilities given by (99) with $A = 1.5, 2, \text{ and } 3$; the inset: zoomed black box that corresponds to small values of the argument given in the double-logarithmic scale.

**Comment 3.** If $A = 1$, then the utility is defined by (9). If $A > 2$ then the utility (108) can be written with hypergeometric functions (98).

**Proof.** By direct calculation we can find

$$u'(x) = (x(x + 2))^{-1/2A}, \quad u''(x) = -\frac{1}{A} (x(x + 2))^{-1/2A-1} (x + 1)$$

$$\sigma(x) = -\frac{u'(x)}{u''(x)x} = A \left(1 + \frac{1}{x + 1}\right). \quad (109)$$

We are going to check now that the condition (30) is satisfied. From (109) we can find the following expression for the aggregate elasticity

$$\mathcal{S}_i = \sum_{j=0}^{n+1} \frac{q_{ij} L_j}{Q_i} \sigma_i(q_{ij}) = A \sum_{j=0}^{n+1} \frac{q_{ij} L_j}{Q_i} \left(1 + \frac{1}{1 + q_{ij}}\right) = A + \frac{A}{Q_i} \sum_{j=0}^{n+1} \frac{q_{ij} L_j}{1 + q_{ij}} =$$

$$= A + \frac{A}{Q_i} \sum_{j=0}^{n+1} \frac{(1 + q_{ij}) - 1}{1 + q_{ij}} L_j = A + \frac{A}{Q_i} \left(\mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1 + q_{ij}}\right). \quad (110)$$

Substituting to the demand equation

$$Q_i = C_i(\mathcal{S}_i - 1)$$

we get

$$Q_i = \frac{AC_i}{Q_i} \left(\mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1 + q_{ij}}\right) + C_i(A - 1) \quad (111)$$
Let now $\delta$ is a number satisfying the condition $0 < \delta < 1$ and individual demand

$$q_{ij} < \frac{1 - \delta}{\delta}. \quad (112)$$

Then it is easy to check that

$$1 - q_{ij} < \frac{1}{1 + q_{ij}} < 1 - \delta q_{ij}. \quad (113)$$

Returning to $S_i$, we get step by step

$$\sum_{j=0}^{n+1} L_j (1 - q_{ij}) < \sum_{j=0}^{n+1} \frac{L_j}{1 + q_{ij}} < \sum_{j=0}^{n+1} L_j (1 - \delta q_{ij}),$$

$$\mathcal{L} - \sum_{j=0}^{n+1} L_j (1 - q_{ij}) > \mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1 + q_{ij}} > \mathcal{L} - \sum_{j=0}^{n+1} L_j (1 - \delta q_{ij}),$$

$$\delta \sum_{j=0}^{n+1} q_{ij} L_j < \mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1 - q_{ij}} < \sum_{j=0}^{n+1} q_{ij} L_j,$$

Using $Q_i = \sum_{j=0}^{n+1} L_j q_{ij}$, we continue

$$\delta Q_i < \mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1 - q_{ij}} < Q_i,$$

$$\delta AC_i < \frac{AC_i}{Q_i} \left( \mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1 - q_{ij}} \right) < AC_i,$$

$$AC_i (1 + \delta) - C_i < \frac{AC_i}{Q_i} \left( \mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1 - q_{ij}} + C_i (A - 1) \right) < 2AC_i - C_i.$$

According to (111), we have

$$AC_i (1 + \delta) - C_i < Q_i < 2AC_i - C_i.$$

and

$$A(1 + \delta) < \mathcal{G}_1 < 2A.$$

As $\frac{S_{i+1}}{S_i}$ is decreasing function of $\mathcal{G}_i$, we have the following double inequality

$$\frac{2A + 1}{2A} < \frac{\mathcal{G}_i}{\mathcal{G}_i - 1} < \frac{A(1 + \delta) + 1}{A(1 + \delta)}$$

Next, since

$$q_{ij} = \frac{Q_i (\mathcal{G}_j + 1)}{L\mathcal{G}_j} \quad i, j = 1 \ldots, n$$
we can use the inequalities $S_{i+1} < \frac{A(1+\delta)+1}{A(1+\delta)}$ and $Q_i < 2AC_i - C_i$ for the following relation
\[ q_{ij} < \frac{(2A - 1)C_i(A(1 + \delta) + 1)}{A(1 + \delta)L} \] (114)

