ASYNMMETRIC LEARNING AND DETERRENCE WHEN THE PROBABILITY OF SANCTION IS UNKNOWN

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Abstract

We analyse the effect on deterrence of uncertainty by potential offenders about the probability of being sanctioned. As lack of information translates into dispersion of beliefs about such probability, some individuals overestimate and some underestimate the expected sanction, and the effect on total deterrence is ambiguous. However, we argue that, under reasonable circumstances, the lack of information is likely to increase deterrence when violations can be repeated. This will be the case when violations convey information about the sanctioning policy only to violators, so that the learning process across the population of potential wrongdoers is not symmetric with respect to the initial beliefs. As a consequence, even when beliefs are initially correct on average, the tend to become biased in the direction of an overestimation of the probability of being sanctioned.

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1. Introduction

The law specifies the size of the sanctions for socially harmful behaviors. However, it is often impossible—and when possible it is in most cases inefficient—to sanction all occurrences of misbehavior, so that the probability of being sanctioned is generally less than one. Scholars have argued that such probability is as important as the size
of the sanction to determine the deterrence effect of a sanctioning policy (Becker, 1968; Polinsky and Shavell, 1979); moreover, uncertainty about being sanctioned can increase deterrence if individuals are risk averse.

While the legal tradition lays emphasis on the importance of specifying precisely the size of the sanction, it is often impossible to be equally precise about the probability that a certain sanction will be actually imposed. Moreover, in many circumstances such probability cannot be observed before the individual decides to commit a violation. As a consequence, in these cases individuals will base behavior on their perception of the probability of being sanctioned. Admittedly, under standard assumptions of maximization of expected utility, more or less uncertainty about the probability of being apprehended and/or convicted should have no effect on the cost of the sanction, as long as the expected value of such probability is unaffected.\footnote{For uncertainty about the probability of being sanctioned to be relevant we need a stronger (and less standard) assumption of ambiguity aversion (Hareland Segal, 1999). Under ambiguity aversion, individuals tend to assign a lower utility to lotteries with unknown probabilities with respect to lotteries whose probability are known, as illustrated by the Ellsberg paradox. Ambiguity averse individuals will be more deterred by a sanctioning policy that does not specify the probability of apprehension. This conclusion has been confirmed by experimental analysis (see for example Baker, Harel and Kugler, 2004).}

However, we should expect that in an uncertain environment, individuals’ estimates of the probability of being sanctioned will be generally inaccurate, and will vary widely among individuals. Indeed, individuals may receive different information about the enforcement policy—they may receive different signals about the “true” probability of being sanctioned—and hence they may have different estimates of this probability. In this sense, more uncertainty will imply that expectations about enforcement will be more disperse across potential offenders. The objective of this paper is to explore the effect on deterrence of different degrees of uncertainty as it translates into different degrees of dispersion of the estimates among potential offenders. We want to analyse whether and under which circumstances the standard economic theory of deterrence supports the generally held view that the sanctioning policy should be made as publicly known as possible. It is not our aim to enter the debate about the desirability in general of certainty in law; we will focus only on the dimension of deterrence, as measured by the number of violations.

To this purpose, we will stick to the standard assumption of rational individual responding to external incentives (i.e. weighing the benefits from violation with the cost of the sanction). Since risk aversion does not play any role in our argument, we further assume that individuals care only for the expected sanction as this is subjectively perceived (risk-neutrality).

Moreover, there is no reason to think that potential offender base their behavior on priors which are systematically biased in one direction or the other. For this reason, a key assumption of our analysis will be that, when the sanctioning policy
is introduced, the estimates of the probability of being sanctioned are correct on average across the population.

We will distinguish between the case of one-shot violation and repeated violations; the latter case refers to situations in which information acquired in one period is useful to assess the probability of being sanctioned in the following periods. We will show that, when violation is one-shot, identifying sufficient conditions for more certainty about the probability of being sanctioned to imply more deterrence is straightforward.

When instead the decision to violate is repeated over time, we must take into account the possibility that individuals learn from past actions, and that learning is affected by previous decisions to violate. It can be convincingly argued that the observation of the sanctioning policy reduces uncertainty and dispersion of beliefs. However, the interesting case is when information about sanctions is not publicly available, so that individuals must update their beliefs based on their own history of violations and sanctions. Indeed, it is possible to identify many cases in which individuals observe the sanctioning policy (more specifically: whether the violation is actually detected and sanctioned) only when they actually commit a violation. This is the case for example when drivers are notified a fine if they have passed a speed limit, but cannot observe sanctions on other individuals passing the limit. In cases like this, committing a violation is itself a way, possibly the only way, to obtain information about the probability of being sanctioned.

This brings about an asymmetry between those who violate and those who do not: the expectations of violators is updated, while individuals who do not violate will continue to base their choices on their initial prior; as time passes and priors are updated, a bias is determined in the distribution of expectations, with the consequence that, on average, individuals in our population will overestimate the probability of being sanctioned. Hence, an important conclusion of our analysis is that, in a multiperiod environment, even if expectations about a new sanctioning policy are initially unbiased, over time asymmetric learning makes them biased. As we will see, this has some consequences on the effect of disclosure policy, and can help explain some puzzles which have been pointed out in the literature on law enforcement.

