Contractual design in agency problems with non-monotonic cost

Daniel Danau* Annalisa Vinella†

Preliminary version

Abstract

In a principal-agent relationship in which the agent’s cost is non-monotonic with respect to type, we introduce the possibility, for the principal, of conditioning the compensation to the agent on the realization of an ex-post signal correlated with type. The possibility of exploiting the correlation between type and signal through lotteries is nonetheless constrained by the agent’s limited liability. We establish conditions for first-best implementation highlighting two peculiar effects on contractual design. First, countervailing incentives arise, with some types being tempted to mimic higher types, others to mimic lower types. Second, for any given type, some lower types gain on cost reimbursement, if they mimic it; others lose, instead. The same occurs with higher types, if they mimic the same given type. The former effect determines what exact lottery each type should be faced with, the optimal lottery differing across types. The latter effect implies restrictions on first-best implementation relative to the case of monotonic cost. This suggests that, unlike in settings without correlated information, countervailing incentives may make screening more difficult, rather than facilitating it.

Keywords: Non-monotonic cost; countervailing incentives; correlated signals; limited liability; first-best implementation

J.E.L. Classification Numbers: D82

*Université de Caen Basse-Normandie - Centre de Recherche en Economie et Management, Esplanade de la Paix, 14032 Caen, (France). E-mail: daniel.danau@unicaen.fr
†Università degli Studi di Bari "Aldo Moro" - Dipartimento di Scienze economiche e metodi matematici, Via C. Rosalba 53, 70124 Bari (Italy). E-mail: annalisa.vinella@uniba.it
1 Introduction

Motivation

In many agency relationships, the agent displays countervailing incentives to misrepresent his private information (the type), which entails that he is tempted either to overstate or to understate type depending on its specific realization. This is mostly the case when trading with the principal occasions an opportunity cost which is higher the better the type in that trade. For instance, when the type is the cost of producing a good for the principal, misrepresentation involves that the difference between false and true production cost and that between false and true opportunity cost are of a different sign. There are thus two opposite effects on incentives. Each agent’s type will be eager to mimic higher or lower types depending on the extent to which the effect associated with the opportunity cost and that associated with the production cost countervail each other. Maggi and Rodriguez-Clare [17] (henceforth, MRC) show that the prevailing effect may differ for any given type according to the shape of the total cost, including both the production cost and the opportunity cost, and characterize the contract for all possible shapes.\(^1\) There are many practical examples of agency problems with countervailing incentives, and it is not surprising that similar findings to those obtained by MRC emerge in essentially all cases explored by the literature.\(^2\)

Hitherto, the literature has focused on situations in which, after receiving the report from the agent in the contracting stage, the principal does not acquire any additional information about the agent’s type during the execution of the contract. In practice, it is very likely that some signal about the agent’s type becomes available after the transaction specified in the contract has taken place. An authority that regulates a monopolist F in some jurisdiction learns something about the production cost by observing the output of a monopolist F’ operating in a different jurisdiction. Under an incentive-compatible regulatory contract, that output reflects the efficiency of F’. Provided the efficiency of two firms operating in the same sector is correlated, the output of F’ is also a signal of the efficiency of F. The same occurs with agencies that regulate more firms at once, all involved in the same activity, as is often the case in water

\(^1\) Jullien [12] extends the analysis of MRC by relaxing the assumption of full participation and exploring common values situations in which the private information parameter has a direct impact on the principal’s welfare.

\(^2\) Examples of countervailing incentives are found: in procurement, when firms are specialized (Boone and Schttmuller [5]) and when they have a privileged knowledge of the quality of a public signal about their production costs (Che and Sappington [6]); in regulation, when the firm incurs a fixed cost inversely related with the privately known marginal cost (Lewis and Sappington [15]), with non-linear pricing under price cap (Jullien [12]), when two-product monopolists face complementary demands (Aguirre and Beïtia [2]) and when utilities are subject to universal service obligations (Poudou et al. [20]); in labour and financial contracts, when the agent’s hidden effort is complementary to his privately known ability in accomplishing the task for the principal (Ollier and Thomas [19]); in vertical relationships, when retailers need to specialize some assets before contracting with the upstream suppliers (Acconcia et al. [1]); in conflicts on investment levels between uninformed shareholders and informed managers (Degryse and de Jong [9]); in landowner-farmer contracts with up-front capital endowments (Lewis and Sappington [16]); in government-taxpayers relationships, when the government wishes to improve the wellbeing of low-skill individuals by taxing high-skill individuals, while being aware that the latter will emigrate if their domestic utility is lower than it would be in other tax jurisdictions (Krause [13]).
and sanitation services. Moreover, sometimes the regulator draws information by auditing the firm’s activity. These are all instances in which the newly acquired information is observable and verifiable and can thus be used in the contractual offer to the agent.