To justify (112) we solve the inequality
\[ \frac{(2A - 1)C_i(A(1 + \delta) + 1)}{A(1 + \delta)L} < \frac{1 - \delta}{\delta} \]
so that
\[ L > \frac{(2A - 1)C_i(A(1 + \delta) + 1)}{A(1 + \delta)(1 - \delta)}, \quad \forall i, j = 1, \ldots, n \] (115)

Now we are going to check Inequality (63). It follows from inequality
\[ \max_{i=1, \ldots, n+1} \max_{\kappa \in [\min_{0 \leq j \leq n} q_{ij}, \max_{0 \leq j \leq n+1} q_{ij}]} \frac{4|\sigma'(\kappa)| C_i M_i}{L} \max_{\kappa' \in [\min_{0 \leq j \leq n} q_{kj}, \max_{0 \leq j \leq n} q_{kj}]} \left( \frac{2n}{\sigma'(\kappa')} + 1 \right) < 1. \] (116)

As $q_{ij} < (1 - \delta)/\delta$ for $i, j = 1, \ldots, n$. Then
\[ |\sigma'(\kappa)| = \frac{A}{(\kappa + 1)^2} > A\delta^2, \quad |\sigma'(\kappa)| < A \]
and
\[ \frac{\sigma'(\kappa')}{\sigma'(\kappa'')} < \frac{1}{\delta^2}. \]
Using $|\sigma'(\kappa)| < A$ we get that (116) is weaker than
\[ \max_{i=1, \ldots, n} \frac{4AC_i M_i}{L} \left( \frac{2n}{\delta^2} + 1 \right) < 1 \] (117)
or,
\[ L > 4AC_i M_i \left( \frac{2n}{\delta^2} + 1 \right) \quad i = 1 \ldots, n \]

\[ \Box \]

B Technical Lemmata

This section contains elementary computations that were used above.

**Lemma 20.** The derivative $\partial \mathcal{S}/\partial p$ is written in the following way:
\[ \frac{\partial \mathcal{S}}{\partial p} = \frac{1}{p} \left( \mathcal{S}^2 - \frac{1}{Q} \sum_{j=0}^{n+1} q_j \sigma_j (q_j + \sigma'_j) L_j \right), \] (118)

where $\sigma_j = \sigma_i(q_{ij})$ for arbitrary fixed $i = 1, \ldots, n$. In more details, Equation (118) is transformed into (34)
Proof. By Equations (6) and (5),
\[
\frac{\partial \mathcal{S}}{\partial p} = \frac{\partial}{\partial p} \left( \frac{\mathcal{Q} \mathcal{L}}{Q} \sum_{j=0}^{n+1} q_j \sigma_j L_j \right) = -\frac{1}{Q^2} \left( -\frac{Q \mathcal{S}}{p} \right) \mathcal{Q} - \frac{1}{Q} \sum_{j=0}^{n+1} q_j \sigma_j^2 L_j - \frac{1}{Q} \sum_{j=0}^{n+1} q_j^2 \sigma_j \sigma_j L_j,
\]
where \( \sigma_j = \sigma(q_j) \), \( \sigma_j' = \sigma'(q_j) \). Simplifications leads to Equation (118). Applying again definition (6) and multiplying both the numerator and denominator of the second term by \( Q = \sum_j q_j L_j \), we have
\[
\frac{\partial \mathcal{S}}{\partial p} = \frac{1}{p Q^2} \sum_{j=0}^{n+1} (q_j \sigma_j L_j) \sum_{j'=0}^{n+1} (q_j' \sigma_j' L_j') - \frac{1}{p Q^2} \sum_{j=0}^{n+1} (\sigma_j^2 q_j L_j) \sum_{j'=0}^{n} (q_j' \gamma_j') - \frac{1}{p Q} \sum_{j=0}^{n} \sigma_j' \sigma_j q_j^2 L_j
\]
We combine the first two terms and group the summands with identical indices to get Equation (34)

Lemma 21. Let \( B \) be the larger root of the equation
\[
B^2 - \left( 4 - \frac{1}{y_0} \right) B + 1 + \frac{1}{y_0} = 0, \tag{119}
\]
and \( \delta \) be some positive number. Then for any \( x \) and \( y \) such that
\[
1 + \delta \leq y_0 < y < x < By \tag{120}
\]
the inequality
\[
x^2 - 4xy + y^2 + (1 + \delta)x + (1 + \delta)y < 0 \tag{121}
\]
is valid. In particular, if \( \delta = 0 \) and \( y_0 = 1 \), then inequality (121) follows from the condition
\[
1 < y < x < 2y.
\]

Comment 4. It is worth noting that the larger root of equation (119) depends on \( y_0 \) but not too much. \( B \) increases from 2 to \( 2 + \sqrt{3} \) when \( y_0 \) changes from 1 to \( +\infty \).

