Prices, schedules, and traveller welfare in multi-service transportation systems

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Abstract

We investigate traveller behaviour in multi-service transportation systems and ascertain how the price and the schedule of the various available services affect traveller choices and welfare. This depends on whether travellers are informed or uninformed about price and schedule upon arrival at the station as well as on their (individual-specific) value of time. Our analysis offers policy-makers a scientifically founded tool to make sensible decisions, based on the correct identification of those who will benefit and those who will lose from policy changes.

Keywords: Travel demand; service scheduling; market segmentation; impact of information on welfare

J.E.L. classification numbers: D01, L91
1 Introduction

Travel behaviour is complex and multi-faceted. Travel demand has mostly a derived nature and its determinants are many, with price being only one of those. Whereas some of the determinants are unrelated to transport operators, others result directly from policy choices, or from decisions made by transport operators. When appraising the performance of a (passenger) transportation system, it is of primary importance to look beyond the sole prices and also consider these other determinants.

Since the first attempts to measure travel demand (McFadden [15]), it has been recognized that two major determinants are (on-vehicle) travel time and service scheduling. Whereas travel time is essentially related to network characteristics and technological choices, and can thus be regarded as a given attribute at least to some extent, service scheduling (hence, frequency) reflects the operator(s)’ decisions nearly in the same fashion as price does. It is thus natural to examine their impact on traveller choices and welfare jointly.

This task is complicated by the fact that, in a number of cases, transportation markets are oligopolies in which travellers are not bound to a single operator but can swing across operators to their best convenience. In any case, the choice to take a given service depends on how the prices differ (along time and possibly across operators) and how the services are scheduled over time (at possibly different prices).

The goal of our paper is to investigate the behaviour of travellers in multi-service transportation systems and assess how the price and/or the schedule of the various services affect their choices and welfare. Because the sequence of available (and differently priced) services matters, the exact time at which travellers arrive at the station of departures is relevant as well, together with the information about prices and schedules. Accordingly, we consider two types of travellers: uninformed and informed.

Uninformed travellers do not know the price and schedule of the available services before reaching the station. Hence, once they are in the station, they can only choose between taking the first next departing service and waiting for the subsequent service. This latter option is attractive only if the subsequent service is sufficiently cheaper to compensate for the disutility associated with the required waiting time.

By contrast, informed travellers know the price and schedule of the available services and decide in advance both when they will reach the station and what exact service they will take. Yet, even the most suitable service, among those that are available, may not exactly depart at their ideal time. When so they will incur a disutility associated with the departure time shift.

Far from adding gratuitous complexity to the model, the introduction of an individual specific value of time plays an important role in our study. The value of time is arguably related to the individual income, hence can be referred to as a reasonably close indicator of traveller earnings. It is also an important determinant of travellers choices, as it reflects
their opportunity cost of waiting.

Not surprisingly, low income travellers tend to privilege relatively cheaper services; high income travellers tend to give more importance to the time schedule rather than to the price. Thus, changes in price and/or time schedule affect differently the various travellers in the population. Even budget-neutral changes, like an increase in service frequency accompanied by an increase in price that compensates for the resulting cost increase, may appear to be welfare-enhancing for some and welfare-downgrading for others. Our detailed model of travel demand offers policy-makers a scientifically founded tool to take informed decisions.

The remainder of the paper is structured as follows. In section 2, we introduce our model. Section 3 and section 4 respectively analyse the demand by uninformed and informed travellers. In these two sections, we also describe the the impact on welfare of marginal changes in price and departure time. Section 5 briefly concludes. All lengthy calculus are relegated to an appendix.

2 The model

Consider an origin-destination pair over which several transportation services are available. Each service \( s \) differs from the others, first, in its departure time \( t \) and, possibly, in its price \( p \). Travellers choose what service to take according to their ideal departure time, denoted \( t^* \), as well as to the price they will have to pay.

The individual demand for transportation services is determined through the maximization of the individual net surplus

\[
S(x, \tilde{p}) = U(x) - \tilde{p}x,
\]

where \( U(x) \) is the gross utility derived from a total of \( x \) travels, \( \tilde{p} = p + \tau \|t^* - t\| \) is the (individual-specific) generalized price, and \( \tau \) is the marginal disutility associated with the departure occurring at a time \( t \neq t^* \).

Because the divergence between the ideal and the true departure time is a part of the generalized price, the individual demand \( x(\tilde{p}) \) depends on the available information about the departure schedule.

Uninformed individuals do not know the schedule before reaching the station. Hence, once they are in the station and learn the schedule, either they take the first available service or, alternatively, they can decide to take a later (but cheaper) service, if available. Thus, for these individuals the difference between the ideal and the true departure time measures the waiting time (henceforth, WT) until departure. Denoting it \( T^W \), and further appending the subscript \( w \) to the marginal disutility, the generalized price of an uninformed individual specifies as \( p + \tau_w T^W \).
Informed individuals know the schedule before reaching the station. They can thus decide to take services departing both earlier and later than their ideal departure time.\(^1\) In their case, the difference between the ideal and the true departure time measures the \textit{departure time shift} (henceforth, DTS). Denoting it \(T^{DS}\), and further appending the subscript \(ds\) to the marginal disutility, the generalized price of an informed individual specifies as \(p + \tau_{ds}T^{DS}\).

For simplicity, we assume that the ideal departure time \(t^*\) is uniformly distributed over the considered time range. We denote \(n\) its density. We further let \(G_w\) and \(G_{ds}\) be the cumulative distribution function, respectively, of the parameters \(\tau_w\) and \(\tau_{ds}\).

3 \hspace{0.5cm} \textbf{Uninformed travellers}

We mentioned that either uninformed travellers take the first available service upon arrival at the station or they opt for the subsequent service. The latter option requires waiting a time interval of \(\Delta T\). Because of this, waiting is not convenient unless it comes along with a price saving, to be denoted \(\Delta p\). Given this saving, an uninformed traveller will decide to wait only if she is sufficiently patient, namely

\[
\tau_w \leq \frac{\Delta p}{\Delta T}.
\]

The attractiveness of a given service depends on how expensive that service is relative to the others. A "cheap" service can attract (patient) travellers, even if the latter reached the station before the departure of the previous service. An "expensive" service can loose some of its potential clients, who may prefer to wait for a subsequent service in order to save some money. Overall, the demand for that and the other services depends on both the pattern of prices and the schedule of departures.

For readability, we focus on situations in which service departures are evenly distributed over time and there are only two levels of price, high \((p_h)\) and low \((p_l)\), so that \(\Delta T\) and \(\Delta p = p_h - p_l\) are both constant. Without loss of generality, we consider a service \(s_1\) that departs at \(t_1\) and is offered at price \(p_1\). Travellers with ideal departure time \(t^* \in [t_1 - 2\Delta T, t_1]\) will allocate across service \(s_1\), the previous service \(s_0\) offered at price \(p_0\), and the subsequent service \(s_2\) offered at price \(p_2\).

There are three possible cases, depending on how the prices of the various services compare. We hereafter describe them, relegating lengthy mathematical details to Appendix A.1.

\(^1\)Informed travellers are typically those facing long-distance journeys (Abrantes and Wardman [1]).
3.1 Case U.1 - \( p_1 = p_l < p_h = p_0 \): some travellers with \( t^* \leq t_0 \) wait for \( s_1 \)

In this case, the demand for \( s_1 \) is greater than the demand for both the previous and the subsequent service. Travellers with \( t^* \in [t_{-1}, t_0] \) and \( \tau_w > \bar{\tau} \) do not find it worth waiting for \( s_1 \), and take \( s_0 \). We can express the demand of those with value of time \( \tau_w \) and their aggregate demand over the relevant values of \( \tau_w \) respectively as

\[
X_0 (\tau_w) = \int_{t_{-1}}^{t_0} x (p_0 + \tau_w \| t_0 - t^* \|) n dt^* \\
\tilde{X}_0 = \int_{\bar{\tau}}^{+\infty} X_0 (\tau_w) dG_W (\tau_w).
\]

This enables us to further express their average value of time as

\[
\bar{\tau}_0 = \int_{\bar{\tau}}^{+\infty} \tau_w \frac{X_0 (\tau_w)}{\tilde{X}_0} dG_W (\tau_w).
\]

Instead, travellers with \( t^* \in [t_{-1}, t_0] \) and \( \tau_w \leq \bar{\tau} \) do find it convenient to wait for \( s_1 \). The demand of those whose value of time is \( \tau_w \) and the aggregate demand over the relevant values of \( \tau_w \) are respectively formulated as

\[
X_1^+ (\tau_w) = \int_{t_{-1}}^{t_0} x (p_1 + \tau_w \| t_1 - t^* \|) n dt^* \\
\tilde{X}_1^+ = \int_{0}^{\bar{\tau}} X_1^+ (\tau_w) dG_W (\tau_w).
\]

Accordingly, the average value of time and the average WT is written as

\[
\bar{\tau}_1^+ = \int_{0}^{\bar{\tau}} \tau_w \frac{X_1^+ (\tau_w)}{\tilde{X}_1^+} dG_W (\tau_w).
\]

Provided that \( p_1 = p_l \leq p_2 \), all travellers with \( t^* \in [t_0, t_{-1}] \) opt for \( s_1 \). The demand of those with a value of time of \( \tau_w \) and the aggregate demand over the relevant values of \( \tau_w \) are now given by

\[
X_1 (\tau_w) = \int_{t_0}^{t_1} x (p_1 + \tau_w \| t_1 - t^* \|) n dt^* \\
\tilde{X}_1 = \int_{0}^{+\infty} X_1 (\tau_w) dG_W (\tau_w),
\]

whereas the average value of time is expressed as

\[
\bar{\tau}_1 = \int_{0}^{+\infty} \tau_w \frac{X_1 (\tau_w)}{\tilde{X}_1} dG_W (\tau_w).
\]
In definitive, we can refer to $X_0$, $X^+_1$ and $\hat{X}_1$ as to the average demand (over all $\tau_w$) of travellers who respectively choose to take $s_0$, prefer to wait for $s_1$ despite they could have taken $s_0$, and take $s_1$ having anyway no interest in waiting longer. As $p_2 \geq p_1$, no traveller with $t^* \in [t_1 - 2\Delta T, t_1]$ chooses to take $s_2$.

3.1.1 The effect of a change in price

We can now look at the impact associated with a change in either $p_0$ or $p_1$. A change in price will affect the distribution of travellers across services. More precisely, a marginal increase in the price saving $p = p_0 - p_1$ that is available to travellers, will induce an increase of $1/\Delta T$ in the cut-off value of the marginal disutility of waiting ($\hat{\tau}$). There are two effects. First, an increase in $p_i$ triggers a decrease in the individual demand of travellers who take service $s_i$. Second, it induces some such travellers to switch to the other service, thus triggering an increase in the demand for the other service. The impact on traveller surplus associated with the concerned services works through these two channels.

Recalling that $S(x, \hat{p}) = U(x) - \hat{p}x$, for any given traveller, we find that

$$\frac{dS(x, \hat{p})}{dp} = (U'(x) - \hat{p}) \frac{dx}{dp} - x = -x.$$ 

This means that, as usual, an infinitesimal increase in price leads to a reduction in individual surplus equal to the demanded quantity. Accordingly, letting $V(\hat{p}) = S(x, \hat{p})$ and using notation along the previous lines, one computes the following aggregate changes:

$$\frac{dS_0}{dp_0} = \frac{d}{dp_0} \left[ \int_{\hat{\tau}}^{+\infty} \left( \int_{t_{l-1}}^{t_0} V(p_0 + \tau_w \| t_0 - t^* \|) \, ndt^* \right) \, dG_W(\tau_w) \right]$$

$$= -\hat{X}_0 - \frac{1}{\Delta T} \left( \int_{t_{l-1}}^{t_0} V(p_0 + \hat{\tau} \| t_0 - t^* \|) \, ndt^* \right) g_W(\hat{\tau}) \tag{2}$$

and

$$\frac{dS_1^+}{dp_0} = \frac{d}{dp_0} \left[ \int_{0}^{\hat{\tau}} \left( \int_{t_{l-1}}^{t_0} V(p_1 + \tau_w \| t_1 - t^* \|) \, ndt^* \right) \, dG_W(\tau_w) \right]$$

$$= \frac{1}{\Delta T} \left( \int_{t_{l-1}}^{t_0} V(p_1 + \hat{\tau} \| t_1 - t^* \|) \, ndt^* \right) g_W(\hat{\tau}) ,$$

whereas $\left( dS_1/dp_0 \right) = 0$. Recalling the definition of $\hat{\tau}$ and observing that $p_0 = p_1 + \Delta p$, it is easy to verify that

$$p_0 + \hat{\tau} \| t_0 - t^* \| = p_1 + \hat{\tau} \| t_1 - t^* \| ,$$

\footnote{All throughout we focus on price changes that do not affect the ranking of prices.}
i.e., an individual with time value of $\hat{\tau}$ faces the same generalized price regardless of whether she takes $s_0$ at a price of $p_0$ or $s_1$ at a price of $p_1$. Being based on this, we can rewrite

$$\frac{dS_1^+}{dp_0} = \frac{1}{\Delta T} \left( \int_{t-1}^{t_0} V (p_0 + \hat{\tau} || t_0 - t^* ||) n dt^* \right) g_W (\hat{\tau}) . \quad (3)$$

The inspection of (2) and (3) is instructive. We see that the effect on surplus of a change in price $p_0$ is twofold. First, it concerns the travellers who still take $s_0$, and is measured by the total number of travels made with that service ($\tilde{X}_0$). Second, it concerns the travellers who are induced to shift from $s_0$ to $s_1$ by the price increase, and is precisely captured by the term that appears in both (2) and (3) with opposite sign. The travellers who were already taking $s_1$ before the price change are not concerned instead. We can thus conclude that a marginal increase in the price of $s_0$ results in a decrease in traveller surplus equal to the demand for that same service:

$$\frac{dS}{dp_0} = \frac{d\tilde{S}_0}{dp_0} + \frac{d\tilde{S}_1^+}{dp_0} + \frac{d\tilde{S}_1}{dp_0}$$

$$= -\tilde{X}_0,$$

with the overall loss in surplus being limited by the availability of service $s_1$, as it attracts some of the travellers who were using $s_0$ prior to the price change.