The literature on mechanism design with correlated information, initiated by Crémer and McLean [7] and Riordan and Sappington [21], has deeply investigated how the principal can take advantage of informative signals to enhance contracting. It is now well known that the use of lotteries, by means of which the compensation assigned to the different types is conveniently linked to the signals, strongly affects the principal’s attainments. In particular, the first-best outcome (full surplus extraction and efficient volume of trade) might be at reach, depending on the agent’s cost and level of liability (Demougin and Garvie [10], Gary-Bobo and Spiegel [11], Danau and Vinella [8]). The existing studies focus on situations in which the agent’s cost is monotonic in type, involving that the potential benefit from private information to the agent is higher the better that the type is. It remains to be clarified how the principal can take advantage of correlated information and how that possibility affects the features of the optimal contract when the cost is non-monotonic, provided in that case the change in production cost exceeds the change in opportunity cost for some types whereas the converse is true for others. The purpose of our study is to shed light on these issues.

Setting and main results

Our model bridges the domain of literature on agency problems with type-dependent opportunity cost with that on the use of correlated information in contractual design. We represent a principal who contracts with an agent for the provision of some good (or service). In the same vein as in MRC, the agent incurs both a variable cost of production, which grows linearly with the agent’s type, and an opportunity cost of trading with the principal, which declines with type and can take any shape. Thus, the shape of the opportunity cost is also the shape of the total cost. A signal correlated with type is publicly observed after the production quantity has been chosen or the good supplied. The signal is hard information, which the principal can use to condition the compensation to the agent. We further assume that the agent is protected by limited liability to take into account that, in practice, agents are generally prevented from incurring serious financial difficulties. For instance, financial distress of regulated firms is deemed to be undesirable in that it could lead to the interruption of services of general interest.

In line with the literature on contractual design with correlated information, we focus on first-best implementation and identify necessary and sufficient conditions for that outcome to be achieved. This requires facing the agent with an appropriate incentive compatible lottery, which is only feasible as long as he can be exposed to sufficiently high deficits. We are thus led to assess how the non-monotonicity of the agent’s cost affects the choice of the optimal lottery

---

3In different situations, correlated information could become available prior to contracting, either naturally or because the perspective of new correlated information appearing in a later stage induces the principal to postpone contracting deliberately. However, this is unfeasible in a number of cases. For instance, regulators cannot afford delaying contracting with firms that provide services of general interest.
under limited liability.

To help comprehension of our results, which we summarize below, it is useful to recall how lotteries should be structured to attain contractual efficiency. Whether the cost is monotonic (the case analyzed in Danau and Vinella [8]) or it is not (the case analyzed in this paper), the lottery must be constructed in such a way that, first, the target type to is an attractive report to neither lower types nor higher types and, second, it retains all surplus. The exact structure of the lottery depends on the agent’s liability and the shape of the cost with respect to type. Intuitively, when the liability is low a lottery which minimizes the risk borne by the agent (i.e., exposes the agent to the lowest feasible deficit) is desirable. In turn, the shape of the cost determines how effective a lottery is at extracting surplus. To illustrate, if the agent’s cost increases with type, then by mimicking some intermediate type $\theta$, a lower type $\theta^{-}$ could appropriate the difference between pretended cost and true cost, whereas a higher type $\theta^{+}$ would lose that difference. What matters is thus how the gain to type $\theta^{-}$ compares with the penalty to type $\theta^{+}$, and hence how great the ratio between the two is. The magnitude of the ratio depends on the shape of the cost, and hence that shape also determines how powerful a lottery designed to prevent type $\theta^{-}$ from reporting $\theta$ is, without rendering that same report attractive to type $\theta^{+}$. When the cost is convex in type, the ratio between gain to type $\theta^{-}$ and penalty to type $\theta^{+}$ is below one, and it is easy to design a little risky but incentive compatible lottery for type $\theta$. Of course, the interest of making the lottery little risky resides in that limited liability constraints are likely to hold. When the cost is concave instead, the ratio between gain to type $\theta^{-}$ and penalty to type $\theta^{+}$ is above one and a riskier lottery must be used, involving that limited liability may well be an issue.