Proof. We fix an arbitrary \( y > y_0 \) and find the roots \( x_\pm \) of the left hand side of equation (121):
\[
x_\pm = \frac{4y - (1 + \delta) \pm \sqrt{12y^2 - 12y(1 + \delta) + (1 + \delta)^2}}{2}.
\]
Direct computation gives evidence that the lesser root \( x_- \) is less that \( y \). Then for all \( x \) such that
\[
y < x < \frac{4y - 1 + \sqrt{12y^2 - 12y(1 + \delta) + (1 + \delta)^2}}{2} \tag{122}
\]
inequality (121) is valid. We intend to derive that inequality (122) follows from (120). It is enough to find the maximal $B$ such that the inequality

$$By \leq \frac{4y - (1 + \delta) + \sqrt{12y^2 - 12y(1 + \delta) + (1 + \delta)^2}}{2}$$

is valid for all $y > y_0$. This inequality is equivalent to

$$(2B - 4)y + 1 + \delta < \sqrt{12y^2 - 12y(1 + \delta) + (1 + \delta)^2}$$

and

$$(B^2 - 4B + 1)y^2 + (B + 1)y < 0.$$ 

It should be valid for all $y$. Then the factor $B^2 - 4B + 1$ is positive, and $B$ should be such that the inequality is valid for the smallest available $y$, i.e., $y_0$.

$$B^2 - \left(4 - \frac{1}{y_0}\right)B + 1 + \frac{1}{y_0} < 0.$$ 

The larger root of the corresponding equation (written in (119)) gives the required $B$. If $y_0 = 1 + \delta$ and $\delta = 0$, then the larger root is equal to 2. 

Lemma 22. Let the tax $\alpha$ satisfy the condition

$$\alpha < \frac{1 - \beta_0}{2B\mathcal{G}^*-\mathcal{G}^*-\beta_0}. \quad (123)$$

Then

$$|\sigma'_k(\varkappa)| |q_{ki} - q_{k0}| > 2\frac{\alpha L |\sigma'_k(q_{k,n+1})|Q_k}{L_{n+1}^2(1 + \mathcal{G}_i)}.$$ 

Proof. According to Lagrange’s formula, the statement of the Lemma is equivalent to

$$|\sigma'_k(\varkappa)| |q_{ki} - q_{k0}| > 2\frac{\alpha L |\sigma'_k(q_{k,n+1})|Q_k}{L_{n+1}^2(1 + \mathcal{G}_i)},$$

where $\varkappa \in (q_{k0}, q_{ki})$. Using expressions for $q_{ki}$ and $q_{k0}$ from Proposition 1 we get

$$|\sigma'_k(\varkappa)| \frac{Q_k}{L\mathcal{G}_i} > 2\frac{\alpha L |\sigma'_k(q_{k,n+1})|Q_k}{L_{n+1}^2(1 + \mathcal{G}_i)}.$$ 

Existence of the constant $B$ from (29) let us can conclude that the last inequality follows from

$$\frac{1}{B} > 2\frac{\alpha L^2\mathcal{G}_i}{L_{n+1}^2(1 + \mathcal{G}_i)},$$

or, as $\alpha L < L_{n+1}$, from

$$\frac{1}{B} > 2\frac{\alpha L\mathcal{G}_i}{L_{n+1}(1 + \mathcal{G}_i)}.$$
Using (26) in the form
\[
\frac{L_{n+1}}{L} = \alpha + (1 - \alpha) \sum_{j=1}^{n} \frac{\beta_j}{\mathcal{S}_j + 1}
\]
we rewrite the last condition in the form
\[
\alpha < \frac{1}{2B} \bigg( \frac{\mathcal{S}_i + 1}{\mathcal{S}_i} \bigg) \bigg( \alpha + (1 - \alpha) \sum_{j=1}^{n} \frac{\beta_j}{\mathcal{S}_j + 1} \bigg)
\]
which is equivalent to
\[
\alpha < \frac{1}{2B} \frac{\mathcal{S}_i + 1}{\mathcal{S}_i} \frac{\sum_{j=1}^{n} \beta_j}{1 - \frac{\mathcal{S}_i + 1}{\mathcal{S}_i} \sum_{j=1}^{n} \beta_j}
\]