We now turn to explore the effect on surplus of an increase in price $p_1$. Proceeding along the same lines, we compute

$$\frac{d\tilde{S}_0}{dp_1} = \frac{d}{dp_1} \left[ \int_{\tau}^{+\infty} \left( \int_{t-1}^{t_0} V (p_0 + \tau_w || t_0 - t^* ||) n dt^* \right) dG_W (\tau_w) \right]$$

$$= \frac{1}{\Delta T} \left( \int_{t-1}^{t_0} V (p_0 + \hat{\tau} || t_0 - t^* ||) n dt^* \right) g_W (\hat{\tau}) \quad (4)$$

and

$$\frac{d\tilde{S}_1^+}{dp_1} = \frac{d}{dp_1} \left[ \int_{\tau}^{+\infty} \left( \int_{t-1}^{t_0} V (p_1 + \tau_w || t_1 - t^* ||) n dt^* \right) dG_W (\tau_w) \right]$$

$$= -\tilde{X}_1 + \frac{1}{\Delta T} \left[ \int_{t-1}^{t_0} V (p_0 + \hat{\tau} || t_0 - t^* ||) n dt^* \right] g_W (\hat{\tau}) \quad (5)$$

and

$$\frac{d\tilde{S}_1}{dp_1} = \frac{d}{dp_1} \left[ \int_{0}^{t_1} V (p_1 + \tau_w || t_1 - t^* ||) n dt^* \right] dG_W (\tau_w)$$

$$= -\tilde{X}_1,$$ 

and we point out that the generalized price $p_0 + \hat{\tau} || t_0 - t^* ||$ appears in (5) because it is equal to $p_1 + \tau_w || t_1 - t^* ||$, as already explained. By looking at (4) to (6), it is immediate
to remark that the increase in $p_1$ affects, first, the travellers who still take $s_1$ despite that the service has become more expensive. They face a surplus loss equal to the number of travels made with that service. Specifically, the loss amounts to $X_1^+$ for those who would have preferred to use $s_0$ but opt for $s_1$ because it is cheaper; it amounts to $X_1$ for those who take $s_1$ because it departures at a convenient time. The price increase also concerns the travellers who are induced to shift from $s_1$ to $s_0$ because the price saving granted by service $s_1$ is now less attractive relative to a earlier departure. Again, this impact on surplus is compensated at the aggregate level, and the overall impact on surplus amounts to

$$\frac{dS}{dp_1} = - \left( X_1^+ + X_1 \right).$$

### 3.1.2 The effect of a change in schedule

Of course, any variation in the service schedule will affect traveller surplus as well. For any given traveller with ideal departure time $t^* \leq \inf \{t, t + dt\}$, the impact on surplus of a change of $dt$ in the departure time amounts to

$$\frac{dS(x, \tilde{p})}{dt} = \left[ (U'(x) - \tilde{p}) \frac{dx}{d\tilde{p}} - x \right] \frac{d\tilde{p}}{dt} = -\tau_w x,$$

meaning that the individual surplus is reduced by the marginal disutility of waiting all over the demanded travels. More generally, a variation in the schedule entails a change in the time interval $\Delta T$ between two subsequent departures, and triggers changes in the distribution of travellers across services. Specifically, a marginal increase in $\Delta T$, to be denoted $dT$, raises the cost of delaying departure and, hence, it reduces the cutoff value $\tilde{r}$. One can verify that

$$\frac{d\tilde{r}}{\tilde{r}} = - \frac{dT}{\Delta T},$$

meaning that the elasticity of $\tilde{r}$ with respect to $\Delta T$ is constant and equal to $1$. By contrast, those who choose a more expensive service have a relatively high cost of waiting. To develop the analysis, we proceed as follows. We consider an increase in the time interval between the departure time of service $s_0$ and the departure times of the services that are scheduled immediately before and immediately after $s_0$. In so doing, we keep the number of travellers constant, to avoid dealing with pure "volume" effects. Formally, we fix $t_0$ and let $t_{-1}$ decrease by $dT$, $t_1$ increase by $dT$, and $t_2$ by $2dT$. Furthermore, we cut out of the surplus variation the terms that are due to the 2nd $dT$ additional travellers whose ideal departure time coincides with any of the two extremes of the time interval (namely, those with $t^* \in [t_{-1} - dT; t_{-1}]$ and $t^* \in [t_1; t_1 + dT]$). This approach will be maintained all throughout.

With the departure time $t_0$ being unchanged, the travellers who were already using $s_0$ are not affected by the change in service schedule. The only effect on the surplus
associated with \( s_0 \) is related to the additional number of travellers taking now the service:

\[
\frac{dS_0}{dT} = \frac{\hat{\tau}}{\Delta T} \left( \int_{t_{-1}}^{t_0} V \left( p_0 + \hat{\tau} \|t^* - t_0\| \right) n dt^* \right) \frac{g_W(\hat{\tau})}{\Delta T}.
\]

As the departure of \( s_1 \) is postponed, the travellers with a value of time of \( b \) shift to the previous service, whereas those who stick to \( s_1 \) all wait longer. There is thus a loss in the surplus associated with \( s_1 \); which is expressed as

\[
\frac{dS_1^+}{dT} = -\tau_1^+ X_1^+ - \frac{\hat{\tau}}{\Delta T} \left( \int_{t_{-1}}^{t_0} V \left( p_1 + \hat{\tau} \|t_1 - t^*\| \right) n dt^* \right) \frac{g_W(\hat{\tau})}{\Delta T}.
\]

\[
\frac{dS_1^-}{dT} = -\tau_1^- X_1^-.
\]

Considering again that \( p_0 + \hat{\tau} \|t^* - t_0\| = p_1 + \hat{\tau} \|t_1 - t^*\| \), the aggregate impact of a variation in service schedule amounts to

\[
\frac{dS}{dT} = \frac{dS_0}{dT} + \frac{dS_1^+}{dT} + \frac{dS_1^-}{dT} = -\left( \tau_1^+ X_1^+ + \tau_1^- X_1^- \right).
\]  \( (7) \)

With \( s_1 \) being cheaper than \( s_0 \), not only does the postponement of \( s_1 \) affect travellers whose ideal departure time belongs to the time interval \([t_0; t_1]\). It also affects some of the travellers with \( t^* \in [t_{-1}; t_0] \), i.e. those whose value of time does not exceed the threshold \( \hat{\tau} \). Note that although the marginal impact of a change in the schedule is proportional to the value of time for all travellers, the cost of WT may sensibly differ across them.

### 3.2 Case U.2 - \( p_0 \leq p_1 \leq p_2 \): all travellers take the first available service

In this case, given the price ordering, there is no advantage to waiting for the subsequent service. Generalizing the notation previously introduced, we let the demand for service \( i = 0, 1 \), given the value of \( \tau_w \) and for all values of \( \tau_w \), be

\[
X_i(\tau_w) = \int_{t_{i-1}}^{t_i} x(p_i + \tau_w \|t_i - t^*\|) n dt^*
\]

\[
\hat{X}_i = \int_{0}^{+\infty} X_i(\tau_w) dG_W(\tau_w),
\]

whereas the average value of time of the travellers taking \( s_i \) is given by

\[
\hat{\tau}_i = \int_{0}^{+\infty} \tau_w \frac{X_i(\tau_w)}{\hat{X}_i} dG_W(\tau_w).
\]
No demand for $s_2$ is expressed by travellers with $t^* \in [t_1 - 2\Delta T, t_1]$.

As long as $p_0 \leq p_1 \leq p_2$, a change in price $p_i$ affects neither the distribution of travellers across services nor their waiting time. It has an impact only on the surplus of those who take $s_i$. Thus, as standard, following an infinitesimal increase in $p_i$, the surplus decreases by the exact size of the demand for that service:

$$\frac{dS}{dp_i} = -\tilde{X}_i.$$

Consider now an infinitesimal increase in the time interval $\Delta T$. Taking $t_0$ to be the reference point, that increase entails that $s_1$ is scheduled at $t_1 + dT$ (instead of $t_1$), whereas the service preceding $s_0$ is scheduled at $t_{-1} - dT$ (instead of $t_{-1}$). Neglecting the effect of gaining $ndT$ travellers at both extremes of the time interval around $t_0$, the surplus decreases by the average value of time of the travellers who use $s_1$ all over the service units they demand:

$$\frac{dS}{dT} = -\tilde{t}_1 \tilde{X}_1. \quad (8)$$

### 3.3 Case U.3 - $p_1 = p_h > p_2 = p_l$: some travellers with $t^* \leq t_1$ prefer $s_2$ to $s_1$

Unlike in the previous case, the demand for $s_1$ is now lower than the demand for the previous (and the subsequent) services. Provided that $p_0 \leq p_1 = p_h$, all travellers with $t^* \in [t_{-1}, t_0]$ take $s_0$. For travellers with $t^* \in [t_0, t_1]$ and $\tau_w > \tilde{\tau}$, it is not convenient to wait until $t_2$, and they take $s_1$. By contrast, travellers with $t^* \in [t_0, t_1]$ and $\tau_w \leq \tilde{\tau}$ do find it worth waiting for $s_2$. Accordingly, for this case we introduce the following notation with similar meaning, *mutatis mutandis*, to the previous cases:

$$X_0 (\tau_w) = \int_{t_{-1}}^{t_0} x (p_0 + \tau_w \| t_0 - t^* \|) \, dt^*$$

$$\tilde{X}_0 = \int_{0}^{+\infty} X_0 (\tau_w) \, dG_W (\tau_w),$$

$$X_1 (\tau_w) = \int_{t_0}^{t_1} x (p_1 + \tau_w \| t_1 - t^* \|) \, dt^*$$

$$\tilde{X}_1 = \int_{\tilde{\tau}}^{+\infty} X_1 (\tau_w) \, dG_W (\tau_w).$$
and

\[ X_2 (\tau_w) = \int_{t_0}^{t_1} x (p_2 + \tau_w \| t_2 - t^* \|) n dt^* \]

\[ X_2 = \int_{0}^{\tilde{\tau}} X_2 (\tau_w) dG_W (\tau_w). \]

Note that, as we do not consider travellers with \( t^* > t_1 \), all those who choose \( s_2 \) could have taken \( s_1 \).

As long as the price ordering is \( p_0 \leq p_1 = p_h \), a change in \( p_0 \) does not affect the distribution of travellers across services. By contrast, a change in either \( p_1 \) or \( p_2 \) does have an impact in that, once again, it induces a change in the cutoff value \( \tilde{\tau} \). For similar reasons as in the previous cases, we still end up with

\[ \frac{dS}{dp_i} = -\tilde{X}_0; \quad \frac{dS}{dp_1} = -\tilde{X}_1; \quad \frac{dS}{dp_2} = -\tilde{X}_2. \]

That is, following an infinitesimal increase in price \( p_i \), \( i \in \{0, 1, 2\} \), traveller surplus is reduced by the exact size of the demand for service \( s_i \).

Again, a variation in the service schedule, hence in the time interval \( \Delta T \), induces a re-distribution of travellers across services, the elasticity of the cutoff value \( \tilde{\tau} \) with respect to \( \Delta T \) being equal to \(-1\). The overall impact on traveller surplus amounts to

\[ \frac{dS}{dT} = - (\tilde{\tau}_1 \tilde{X}_1 + 2 \tilde{\tau}_2 \tilde{X}_2), \quad (9) \]

where

\[ \tilde{\tau}_1 = \int_{\tilde{\tau}}^{+\infty} \tau_w \frac{X_1 (\tau_w)}{\tilde{X}_1} dG_W (\tau_w) \]

\[ \tilde{\tau}_2 = \int_{0}^{\tilde{\tau}} \tau_w \frac{X_2 (\tau_w)}{\tilde{X}_2} dG_W (\tau_w). \]

The only difference with respect to the previous cases lies in the impact on surplus through service \( s_2 \), which accounts doubly. This is easily understood by recalling that, with \( t_0 \) being fixed, as \( \Delta T \) is increased, \( t_1 \) is postponed by \( dT \) and \( t_2 \) by \( 2dT \).

4 Informed travellers

Unlike uninformed travellers, those who are informed may also use one of the services scheduled before their ideal departure time. A traveller with ideal departure time of \( t^* \in [t_i, t_j] \) prefers \( s_i \) to \( s_j \) as long as

\[ p_i + \tau_{ds} (t^* - t_i) < p_j + \tau_{ds} (t_j - t^*). \]
Given $\tau_{ds}$, there exists a critical departure time

$$t^*_c(\tau_{ds}) = \frac{t_i + t_j}{2} + \frac{p_j - p_i}{2\tau_{ds}},$$

such that travellers are split into two groups. Those who would like to leave before $t^*_c(\tau_{ds})$ take $s_i$, the others opt for $s_j$. Notice that the travellers with a high (marginal) disutility of DTS prefer the service which is closer to their ideal departure time, almost neglecting price differences. By contrast, those with relatively low value of $\tau_{ds}$ choose the cheapest service, almost neglecting its departure time.

Focusing on travellers with $t^* \in [t_0, t_2]$, we now turn to analyse how they allocate across the three subsequent services $s_0$, $s_1$ and $s_2$ and how their welfare is affected by changes in price and in service schedule. To better assess the impact of such changes, we will also compute the traveller average DTS. This is the average difference between their ideal departure time and the departure time of the service $s_i$ they take. Again, for readability, we look at situations in which service departures are evenly distributed over time and there are only two levels of price, high ($p_h$) and low ($p_l$), so that $\Delta T$ and $\Delta p = p_h - p_l$ are both constant.

There are four cases to consider. We hereafter describe them, gathering lengthy mathematical details in Appendix A.2.

4.1 Case I.1 - $p_1 = p_l < p_h = p_0 = p_2 : s_1$ is cheaper than any other service

All travellers with $\tau_{ds} < \hat{\tau}$ (where $\hat{\tau} = \Delta p/\Delta T$ as in the case of uninformed travellers) are eager to increase their DTS by more than $\Delta T$ to obtain the price saving $\Delta p$. Therefore, as long as $p_1 = p_l < p_h = p_0 = p_2$, all travellers with $t^* \in [t_0, t_2]$ and $\tau_{ds} < \hat{\tau}$ take $s_1$. Travellers with $\tau_{ds} \geq \hat{\tau}$ respectively use $s_0$, $s_1$ or $s_2$ when their ideal departure time is such that

$$t^* \in \left[t_0; \frac{t_0 + t_1}{2} - \frac{\Delta p}{2\tau_{ds}}\right],$$

$$t^* \in \left[t_0 + t_1; \frac{t_0 + t_2}{2} - \frac{\Delta p}{2\tau_{ds}}\right] + \left[\frac{\Delta p}{2\tau_{ds}}; t_2\right],$$

$$t^* \in \left[t_1 + t_2; \frac{\Delta p}{2\tau_{ds}}\right].$$
Accordingly, the aggregate demand for \( s_0, s_1 \) and \( s_2 \) is respectively written as

\[
X_0 = \int_{\tau}^{+\infty} X_0 (\tau_{ds}) \, dG_{DS} (\tau_{ds})
\]
\[
\hat{X}_1 = X_1 + X_1^+
\]
\[
= \int_{\tau}^{+\infty} X_1 (\tau_{ds}) \, dG_{DS} (\tau_{ds}) + \int_{\tau}^{+\infty} X_1^+ (\tau_{ds}) \, dG_{DS} (\tau_{ds})
\]
\[
X_2 = \int_{\tau}^{+\infty} X_2 (\tau_{ds}) \, dG_{DS} (\tau_{ds})
\]

where

\[
X_0 (\tau_{ds}) = \int_{t_0}^{t_0+\Delta t} x \left( p_0 + \tau_{ds} \left\| t^* - t_0 \right\| \right) \, n dt^*
\]
\[
X_1 (\tau_{ds}) = \int_{t_0}^{t_1} x \left( p_1 + \tau_{ds} \left\| t^* - t_1 \right\| \right) \, n dt^*
\]
\[
X_1^+ (\tau_{ds}) = \int_{t_1+\Delta t}^{t_1+\Delta t+\Delta T} x \left( p_1 + \tau_{ds} \left\| t^* - t_1 \right\| \right) \, n dt^*
\]
\[
X_2 (\tau_{ds}) = \int_{t_1+\Delta t}^{t_1+\Delta t+\Delta T} x \left( p_2 + \tau_{ds} \left\| t^* - t_2 \right\| \right) \, n dt^*.
\]