First and foremost, there is a general lesson to be derived from our study. In settings with non-monotonic cost the use of correlated information has two effects on contractual design. The first effect is understood being based on the result, derived by MRC, that the incentives of any given type to mimic higher or lower types depend on the shape of the opportunity cost, which reflects in what direction and to what extent the cost varies with type. When the opportunity cost is either concave or little convex, the total cost increases with type if $\theta$ is low, it decreases with type if $\theta$ is high. Hence, low types obtain a bonus if they overstate information, high types obtain it if they understate information. The converse occurs when the opportunity cost is sufficiently convex instead. It follows that when the principal uses informative signals in contractual design a lottery must be targeted to any given $\theta$ depending on whether, for that particular $\theta$, the total cost increases or decreases with type. The second effect is that, for any given $\theta$, some lower types might gain in cost reimbursement by claiming $\theta$, some others might lose instead. Therefore, a lottery designed to extract the bonus available to some type below $\theta$ will actually grant a benefit to some other type below $\theta$. Analogous problem may arise with types above $\theta$. Both effects are specific to settings with correlated information, provided that lotteries are not used otherwise. Whereas the first effect determines how correlated information should be exploited in presence of countervailing incentives, the second effect makes screening more difficult for the principal.
The first specific result of our study concerns the case in which the opportunity cost is convex. In a setting without informative signals, MRC find that when the opportunity cost is little convex a range of intermediate types obtain no rent, whereas rents accrue to nearly all types when the opportunity cost is sufficiently convex. Indeed, in the former case, countervailing incentives are strong and there is little to gain from private information. By contrast, in the latter case the potential gain is high and incentives are more clear-cut. Specifically, low types are willing to understate as the bonus on the opportunity cost they would obtain is great; high types are willing to overstate as the penalty on the opportunity cost they would incur is little. A substantial conclusion to the analysis of MRC is that countervailing incentives are helpful in that they enable the principal to rely on output distortions to retain surplus, thus making screening less costly. Our analysis delivers a different conclusion with regards to correlated information settings. We find that when the opportunity cost is sufficiently convex, and hence the agent displays clear-cut incentives, it is easier for the principal to construct powerful lotteries because the second effect is not at work in that case. This suggests that strong countervailing incentives make screening more difficult, rather than facilitating it, when informative signals are used in the contractual offer to the agent.

Overall, as long as the agent’s level of liability is high enough to satisfy some necessary condition, first best is effected both when the opportunity cost is convex and when it is little concave. By contrast, this is not the case when the opportunity cost is sufficiently concave, except in the extreme case in which the agent could be exposed to unbounded deficits. Indeed, when the concavity is sufficiently pronounced the ratio between gain to type $\theta^+$ and penalty to type $\theta^-$ is above one, and the bound on the agent’s liability prevents the principal from making the lottery sufficiently risky to retain surplus without inducing distortions. Noticeably, whereas this is also true when the total cost is monotonic (compare Danau and Vinella [8]), non-monotonicity further restricts the degrees of concavity for which first best is at reach. This is because the second effect is relevant and the lottery is less powerful at extracting surplus.

Remark that strong concavity of the opportunity cost involves weak countervailing incentives for a wide range of types, and weaker and weaker such incentives moving towards the extremes of the feasible set. This suggests that when the opportunity cost is concave, unlike in the convex case, the principal prefers facing strong countervailing incentives because agency costs are lower in that case. This is in line with situations without informative signals in which, as Lewis and Sappington [15] - [16] emphasize, the principal may even want to create/reinforce countervailing incentives for an agent with concave opportunity cost, at the aim of enhancing contracting. We can thus conclude that, in correlated information settings, as long as the opportunity cost is convex, the principal dislikes strong countervailing incentives in that they make the lottery design more complicated. However, when the opportunity cost is concave the principal would rather like to face such incentives in order to afford lower agency costs.
1.1 Outline

The reminder of the article is organized as follows. In section 2 we describe the model, we state the principal’s programme and we characterize the first-best allocation. In section 3 we recall the principal’s attainment in the absence of informative signals, as characterized by MRC. Section 4 is devoted to the analysis of first-best implementation. After describing the optimal lottery design, we state conditions under which first best is attained according to the shape of the agent’s opportunity cost. Mathematical details are relegated to an appendix.