A few other papers have considered the effect on enforcement of individuals being imperfectly informed about the probability of being sanctioning. Bebchuk and Kaplow (1992) consider how imperfect information can affect the optimal size of the sanction, and show that, when errors are independent of the probability, the optimal sanction is not maximal. Garoupa (1999) extends this result by assuming different possible relations between the probability and errors and explicitly con-

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2The one-shot period case extends to circumstances in which we have multiple rounds of violations but the probability of sanctioning is decided independently in each period.
siders disclosure policy. Sah (1991) and Ben-Shahar (1997), have considered the possibility that wrongdoers learn from past actions, and have discussed how this affects the optimal enforcement policy.\(^3\)

However, we depart from previous contributions in several regards. First of all, this paper has a different focus: we are not concerned with the problem of setting the optimal \(p\) and \(S\), which we take as given. Instead, our primary interest is in the effect of different degrees of uncertainty (as it translates into dispersion of expectations across the population) on deterrence (as measured by the frequency of violations). We think that this problem has an interest of its own for the policymaker and for the analyst concerned with the effect of different sanctioning policies on the crime rate.

Additionally, our paper is more general than previous contributions in the way uncertainty is modelized. Here, it suffices to say that in previous contributions errors are usually assumed to be symmetric around the “true” probability and the distribution of benefit is assumed to be uniform; by removing these restrictions we allow for a wider array of possibilities in the relation between uncertainty and deterrence.

The paper is organized as follows. In section 2 we will consider a simple model where potential wrongdoers choose only once, conditional on their idiosyncratic information, whether to violate or not. In section 3 we extend the model to the case of repeated decision to violate and introduce the possibility that the individuals learn from past actions. In section 4 we consider some implications of our model, namely whether disclosure of information on the sanctioning policy can increase deterrence. Section 5 concludes.

2. One-shot violations

Let \(V^h \geq 0\) be the benefit from wrongdoing (committing an offence) for individual \(h\). Let \(S\) be the sanction and \(p\) the probability of apprehension; \(S\) and \(p\) are exogenously given and assumed to be fixed throughout the paper.\(^4\)

Probability \(p\) is unknown to the individual, and \(G^h\) is the probability distribution representing \(h\)’s uncertainty about \(p\); \(G^h\) differs across individuals, reflecting different information they may have received about \(p\).

More specifically, we assume that some individuals have received a signal suggesting that the probability of sanction is low, while the others have received a signal

\(^3\)Instead, Garoupa and Jellal (2004) consider the effect of learning by the enforcer on the optimal sanctioning policy.

\(^4\)The model can be easily reinterpreted to include the case in which uncertainty is about the size of the sanction, provided this has an upper bound. Let the sanction be \(R \leq S\) where \(S\) is the upper bound, so that we can write \(R = rS\) with \(r \in [0, 1]\). If \(q\) is the probability that there is a sanction, the expected sanction is \(qrS\), where \(qr \in [0, 1]\). If we set \(p = rq\), under risk neutrality this case is encompassed by our formalization.
suggesting the probability is high. In other words, the probability distribution $G^h$ can be either $G_L$ or $G_H$, with $G_L(p) > G_H(p)$ for $p \in (0, 1)$, so that $L$ and $H$ identify two “types” of individuals, who we will be named respectively “optimists” and “pessimists”. The corresponding expected values $\hat{p}_L$ and $\hat{p}_H$ satisfy $\hat{p}_L < \hat{p}_H$.

We assume that $\hat{p}_L < p < \hat{p}_H$, and we rule out the case that they underestimate or overestimate $p$ systematically in the aggregate by assuming that on average they have a correct perception of $p$:

$$\eta \hat{p}_H + (1 - \eta) \hat{p}_L = p$$

(1)

where $\eta$ is the share of individuals whose prior is $G_H$. This setting can be easily rationalized in a bayesian setting as follows. Consider a population of individuals with a common unbiased prior on $p$ represented by a distribution $\tilde{G}$. Each individual privately has access to information about whether an occurred violation has been sanctioned or not. Therefore, a fraction $\eta = p$ of the individuals will observe a sanction and will update their prior upward ($G_H$), while the others will update it downward ($G_L$).

This setting could be made richer (and more complicated) by considering $n > 1$ observations, in which case the number of different “groups” of individuals will be equal to $n + 1$.

As already stated above, we assume that individuals are risk-neutral. This implies that $h$ will commit the offence if

$$V^h > E^h(p)S.$$  

(2)

Individuals differ as to the benefit from wrongdoing. We assume $V^h$’s are drawn from a continuous random variable $V \geq 0$ whose distribution is independent of $G^h$; the cumulative distribution function of $V$ is $F$, and $f$ is the corresponding density function.

We will compare from the point of view of (total or aggregate) deterrence the case in which $p$ is unknown with the case in which all individuals know $p$ with certainty. In the former case, the total probability of violation is

$$\phi_U(S) \equiv \Pr\{ V > \hat{p}S \} = (1 - \eta) \int_{\hat{p}S}^{\infty} f(V)dV + \eta \int_{\hat{p}_H S}^{\hat{p}_L S} f(V)dV$$

(3)

while in the latter case (when $p$ is known to everybody):

$$\phi_I(S) \equiv \Pr\{ V > pS \} = \int_{pS}^{\infty} f(V)dV.$$  

(4)

Specifically, if the prior distribution is represented by a Beta$(a, b)$ with mean $p = a/(a + b)$, the posterior distributions will be respectively Beta$(a + 1, b)$ and Beta$(a, b + 1)$, with $\hat{p}_H = (a + 1)/(a + b + 1)$ and $\hat{p}_L = a/(a + b + 1)$, and it is easy to check that equation (1) will be satisfied.
The difference $\phi_U(S) - \phi_I(S)$ can be written
\[
(1 - \eta) \int_{\hat{p}_L S}^{p_S} f(V) dV - \eta \int_{p_S}^{\hat{p}_H S} f(V) dV
\]
when this is positive, lack of information increases the probability of an offence (it decreases deterrence).