Let us begin by considering an infinitesimal change in price \( p_i, i = 0, 1, 2 \). First, this affects all the travellers who take \( s_i \) both before and after the change (the intensive margin). It also affects the distribution of travellers across services through both the critical departure time \( t^*_c (\tau_{ds}) \) and the cutoff value of time \( \tau_c \) (the extensive margin). However, it is easy to verify that, as standard, a price increase of \( dp_i \) triggers a surplus reduction equal to the aggregate demand for service \( s_i \), namely

\[
\frac{dS}{dp_0} = -X_0; \quad \frac{dS}{dp_1} = -\hat{X}_1; \quad \frac{dS}{dp_2} = -X_2.
\]

We next consider an infinitesimal change in the infra-service time interval \( \Delta T \). Defining

\[
\bar{\tau}_0 = \int_{\tau}^{+\infty} \tau_{ds} \, \frac{X_0 (\tau_{ds})}{X_0} \, dG_{DS} (\tau_{ds})
\]
\[
\bar{\tau}_2 = \int_{\tau}^{+\infty} \tau_{ds} \, \frac{X_2 (\tau_{ds})}{X_2} \, dG_{DS} (\tau_{ds}),
\]

the average value of time of the travellers respectively using \( s_0 \) and \( s_2 \), and netting out any volume effects, the impact of an increase of \( dT \) on the surplus respectively associated
with \( s_0, s_1 \) and \( s_2 \) is given by

\[
\frac{dS_0}{dT} = -\bar{\tau}_0 \bar{X}_0 - \frac{n}{2} \int_0^{+\infty} V \left( p_0 + (\tau_{ds} - \bar{\tau}) \frac{\Delta T}{2} \right) dG_{DS} (\tau_{ds}) \tag{10a}
\]

\[
\frac{dS_1}{dT} = n \int_0^{+\infty} \left[ V (p_1 + \tau_{ds} |t_2 - t_1|) + V (p_1 + \tau_{ds} |t_0 - t_1|) \right] dG_{DS} (\tau_{ds}) \tag{10b}
\]

\[
+ n \int_0^{+\infty} V \left( p_1 + \tau_{ds} \left( 1 + \frac{\bar{\tau}}{\tau_{ds}} \right) \frac{\Delta T}{2} \right) dG_{DS} (\tau_{ds})
\]

\[
\frac{dS_2}{dT} = -\bar{\tau}_2 \bar{X}_2 - \frac{n}{2} \int_0^{+\infty} V \left( p_2 + (\tau_{ds} - \bar{\tau}) \frac{\Delta T}{2} \right) dG_{DS} (\tau_{ds}) \tag{10c}
\]

Therefore, once it is considered that

\[
p_0 + (\tau_{ds} - \bar{\tau}) \frac{\Delta T}{2} = p_1 + (\tau_{ds} + \bar{\tau}) \frac{\Delta T}{2}
\]

\[
p_2 + (\tau_{ds} - \bar{\tau}) \frac{\Delta T}{2} = p_1 + (\tau_{ds} + \bar{\tau}) \frac{\Delta T}{2},
\]

the overall impact on traveller welfare amounts to

\[
\frac{dS}{dT} = - \left( \bar{\tau}_0 \bar{X}_0 + \bar{\tau}_2 \bar{X}_2 \right). \tag{11}
\]

From (11) we see that as the service frequency is reduced and travellers using \( s_0 \) or \( s_2 \) are faced with a greater DTS, welfare decreases exactly by an aggregate measure of the value of time of those same travellers. Travellers using \( s_1 \) are not concerned in this respect, provided that service still departs at \( t_1 \) after the schedule variation.

### 4.2 Case I.2 - \( p_0 = p_h > p_l = p_2 = p_1 \) (or \( p_2 = p_h > p_l = p_0 = p_1 \))

\( s_1 \) is one of the cheap services

We now turn to explore a case where \( s_1 \) is cheap, together with \( s_2 \), whereas \( s_0 \) is expensive. The analysis is analogous, *mutatis mutandis*, to that of a case where \( s_1 \) is cheap, together with \( s_0 \), whereas \( s_2 \) is expensive. To avoid redundancy, we shall omit the presentation of this latter case.

First consider travellers with \( \tau_{ds} < \bar{\tau} \). Again, they are eager to increase their DTS by more than \( \Delta T \) to take advantage of the price saving. Thus, all such travellers who also have \( t^* \in [t_0, t_1] \) use \( s_1 \). Next consider travellers with \( \tau_{ds} \geq \bar{\tau} \). Those with \( t^* \in \left[ t_0, \frac{t_0 + t_1}{2} - \frac{\Delta p}{2\tau_{ds}} \right] \) use \( s_0 \), whereas those with \( t^* \in \left[ \frac{t_0 + t_1}{2} - \frac{\Delta p}{2\tau_{ds}}, t_1 \right] \) opt for \( s_1 \). Clearly, regardless of the value of \( \tau_{ds} \), all travellers with \( t^* \in \left[ t_1, \frac{t_1 + t_2}{2} \right] \) choose \( s_1 \), whereas those with \( t^* \in \left[ \frac{t_1 + t_2}{2}, t_2 \right] \) prefer \( s_2 \). Accordingly, the aggregate demand for \( s_0, s_1 \) and \( s_2 \) is
respectively written as

\[
\begin{align*}
\tilde{X}_0 &= \int_{\tau}^{+\infty} X_0(\tau_{ds}) \, dG_{DS}(\tau_{ds}) \\
\tilde{X}_1 &= \tilde{X}_{0} + \tilde{X}_1 \\
&= \int_{0}^{\tau} X_1(\tau_{ds}) \, dG_{DS}(\tau_{ds}) + \int_{\tau}^{+\infty} X_1(\tau_{ds}) \, dG_{DS}(\tau_{ds}) \\
\tilde{X}_2 &= \int_{0}^{+\infty} X_2(\tau_{ds}) \, dG_{DS}(\tau_{ds}),
\end{align*}
\]

where

\[
\begin{align*}
X_0(\tau_{ds}) &= \int_{t_0}^{t_0+\frac{\Delta p_{\tau_{ds}}}{\Delta \tau_{ds}}} x(p_0 + \tau_{ds} \| t^* - t_0\|) \, n dt^* \\
X_1(\tau_{ds}) &= \int_{t_0}^{t_0+\Delta p_{\tau_{ds}}} x(p_1 + \tau_{ds} \| t^* - t_1\|) \, n dt^* \\
X_1^-(\tau_{ds}) &= \int_{t_0+\frac{\Delta p_{\tau_{ds}}}{\Delta \tau_{ds}}}^{t_0+\frac{\Delta p_{\tau_{ds}}}{\Delta \tau_{ds}}} x(p_1 + \tau_{ds} \| t^* - t_1\|) \, n dt^* \\
X_2(\tau_{ds}) &= \int_{t_0+\frac{\Delta p_{\tau_{ds}}}{\Delta \tau_{ds}}}^{t_2} x(p_2 + \tau_{ds} \| t^* - t_2\|) \, n dt^*,
\end{align*}
\]

and the impact on surplus of a price increase is analogous to the other cases, namely

\[
\frac{dS}{dp_0} = -\tilde{X}_0; \quad \frac{dS}{dp_1} = -\tilde{X}_1; \quad \frac{dS}{dp_2} = -\tilde{X}_2.
\]

Consider now an infinitesimal change in the infra-service time interval \(\Delta T\). In the same vein as above, it is useful to define

\[
\begin{align*}
\tilde{\tau}_0 &= \int_{\tau}^{+\infty} \frac{X_0(\tau_{ds})}{X_0} \, dG_{DS}(\tau_{ds}), \\
\tilde{\tau}_2 &= \int_{0}^{+\infty} \frac{X_2(\tau_{ds})}{X_2} \, dG_{DS}(\tau_{ds})
\end{align*}
\]

the average value of time of the travellers respectively using \(s_0\) and \(s_2\). Neglecting any volume effects, we compute the impact of an increase of \(dT\) on the surplus respectively
associated with \( s_0, s_1 \) and \( s_2 \) as follows:

\[
\frac{dS_0}{dT} = -\int_0^{+\infty} \tau_{ds} \left( \int_{t_1-T}^{t_1} x(t) \frac{dT}{2} \right) dG_{DS}(\tau_{ds}) \tag{12a}
\]

\[
\frac{d\tilde{S}_1}{dT} = \frac{n}{2} \int_0^{+\infty} V(t) \left( p_1 + \tau_{ds} \frac{\Delta T}{2} \right) dG_{DS}(\tau_{ds}) \tag{12b}
\]

\[
\frac{d\tilde{S}_2}{dT} = -\int_0^{+\infty} \tau_{ds} \left( \int_{t_1+\Delta T}^{t_1} x(t) \frac{dT}{2} \right) dG_{DS}(\tau_{ds}) \tag{12c}
\]

Summing up with similar manipulations to those previously used (detailed upon in Appendix A.2.2), we find that the overall impact on traveller welfare is given by

\[
\frac{dS}{dT} = -\left( \tilde{\tau}_{ds,0} \tilde{S}_0 + \tilde{\tau}_{ds,2} \tilde{S}_2 \right). \tag{13}
\]

Again, as the service becomes less frequent and travellers using \( s_0 \) or \( s_2 \) are faced with a greater DTS, welfare reduces by an aggregate measure of the value of time of these latter travellers.

**4.3 Case I.3 -** \( p_1 = p_0 = p_h > p_l = p_2 \) (or \( p_1 = p_2 = p_h > p_l = p_0 \)) : \( s_1 \) is one of the expensive services

The next case to consider is one where \( s_1 \) is an expensive service, together with \( s_0 \), whereas \( s_2 \) is cheap. Again, the analysis is analogous, *mutatis mutandis*, to that of a case where \( s_1 \) is expensive, together with \( s_2 \), whereas \( s_0 \) is cheap. We shall thus omit the presentation of this latter case.

Because \( p_0 = p_1 \), all travellers with \( t^* \in [t_0, t_0 + \frac{t_1}{2}] \) are better off if they take \( s_0 \) rather than \( s_1 \). However, it can be more advantageous to take \( s_2 \), if \( \tau_{ds} (t_2 - t^*) \leq p_0 + \tau_{ds} (t^* - t_0) \), i.e., if

\[
t^* \geq t_0 + \Delta T - \frac{\Delta p}{2\tau_{ds}}.
\]

First take \( \tau_{ds} \leq \frac{\hat{\tau}}{2} \). Then, the inequality above holds true for all travellers with \( t^* \in [t_0, t_2] \).

Hence, all travellers prefer to use \( s_2 \). Next take \( \tau_{ds} \in \left( \frac{\hat{\tau}}{2}, \hat{\tau} \right) \). Then, all travellers with \( t^* \in \left[ t_0, t_1 - \frac{\Delta p}{2\tau_{ds}} \right] \) choose \( s_0 \), whereas those with \( t^* \in \left[ t_1 - \frac{\Delta p}{2\tau_{ds}}, t_2 \right] \) opt for \( s_2 \). No traveller uses \( s_1 \). Last take \( \tau_{ds} \geq \hat{\tau} \). Then, travellers with \( t^* \in \left[ t_0, t_0 + \frac{t_1}{2} \right] \) choose \( s_0 \), travellers with
\( t^* \in \left[ \frac{t_0 + t_1}{2}, \frac{t_1 + t_2 - \Delta p}{2r_{ds}} \right] \supset \left[ \frac{t_0 + t_1}{2}, t_1 \right] \) opt for \( s_1 \), and only those with \( t^* \in \left[ \frac{t_1 + t_2}{2} - \frac{\Delta p}{2r_{ds}}, t_2 \right] \) use the cheap service \( s_2 \).

The demand for service \( s_0, s_1 \) and \( s_2 \) is respectively written as

\[
X_0 = \tilde{X}_0 + \overline{X}_0 = \int_{\frac{t_0}{2}}^T \tilde{X}_0 (\tau_{ds}) \, dG_{DS} (\tau_{ds}) + \int_{\frac{t_1}{2}}^T \tilde{X}_0 (\tau_{ds}) \, dG_{DS} (\tau_{ds})
\]
\[
\overline{X}_1 = \int_{\frac{t_1}{2}}^T X_1 (\tau_{ds}) \, dG_{DS} (\tau_{ds})
\]
\[
X_2 = \tilde{X}_2 + \overline{X}_2 = \int_{t_0}^{t_0 + \frac{t_1}{2}} X_2 (\tau_{ds}) \, dG_{DS} (\tau_{ds}) + \int_{\frac{t_1}{2}}^T \tilde{X}_2 (\tau_{ds}) \, dG_{DS} (\tau_{ds}) + \int_{\frac{t_1}{2}}^T \overline{X}_2 (\tau_{ds}) \, dG_{DS} (\tau_{ds})
\]

where now

\[
\tilde{X}_0 (\tau_{ds}) = \int_{t_0}^{t_1 - \frac{\Delta p}{2r_{ds}}} x (p_0 + \tau_{ds} || t^* - t_0 ||) \, ndt^*
\]
\[
\overline{X}_0 (\tau_{ds}) = \int_{t_0}^{t_0 + \frac{t_1}{2}} x (p_0 + \tau_{ds} || t^* - t_0 ||) \, ndt^*
\]

together with

\[
X_1 (\tau_{ds}) = \int_{\frac{t_0}{2}}^{\frac{t_1 + t_2}{2} - \frac{\Delta p}{2r_{ds}}} x (p_1 + \tau_{ds} || t^* - t_1 ||) \, ndt^*
\]

and with

\[
\tilde{X}_2 (\tau_{ds}) = \int_{t_1 - \frac{\Delta p}{2r_{ds}}}^{t_2} x (p_2 + \tau_{ds} || t^* - t_2 ||) \, ndt^*
\]
\[
\overline{X}_2 (\tau_{ds}) = \int_{\frac{t_1}{2}}^{t_2} x (p_2 + \tau_{ds} || t^* - t_2 ||) \, ndt^*
\]
\[
\overline{X}_2 (\tau_{ds}) = \int_{\frac{t_1}{2}}^{t_2} x (p_2 + \tau_{ds} || t^* - t_2 ||) \, ndt^*
\]

As usual, an infinitesimal increase in price \( p_i \), where \( i = 0, 1 \) or \( 2 \), will determine a reduction in surplus exactly equal to the aggregate demand for service \( i \).

\(^3\)There is a discontinuity at \( \tau_{ds} = \tilde{\tau} \), in the following sense. For \( \tau_{ds} > \tilde{\tau} \) there is a set of travellers of strictly positive measure using \( s_1 \) (the set must contain all travellers with \( t^* \in \left[ \frac{t_0 + t_1}{2}, t_1 \right] \)). For \( \tau_{ds} < \tilde{\tau} \) no traveller uses \( s_1 \) regardless of \( t^* \). Further details on the distribution of travellers across services are found in Appendix A.2.3.
Consider next an infinitesimal change in the infra-service time interval $\Delta T$. Denote as

\[
\begin{align*}
\hat{\tau}_0 &= \int_{\frac{\tau}{\tau_0}}^{+\infty} \frac{X_0(t)}{X_0} dG_{DS}(t) \\
\bar{\tau}_0 &= \int_{\frac{\tau}{\tau_0}}^{+\infty} \frac{X_0(t)}{X_0} dG_{DS}(t) \\
\bar{\tau}_2 &= \int_{\frac{\tau}{\tau_2}}^{+\infty} \frac{X_2(t)}{X_2} dG_{DS}(t) \\
\hat{\tau}_2 &= \int_{\frac{\tau}{\tau_2}}^{+\infty} \frac{X_2(t)}{X_2} dG_{DS}(t) \\
\bar{\tau}_2 &= \int_{\frac{\tau}{\tau_2}}^{+\infty} \frac{X_2(t)}{X_2} dG_{DS}(t)
\end{align*}
\]

the average value of time of more to less patient travellers respectively using $s_0$ and $s_2$. Although, in this case, the exact expressions that capture the impact of an increase of $dT$ on the surplus of travellers using the different services are too long to be informative and, hence, they are only reported in Appendix A.2.3, the overall impact admits the compact and easily interpretable formulation here below, once volume effects are neglected:

\[
\frac{dS}{dT} = -\left( \hat{\tau}_0 \hat{X}_0 + \bar{\tau}_0 \bar{X}_0 + \bar{\tau}_2 \bar{X}_2 + \hat{\tau}_2 \hat{X}_2 \right). \tag{14}
\]

In line with the previous findings, (14) says that, as the service becomes less frequent and travellers using $s_0$ or $s_2$ are faced with a greater DTS, welfare reduces by an aggregate measure of the value of time of those travellers. However, within the overall impact, we can here disentangle the quota pertaining to the travellers using $s_0$ with low and high value of time, and the quota pertaining to the travellers using $s_2$ with low, medium and high value of time.