From the definition (31) of \( \mathcal{S}^* \) and auxiliary inequality
\[
\sum_{j=1}^{n} \frac{\beta_j}{\mathcal{S}_j + 1} < \frac{1}{\mathcal{S}^* + 1} \sum_{j=1}^{n} \beta_j = \frac{1 - \beta_0}{\mathcal{S}^* + 1}
\]
we obtain that the last condition on \( \alpha \) is satisfied if
\[
\alpha < \frac{1}{2B} \frac{\mathcal{S}_i + 1}{\mathcal{S}_i} \frac{1 - \beta_0}{1 - \frac{\mathcal{S}_i + 1}{\mathcal{S}^* + 1}}
\]

Considering the right hand side of the last inequality as a function of \( \mathcal{S}_i = \mathcal{S}^* \) we can conclude that its minimum value is archived at \( \mathcal{S}_i = \mathcal{S}^* \) so that it can be rewritten in the form
\[
\alpha < \frac{1}{2B} \frac{\mathcal{S}_i + 1}{\mathcal{S}_i} \frac{1 - \beta_0}{2B \mathcal{S}^* \left( 1 - \frac{1 - \beta_0}{2B \mathcal{S}^* + 1} \right)}
\]
which is equivalent to (123). \( \square \)

**Lemma 23.** Let
\[
A = \begin{pmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
    a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
    a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{pmatrix}
\]
with \( a_{ii} > 0, \ i = 1, \ldots, n; \)

\[
\tilde{A} = \begin{pmatrix}
    b_{1} & a_{12} & a_{13} & \cdots & a_{1n} \\
    b_{2} & a_{22} & a_{23} & \cdots & a_{2n} \\
    b_{3} & a_{32} & a_{33} & \cdots & a_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    b_{n} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{pmatrix}
= \begin{pmatrix}
    \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \cdots & \tilde{a}_{1n} \\
    \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} & \cdots & \tilde{a}_{2n} \\
    \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} & \cdots & \tilde{a}_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \tilde{a}_{n1} & \tilde{a}_{n2} & \tilde{a}_{n3} & \cdots & \tilde{a}_{nn}
\end{pmatrix}
\]
The system of the linear equations
\[ Ax = b \]
is considered. We assume that first, the right hand side is not dispersed:
\[
|b_{j_1}a_{ij_2}| < |B\gamma x_{j_1}x_{j_2}| \quad j_1 \neq i, \; j_2 \neq i,
\]
and, second, the elements \( a_{ii}, \; i = 1, \ldots, n, \) being positive, dominate the other elements:
\[
|a_{ij}| < \gamma a_{jj}, \quad i \neq j, \quad \gamma < \frac{1}{(2B + 1)n}.
\]
for some positive numbers \( \kappa_1, \ldots, \kappa_n \). Then the sign of each root \( x_i \) follows the sign of \( b_i \), \( i = 1, \ldots, n, \) and
\[
\frac{1}{2} \frac{|b_i|}{a_{ii}} < |x_i| < \frac{3}{2} \frac{|b_i|}{a_{ii}}.
\]

Proof. The roots \( x_1, x_2, \ldots, x_n \) of the equation \( Ax = b \) are determined by Cramer’s rule:
\[
x_1 = \frac{|\tilde{A}|}{|A|}.
\]
For simplicity, we assume that \( b_1 > 0 \). The determinant \( |\tilde{A}| \) is defined as the sum of the terms \( a_{i_1 j_1}a_{i_2 j_2} \ldots a_{i_n j_n} \), where the sequences \( \{i_k\}_{k=1}^n \) and \( \{j_k\}_{k=1}^n \) are permutations of \( \{1, \ldots, n\} \). We are going to establish that the sign of the determinant follows the sign of the main diagonal term:
\[
D_b = b_1a_{22}a_{33} \ldots a_{nn}.
\]
Let \( k \) be an arbitrary integer between 0 and \( k - 2 \), and \( T_k \) be the sum of the terms
\[
\tilde{a}_{i_1 j_1} \ldots \tilde{a}_{i_{n-k} j_{n-k}} \ldots \tilde{a}_{i_{n-k+1} j_{n-k+1}} \ldots \tilde{a}_{i_{n-k} i_{n-k}}, \quad i_l \neq j_l \text{ for } l = 1, \ldots, n - k
\]
that contain merely \( k \) diagonal elements of the matrix \( \tilde{A} \). This \( k \) obviously cannot be equal to \( n - 1 \) because in this case the \( n \)-th multiplier in \( T_{n-1} \), “choosing” its own row and column, ultimately coincides with the last main diagonal element. We reconsider the multipliers in (126) in such a way that the first index (= the row of the matrix element) of the current multiplier coincides with the second index (= the column) of the following multiplier. Given term (126), we define a map \( r \) transforming the first index of each matrix element in (126) into the second index: \( r(i_l) = j_l \). Then starting from \( \tilde{a}_{i_1}r(i_1) \) we select \( \tilde{a}_{r(i_1)}rr(i_1), \tilde{a}_{rr(i_1)}rrr(i_1), \) and so forth. At some step \( \nu \), the sequence of indices \( i_1, r(i_1), rr(i_1), \ldots, \) forms a cycle: the last index coincides with the first index, \( r \ldots r(i_1) = i_1 \). If \( \nu = n - k \), then the non-diagonal multipliers in (126) form
a single cycle. If \( \nu < n - k \), the non-diagonal multipliers form several cycles. We distinguish terms (126) with and without \( b_{i_1} \neq b_1 \). Let \( b_{i_1} \neq b_1 \) be in term (126). Then, by (124),