When individuals learn the true $p$ there are two effects on deterrence: on the one hand, individuals with a $V$ higher than $pS$ who overestimated the probability of sanction will be now induced to violate. On the other, those with a lower valuation who underestimated $pS$ will be deterred. The net effect on the total amount of offences depends on the numerosity of the two groups of individuals, i.e. on the distribution of $V$ and on $\eta$.

This comparison can be illustrated graphically in figure 1 where the curve represents the density function $f$. The area below the function on the right of the line $V > pS$ gives the probability of violation when individuals know $p$, while the area on the right of the bisetrix gives the probability of violation under the two possible estimates $\hat{p}_H$ and $\hat{p}_L$. Thus, the increase in probability of violation in the first group ($\hat{p}_H$) is represented by area B, while the decrease in the probability in the second group ($\hat{p}_L$) is represented by area A. Whenever the (weighted) difference between A and B is positive, we have $\phi_U(S) > \phi_I(S)$, so that learning about $p$ brings about an increase in deterrence.

Although it is not possible to say in general whether there is an increase or a decrease in the frequency of offences, some conclusions can be reached if we restrict the class of admissible distributions by requiring that $f$ is either decreasing.
or unimodal (i.e. single-peaked) in the open interval \( V > 0 \). We are able to prove the following:

**Proposition 1.** (1) Assume \( f \) is always decreasing for \( V \geq 0 \). Then \( \phi_U(S) > \phi_I(S) \) for all \( S > 0 \). (2) Assume \( f \) is unimodal; then there exist \( \hat{S} \geq 0 \) such that \( \phi_U(S) > \phi_I(S) \) for \( S > \hat{S} \), and \( \phi_U(S) < \phi_I(S) \) otherwise.

**Proof.** See Appendix.

Decreasing distributions and single peaked distributions include most of continuous distributions on \( V > 0 \). It is possible in abstract to imagine distributions having higher frequencies at low and high values, but it is not clear if this may correspond to any realistic situation.

When the frequency is lower the higher is the benefit, the clear cut conclusion is that there is no gain from uncertainty about \( p \). For single peaked distributions (first increasing and then decreasing) we have that there always exists a \( S \) high enough that uncertainty brings no benefits.

The only circumstance in which uncertainty might correspond to higher deterrence is when \( S \) is constrained to be low and the distribution is unimodal (so that \( f \) is increasing at \( pS \)). In this case, by leaving some uncertainty on \( p \) and letting individuals rely on their idiosyncratic information, the enforcer can improve deterrence. Therefore, the possibility to benefit from uncertainty is associated with low powered sanctions.

It is less straightforward to analyze the effect of different values of \( p \) on the sign of expression (5), since in this case we should also specify how the expectations \( \hat{p}_L \) and \( \hat{p}_H \) vary across the population as \( p \) varies—in our setting the expectation are taken to be endogenous and satisfy the unbiasedness condition (1). In this regard, a minimum reasonable assumption is that, when we consider a different probability \( p' > p \), the corresponding expectations \( \hat{p}'_L \) and \( \hat{p}'_H \) satisfy \( \hat{p}'_L \geq \hat{p}_L \) and \( \hat{p}'_H \geq \hat{p}_H \). This turns out to be sufficient to state the following

**Proposition 2.** Consider \( p' > p \) and assume that the corresponding expectations satisfy \( \hat{p}'_L \geq \hat{p}_L \) and \( \hat{p}'_H \geq \hat{p}_H \). Then, if \( f \) is single-peaked, \( \hat{S}' \leq \hat{S} \), with strict inequality when \( \hat{S} > 0 \).

\footnote{If \( \hat{p}_H \) and \( \hat{p}_L \) are derived by bayesian update from a common prior, the way they are affected by a change of \( p \) depends on such prior. Moreover, in order to grant that condition (1) is satisfied for different values of \( p \), we must assume that the priors are different too: only when the prior gives an unbiased estimate of \( p \) we have that posteriors are unbiased on average. However, there are many way to modify the prior so that the expect under risk neutralized value is kept equal to \( p \), and this introduces a degree of indeterminateness in the procedure. A possible way out is to assume that the prior is completely non-informative so that \( \hat{p}_H \) and \( \hat{p}_L \) only reflects past observation of sanctioning policy. In this case, it will be \( \hat{p}_L = 0 \), \( \hat{p}_H = 1 \) and \( \eta = p \).}
Proof. See Appendix.

This is related to the previous result: with a single-peaked distribution, a higher $p$ is substitute for a higher $S$ in making sure that information about $p$ increases deterrence. However, note that the two propositions together do not imply that information on $p$ is beneficial above a specific level of the expected sanction $pS$, since from our point of view an increase in $p$ is not equivalent to an increase in $S$ in the same proportion.\footnote{Namely, an increase in $p$ affects the distribution of perceptions of the expected sanction without changing the support (e.g. minumum and maximum expected sanctions are unaffected) while an increase in $S$ scales the whole suport of the distribution.}

2.1. Welfare analysis

We stated in the introduction that our focus is in positive rather than normative analysis. However, it is useful to relate our result to the analysis of optimal enforcement in terms of welfare. Looking at welfare rather than simply at deterrence, we should weight the cost and benefit of offences: most of the literature on optimal sanctions considers that $V$, the benefit of the offender, should be counted as social benefit. It is easy to conclude that this reinforces the case for disclosing information on $p$ to potential wrongdoers.