### Case I.4 - $p_1 = p_h > p_l = p_0 = p_2$ : $s_1$ is more expensive than any other service

In this case, the only travellers who might renounce to take advantage of the price saving of $\Delta p$ are those with $\tau_{ds} \geq \hat{\tau}$. Therefore, there is no traveller with $\tau_{ds} < \hat{\tau}$ who takes $s_1$ : those with $t^* \in [t_0; t_1]$ opt for $s_0$, those with $t^* \in [t_1; t_2]$ prefer $s_2$. Travellers with $\tau_s \geq \hat{\tau}_s$ choose to take $s_1$ only when $t_1$ is sufficiently close to their ideal departure
time. Particularly, they respectively use $s_0$, $s_1$ and $s_2$ when

$$t^* \in \left[ t_0; \frac{t_0 + t_1}{2} + \frac{\Delta p}{2\tau_s} \right]$$

$$t^* \in \left[ \frac{t_0 + t_1}{2} + \frac{\Delta p}{2\tau_s}; \frac{t_1 + t_2}{2} - \frac{\Delta p}{2\tau_s} \right]$$

$$t^* \in \left[ \frac{t_1 + t_2}{2} - \frac{\Delta p}{2\tau_s}; t_2 \right].$$

Accordingly, we can respectively write the demand for service $s_0$, $s_1$ and $s_2$ as

$$\tilde{X}_0 = X_0 + X_0^+$$

$$= \int_0^{t_1} x (p_0 + \tau_{ds} \|t^* - t_0\|) \, ndt^*$$

$$X_0^+ (\tau_{ds}) = \int_0^{t_0+t_1 + \frac{\Delta p}{2\tau_s}} x (p_0 + \tau_{ds} \|t^* - t_0\|) \, ndt^*,$$

together with

$$X_1 (\tau_{ds}) = \int_{t_0}^{t_1 + \frac{t_1 + t_2}{2} - \frac{\Delta p}{2\tau_s}} x (p_1 + \tau_{ds} \|t^* - t_1\|) \, ndt^*$$

and with

$$X_2^- (\tau_{ds}) = \int_{t_1}^{t_2} x (p_2 + \tau_{ds} \|t^* - t_2\|) \, ndt^*$$

$$X_2 (\tau_{ds}) = \int_{t_1 + \frac{t_1 + t_2}{2} - \frac{\Delta p}{2\tau_s}} x (p_2 + \tau_{ds} \|t^* - t_2\|) \, ndt^*.$$

Analogously to the other cases, an increase in the price of a given service induces a reduction in surplus equal to the demand for that service:

$$\frac{dS}{dp} = -\tilde{X}_0; \quad \frac{dS}{dp_1} = -X_1; \quad \frac{dS}{dp_2} = -\tilde{X}_2.$$

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We now turn to consider a change of \( dT \) in the infra-service time interval \( \Delta T \). Let

\[
\tau_0 = \int_0^\tau \frac{X_0(\tau_{ds})}{X_0} dG_{DS}(\tau_{ds}) \\
\tau_0^+ = \int_0^{+\infty} \frac{X_0^+(\tau_{ds})}{X_0^+} dG_{DS}(\tau_{ds}) \\
\tau_2^- = \int_0^\tau \frac{X_2^-(\tau_{ds})}{X_2^-} dG_{DS}(\tau_{ds}) \\
\tau_2 = \int_0^{+\infty} \frac{X_2(\tau_{ds})}{X_2} dG_{DS}(\tau_{ds})
\]

the average value of time of more to less patient travellers respectively using \( s_0 \) and \( s_2 \). Omitting again, for shortness, the single expressions that capture the impact of an increase of \( dT \) on the surplus of travellers using the different services, the overall impact on traveller surplus (net of the volume effects) amounts to

\[
\frac{dS}{dT} = - \left( \tau_0 X_0 + \tau_0^+ X_0^+ + \tau_2^- X_2^- + \tau_2 X_2 \right). \tag{15}
\]

As expected, also in this case welfare reduces by an aggregate measure of the value of time of the travellers using \( s_0 \) or \( s_2 \), as transportation services become less frequent. Within the overall impact on welfare, we disentangle the quota pertaining to the travellers with low and high value of time. Once more this emphasizes the benefits of the modelling approach here adopted. Whereas it provides one with an aggregate measure of the impact of changes in service scheduling on the overall population of travellers, it also allows for a useful segmentation of the market, based on an individual characteristics, namely the value of time, which is a valuable indicator of the traveller economic conditions and preferences.

5 Conclusion

We constructed a micro-founded traveller demand model that allows for two dimensions of heterogeneity. Travellers differ both in their ideal departure time (horizontal differentiation) and in their value of time (vertical differentiation).

We showed that either aspect is relevant in terms of the choices made by the travellers and of the impact of changes in price and/or schedule on welfare. If policy-makers are to account for these effects, then our model shall be precious to enable them to make informed decisions.

The model also sheds light on the role and the importance of information. *Ceteris paribus*, information on prices and service schedules enhances traveller welfare. Again, the impact of information varies across travellers. Some are unaffected, others may derive significant benefits from it - and their behaviour may be altered radically, if this
information is made available to them.

Whereas our paper still falls short of providing a full analysis of the impact of information on traveller welfare, we believe it paves the way for that. We leave it for further research.

References


A The effect of a variation in schedule on surplus

A.1 Uninformed travellers

A.1.1 Case U.1 - $p_1 = p_l < p_h = p_0$

The surplus respectively associated with $s_0$ and $s_1$ is given by

$$S_0 = \int_{\tilde{f}}^{+\infty} \left( \int_{t-1}^{t_0} V (p_0 + \tau_w \mid t_0 - t^*\mid) n dt^* \right) dG_W (\tau_w)$$

and

$$S_1^+ + \tilde{S}_1 = \int_{\tilde{f}}^{+\infty} \left( \int_{t-1}^{t_0} V (p_1 + \tau_w \mid t_1 - t^*\mid) n dt^* \right) dG_W (\tau_w) + \int_{0}^{+\infty} \left( \int_{t_0}^{t_1} V (p_1 + \tau_w \mid t_1 - t^*\mid) n dt^* \right) dG_W (\tau_w).$$
We compute
\[
\frac{1}{n} \frac{d \tilde{S}_0}{dT} = -\frac{d \tilde{\tau}}{dT} \left( \int_{t_{t-1}}^{t_0} V(p_0 + \tilde{\tau} \| t_0 - t^*\|) \, dt^* \right) g_W(\tilde{\tau}) \\
+ \int_{\tilde{\tau}}^{+\infty} \left( \frac{dt_0}{dT} V(p_0) - \frac{dt_{t-1}}{dT} V(p_0 + \tau_w \Delta T) \right) dG_W(\tau_w) \\
- \int_{\tilde{\tau}}^{+\infty} \left( \int_{t_{t-1}}^{t_0} \frac{dt_0}{dT} \tau_w x(p_0 + \tau_w \| t_0 - t^*\|) \, dt^* \right) dG_W(\tau_w).
\]

Under the assumption (to be maintained throughout) that \( t_0 \) is the reference point, \( t_{t-1} \) decreases by \( dT \), \( t_1 \) increases by \( dT \) (and \( t_2 \) by \( 2dT \)), this further becomes
\[
\frac{d \tilde{S}_0}{dT} = \frac{\tilde{\tau}}{\Delta T} \left( \int_{t_{t-1}}^{t_0} V(p_0 + \tilde{\tau} \| t_0 - t^*\|) \, n dt^* \right) g_W(\tilde{\tau}) \\
+ \int_{\tilde{\tau}}^{+\infty} V(p_0 + \tau_w \Delta T) n dG_W(\tau_w),
\]

where the last term is the increase in surplus associated with the additional travellers that use \( s_0 \) as the time interval between subsequent services is widened. Similarly, we compute
\[
\frac{1}{n} \left( \frac{d \tilde{S}_1^+}{dT} + \frac{d \tilde{S}_1^-}{dT} \right) = \frac{d \tilde{\tau}}{dT} \left( \int_{t_{t-1}}^{t_0} V(p_1 + \tilde{\tau} \| t_1 - t^*\|) \, dt^* \right) g_W(\tilde{\tau}) \\
+ \int_{\tilde{\tau}}^{+\infty} \left( \int_{t_{t-1}}^{t_0} \frac{dt_1}{dT} \tau_w x(p_1 + \tau_w \| t_1 - t^*\|) \, dt^* \right) dG_W(\tau_w) \\
+ 0 + \int_{0}^{+\infty} \left( \int_{t_{t-1}}^{t_0} \frac{dt_0}{dT} V(p_1) - \frac{dt_{t-1}}{dT} V(p_1 + \tau_w \Delta T) \right) dG_W(\tau_w) \\
- \int_{0}^{+\infty} \left( \int_{t_{t-1}}^{t_0} \frac{dt_1}{dT} \tau_w x(p_1 + \tau_w \| t_1 - t^*\|) \, dt^* \right) dG_W(\tau_w).
\]

Using the notation introduced in the main text, this further becomes
\[
\frac{d \tilde{S}_1^+}{dT} + \frac{d \tilde{S}_1^-}{dT} = -\tilde{\tau} \Sigma_1 + \tilde{\tau} \tilde{X}_1 - \frac{\tilde{\tau}}{\Delta T} \left( \int_{t_{t-1}}^{t_0} V(p_1 + \tilde{\tau} \| t_1 - t^*\|) \, n dt^* \right) g_W(\tilde{\tau}) \\
+ n V(p_1) + \int_{0}^{+\infty} V(p_1 + 2\tau_w \Delta T) n dG_W(\tau_w),
\]

where again the last two terms capture the increase in surplus associated with the additional travellers who take \( s_1 \) as the time interval between services is widened. The total
effect on surplus is given by

\[
\frac{dS}{dT} = \frac{d\tilde{S}_0}{dT} + \frac{d\tilde{S}_1}{dT} + \frac{d\hat{S}_1}{dT}
= -X_1^+ \tilde{X}_1 + \tilde{\gamma}_1 \tilde{X}_1
+ \frac{\tilde{\gamma}}{\Delta T} \left( \int_{t-1}^{t} V \left( p_0 + \tilde{\tilde{\gamma}} \parallel t_0 - t^* \parallel \right) n dt^* \right) g_W(\tilde{\tilde{\gamma})}
- \frac{\tilde{\gamma}}{\Delta T} \left( \int_{t-1}^{t} V \left( p_1 + \tilde{\tilde{\gamma}} \parallel t_1 - t^* \parallel \right) n dt^* \right) g_W(\tilde{\tilde{\gamma})}
+ \int_{\tilde{\tilde{\gamma}}}^{+\infty} V \left( p_0 + \tau_w \Delta T \right) ndG_W(\tau_w) + \int_{0}^{\tilde{\tilde{\gamma}}} V \left( p_1 + 2\tau_w \Delta T \right) ndG_W(\tau_w) + nV(p_1).
\]

Relying on the equality \( p_0 + \tilde{\gamma} || t_0 - t^* || = p_1 + \tilde{\gamma} || t_1 - t^* || \) and neglecting the last three terms (to avoid accounting for the "volume" effect) yield (7).

A.1.2 Case U.2 - \( p_0 \leq p_1 \leq p_2 \)

The surplus associated with \( s_i, i = 0, 1 \), is

\[
\tilde{S}_i = \int_{0}^{+\infty} \left( \int_{t_{i-1}}^{t_i} V \left( p_i + \tau_w \parallel t_i - t^* \parallel \right) n dt^* \right) dG_W(\tau_w).
\]

First take \( i = 0 \) and compute

\[
\frac{1}{n} \frac{d\tilde{S}_0}{dT} = \int_{0}^{+\infty} \left( \int_{t_{i-1}}^{t_i} \frac{dt_0}{dT} V(\tau_0) - \frac{dt_{i-1}}{dT} V(\tau_0 + \tau_w \Delta T) \right) dG_W(\tau_w)
- \int_{0}^{+\infty} \left( \int_{t_{i-1}}^{t_i} \frac{dt_0}{dT} \tau_w x(\tau_0 + \tau_w (t^* - t_0)) dt^* \right) dG_W(\tau_w).
\]

With \( t_0 \) the reference point, we have \( (dt_0/dT) = 0 \) and \( (dt_{i-1}/dT) = -1 \) so that

\[
\frac{d\tilde{S}_0}{dT} = \int_{0}^{+\infty} V(\tau_0 + \tau_w \Delta T) ndG_W(\tau_w).
\]

Next take \( i = 1 \) and compute

\[
\frac{1}{n} \frac{d\tilde{S}_1}{dT} = \int_{0}^{+\infty} \left( \int_{t_{i-1}}^{t_i} \frac{dt_1}{dT} V(\tau_0) - \frac{dt_0}{dT} V(\tau_0 + \tau_w \Delta T) \right) dG_W(\tau_w)
- \int_{0}^{+\infty} \left( \int_{t_{i-1}}^{t_i} \frac{dt_1}{dT} \tau_w x(\tau_0 + \tau_w (t_1 - t^*)) dt^* \right) dG_W(\tau_w).
\]

With the notation introduced in the main text and \( t_0 \) the reference point, this further yields

\[
\frac{d\tilde{S}_1}{dT} = nV(p_1) - \tilde{\gamma}_1 \tilde{X}_1.
\]
Overall:

\[
\frac{dS}{dT} = \frac{d\tilde{S}_0}{dT} + \frac{d\tilde{S}_1}{dT} = \int_{0}^{+\infty} V (p_0 + \tau_w \Delta T) \, dG_W (\tau_w) + nV (p_1) - \tilde{\tau}_1 \tilde{X}_1.
\]

Neglecting the first two terms, which capture the volume effect induced by the additional travellers who use the services as the time interval is widened, (8) is obtained.