\[
|b_{i_1}a_{i_2}| < |b_1a_{i_2}| < B\gamma \frac{x_{i_1}}{x_{i_2}},
\]

where \( i_2 = r(1) = rr(i_1) \). Other multipliers in (126) is estimated with (125). Eventually,

\[
|\tilde{a}_{i_1j_1} \cdots \tilde{a}_{i_{n-k}j_{n-k}} \tilde{a}_{i_{n-k+1}j_{n-k+1}} \cdots \tilde{a}_{i_nj_n}| < D_b\gamma^{n-k-1}.
\]

The factors \( x_{i_1}, x_{i_2}, \ldots \), disappear altogether because the indices in (126) are split into cycles. Given indices \( i_{n-k+1}, \ldots, i_n \), the number of different terms (126) is equal to \((n - k - 1)!\) (this number coincides with the number of the terms in the determinant of the \((n - k) \times (n - k)\)-matrix, when the main diagonal has been already chosen). The number of possibilities to choose indices \( i_{n-k+1}, \ldots, i_n \) among numbers \( 2, \ldots, n \) (the first multiplier in the product is chosen among \( b_1, i = 1, \ldots, n \)) is \( \binom{n-1}{k} \). Let \( T_{k1} \) be the sum of all terms (126) with \( b_{i_1}, i_1 \neq 1 \). Then

\[
\sum_{k=0}^{n-2} |T_{k1}| < \sum_{k=0}^{n-2} D_b\gamma^{n-k-1}(n-k-1)! \binom{n-1}{k} = D_b B \sum_{k=0}^{n-2} \gamma^{n-k-1}(n-1)(n-2) \cdots (k+1). \tag{127}
\]

If term (126) contains \( b_1 \) (instead of \( b_{i_1} \) with \( i_1 \neq 1 \)), then

\[
\tilde{a}_{i_1j_1} \cdots \tilde{a}_{i_{n-k}j_{n-k}} \tilde{a}_{i_{n-k+1}j_{n-k+1}} \cdots \tilde{a}_{i_nj_n} < D_b\gamma^{n-k}.
\]

Let \( T_{k2} \) be the sum of all terms (126) with \( b_1 \). The same arguments as above lead to

\[
\sum_{k=0}^{n-2} |T_{k2}| < \sum_{k=0}^{n-2} D_b\gamma^{n-k-1}(n-k-1)! \binom{n-1}{k-1} = D_b \sum_{k=0}^{n-2} \gamma^{n-k} \frac{(n-1)(n-2) \cdots k}{n-k}. \tag{128}
\]

Let \( c = 1/(3B) \). Then combining (127) and (128), we end up with \( T_k = T_{k1} + T_{k2} \) given by the following equation:

\[
\sum_{k=0}^{n-2} |T_k| < D_b \left( \frac{cB}{1-c} + \frac{c^2}{2(1-c)} \right).
\]

If \( B > 2 \), the last inequality can be simplified into

\[
\sum_{k=0}^{n-2} |T_k| < \frac{1}{2} D_b B.
\]

In the same manner one can argue that the determinant \( |A| \) is greater than \( a_{11}a_{22} \cdots a_{nn}/2 \). Then the sign of \( x_1 \) coincides with the sign of \( b_1 \). The same arguments are valid for other variables \( x_2, \ldots, x_n \).
References


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