The reason is that even if the number of those who overestimate $p$ (pessimists, or area B in figure 1) is higher than the number of those underestimating it (optimists, or area A), so that the number of violation is lower with uncertainty, the benefit from violation of pessimists is higher than the the benefit of optimists, hence their violations should be weighted differently in terms of welfare.

Moreover, if the expected sanction reflects the social cost of a violation, uncertainty may result in overdeterrence, i.e. some individuals are deterred whose benefit from violation is larger than the social cost. This will be the case for those $h$ with $\hat{p}_H > V^h > C$, where $C$ is the social cost of offence.

In formal terms, assuming that $C$ is the same for all violations, the change in social welfare from eliminating uncertainty on $p$ is:

$$ (1 - \eta) \int_{\hat{p}_S}^{pS} (C - V) f(V) dV - \eta \int_{pS}^{\hat{p}_H} (C - V) f(V) dV. \tag{6} $$

We assume $pS \leq C$. In doing so we rule out the case that the sanction is so high that there is overdeterrence of violations, as this cannot be part of an optimal enforcement strategy, nor it is reasonable to assume that either $p$ or $S$ is constrained from below. On the contrary, it is well possible that the optimal strategy implies $pS < C$ or that the size of the sanction and the probability is constrained from above.

We state the following:
Proposition 3. A necessary condition for welfare not to be higher when \( p \) is known is that \( p_S < C \) and deterrence is higher when \( p \) is unknown.

Proof. It must be shown that \( p_S = C \) and a nonnegative (5) are each a sufficient condition for (6) to be positive. This is shown in the Appendix.

The proposition implies that when \( p_S = C \) knowledge of \( p \) is always beneficial in terms of welfare, as both terms in (6) are positive. If the expected sanction equals the negative externality produced by the offence, the sanction can be thought of as a “price” to guide individuals’ choices, and it is important that it provides a signal as free of noise as possible of the social cost of each action.

Under \( p_S < C \), it is always better in terms of welfare that individuals are informed about \( p \) as long as uncertainty does not increase deterrence. On the other hand, when uncertainty increases deterrence, the sign of the welfare change depends on \( f \). Note that, though we expect this is not the case in most circumstances, in principle it is always possible to find a density function \( f \) such that welfare is higher with uncertainty. This is to say that without further restrictions we cannot be sure that less uncertainty about \( p \) is welfare reducing.

3. Repeated violations and learning

We will now consider the case in which the decision to violate is repeated over time, and individuals can learn from the consequences of past violations. We will show that in the case of repeated violations, under reasonable circumstances, there is a tendency of expectations to be biased towards more pessimism—i.e. in the population of potential wrongdoers we have an overestimation of the probability of being sanctioned. As a consequence, the case for more uncertainty is reinforced.

The crucial assumption here is that violating provides information to violators about the sanctioning policy which is not available to non-violators. A violator, whether he is caught or not, receives a signal on the effectiveness of enforcement (although of course such signal will be different whether he is or he is not caught and sanctioned). To illustrate, think of speed cameras: it is well known that a number of them are turned off, but a car driver doesn’t know, until she makes a “test” by passing the speed limit in front of it, whether a camera is actually working or not; she will know when she receives (or else if, after a reasonable time, she does not receive) the ticket from the police. In contrast, a driver who does not violate is not able to infer, from the fact he hasn’t been sanctioned, whether his behavior has been monitored or not.

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8For welfare to increase with uncertainty, \( f \) must assign a high probability that \( V \) is between \( p_S \) and \( \min\{\hat{p}_H, S, C\} \).
Violators can experience directly the intensity of controls and detection effort, while those who do not violate will have no way to improve their prior about the probability \( p \) of being sanctioned. This seems reasonable in many instances similar to the case of a speed camera; namely, when only a subset of individuals is subject to inspection or control by the authority, and the individual is not aware of being monitored until he or she is notified a violation. It is crucial to assume that, additionally, individuals observe only their own past violations and sanctions, and not violations and sanctions by others.\(^9\)

To summarize, we will consider here the case that individuals obtain additional information \emph{if and only if} they violate. This asymmetry in learning reminds the one described in Ben-Shahar (1997), although there is an important difference: in that model the individual learns the true probability only if he is caught, while we assume that violation is enough to observe the sanctioning policy. It is certainly possible to find circumstances when being caught and sanctioned gives the offender additional information, not available if he violates but is not caught. However, this seems more relevant when uncertainty is about the size of the sanction than when it is about the probability of being sanctioned, which is the focus of our analysis.

Since we expect to find a higher share of wrongdoers among those who underestimate \( p \) (pessimist), the speed of learning will be different among them than among those who overestimate \( p \) (optimists). As a consequence, learning induces a bias in expectation. Namely, it decrease the number of optimists more than it decreases the number of pessimists. Hence, even if initial expectation are different, the learning-by-violating process makes the distribution of expectations about the sanctioning policy biased towards pessimism. Because such a bias increases the average expectation of being sanctioned, the effect is an increase in deterrence.