A.1.3 Case U.3 - \( p_1 = p_h > p_2 = p_l \)

The surplus respectively associated with \( s_0, s_1 \) and \( s_2 \) is given by

\[
\tilde{S}_0 = \int_{0}^{+\infty} \left( \int_{t_0}^{t_{t\_1}} V (p_0 + \tau_w \| t_0 - t^* \|) \, n dt^* \right) \, dG_W (\tau_w)
\]
\[
\tilde{S}_1 = \int_{\tilde{\tau}}^{+\infty} \left( \int_{t_0}^{t_{t\_1}} V (p_1 + \tau_w \| t_1 - t^* \|) \, n dt^* \right) \, dG_W (\tau_w)
\]
\[
\tilde{S}_2 = \int_{0}^{\tilde{\tau}} \left( \int_{t_0}^{t_{t\_1}} V (p_2 + \tau_w \| t_2 - t^* \|) \, n dt^* \right) \, dG_W (\tau_w)
\]

We compute

\[
\frac{1}{n} \frac{d\tilde{S}_0}{dT} = 0 + \int_{0}^{+\infty} \left( \frac{dt_0}{dT} V (p_0) - \frac{dt_{t\_1}}{dT} V (p_0 + \tau_w \Delta T) \right) \, dG_W (\tau_w)
\]
\[
- \int_{0}^{+\infty} \left( \int_{t_0}^{t_{t\_1}} \frac{dt_0}{dT} \tau_w x (p_0 + \tau_w (t^* - t_0)) \, dt^* \right) \, dG_W (\tau_w)
\]

With \( t_0 \) the reference point, \( (dt_0/dT) = 0 \) and \( (dt_{t\_1}/dT) = -1 \) and we have

\[
\frac{d\tilde{S}_0}{dT} = \int_{0}^{+\infty} V (p_0 + \tau_w \Delta T) \, n dG_W (\tau_w)
\]

We next compute

\[
\frac{1}{n} \frac{d\tilde{S}_1}{dT} = - \frac{d\tilde{\tau}}{dT} \left( \int_{t_0}^{t_{t\_1}} V (p_1 + \tilde{\tau} \| t_1 - t^* \|) \, dt^* \right) \, g_W (\tilde{\tau})
\]
\[
+ \int_{\tilde{\tau}}^{+\infty} \left( \frac{dt_1}{dT} V (p_1) - \frac{dt_0}{dT} V (p_1 + \tau_w \Delta T) \right) \, dG_W (\tau_w)
\]
\[
- \int_{\tilde{\tau}}^{+\infty} \left( \int_{t_0}^{t_{t\_1}} \frac{dt_1}{dT} \tau_w x (p_1 + \tau_w (t_1 - t^*)) \, dt^* \right) \, dG_W (\tau_w)
\]
Recalling the definitions of $\bar{X}_1$ and $\tilde{\tau}_1$ and considering that $t_0$ is the reference point, we can further write

$$\frac{dS_1}{dT} = -\frac{d\tilde{\tau}}{dT} \left( \int_{t_0}^{t_1} V \left( p_1 + \tilde{\tau} \| t_1 - t^* \| \right) n dt \right) g_W(\tilde{\tau}) - \tilde{\tau}_1 \bar{X}_1 + n \left[ 1 - G_W(\tilde{\tau}) \right] V(p_1).$$

We complete by computing

$$\frac{1}{n} \frac{dS_2}{dT} = \frac{d\tilde{\tau}}{dT} \left( \int_{t_0}^{t_1} V \left( p_2 + \tilde{\tau} (t_2 - t^*) \right) dt^* \right) g_W(\tilde{\tau})$$

$$+ \int_{t_0}^{\tilde{\tau}} \left( \frac{dt_1}{dT} V(p_2 + \tau_w \Delta T) - \frac{dt_2}{dT} V(p_2 + 2\tau_w \Delta T) \right) dG_W(\tau_w)$$

$$- \int_{t_0}^{\tilde{\tau}} \left( \int_{t_0}^{t_1} \frac{dt_2}{dT} \tau_w x(p_2 + \tau_w (t_2 - t^*)) dt^* \right) dG_W(\tau_w).$$

With $t_0$ as a reference point, $(dt_1/dT) = 1$ and $(dt_2/dT) = 2$ and we have

$$\frac{dS_2}{dT} = \frac{d\tilde{\tau}}{dT} \left( \int_{t_0}^{t_1} V \left( p_2 + \tilde{\tau} (t_2 - t^*) \right) n dt^* \right) g_W(\tilde{\tau}) - 2\tilde{\tau}_2 \bar{X}_2$$

$$+ \int_{0}^{\tilde{\tau}} V(p_2 + \tau_w \Delta T) dG_W(\tau_w),$$

where we have used the notation introduced in the main text. Overall:

$$\frac{dS}{dT} = \frac{d\tilde{\tau}}{dT} \left( \int_{t_0}^{t_1} V(p_2 + \tilde{\tau} (t_2 - t^*)) n dt^* \right) g_W(\tilde{\tau})$$

$$+ \int_{0}^{\tilde{\tau}} V(p_0 + \tau_w \Delta T) n dG_W(\tau_w) + n \left[ 1 - G_W(\tilde{\tau}) \right] V(p_1)$$

$$+ \int_{0}^{\tilde{\tau}} V(p_2 + \tau_w \Delta T) n dG_W(\tau_w).$$

By definition, $p_2 + \tilde{\tau} (t_2 - t^*) = p_2 + \tilde{\tau} \Delta T + \tilde{\tau} (t_1 - t^*) = p_1 + \tilde{\tau} (t_1 - t^*)$. Hence, this is further rewritten as

$$\frac{dS}{dT} = -\tilde{\tau}_1 \bar{X}_1 - 2\tilde{\tau}_2 \bar{X}_2 + n \left[ 1 - G_W(\tilde{\tau}) \right] V(p_1)$$

$$+ \int_{0}^{\tilde{\tau}} V(p_0 + \tau_w \Delta T) n dG_W(\tau_w) + \int_{0}^{\tilde{\tau}} V(p_2 + \tau_w \Delta T) n dG_W(\tau_w).$$

Neglecting the last three terms, which only capture a volume effect, yields (9).
A.2 Informed travellers

A.2.1 Case I.1 - $p_1 = p_t < p_h = p_0 = p_2$

The surplus respectively associated with $s_0$, $s_1$ and $s_2$ is written as

$$\overline{S}_0 = \int_{\tau}^{+\infty} \left( \int_{t_0}^{t_0+\Delta t} V(p_0 + \tau \|t^* - t_0\|) \, dt^* \right) \, dG_D S(\tau_ds),$$

$$\overline{S}_1 = \overline{S}_1 + \overline{S}_1^+ = \int_{\tau}^{+\infty} \left( \int_{t_0}^{t_1} V(p_0 + \tau \|t^* - t_1\|) \, dt^* \right) \, dG_D S(\tau_ds)$$

$$+ \int_{\tau}^{+\infty} \left( \int_{t_0+\Delta t}^{t_1+\Delta t} V(p_0 + \tau \|t^* - t_1\|) \, dt^* \right) \, dG_D S(\tau_ds),$$

$$\overline{S}_2 = \int_{\tau}^{+\infty} \left( \int_{t_1+\Delta t}^{t_2} V(p_2 + \tau \|t^* - t_2\|) \, dt^* \right) \, dG_D S(\tau_ds).$$

We begin by computing

$$\frac{1}{n} \frac{d\overline{S}_0}{dT} = -\frac{d\hat{T}}{dT} \left( \int_{t_0}^{t_0+\Delta T} V(p_0 + \hat{T} \|t^* - t_0\|) \, dt^* \right)$$

$$- \int_{\tau}^{+\infty} \left( \int_{t_0}^{t_0+\Delta t} \tau_{ds} \, dT \, \frac{d\|t^* - t_0\|}{dT} \, x(p_0 + \tau \|t^* - t_0\|) \, dt^* \right) \, dG_D S(\tau_ds)$$

$$+ \int_{\tau}^{+\infty} \frac{1}{2} \frac{d\tau_0}{dT} V(p_0 + \tau \|t_0 + t_1 \| - \frac{\Delta p}{2\tau_{ds}} - T_0) \, dG_D S(\tau_ds)$$

$$- \int_{\tau}^{+\infty} \frac{d\tau_0}{dT} V(p_0) \, dG_D S(\tau_ds),$$

which is rewritten as

$$\frac{1}{n} \frac{d\overline{S}_0}{dT} = -\int_{\tau}^{+\infty} \tau_{ds} \left( \int_{t_0}^{t_0+\Delta t} x(p_0 + \tau \|t^* - t_0\|) \, dt^* \right) \, dG_D S(\tau_ds)$$

$$- \frac{1}{2} \int_{\tau}^{+\infty} V(p_0 + (\tau_{ds} - \hat{T}) \frac{\Delta T}{2}) \, dG_D S(\tau_ds) + (1 - G(\hat{T})) V(p_0).$$

Using the notation in the main text, this is further written as

$$\frac{d\overline{S}_0}{dT} = -\tau_{ds} \frac{X_0 - \frac{1}{2}}{n} \int_{\tau}^{+\infty} V(p_0 + (\tau_{ds} - \hat{T}) \frac{\Delta T}{2}) \, dG_D S(\tau_ds) + n (1 - G(\hat{T})) V(p_0),$$

and ultimately yields (10a) once the volume effect, as captured by the last term, is net out. We next compute
Using the notation introduced in the main text, this further becomes

\[
\frac{1}{n} \frac{d\tilde{S}_1}{dT} = \frac{d\tilde{\tau}}{dT} \int_{t_0}^{t_2} V \left( p_1 + \tilde{\tau} \| t^* - t_1 \| \right) dt^* - \frac{d\tilde{\tau}}{dT} \int_{t_0}^{t_1/2 + \Delta T} V \left( p_1 + \tilde{\tau} \| t^* - t_1 \| \right) dt^*
\]

\[
- \int_{\tilde{\tau}}^{+\infty} \left( \int_{t_0}^{t_2} \frac{d \| t^* - t_1 \|}{dT} x \left( p_1 + \tau_{ds} \| t^* - t_1 \| \right) dt^* \right) dG_{DS} (\tau_{ds})
\]

\[
- \int_{\tilde{\tau}}^{+\infty} \left( \int_{t_0}^{t_1/2 + \Delta \tau_{ds}} \frac{d \| t^* - t_1 \|}{dT} x \left( p_1 + \tau_{ds} \| t^* - t_1 \| \right) dt^* \right) dG_{DS} (\tau_{ds})
\]

\[
+ \int_{0}^{\tilde{\tau}} \left[ V \left( p_1 + \tau_{ds} \| t_2 - t_1 \| \right) + V \left( p_1 + \tau_{ds} \| t_0 - t_1 \| \right) \right] dG_{DS} (\tau_{ds})
\]

\[
+ \frac{1}{2} \int_{\tilde{\tau}}^{+\infty} \frac{dt_2}{dT} V \left( p_1 + \tau_{ds} \left\| \frac{t_1 + t_2}{2} + \frac{\Delta p}{2\tau_{ds}} - t_1 \right\| \right) dG_{DS} (\tau_{ds})
\]

\[
- \frac{1}{2} \int_{\tilde{\tau}}^{+\infty} \frac{dt_2}{dT} V \left( p_1 + \tau_{ds} \left\| \frac{t_0 + t_1}{2} - \frac{\Delta p}{2\tau_{ds}} - t_1 \right\| \right) dG_{DS} (\tau_{ds})
\]

This is rearranged to obtain (10b). We lastly compute

\[
\frac{1}{n} \frac{d\tilde{S}_2}{dT} = \frac{d\tilde{\tau}}{dT} \left( \int_{t_0}^{t_2} V \left( p_2 + \tilde{\tau} \| t^* - t_2 \| \right) dt^* \right)
\]

\[
+ \int_{\tilde{\tau}}^{+\infty} \left( \int_{t_0}^{t_2} \tau_{ds} \frac{d \| t^* - t_2 \|}{dT} x \left( p_2 + \tau_{ds} \| t^* - t_2 \| \right) dt^* \right) dG_{DS} (\tau_{ds})
\]

\[
- \int_{\tilde{\tau}}^{+\infty} \frac{1}{2} \frac{dt_2}{dT} V \left( p_2 + \tau_{ds} \left\| \frac{t_1 + t_2}{2} + \frac{\Delta p}{2\tau_{ds}} - t_2 \right\| \right) dG_{DS} (\tau_{ds})
\]

\[
+ \int_{\tilde{\tau}}^{+\infty} \frac{dt_2}{dT} V \left( p_2 \right) dG_{DS} (\tau_{ds})
\]

\[
= 0 - \int_{\tilde{\tau}}^{+\infty} \tau_{ds} \left( \int_{t_0}^{t_2} x \left( p_2 + \tau_{ds} \| t^* - t_2 \| \right) dt^* \right) dG_{DS} (\tau_{ds})
\]

\[
- \frac{1}{2} \int_{\tilde{\tau}}^{+\infty} V \left( p_2 + (\tau_{ds} - \tilde{\tau}) \left\| \frac{\Delta T}{2} \right\| \right) dG_{DS} (\tau_{ds}) + \frac{1}{2} \int_{\tilde{\tau}}^{+\infty} V \left( p_2 + (\tau_{ds} - \tilde{\tau}) \left\| \frac{\Delta T}{2} \right\| \right) dG_{DS} (\tau_{ds})\]

Using the notation introduced in the main text, this further becomes

\[
\frac{d\tilde{S}_2}{dT} = -\tau_{ds}^2 X_2 + n \left[ 1 - G \left( \tilde{\tau} \right) \right] V \left( p_2 \right) - \frac{1}{2} \int_{\tilde{\tau}}^{+\infty} V \left( p_2 + (\tau_{ds} - \tilde{\tau}) \left\| \frac{\Delta T}{2} \right\| \right) ndG_{DS} (\tau_{ds})
\]
which ultimately yields (10c) once the volume effect, as captured by the last term, is net out. Summing up we obtain

\[
\frac{dS}{dT} = \frac{dS_0}{dT} + \frac{dS_1}{dT} + \frac{dS_2}{dT}
\]

\[
= -\tilde{\tau}_{ds}^0 \overline{X}_0 - \tilde{\tau}_{ds}^2 \overline{X}_2 - \frac{n}{2} \int_{\tilde{\tau}}^{+\infty} \left( p_0 + (\tau_{ds} - \tilde{\tau}) \frac{\Delta T}{2} \right) dG_{DS} (\tau_{ds})
\]

\[
+ n \int_{\tilde{\tau}}^{+\infty} \left( p_1 + (\tau_{ds} + \tilde{\tau}) \frac{\Delta T}{2} \right) dG_{DS} (\tau_{ds})
\]

\[
- \frac{n}{2} \int_{\tilde{\tau}}^{+\infty} \left( p_2 + (\tau_{ds} - \tilde{\tau}) \frac{\Delta T}{2} \right) dG_{DS} (\tau_{ds})
\]

\[
+ 2n \int_0^{\tilde{\tau}} (V(p_1 + \tau_{ds}\Delta T)) dG_{DS} (\tau_{ds}) + n \left[ 1 - G(\tilde{\tau}) \right] [V(p_0) + V(p_2)].
\]

By the definition of \( \tilde{\tau} \), we have

\[
p_0 + (\tau_{ds} - \tilde{\tau}) \frac{\Delta T}{2} = p_0 + \left( \tau_{ds} - \frac{\Delta p}{\Delta T} \right) \frac{\Delta T}{2}
\]

\[
= p_0 - \frac{\Delta p}{2} + \tau_{ds} \frac{\Delta T}{2}
\]

\[
= p_1 + \tau_{ds} \frac{\Delta T}{2} + \frac{\Delta p}{2}
\]

\[
= p_1 + (\tau_{ds} + \tilde{\tau}) \frac{\Delta T}{2}
\]

and

\[
p_2 + (\tau_{ds} - \tilde{\tau}) \frac{\Delta T}{2} = p_2 + \left( \tau_{ds} - \frac{\Delta p}{\Delta T} \right) \frac{\Delta T}{2}
\]

\[
= p_2 - \frac{\Delta p}{2} + \tau_{ds} \frac{\Delta T}{2}
\]

\[
= p_1 + \tau_{ds} \frac{\Delta T}{2} + \frac{\Delta p}{2}
\]

\[
= p_1 + (\tau_{ds} + \tilde{\tau}) \frac{\Delta T}{2}
\]

so that

\[
V \left( p_0 + (\tau_{ds} - \tilde{\tau}) \frac{\Delta T}{2} \right) = V \left( p_1 + (\tau_{ds} + \tilde{\tau}) \frac{\Delta T}{2} \right)
\]

\[
V \left( p_2 + (\tau_{ds} - \tilde{\tau}) \frac{\Delta T}{2} \right) = V \left( p_1 + (\tau_{ds} + \tilde{\tau}) \frac{\Delta T}{2} \right),
\]
and we can rewrite
\[
\frac{dS}{dT} = -\tilde{\tau}_{dS}^0 \nabla_0 - \tilde{\tau}_{dS}^2 \nabla_2 + 2n \int_0^{\tilde{\tau}} (V(p_1 + \tau_{ds} \Delta T)) dG_{DS} (\tau_{ds}) + n [1 - G(\tilde{\tau})] [V(p_0) + V(p_2)].
\]

This yields (11) once the volume effects (the last two terms) are net out.

**A.2.2 Case I.2 -** \(p_0 = p_h > p_1 = p_2 = p_1\) (or \(p_2 = p_h > p_1 = p_0 = p_1\))

The surplus respectively associated with \(s_0, s_1\) and \(s_2\) is given by
\[
\tilde{S}_0 = \int_{\tilde{\tau}}^{+\infty} (\int_{t_0}^{t_0+t_1} - \frac{\Delta p}{\tau_{dS}} V(p_0 + \tau_{ds} \||t^* - t_0\||) dt^*) dG_{DS}(\tau_{ds})
\]
\[
\tilde{S}_1 = \tilde{S}_1 + \tilde{S}_0
\]
\[
= \int_{\tilde{\tau}}^{+\infty} (\int_{t_0}^{t_0+t_1} V(p_1 + \tau_{ds} \||t^* - t_1\||) dt^*) dG_{DS}(\tau_{ds})
+ \int_{\tilde{\tau}}^{+\infty} (\int_{t_0+\frac{t_1+t_2}{2}}^{t_0+t_1} \frac{\Delta p}{\tau_{dS}} V(p_1 + \tau_{ds} \||t^* - t_1\||) dt^*) dG_{DS}(\tau_{ds})
\]
\[
\tilde{S}_2 = \int_{\tilde{\tau}}^{+\infty} (\int_{t_0+t_1+t_2}^{t_0+t_1+t_2} V(p_2 + \tau_{ds} \||t^* - t_2\||) dt^*) dG_{DS}(\tau_{ds}).
\]

We first compute
\[
\frac{1}{n} \frac{d\tilde{S}_0}{dT} = \frac{d\tilde{\tau}}{dT} (\int_{t_0}^{t_0+t_1} V(p_0 + \tilde{\tau} \||t^* - t_0\||) dt^*) g_{DS}(\tilde{\tau})
- \int_{\tilde{\tau}}^{+\infty} \tau_{ds} (\int_{t_1-\Delta T}^{t_1-\frac{\Delta T}{2}} x (p_0 + \tau_{ds} \||t^* - t_0\||) dt^*) dG_{DS}(\tau_{ds})
+ [1 - G(\tilde{\tau})] V(p_0) - \frac{1}{2} \int_{\tilde{\tau}}^{+\infty} V(p_0 + (\tau_{ds} - \tilde{\tau}) \frac{\Delta T}{2}) dG_{DS}(\tau_{ds}),
\]
from which we obtain
\[
\frac{d\tilde{S}_0}{dT} = \int_{\tilde{\tau}}^{+\infty} \tau_{ds} (\int_{t_1-\Delta T}^{t_1-\frac{\Delta T}{2}} x (p_0 + \tau_{ds} \||t^* - t_0\||) dt^*) dG_{DS}(\tau_{ds})
+ [1 - G(\tilde{\tau})] V(p_0) - \frac{1}{2} n \int_{\tilde{\tau}}^{+\infty} V(p_0 + (\tau_{ds} - \tilde{\tau}) \frac{\Delta T}{2}) dG_{DS}(\tau_{ds}).
\]
We then compute

\[
\frac{1}{n} \frac{d \tilde{S}_1}{dT} = \frac{d \tilde{T}}{dT} \left( \int_{-T_0}^{t_1 + t_2} V \left(p_1 + \tilde{r} \parallel t^* - t_1\right) dt^* \right) g_D (\tilde{T})
\]

\[
- \frac{d \tilde{T}}{dT} \left( \int_{-T_0 + t_1}^{t_1 + t_2} V \left(p_1 + \tilde{r} \parallel t^* - t_1\right) dt^* \right) g_D (\tilde{T})
\]

\[+ \frac{1}{2} \int_{-T_0}^{\tilde{T}} V \left(p_1 + \tau ds \parallel t_1 + t_2 - t_1\right) dG_D (\tau ds) + \int_{-T_0}^{\tilde{T}} V \left(p_1 + \tau ds \parallel t_0 - t_1\right) dG_D (\tau ds)
\]

\[+ \int_{-T_0}^{\tilde{T}} \left( \int_{-T_0}^{t_1 + t_2} \tau ds \frac{d \parallel t^* - t_1\parallel}{dT} V' \left(p_1 + \tau ds \parallel t^* - t_1\right) dt^* \right) dG_D (\tau ds)
\]

\[+ \frac{1}{2} \int_{-T_0}^{\tilde{T}} V \left(p_1 + \Delta T \frac{\Delta T}{2}\right) dG_D (\tau ds)
\]

\[+ \frac{1}{2} \int_{-T_0}^{\tilde{T}} V \left(p_1 + \tau ds \parallel \frac{\Delta T}{2} \left(1 + \frac{\tilde{r}}{\tau ds}\right)\right) dG_D (\tau ds)
\]

\[+ n \int_{-T_0}^{\tilde{T}} \left( \int_{-T_0}^{t_1 + t_2} \tau ds \frac{d \parallel t^* - t_1\parallel}{dT} V' \left(p_1 + \tau ds \parallel t^* - t_1\right) dt^* \right) dG_D (\tau ds),
\]

which is rearranged as

\[
\frac{1}{n} \frac{d \tilde{S}_1}{dT} = - \frac{\tilde{r}}{\Delta T} \left( \int_{-T_0}^{t_1 + t_2} V \left(p_1 + \tilde{r} \parallel t^* - t_1\right) dt^* \right) g_D (\tilde{T})
\]

\[+ \frac{\tilde{r}}{\Delta T} \left( \int_{-T_0}^{t_1 + t_2} V \left(p_1 + \tilde{r} \parallel t^* - t_1\right) dt^* \right) g_D (\tilde{T})
\]

\[+ \frac{1}{2} \int_{-T_0}^{\tilde{T}} V \left(p_1 + \tau ds \parallel \frac{\Delta T}{2}\right) dG_D (\tau ds) + \int_{-T_0}^{\tilde{T}} V \left(p_1 + \tau ds \Delta T\right) dG_D (\tau ds)
\]

\[+ \frac{1}{2} \int_{-T_0}^{\tilde{T}} V \left(p_1 + \tau ds \parallel \frac{\Delta T}{2}\right) dG_D (\tau ds) + \frac{1}{2} \int_{-T_0}^{\tilde{T}} V \left(p_1 + \tau ds \left(1 + \frac{\tilde{r}}{\tau ds}\right) \frac{\Delta T}{2}\right) dG_D (\tau ds).
\]

This ultimately yields

\[
\frac{d \tilde{S}_1}{dT} = \frac{n}{2} \int_{-T_0}^{\tilde{T}} V \left(p_1 + \tau ds \parallel \frac{\Delta T}{2}\right) dG_D (\tau ds) + \int_{-T_0}^{\tilde{T}} V \left(p_1 + \tau ds \Delta T\right) dG_D (\tau ds)
\]

\[+ \frac{n}{2} \int_{-T_0}^{\tilde{T}} V \left(p_1 + \tau ds \left(1 + \frac{\tilde{r}}{\tau ds}\right) \frac{\Delta T}{2}\right) dG_D (\tau ds).
\]
We also compute

\[
\frac{1}{n} \frac{dS_2}{dT} = \int_0^{+\infty} V(p_2) dG_{DS}(\tau_{ds}) \bigg[ -\frac{1}{2} \int_0^{+\infty} V\left(p_2 + \tau_{ds} \left| t_1 + \frac{t_2}{2} - t_2 \right|\right) dG_{DS}(\tau_{ds}) \\
+ \int_0^{+\infty} \left( \int_{\frac{t_1}{2} + t_2}^{t_2} \tau_{ds} \frac{d||t^* - t_2||}{dT} V'(p_2 + \tau_{ds} ||t^* - t_2||) dt^* \right) dG_{DS}(\tau_{ds}) \bigg].
\]

Using \( t_2 = t_1 + \Delta T \) and \( \frac{t_1 + t_2}{2} = t_1 + \frac{\Delta T}{2} \), this is rewritten as

\[
\frac{d\tilde{S}_2}{dT} = -\int_0^{+\infty} \tau_{ds} \left( \int_{\frac{t_1 + \Delta T}{2}}^{t_1 + \Delta T} x \left( p_0 + \tau_{ds} ||t^* - t_0|| \right) n dt^* \right) dG_{DS}(\tau_{ds}) \\
+ nV(p_2) - \frac{n}{2} \int_0^{+\infty} V\left(p_2 + \tau_{ds} \frac{\Delta T}{2} \right) dG_{DS}(\tau_{ds}).
\]

As a result

\[
\frac{dS}{dT} = \frac{dS_0}{dT} + \frac{d\tilde{S}_1}{dT} + \frac{d\tilde{S}_2}{dT} \\
= -\int_0^{+\infty} \tau_{ds} \left( \int_{\frac{t_1 - \Delta T}{2}}^{t_1 - \Delta T} x \left( p_0 + \tau_{ds} ||t^* - t_0|| \right) n dt^* \right) dG_{DS}(\tau_{ds}) \\
- \frac{n}{2} \int_0^{+\infty} V\left(p_0 + \left( \tau_{ds} - \frac{\Delta T}{2} \right) \frac{\Delta T}{2} \right) dG_{DS}(\tau_{ds}) + n \left[ 1 - G(\hat{\tau}) \right] V(p_0) \\
+ \frac{n}{2} \int_0^{+\infty} V\left(p_1 + \tau_{ds} \frac{\Delta T}{2} \right) dG_{DS}(\tau_{ds}) + n \int_{\frac{\hat{\tau}}{2}}^{+\infty} V\left(p_1 + \tau_{ds} \Delta T \right) dG_{DS}(\tau_{ds}) \\
+ \frac{n}{2} \int_{\hat{\tau}}^{+\infty} V\left(p_1 + \tau_{ds} \frac{1 + \frac{\tau}{\tau_{ds}}}{2} \frac{\Delta T}{2} \right) dG_{DS}(\tau_{ds}) \\
+ nV(p_2) - \frac{n}{2} \int_0^{+\infty} V\left(p_2 + \tau_{ds} \frac{\Delta T}{2} \right) dG_{DS}(\tau_{ds}) \\
- \int_0^{+\infty} \tau_{ds} \left( \int_{\frac{t_1 + \Delta T}{2}}^{t_1 + \Delta T} x \left( p_2 + \tau_{ds} ||t^* - t_2|| \right) n dt^* \right) dG_{DS}(\tau_{ds}),
\]
from which we further obtain

\[
\frac{dS}{dT} = - \int_{\tau_{\Delta T}}^{+\infty} \tau_{ds} \left( \int_{t_{1-\Delta T}}^{t_{1-\Delta T}+\frac{\Delta T}{\tau_{ds}}} x \left( p_0 + \tau_{ds} \|t^* - t_0\| \right) n dt^* \right) dG_{DS} (\tau_{ds})
\]

\[
- \int_{0}^{+\infty} \tau_{ds} \left( \int_{t_{1+\Delta T}}^{t_{1+\Delta T}+\Delta T} x \left( p_2 + \tau_{ds} \|t^* - t_2\| \right) n dt^* \right) dG_{DS} (\tau_{ds})
\]

\[
- \frac{n}{2} \int_{\tau}^{+\infty} V \left( p_0 + (\tau_{ds} - \hat{\tau}) \frac{\Delta T}{2} \right) dG_{DS} (\tau_{ds})
\]

\[
+ n \int_{0}^{\hat{\tau}} V (p_1 + \tau_{ds} \Delta T) dG_{DS} (\tau_{ds})
\]

\[
+ n \left[ (1 - G (\hat{\tau})) V (p_0) + V (p_2) \right].
\]

Recall that when \( t^* = t_0 + t_1 - \frac{\Delta p}{2 \tau_{ds}} \) we have

\[
p_0 + \tau_{ds} \|t^* - t_0\| = p_1 + \tau_{ds} \|t^* - t_1\|
\]

so that

\[
p_0 + \tau_{ds} \left\| \frac{\Delta T}{2} - \frac{\Delta p}{2 \tau_{ds}} \right\| = p_1 + \tau_{ds} \left\| \frac{\Delta T}{2} + \frac{\Delta p}{2 \tau_{ds}} \right\|
\]

which further yields

\[
p_0 + (\tau_{ds} - \hat{\tau}) \frac{\Delta T}{2} = p_1 + (\tau_{ds} + \hat{\tau}) \frac{\Delta T}{2}.
\]

Observing also that \( p_0 = p_1 + \hat{\tau} \Delta T \), we end up with

\[
V \left( p_0 + (\tau_{ds} - \hat{\tau}) \frac{\Delta T}{2} \right) = V \left( p_1 + (\tau_{ds} + \hat{\tau}) \frac{\Delta T}{2} \right) = V \left( p_2 + (\tau_{ds} + \hat{\tau}) \frac{\Delta T}{2} \right)
\]

so that we can write

\[
\frac{dS}{dT} = - \int_{\tau_{ds}}^{+\infty} \tau_{ds} \left( \int_{t_{1-\Delta T}}^{t_{1-\Delta T}+\frac{\Delta T}{\tau_{ds}}} x \left( p_0 + \tau_{ds} \|t^* - t_0\| \right) n dt^* \right) dG_{DS} (\tau_{ds})
\]

\[
- \int_{0}^{+\infty} \tau_{ds} \left( \int_{t_{1+\Delta T}}^{t_{1+\Delta T}+\Delta T} x \left( p_2 + \tau_{ds} \|t^* - t_2\| \right) n dt^* \right) dG_{DS} (\tau_{ds})
\]

\[
+ n \left[ (1 - G (\hat{\tau})) V (p_0) + V (p_2) \right] + n \int_{0}^{\hat{\tau}} V (p_1 + \tau_{ds} \Delta T) dG_{DS} (\tau_{ds}).
\]

Using

\[
\frac{t_0 + t_1}{2} - \frac{\Delta p}{2 \tau_{ds}} = t_1 - \frac{\Delta T}{2} \left( 1 + \frac{\hat{\tau}}{\tau_{ds}} \right)
\]

\[
\frac{t_1 + t_2}{2} = t_1 + \frac{\Delta T}{2}.
\]
we can rely on the notation introduced in the text to further express the previous derivative as

\[
\frac{dS}{dT} = -\gamma_1 - \gamma_2 + n \left[ (1 - G (\tau)) V (p_0) + V (p_2) \right] + n \int_0^\tau V (p_1 + \tau \Delta T) dG_D S (\tau ds).
\]

Neglecting the volume effect (the terms in the second line), we obtain (13).