### 3.1. Numerical simulations

The study of learning process can easily become extremely difficult from an analytical point of view. Therefore, we illustrate the effect of learning through numerical simulations. To make the framework as simple as possible, we consider

- a population of \( N \) individuals, whose initial priors are heterogeneous (but unbiased in the aggregate, as discussed in the previous sections), as described in the one-shot model above;

- in each period \( t \), each individual \( h \) has an opportunity to gain from violating a norm; his/her benefit from violation in that period, \( V_t^h \), is drawn from the distribution \( F \),

\(^9\)Of course, there are cases in which the individual is aware of the inspection whether he/she has violated or not, or the individual can observe whether other individuals violate and are sanctioned or escape the sanction. Our analysis does not apply to these cases.
which is the same for all individuals (i.e. it is independent on their beliefs on the sanction, as in the one-shot model) and for all periods (i.e. it is independent on passed behavior and sanctions received);

- in each period $t$, an individual $h$ violates if and only if expected benefit is larger than expected sanction, or $V^h_t > p^h_t S$, where $p^h_t$ is the expected probability of being sanctioned, which depends on the initial prior of the individual and the past history of violations and sanctions (see below);

- individuals who decided to violate, whether they are sanctioned or not, update (through Bayes’ rule) their beliefs about the sanctioning probability, and each bases their decision to violate on the updated value of $p^h_t$;

- in the first period, the beliefs on $p$ are based on an exogenous idiosyncratic signal received by the individual, as in the one-shot case.

As we have already argued in section 2, a consistent way to formalize heterogeneity of beliefs is to assume that they are derived by Bayesian updating from a common prior distribution beta($a$, $b$) with mean equal to the true probability $p$ (i.e. $a/(a + b) = p$). Depending on the signal they have received, individuals will be either “pessimists” with posterior distribution beta($a+1$, $b$), or “optimists” with distribution beta($a$, $b+1$). The initial probabilities on which individuals will base their decisions will be respectively $\hat{p}_L = a/(a + b + 1)$ and $\hat{p}_H = (a + 1)/(a + b + 1)$.

Because the posterior probability is still a beta, we describe the distribution (of $p$) of individual $h$ in period $r$ by two parameters $a^h_r$ and $b^h_r$, whose evolution, calculated using the Bayes’ rule, depends on previous decision to violate and the sanctioning policy as follows:

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10See discussion in section 2 on this point.
Recalling that the expected value of $p$ is $p^h = a^h / (a^h + b^h)$, it is immediate to check that if $h$ does not violate it is $p^h = p^h_{t-1}$ (beliefs about the probability of being sanctioned is unchanged), while, if $h$ violates, it is $p^h_t > p^h_{t-1}$ if $h$ is sanctioned, and $p^h_t < p^h_{t-1}$ if $h$ is not sanctioned.

In order to carry out our simulation, we assume that the initial (period zero) distributions of $p$ for optimists and pessimists are described respectively by probabilities distributions beta(.8, 1.2) and a beta(1.8, .2). It follows that the expected values of $p$ are .4 and .9 respectively for optimists and pessimists, whose shares in the population are .2 and .8 (consistent with the true probabilities).\footnote{As explained in section 2, this initial setting can be thought of as derived from a common “true” distribution beta(.8, .2), with 80% of the population observing a violation which is sanctioned (hence becoming “pessimist”), and 20% observing a violation which is not sanctioned (“optimists”).}

To calculate the frequency of violations we need to know the distribution of $V$, or equivalently (for $S$ fixed) the distribution of $V/S$. For simplicity, $V$ is assumed to be uniformly distributed on the interval between zero and $S$ (so that the probability of violation is always positive for $p < 1$). This implies that the frequency of violations in the population at time $t$ will be simply $1 - p_t$, where $p_t$ is the average expected value of $p$ (which can differ from the “true” $p$).

If individuals knew from the start the true probability of being sanctioned ($p = .8$), the violation rate would be always 20% (corresponding, under our assumption, to the probability that $p < V/S \leq 1$).

To see the effect of uncertainty and learning, we run the simulation with a population of 10,000 individuals “playing” 100 rounds, and calculate:

1. total number of violations in the first $T$ rounds, with $T = 10$, $T = 30$ and $T = 100$. In parenthesis we indicate the frequency as a percentage of total decisions taken. We compare the result of our simulation with the “certainty” case in which individuals know the true probability $p$, and calculate the cumulated difference in violations. This is negative, meaning that deterrence is increases by uncertainty.

<table>
<thead>
<tr>
<th>$T$</th>
<th># violations</th>
<th>$\Delta$% with respect to certainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>17144 (17.1%)</td>
<td>-14.5%</td>
</tr>
<tr>
<td>20</td>
<td>32834 (16.4%)</td>
<td>-17.9%</td>
</tr>
<tr>
<td>100</td>
<td>154851 (15.5%)</td>
<td>-22.6%</td>
</tr>
</tbody>
</table>

2. The distribution of expected $p$ across the population of potential wrongdoers at different times (namely: $t = 1, 2, 5, 10, 20$ and in our final round $t = 100$) is illustrated in figure 2. In the first round the population is divided between optimists (expecting $p = .4$) and pessimists (expecting $p = .9$),
Figure 2 How the distribution of expectations on $p$ across the population evolves over time
As times passes and people decides to violate and observe the sanctioning policy, expectations are updated and tend to converge to the “true” probability $p$. Note however that such convergence is faster for optimists, who tend to violate more than pessimists, and this produces a skewed distribution. In our example, being the probability of sanctioning very high and the probability that a pessimist violates low, the convergence of pessimist towards $.8$ (the true value) is extremely slow.