**A.2.3 Case I.3 -** \( p_1 = p_0 = p_h > p_t = p_2 \)

(I) We first prove the distribution of travellers across services.

For all travellers with \( t^* \in [t_0, \frac{t_0 + t_1}{2}] \) it is worth taking \( s_0 \) rather than \( s_1 \). It can be more advantageous to take \( s_2 \) if

\[
p_t + \tau_s (t_2 - t^*) \leq p_h + \tau_s (t^* - t_0) \iff t^* \geq t_0 + \Delta T - \frac{\Delta p}{2 \tau_s}.
\]

When \( \tau_s \leq \frac{t_2}{t} \) the latter inequality holds true for all travellers with \( t^* \in [t_0, t_2] \); hence, all travellers prefer to take \( s_2 \). When \( \tau_s > \frac{t_2}{t} \) there exists \( t_c \in [t_0, t_2] \) such that all travellers with \( t^* \in [t_0, t_c] \) prefer to take \( s_0 \) rather than \( s_2 \). By definition

\[
t_c = t_0 + \Delta T - \frac{\Delta p}{2 \tau_s} = t_1 - \frac{\Delta p}{2 \tau_s},
\]

When \( t_c \in [t_0, \frac{t_0 + t_1}{2}] \) travellers with \( t^* \in [t_0, t_c] \) also prefer to take \( s_0 \) rather than \( s_1 \), hence they choose \( s_0 \). For this to occur it must be the case that

\[
t_1 - \frac{\Delta p}{2 \tau_s} < \frac{t_0 + t_1}{2},
\]

that is \( \frac{t_2}{2} < \tau_s < \hat{\tau}_s \). Travellers with \( t^* \in \left[ t_c, \frac{t_0 + t_1}{2} \right] \) prefer \( s_2 \) to \( s_0 \) and \( s_0 \) to \( s_1 \); hence, they use \( s_2 \).

For travellers with \( t^* \geq \frac{t_0 + t_1}{2} \), \( s_0 \) is never convenient. Travellers with \( t^* \in \left[ \frac{t_0 + t_1}{2}, t_1 \right] \) find \( s_2 \) more convenient than \( s_1 \) if

\[
p_t + \tau_s (t_2 - t^*) \leq p_h + \tau_s (t_1 - t^*),
\]

which is rewritten as \( \tau_s \leq \hat{\tau}_s \). In turn, travellers with \( t^* \in [t_1, t_2] \) find \( s_2 \) more convenient than \( s_1 \) if

\[
p_t + \tau_s (t_2 - t^*) \leq p_h + \tau_s (t^* - t_1),
\]

hence if

\[
t^* \geq t_1 + \frac{1}{2} \left( \Delta T - \frac{\Delta p}{\tau_s} \right),
\]

which is the case when \( \tau_s \leq \hat{\tau}_s \).

To sum up, when \( \frac{t_2}{2} < \tau_s < \hat{\tau}_s \), all travellers with \( t^* \in \left[ \frac{t_0 + t_1}{2}, t_2 \right] \) prefer \( s_2 \) to \( s_1 \) and \( s_1 \) to \( s_0 \); hence, they take \( s_2 \). When \( \tau_s > \hat{\tau}_s \) all travellers with \( t^* \in \left[ t_1, \frac{t_1 + t_2}{2} - \frac{\Delta p}{2 \tau_s} \right] \) find \( s_1 \) more convenient, whereas those with \( t^* \in \left[ \frac{t_1 + t_2}{2} - \frac{\Delta p}{2 \tau_s}, t_2 \right] \) find \( s_2 \) more convenient.
Moreover, all travellers with \( t^* \in \left[ \frac{t_0 + t_1}{2}, t_1 \right] \) prefer \( s_1 \) to \( s_2 \). As they also prefer \( s_1 \) to \( s_0 \), they take \( s_1 \). There is, thus, a discontinuity at \( \tau_s = \tilde{\tau}_s \). When \( \tau_s > \tilde{\tau}_s \) travellers with \( t^* \in \left[ t_0, \frac{t_0 + t_1}{2} \right] \) choose \( s_0 \), travellers with \( t^* \in \left[ \frac{t_0 + t_1}{2} - \frac{\Delta p}{2 \tau_s}, \frac{t_0 + t_1}{2} \right] \) choose \( s_1 \), and travellers with \( t^* \in \left[ \frac{t_1 + t_2}{2} - \frac{\Delta p}{2 \tau_s}, t_2 \right] \) choose \( s_2 \).

\((II)\) We next determine the impact of a variation in schedule on welfare. The surplus respectively associated with \( s_0, s_1 \) and \( s_2 \) is given by

\[
S_0 = \tilde{S}_0 + \tilde{S}_0 \\
= \int_{\frac{t_1}{2}}^{+\infty} \left( \int_{t_0}^{t_1 - \frac{\Delta p}{2 \tau_s}} V \left( p_0 + \tau_{ds} \| t^* - t_0 \| \right) n dt^* \right) dG_{DS} (\tau_{ds}) \\
+ \int_{\frac{t_1}{2}}^{+\infty} \left( \int_{t_0}^{t_1 + t_2 - \frac{\Delta p}{2 \tau_s}} V \left( p_0 + \tau_{ds} \| t^* - t_0 \| \right) n dt^* \right) dG_{DS} (\tau_{ds})
\]

\[
S_1 = \int_{\frac{t_1}{2}}^{+\infty} \left( \int_{t_0}^{t_1 + t_2 - \frac{\Delta p}{2 \tau_s}} V \left( p_0 + \tau_{ds} \| t^* - t_0 \| \right) n dt^* \right) dG_{DS} (\tau_{ds})
\]

\[
S_2 = \tilde{S}_2 + \tilde{S}_2 + \tilde{S}_2 \\
= \int_{\frac{t_1}{2}}^{t_2} \left( \int_{t_0}^{t_2} V \left( p_0 + \tau_{ds} \| t^* - t_0 \| \right) n dt^* \right) dG_{DS} (\tau_{ds}) \\
+ \int_{\frac{t_1}{2}}^{+\infty} \left( \int_{t_0}^{t_1 - \frac{\Delta p}{2 \tau_s}} V \left( p_0 + \tau_{ds} \| t^* - t_0 \| \right) n dt^* \right) dG_{DS} (\tau_{ds}) \\
+ \int_{\frac{t_1}{2}}^{+\infty} \left( \int_{t_0}^{t_1 + t_2 - \frac{\Delta p}{2 \tau_s}} V \left( p_0 + \tau_{ds} \| t^* - t_0 \| \right) n dt^* \right) dG_{DS} (\tau_{ds})
\]

We first compute

\[
\frac{1}{n} \frac{dS_0}{dT} = \frac{d\tilde{\tau}}{dT} \left( \int_{t_0}^{t_1 - \frac{\Delta p}{2 \tau_s}} V \left( p_0 + \tilde{\tau} \| t^* - t_0 \| \right) dt^* \right) g_{DS} (\tilde{\tau}) \]

\[
- \frac{1}{2} \frac{d\tilde{\tau}}{dT} \left( \int_{t_0}^{t_1 - \frac{\Delta p}{2 \tau_s}} V \left( p_0 + \frac{\tau_{ds}}{2} \| t^* - t_0 \| \right) dt^* \right) g_{DS} \left( \frac{\tau_{ds}}{2} \right)
\]

\[
+ \int_{\frac{t_1}{2}}^{+\infty} V \left( p_0 \right) dG_{DS} (\tau_{ds}) - \int_{\frac{t_1}{2}}^{+\infty} \tau_{ds} \left( \int_{t_0}^{t_1 - \frac{\Delta p}{2 \tau_s}} x \left( p_0 + \tau_{ds} \left( t^* - t_0 \right) \right) dt^* \right) dG_{DS} (\tau_{ds}) \]

\[
- \frac{d\tilde{\tau}}{dT} \left( \int_{t_0}^{t_1 + t_2 - \frac{\Delta p}{2 \tau_s}} V \left( p_0 + \tilde{\tau} \| t^* - t_0 \| \right) dt^* \right) g_{DS} (\tilde{\tau}) \\
+ \int_{\frac{t_1}{2}}^{+\infty} \left( \int_{t_0}^{t_1 + t_2 - \frac{\Delta p}{2 \tau_s}} V \left( p_0 + \tau_{ds} \| t^* - t_0 \| \right) dt^* \right) dG_{DS} (\tau_{ds}) \]

\[
- \int_{\frac{t_1}{2}}^{+\infty} \tau_{ds} \left( \int_{t_0}^{t_1 + t_2 - \frac{\Delta p}{2 \tau_s}} x \left( p_0 + \tau_{ds} \left( t^* - t_0 \right) \right) dt^* \right) dG_{DS} (\tau_{ds}),
\]
from which we then obtain

\[
\frac{1}{n} \left\{ ds_0 \right\} = - \int_{t_0}^t \tau_{ds} \left( \int_{t_0}^{t_1 - \frac{\Delta p}{\tau_{ds}}} x (p_0 + \tau_{ds} (t^* - t_0)) \, dt^* \right) \, dG_{DS} (\tau_{ds}) \\
- \int_{\frac{t_0 + t_1}{2}}^{\infty} \tau_{ds} \left( \int_{t_0}^{t_0 + t_1} x (p_0 + \tau_{ds} (t^* - t_0)) \, dt^* \right) \, dG_{DS} (\tau_{ds}) \\
+ \int_{\frac{t_0 + t_1}{2}}^{\infty} \frac{1}{2} \left( \int_{t_0}^{t_1} V (p_0) \, dG_{DS} (\tau_{ds}) \right) \\
- \frac{1}{2} \int_{\frac{t_0 + t_1}{2}}^{\infty} V \left( p_0 + \tau_{ds} \left\| \frac{t_0 + t_1}{2} - t_0 \right\| \right) \, dG_{DS} (\tau_{ds}).
\]

We next compute

\[
\frac{1}{n} \left\{ ds_1 \right\} = - \frac{d\hat{\tau}}{dT} \left( \int_{t_0 + t_1}^{t_1 + t_2 - \frac{\Delta p}{\tau_{ds}}} V (p_1 + \hat{\tau} \left\| t^* - t_1 \right\|) \, dt^* \right) \, g_{DS} (\hat{\tau}) \\
+ \frac{1}{2} \int_{\frac{t_0 + t_1}{2}}^{\infty} V \left( p_1 + \tau_{ds} \left\| \frac{t_1 + t_2}{2} - \frac{\Delta p}{2\tau_{ds}} - t_1 \right\| \right) \, dG_{DS} (\tau_{ds}) \\
+ \frac{1}{2} \int_{\frac{t_0 + t_1}{2}}^{\infty} V \left( p_1 + \tau_{ds} \left\| \frac{t_0 + t_1}{2} - t_1 \right\| \right) \, dG_{DS} (\tau_{ds}),
\]

which is rearranged as

\[
\frac{1}{n} \left\{ ds_1 \right\} = - \frac{d\hat{\tau}}{dT} \left( \int_{t_0 + t_1}^{t_1} V (p_1 + \hat{\tau} \left\| t^* - t_1 \right\|) \, dt^* \right) \, g_{DS} (\hat{\tau}) \\
+ \frac{1}{2} \int_{\frac{t_0 + t_1}{2}}^{\infty} V \left( p_1 + \Delta T \left( \frac{\tau_{ds}}{2} - \hat{\tau} \right) \right) \, dG_{DS} (\tau_{ds}) \\
+ \frac{1}{2} \int_{\frac{t_0 + t_1}{2}}^{\infty} V \left( p_1 + \frac{1}{2} \tau_{ds} \Delta T \right) \, dG_{DS} (\tau_{ds}).
\]
We also compute

\[
\frac{1}{n} \frac{dS_2}{dT} = \frac{1}{2} \frac{d\hat{\tau}}{dT} \left( \int_{t_0}^{t_2} V \left( p_2 + \frac{\hat{\tau}}{2} \| t^* - t_2 \| \right) dt^* \right) g_{DS} \left( \frac{\hat{\tau}}{2} \right) \\
+ \int_{0}^{\frac{\hat{\tau}}{2}} (V(p_2) + V(p_2 + \tau_{ds} \| t_0 - t_2 \|)) dG_{DS} (\tau_{ds}) \\
- \int_{0}^{\frac{\hat{\tau}}{2}} \left( \tau_{ds} \int_{t_0}^{t_2} x \left( p_2 + \tau_{ds} \| t^* - t_2 \| \right) dt^* \right) dG_{DS} (\tau_{ds}) \\
+ \frac{d\hat{\tau}}{dT} \left( \int_{t_1 - \frac{\Delta \rho}{2\rho}}^{t_2} V \left( p_2 + \hat{\tau} \| t^* - t_2 \| \right) dt^* \right) g_{DS} (\hat{\tau}) \\
- \frac{1}{2} \frac{d\hat{\tau}}{dT} \left( \int_{t_1 - \frac{\Delta \rho}{2\rho}}^{t_2} V \left( p_2 + \frac{\hat{\tau}}{2} \| t^* - t_2 \| \right) dt^* \right) g_{DS} \left( \frac{\hat{\tau}}{2} \right) \\
+ \int_{\frac{\hat{\tau}}{2}}^{\hat{\tau}} V(p_2) dG_{DS} (\tau_{ds}) \\
- \int_{\frac{\hat{\tau}}{2}}^{\hat{\tau}} \left( \tau_{ds} \int_{t_{1 - \frac{\Delta \rho}{2\rho}}}^{t_2} x \left( p_2 + \tau_{ds} \| t^* - t_2 \| \right) dt^* \right) dG_{DS} (\tau_{ds}) \\
- \frac{d\hat{\tau}}{dT} \left( \int_{t_{1 - \frac{\Delta \rho}{2\rho}}}^{t_2} V \left( p_2 + \hat{\tau} \| t^* - t_2 \| \right) dt^* \right) g_{DS} (\hat{\tau}) \\
+ \int_{\hat{\tau}}^{+\infty} \left( V(p_2) - \frac{1}{2} V \left( p_2 + \tau_{ds} \left\| \frac{\Delta \rho}{2\tau_{ds}} + \frac{t_2 - t_1}{2} \right\| \right) \right) dG_{DS} (\tau_{ds}) \\
- \int_{\hat{\tau}}^{+\infty} \left( \tau_{ds} \int_{t_{1 - \frac{\Delta \rho}{2\rho}}}^{t_2} x \left( p_2 + \tau_{ds} \| t^* - t_2 \| \right) dt^* \right) dG_{DS} (\tau_{ds}) .
\]

This simplifies to

\[
\frac{1}{n} \frac{dS_2}{dT} = \frac{d\hat{\tau}}{dT} \left( \int_{t_0 + \frac{\hat{\tau}}{2}}^{t_1} V \left( p_2 + \frac{\hat{\tau}}{2} \| t^* - t_2 \| \right) dt^* \right) g_{DS} (\hat{\tau}) \\
+ V(p_2) + \int_{0}^{\frac{\hat{\tau}}{2}} V(p_2 + \tau_{ds} \| t_0 - t_2 \|) dG_{DS} (\tau_{ds}) \\
- \frac{1}{2} \int_{\hat{\tau}}^{+\infty} V \left( p_2 + \Delta T \left( \frac{\hat{\tau}}{2} + \tau_{ds} \right) \right) dG_{DS} (\tau_{ds}) \\
- \int_{0}^{\frac{\hat{\tau}}{2}} \left( \tau_{ds} \int_{t_0}^{t_2} x \left( p_2 + \tau_{ds} \| t^* - t_2 \| \right) dt^* \right) dG_{DS} (\tau_{ds}) \\
- \int_{\frac{\hat{\tau}}{2}}^{\hat{\tau}} \left( \tau_{ds} \int_{t_{1 - \frac{\Delta \rho}{2\rho}}}^{t_2} x \left( p_2 + \tau_{ds} \| t^* - t_2 \| \right) dt^* \right) dG_{DS} (\tau_{ds}) \\
- \int_{\hat{\tau}}^{+\infty} \left( \tau_{ds} \int_{t_{1 - \frac{\Delta \rho}{2\rho}}}^{t_2} x \left( p_2 + \tau_{ds} \| t^* - t_2 \| \right) dt^* \right) dG_{DS} (\tau_{ds}) .
\]

As a result
Given that \( p_1 = p_0 = p_h > p_l = p_2 \), we have \( p_1 = p_2 + \Delta p \) and, hence, \( p_1 = p_2 + \hat{\tau}\Delta T \). We can thus write
\[
p_1 + \hat{\tau} t_1 = p_2 + \hat{\tau} (\Delta T + t_1) = p_2 + \hat{\tau} t_2
\]
and so
\[
p_1 - \frac{1}{2} \hat{\tau}\Delta T = p_2 + \frac{1}{2} \hat{\tau}\Delta T.
\]
Using this equality in the previous computation, we further obtain

\[
\frac{1}{n} \frac{dS}{dT} = \left(1 - G_{DS} \left(\frac{\hat{\tau}}{2}\right)\right) V(p_0) + V(p_2) + \int_0^{\hat{\tau}} V(p_2 + \tau_{ds} 2\Delta T) dG_{DS}(\tau_{ds}) \nonumber \\
- \int_{\hat{\tau}}^{\tau_{ds}} \left(\int_{t_0}^{t_1} x(p_0 + \tau_{ds} (t_* - t_0)) \, dt^* \right) dG_{DS}(\tau_{ds}) 
- \int_{\hat{\tau}}^{+\infty} \tau_{ds} \left(\int_{t_0}^{t_0 + t_1} x(p_0 + \tau_{ds} (t_* - t_0)) \, dt^* \right) dG_{DS}(\tau_{ds}) 
- \int_{\hat{\tau}}^{\tau_{ds}} \tau_{ds} \left(\int_{t_1}^{t_2} x(p_2 + \tau_{ds} \|t_* - t_2\|) \, dt^* \right) dG_{DS}(\tau_{ds}) 
- \int_{\hat{\tau}}^{+\infty} \tau_{ds} \left(\int_{t_1 + t_2}^{+\infty} x(p_2 + \tau_{ds} (t_2 - t^*)) \, dt^* \right) dG_{DS}(\tau_{ds})
\]

Relying on the notation introduced in the text, this is rewritten as

\[
\frac{dS}{dT} = n \left[ \left(1 - G_{DS} \left(\frac{\hat{\tau}}{2}\right)\right) V(p_0) + V(p_2) \right] 
+ n \int_0^{\hat{\tau}} V(p_2 + \tau_{ds} 2\Delta T) dG_{DS}(\tau_{ds}) 
- \frac{\tau_{ds}}{2} \tilde{X}_0 - \frac{\tau_{ds}}{2} \tilde{X}_0 - \frac{\tau_{ds}}{2} \tilde{X}_2 - \frac{\tau_{ds}}{2} \tilde{X}_2
\]

Neglecting the first three terms, which represent volume effects, this ultimately yields (14).

**A.2.4 Case I.4 -** \(p_1 = p_h > p_l = p_0 = p_2\)

The surplus respectively associated with \(s_0\), \(s_1\) and \(s_2\) is given by

\[
\tilde{S}_0 = \int_0^{\hat{\tau}} \left(\int_{t_0}^{t_1} V(p_0 + \tau_{ds} \|t_* - t_0\|) \, n dt^* \right) dG_{DS}(\tau_{ds}) 
+ \int_{\hat{\tau}}^{+\infty} \left(\int_{t_0}^{t_0 + t_1} V(p_0 + \tau_{ds} \|t_* - t_0\|) \, n dt^* \right) dG_{DS}(\tau_{ds})
\]

\[
S_1 = \int_{\hat{\tau}}^{+\infty} \left(\int_{t_0}^{t_0 + t_1 + \frac{\Delta \tau_{ds}}{2}} V(p_1 + \tau_{ds} \|t_* - t_1\|) \, n dt^* \right) dG_{DS}(\tau_{ds})
\]

\[
\tilde{S}_2 = \int_0^{\hat{\tau}} \left(\int_{t_1}^{t_2} V(p_2 + \tau_{ds} \|t_* - t_2\|) \, n dt^* \right) dG_{DS}(\tau_{ds}) 
+ \int_{\hat{\tau}}^{+\infty} \left(\int_{t_1 + t_2}^{+\infty} V(p_2 + \tau_{ds} \|t_* - t_2\|) \, n dt^* \right) dG_{DS}(\tau_{ds})
\]

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We begin by computing
\[
\frac{1}{n} \frac{d\tilde{S}_0}{dT} = \frac{d\tilde{T}}{dT} \left( \int_{t_0}^{t_1} V \left( p_0 + \tilde{T} \|t^* - t_0\| \right) dt^* \right) g_{DS} (\tilde{T})
\]
\[+ \int_{0}^{t_1} V (p_0) \, dG_{DS} (\tau_{ds})
- \int_{0}^{t_1} \tau_{ds} \left( \int_{t_0}^{t_1} x \left( p_0 + \tau_{ds} (t^* - t_0) \right) dt^* \right) dG_{DS} (\tau_{ds})
- \frac{d\tilde{T}}{dT} \left( \int_{t_0}^{t_1} V \left( p_0 + \tilde{T} \|t^* - t_0\| \right) dt^* \right) g_{DS} (\tilde{T})
+ \int_{t_1}^{+\infty} \left( -\frac{1}{2} V \left( p_0 + \frac{1}{2} (\tau_{ds} + \tilde{T}) \Delta T \right) + V (p_0) \right) dG_{DS} (\tau_{ds})
- \int_{t_1}^{+\infty} \tau_{ds} \left( \int_{t_0}^{t_1} x \left( p_0 + \tau_{ds} (t^* - t_0) \right) dt^* \right) dG_{DS} (\tau_{ds}) \right),
\]
which is rearranged as
\[
\frac{1}{n} \frac{d\tilde{S}_0}{dT} = V (p_0) - \frac{1}{2} \int_{t_1}^{+\infty} V \left( p_0 + \frac{1}{2} (\tau_{ds} + \tilde{T}) \Delta T \right) dG_{DS} (\tau_{ds})
- \int_{0}^{t_1} \tau_{ds} \left( \int_{t_0}^{t_1} x \left( p_0 + \tau_{ds} (t^* - t_0) \right) dt^* \right) dG_{DS} (\tau_{ds})
- \int_{t_1}^{+\infty} \tau_{ds} \left( \int_{t_0}^{t_1} x \left( p_0 + \tau_{ds} (t^* - t_0) \right) dt^* \right) dG_{DS} (\tau_{ds}) \right).
\]
We next compute
\[
\frac{1}{n} \frac{dS_1}{dT} = -\frac{d\tilde{T}}{dT} \left( \int_{t_1}^{t_{11}} V \left( p_1 + \tilde{T} \|t^* - t_{11}\| \right) dt^* \right) g_{DS} (\tilde{T})
+ \frac{1}{2} \int_{\tilde{T}}^{+\infty} V \left( p_1 + \frac{\Delta T}{2} (\tau_{ds} - \tilde{T}) \right) dG_{DS} (\tau_{ds})
+ \frac{1}{2} \int_{\tilde{T}}^{+\infty} V \left( p_1 + \frac{\Delta T}{2} \|\tilde{T} - \tau_{ds}\| \right) dt^* dG_{DS} (\tau_{ds})
\]
which yields
\[
\frac{1}{n} \frac{dS_1}{dT} = \int_{\tilde{T}}^{+\infty} V \left( p_1 + \frac{\Delta T}{2} (\tau_{ds} - \tilde{T}) \right) dG_{DS} (\tau_{ds}).
\]
We lastly compute
\[
\frac{1}{n} \frac{d \tilde{S}_2}{d T} = \frac{d \tilde{\tau}}{d T} \left( \int_{t_1}^{t_2} V (p_2 + \tilde{\tau} \| t^* - t_2 \|) \, dt^* \right) g_{DS} (\tilde{\tau}) \\
+ \int_0^{\tilde{\tau}} V (p_2) \, dG_{DS} (\tau_{ds}) - \int_0^{\tilde{\tau}} \tau_{ds} \left( \int_{t_1}^{t_2} x (p_2 + \tau_{ds} (t_2 - t^*)) \, dt^* \right) \, dG_{DS} (\tau_{ds}) \\
- \frac{d \tilde{\tau}}{d T} \left( \int_{t_1}^{t_2} V (p_2 + \tilde{\tau} \| t^* - t_2 \|) \, dt^* \right) g_{DS} (\tilde{\tau}) \\
+ \int_{\tilde{\tau}}^{+\infty} V (p_2) - \frac{1}{2} V \left( p_2 + (\tilde{\tau} + \tau_{ds}) \frac{\Delta T}{2} \right) \, dG_{DS} (\tau_{ds}) \\
- \int_{\tilde{\tau}}^{+\infty} \tau_{ds} \left( \int_{t_1}^{t_2} x (p_2 + \tau_{ds} (t_2 - t^*)) \, dt^* \right) \, dG_{DS} (\tau_{ds}) \\
- \int_{\tilde{\tau}}^{+\infty} \tau_{ds} \left( \int_{t_1}^{t_2} x (p_2 + \tau_{ds} (t_2 - t^*)) \, dt^* \right) \, dG_{DS} (\tau_{ds}),
\]
which is rearranged to obtain
\[
\frac{1}{n} \frac{d \tilde{S}_2}{d T} = V (p_2) - \frac{1}{2} \int_{\tilde{\tau}}^{+\infty} V \left( p_2 + (\tilde{\tau} + \tau_{ds}) \frac{\Delta T}{2} \right) \, dG_{DS} (\tau_{ds}) \\
- \int_{\tilde{\tau}}^{\tilde{\tau}} \tau_{ds} \left( \int_{t_1}^{t_2} x (p_2 + \tau_{ds} (t_2 - t^*)) \, dt^* \right) \, dG_{DS} (\tau_{ds}) \\
- \int_{\tilde{\tau}}^{+\infty} \tau_{ds} \left( \int_{t_1}^{t_2} x (p_2 + \tau_{ds} (t_2 - t^*)) \, dt^* \right) \, dG_{DS} (\tau_{ds}).
\]

Summing up yields
\[
\frac{1}{n} \frac{d S}{d T} = 1 \left( \frac{d \tilde{S}_0}{d T} + \frac{d S_1}{d T} + \frac{d \tilde{S}_2}{d T} \right) \\
= V (p_0) - \frac{1}{2} \int_{\tilde{\tau}}^{+\infty} V \left( p_0 + \frac{\Delta T}{2} (\tau_{ds} + \tilde{\tau}) \right) \, dG_{DS} (\tau_{ds}) \\
+ \int_{\tilde{\tau}}^{+\infty} V \left( p_1 + \frac{\Delta T}{2} (\tau_{ds} - \tilde{\tau}) \right) \, dG_{DS} (\tau_{ds}) \\
+ V (p_2) - \frac{1}{2} \int_{\tilde{\tau}}^{+\infty} V \left( p_2 + \frac{\Delta T}{2} (\tau_{ds} + \tilde{\tau}) \right) \, dG_{DS} (\tau_{ds}) \\
- \int_0^{\tilde{\tau}} \tau_{ds} \left( \int_{t_0}^{t_1} x (p_0 + \tau_{ds} (t^* - t_0)) \, dt^* \right) \, dG_{DS} (\tau_{ds}) \\
- \int_{\tilde{\tau}}^{+\infty} \tau_{ds} \left( \int_{t_0}^{t_1 + t_2} x (p_0 + \tau_{ds} (t^* - t_0)) \, dt^* \right) \, dG_{DS} (\tau_{ds}) \\
- \int_0^{\tilde{\tau}} \tau_{ds} \left( \int_{t_1}^{t_2} x (p_2 + \tau_{ds} (t_2 - t^*)) \, dt^* \right) \, dG_{DS} (\tau_{ds}) \\
- \int_{\tilde{\tau}}^{+\infty} \tau_{ds} \left( \int_{t_1 + t_2}^{t_2} x (p_2 + \tau_{ds} (t_2 - t^*)) \, dt^* \right) \, dG_{DS} (\tau_{ds}).
\]
Considering that \( p_1 = p_h > p_t = p_0 = p_2 \), we can write

\[
p_0 + \frac{\Delta T}{2} (\tau_{ds} + \tilde{\tau}) = p_1 - \Delta p + \frac{\Delta T}{2} (\tau_{ds} + \tilde{\tau})
\]

\[
= p_1 - \tilde{\tau} \Delta T + \frac{\Delta T}{2} (\tau_{ds} + \tilde{\tau})
\]

\[
= p_1 + \frac{\Delta T}{2} (\tau_{ds} - \tilde{\tau})
\]

Using this result in the expression above, we can reformulate to obtain

\[
\frac{dS}{dT} = n \left( V(p_0) + V(p_2) \right) - \int_0^{t_1} \tau_{ds} \left( x(p_0 + \tau_{ds} (t^* - t_0)) \right) n dt^* dG_{DS}(\tau_{ds})
\]

\[
- \int_{\tilde{\tau}}^{+\infty} \tau_{ds} \left( \int_{t_0}^{t_1 + t_1 + \tau_{ds} \frac{\Delta p}{\tau_{ds}}} x(p_0 + \tau_{ds} (t^* - t_0)) \right) n dt^* dG_{DS}(\tau_{ds})
\]

\[
- \int_0^{\tilde{\tau}} \tau_{ds} \left( \int_{t_1}^{t_2} x(p_2 + \tau_{ds} (t_2 - t^*)) \right) n dt^* dG_{DS}(\tau_{ds})
\]

\[
- \int_{\tilde{\tau}}^{+\infty} \tau_{ds} \left( \int_{t_1 + t_2 + \tau_{ds} \frac{\Delta p}{\tau_{ds}}} x(p_2 + \tau_{ds} (t_2 - t^*)) \right) n dt^* dG_{DS}(\tau_{ds})
\]

Making use of the notation introduced in the text, we further rewrite

\[
\frac{dS}{dT} = -\tau_0 X_0 - \tau_0 X_0^+ - \tau_2^- X_2^- - \tau_2 X_2 + n \left( V(p_0) + V(p_2) \right)
\]

Neglecting the bracketed term, which captures the volume effects, (15) is derived.