In our pictures, the grey part of each bar represents individuals who decide to violate at that round. The vertical dashed line identifies the average expected value of $p$ in the population in that period (after the first period, it is always larger than the “true” probability $.8$).

3. Figure 3 shows the evolution of the average expected $p$ across the population over time. We see that this value departs quickly from the true value $.8$, it increases by five percent points and then declines slowly.

4. Finally, figure 4 illustrates the number of violations at each round. This is always below the level we have with certainty $(2,000, or 20\% of the population). The shape of the curve representing violations mirrors in some way the curve of average probabilities in figure 3, although violations depend also on the dispersion of expected probability, not only on their average in the population.

As time goes by, individuals will collect signals that make their expectations more and more accurate;\footnote{This requires that there is a positive probability of violation even when the expected probability of being sanctioned is close to one. Otherwise, there might be individuals who never violate and never update their beliefs.} this process can be very slow if the frequency of violation is very low for “pessimist” individuals, but for high enough number of rounds, all expectations will converge to the true probability $p$. Indeed, the signals received when violation takes place are unbiased for all individuals; aggregate beliefs are biased because such convergence takes place at different speed depending on the initial expectation of each individual, with higher speed for those who assign a low probability of being sanctioned.

### 3.2. Qualifications

There are a number of qualification to this result.

**Distribution of benefits from violation.** The assumption of uniform distribution has the implication that a symmetric spread of expectations $p^h_t$ around the true value $p$ leaves the frequency of violations unchanged. With other distributions, the effect
Figure 3  How the average expectation of $p$ in the population evolves over time

Figure 4  Frequency of violations over time
may be different due to the asymmetric effect on violation of an increase and a decrease of $p^h$.

For example, by repeating the simulation with a distribution which is decreasing around $p$, we see that the frequency of violation increases with uncertainty.

What we observe in this case is that in the initial stages the frequency of violation is higher than in the initial rounds, while it is lower later on, and it might finally approaches the frequency in the case of certainty. This is illustrated by figure 5, which differs from figure 4 in that we have replaces the uniform distribution with a beta defined on the same interval. In this case, we have a higher frequency of violations in the first periods, but after some learning the frequency is well below the level with certainty.

In general, whether asymmetric learning due to violation is still able to determine an overall reduction of violations over time depends on the parameters of the model, on the specific shape of the distributions.

*Observing others’ offenses* As we have already emphasized, the argument above relies on the assumption that the individual cannot observe the sanctioning policy with regard to other people (i.e. he cannot observe either the application of sanctions or the number of total offenses). Though reasonable in many cases, this assumption

*Figure 5* Frequency of violations when $V^h_i$ are extracted from a beta(2, 4) distribution
might be too strong in other cases where people interact and observe one another.

However, relaxing this assumption does not necessarily weaken our claim. It might be that the individual is closer to individuals with the same or similar $V^h$ and $E^h(p)$. In this case, violators will tend to observe the behavior of other violators. This has a reinforcing effect on our argument, as (1) uncertainty about $p$ will be more effectively reduced relying on the information about many violators, and (2) violators will free ride on the investment in information.

Hence, it is safe to conclude that the asymmetry of the effect is reinforced in the case individuals can observe the behavior of other individuals with similar characteristics.

**Strategic violation in the first period** The increase in deterrence in the second and following periods that we have described should be balanced against an additional negative effect of uncertainty we may have in the first period. The fact that by violating in the first period the individual has the opportunity to know the true $p$ when this is uncertain can encourage violation: the reason is that by violating in the first period, the individual will make an “investment” whose return is a better informed choice in the following periods. The value of information is the difference between the expected value from the violation using the prior and the posterior estimates.

This effect has not been considered in the simulation. We expect that this additional return from violation can increase violations (reduce deterrence) in the first periods.

4. **The effect of information disclosure at later periods**

An important implication of our model is that the disclosure of information about the probability of sanctioning can have different effects on deterrence depending on when it is done.

Once investments in information through violation have been made by individuals, there is not much gain from disclosing the true value of $p$, since the main effect will be to induce pessimistic individuals to revise downward their expectations on the probability of being sanctioned. This will reduce deterrence.

Therefore, information disclosure is most effective in increasing deterrence when it is done before an asymmetry of information between pessimists and optimists is generated by the differential effect of learning.

This conclusion can help us understand the allegedly paradoxical effect of substituting explicit sanctions for informal ones, which has been noted in many studies on deterrence. In a celebrated paper, Gneezy and Rustichini (2000) reported the case of a group of day-care centers where parents used to arrive late to collect their chil-
dren. Contrary to what standard theory of deterrence predicts, the introduction of a (mild) monetary fine for late-comers increased the number of violations; moreover, the subsequent removal of the fine leaved the frequency of violations unchanged. Gneezy and Rustichini offer two possible explanations for this result: first, and more consistent with standard economic reasoning, the imposition of a fine makes sure the consequences of parents’ action, which had been left unspecified in the initial (incomplete) contract with the school. As an alternative, the fine reshapes agents’ perception of the environment in which they operate, and causes them to adhere to a different social norm.

The interpretation suggested by our model is similar in its spirit to the first of the two explanations (although it is possibly simpler in how the interaction among violators and the authority is characterized). We can interpret the change in sanctioning policy as a substitution of a formal fine for an informal sanction (reproval, a possible bad attitude of teachers towards the child’s family, etc) whose actual application was uncertain for parents. Our claim that the fine replaces informal sanctions rather than supplementing it is simply a conjecture, but we think it is reasonable to assume that teachers reduce their informal sanctions to parents when the fine is introduced.\textsuperscript{13}

As long as the specification of the fine makes it clear that there will be a sanction, while it does not increase it (or does not increase it too much) with respect to the previous informal and uncertain sanction, our model predicts that deterrence may decrease. By making the sanction publicly known at later periods, school authorities reduce deterrence among pessimistic parents—those who expected a high probability of reproval by teachers for late arrivals, and hence believed the sanction was higher than it actually was. On the other hand, deterrence of optimistic parents will not change much if such parents, by violating in previous period, observed (and hence learned) whether an informal sanction was levied and how large it was.

5. Conclusions

Within the standard model of rational agents we have developed it is not possible to draw neat general conclusions on whether information on $p$ will increase or decrease deterrence. According to our analysis, it is likely that in most cases information reduces the frequency of violation on average. However, we have tried to give some hints on circumstances that make lack of information beneficial from the point of view of deterrence.

\textsuperscript{13}This can be rationalized by thinking of deterrence of latecomers as a public good from the point of view of teachers, to which they voluntarily contribute by imposing informal sanctions; crowding out of voluntary sanctions by public sanctioning is consistent with what economic analysis would predict in this case.
A first aspect to consider is whether \( p \) and \( S \) are low or high with respect to the distribution of \( V \).

A second important aspect has to do with the circumstance that the offence can be repeated rather than being a one-shot decision. In this case, first period decisions convey information useful for future periods. We have emphasized that the opportunity to gain additional information may not be symmetric among individuals, as offenders have the opportunity to learn \( p \) while non-offenders have not. This results in a positive effect of uncertainty on deterrence, depending on the fact that with uncertainty a share of those who overestimate \( p \) will not violate even if it would be in their interest should they know the true \( p \).

Although depending on the distribution of the variables and expectations we can have different results, a prediction of this model is that, in a multi-period environment with a stable population, the effect of an unanticipated decrease in uncertainty in later periods will be a decrease in deterrence. In this case, the effect of an announce of the “true” \( p \) will affect only the group of those who do not violate because they overestimate \( p \). The revision of their expectation on \( p \) will cause an increase in the number of offences.

A third aspect has to do with whether individuals can observe sanctions imposed on others and whether others are “similar” in terms of expectation about the probability of being sanctioned.

Our model may be also useful to study the dynamics of deterrence when information is provided in later periods. In particular, it can provide an explanation of the well known result found by Gneezy and Rustichini (2000), where it is shown that when a fine is introduced in a context where the sanctioning policy is undefined, the number of violation may increase.

Although some of our results may suggest what ist he optimal disclosure policy by a welfare maximizing government, there may some difficulties in making the current model a normative one. The reason is that, if disclosure is a policy variable, a sophisticated individual may be able to make some inference on the optimal choice of \( p \) by the government. Namely, as long as the choice to disclose is related to the true value of the probability, a sophisticated individual could infer some information on \( p \) from the fact that the government chose not to disclose this value. For example, if the individual forms some expectations on the distribution of \( p \), and according to her expectation the government should disclose, the fact that the information is not disclosed should force her to modify her expectation. It is extremely complex to take into account such effects, and if the individuals have limited information about the payoff function of the government and its ability to decide whether to disclose or not, the inference may be very poor. However, a more thorough exploration of what the strategic interaction between an informed government and a sophisticated individual might be worthwhile, and be the focus for future analysis.
Appendix. Proof of propositions

We begin with the following

**Lemma 1.** Let \( f \) be a unimodal density function defined on \( x \geq 0 \). Let \( 0 \leq a < b < c \) such that

\[
\frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{c-b} \int_b^c f(x)dx = 0 \tag{7}
\]

Then

i. \( f(b) \geq \max\{f(a), f(c)\} \);
ii. the left hand side of (7) is increasing in \( a, b \) and \( c \);
iii. if \( a' \geq a, b' > b \) and \( c' > c \), it is

\[
\frac{1}{b'-a'} \int_{a'}^{b'} f(x)dx - \frac{1}{c'-b'} \int_{b'}^{c'} f(x)dx > 0 \tag{8}
\]

\( \square \)

**Proof.** From the fact that \( f \) is unimodal follows that, for any \( z_1 \) and \( z_2 \),

\[
f(x) \geq \min\{f(z_1), f(z_2)\} \quad \text{for all } x \in [z_1, z_2]. \tag{9}
\]

Moreover, for any \( z_1 < z_2 < z_3 \),

\[
f(z_1) < f(z_2) < f(z_3) \implies \begin{cases} f(x) \geq f(z_2) & \text{for } x \in [z_2, z_3] \\ f(z_1) \leq f(x) \leq f(z_2) & \text{for } x \in [z_1, z_2] \end{cases} \tag{10}
\]

and, similarly,

\[
f(z_1) > f(z_2) > f(z_3) \implies \begin{cases} f(x) \geq f(z_2) & \text{for } x \in [z_1, z_2] \\ f(z_2) \leq f(x) \leq f(z_3) & \text{for } x \in [z_2, z_3]. \end{cases} \tag{11}
\]

Hence, if \( f \) were such that \( f(a) < f(b) < f(c) \), we would have \( f(x) \geq f(b) \) for all \( x \in [b, c] \) and \( f(a) \leq f(x) \leq f(b) \) for all \( x \in [a, b] \). Similarly, if it were \( f(a) > f(b) > f(c) \), we would have \( f(x) \geq f(b) \) for all \( x \in [a, b] \) and \( f(c) \leq f(x) \leq f(b) \) for all \( x \in [b, c] \). In neither case could condition (7) be verified. In other words, condition (7) requires that \( f(b) > f(a) \) and \( f(b) > f(c) \).

This proves point i.

From (9) follows that \( f(x) \geq f(a) \) for all \( x \in [a, b] \), hence \( \int_a^b f(x)dx/(b-a) > f(a) \). Similarly, \( f(x) \geq f(c) \) for all \( x \in [b, c] \), so that \( \int_b^c f(x)dx/(c-b) > b(c) \).

We know that

\[
\frac{1}{y-z} \int_z^y f(x)dx \tag{12}
\]
is increasing in $z$ if
\[
\frac{1}{y - z} \int_z^y f(x) dx - f(z) > 0 \tag{13}
\]
(we differentiated (12) w.r.t. $z$) while it is decreasing in $y$ if
\[
\frac{1}{y - z} \int_z^y f(x) dx - f(y) < 0. \tag{14}
\]
Therefore, in (7) the first member is increasing in $a$ while the second member is decreasing in $c$.

As to the effect of an increase in $b$, consider first the case that the maximum (mode) of $f(x)$ is in the interval $(a, b]$. This implies that $f(b) \geq f(x)$ for all $x \in (b, c]$, so that $f(b) > \int_b^c f(x) dx/(c - b)$. It follows that the second member in (7) is decreasing in $b$. On the other hand, $\int_a^b f(x) dx/(b - a) = \int_b^c f(x) dx/(c - b)$, hence we have $f(b) > \int_a^b f(x) dx/(b - a)$, and we can conclude that the first member is increasing in $b$.

A similar argument applies for the case that the maximum of $f(x)$ lies in the interval $[b, c)$, and this proves point $ii$.

Given $ii$, in order to prove point $iii$ we must only consider the case that $f(a') > f(b') > f(c')$. But because $f(x) \geq f(b')$ for all $x \in [a', b']$ and $f(b') \geq f(x) \geq f(c')$ for all $x \in [b', c']$, condition (8) is always satisfied in this case. 

\textbf{Proof of Proposition 1}

We first show that there exist $\hat{S}$ such that
\[
(1 - \eta) \int_{\hat{p}_L \hat{S}}^\hat{p}_S f(V) dV - \eta \int_{\hat{p}_L \hat{S}}^\hat{p}_H \hat{S} f(V) dV = 0; \tag{15}
\]
dividing by $\eta (\hat{p}_H - p) S$ and using condition (1) the left hand side becomes
\[
\frac{1}{(p - \hat{p}_L) S} \int_{\hat{p}_L \hat{S}}^{\hat{p}_S} f(V) dV - \frac{1}{(\hat{p}_H - p) S} \int_{\hat{p}_S}^{\hat{p}_H \hat{S}} f(V) dV. \tag{16}
\]
Let $V^*$ be the mode of the distribution $f$. There exists $S_1 > 0$ small enough that $0 < \hat{p}_H S_1 \leq V^*$. Since $f$ is increasing on the interval $[\hat{p}_L S_1, \hat{p}_H S_1]$, the second member in expression (16) is larger than the first, so the difference is negative for $S = S_1$. On the other hand, there exists $S_2 > S_1$ such that $\hat{p}_L S_2 \geq V^*$ and expression (16) is positive. For continuity of the integrals, there must exist at least one value in $[S_1, S_2]$ such that such that (16) is zero. Let
\[
\hat{S} \equiv \inf \{S| \text{expression (16) is zero}\} \tag{17}
\]
i.e. (16) is negative for all $S < \hat{S}$.

Using Lemma 1 we have that (16) is positive [nonnegative] for all $S > \hat{S}$.
Proof of Proposition 2

Consider $\hat{p}_L, \hat{p}_H$, $p$, $\eta$ and $S$ satisfying (15). Under condition (1), the expression (16) is zero. As shown by Lemma 1, it must be $f(pS) > \min\{f(\hat{p}_LS), f(\hat{p}_HS)\}$ and expression (16) is increasing in $pS$, $\hat{p}_LS$ and $\hat{p}_HS$.

Therefore, if $\hat{p}_L = \hat{p}_L(p)$ and $\hat{p}_H = \hat{p}_H(p)$ are nondecreasing functions of $p$, expression (16) is decreasing both in $p$ and in $S$, so that it implicitly defines a decreasing function $\hat{S} = \hat{S}(p)$.

Proof of Proposition 3

Welfare is higher when $p$ is known if

$$
(1 - \eta) \int_{pS}^{pS} (C - V) f(V) dV > \eta \int_{\hat{p}H}^{\hat{p}H} (C - V) f(V) dV.
$$

Since $V \geq \hat{p}_H$ on the right hand side and $V \leq \hat{p}_H$ on the left hand side, the inequality is always satisfied when expression (5) is nonnegative.

Moreover, when $pS = C$, $V \geq C$ on the right hand side and $V \leq C$ on the left hand side, so that the first integral is positive and the second is negative.

References