2 The model

A risk-neutral principal contracts with a risk-neutral agent for the provision of \( q \) units of some good (or service).

Consumption of \( q \) units of the good yields a gross utility of \( S(q) \). We assume that the function \( S(\cdot) \) is twice continuously differentiable and such that \( S'(\cdot) > 0 \) and \( S''(\cdot) < 0 \). Furthermore, \( S(0) = 0 \) and the Inada’s conditions are satisfied.

To trade with the principal and accomplish his task, the agent incurs a total cost of

\[
C(q, \theta) = \theta q + K(\theta) \quad \text{The marginal cost} \quad \theta \in \Theta \equiv [\theta_l, \theta_u] \quad \text{where} \quad \theta_l < \theta_u > 0 \quad \text{captures the} \quad \text{agent’s efficiency in the production activity.} \quad K(\theta) \quad \text{represents the agent’s opportunity cost of renouncing to other businesses, which depends on the agent’s efficiency. Taking} \quad K(\cdot) \quad \text{to be} \quad \text{twice continuously differentiable, we assume that} \quad K'(\theta) < 0 \quad \text{for all} \theta \quad \text{because the less efficient that the agent is in the relationship with the principal, the worse the outside opportunity that he faces. Moreover, we take} \quad K''(\cdot) \quad \text{to have a constant sign across types, which will be functional to the exposition of results below. With these properties,} \quad K(\cdot) \quad \text{is the exact counterpart of the agent’s reservation utility in MRC and our analysis is thus comparable with theirs.}
\]

Information structure  Nature draws \( \theta \) and the agent observes its realization (his type) before receiving the contractual offer. The public beliefs about \( \theta \) are reflected in the continuously differentiable density function \( f(\theta) \). The associated cumulative distribution function is denoted \( F(\theta) \). The agent’s marginal cost is correlated with a random signal \( s \) (the "state" of nature). This is hard information and can be included in a legally enforceable contract. For instance, in regulatory settings the signal can be the behaviour or the market performance of another firm, operating either in the same sector or in an analogous (possibly regulated) sector placed in a neighboring economy, which conveys information about production costs. Alternatively, the signal can be the outcome of an audit of the activity. We assume that it is drawn from the discrete support \( N \equiv \{1, \ldots, n\} \), where \( n \geq 3 \), and publicly observed after the contract has been signed and the level of output has been chosen (or the output has been delivered). The degree of correlation between type and signal is commonly known prior to the contractual offer being made. It is measured by the probability \( p_s(\theta) \) of observing signal \( s \) conditional on the type being \( \theta \), which we assume to be strictly positive for all \( s \in N \). We
also take the function $p_s(\cdot)$ to be twice continuously differentiable for all types. For reasons that will become apparent soon, we assume that there exist two signals, taken to be $1$ and $n$, for simplicity, such that the monotonic likelihood property, which is standard in mechanism design, is partially satisfied:

$$\frac{p_1(\theta)}{p_1(\theta')} > \frac{p_s(\theta)}{p_s(\theta')} > \frac{p_n(\theta)}{p_n(\theta')}, \quad \forall \theta, \theta' \text{ such that } \theta \geq \theta', \quad \forall s \notin \{1, n\}. \quad (1)$$

Furthermore, the conditional probabilities of the two signals are both concave in type, but they display a different degree of concavity, namely

$$p''_n(\theta) < p''_1(\theta) < 0, \quad \forall \theta \in \Theta. \quad (2)$$

The contractual offer The Revelation Principle applies and the principal can restrict attention to direct mechanisms such that the agent reports truthfully (or, equivalently, picks the contract targeted to his type within the menu offered by the principal). As the signal is publicly observed ex post, it can be used to condition the compensation to the agent. For instance, when a regulator deals with two firms with correlated costs, the compensation to the first firm can be made contingent not only on its own report (or contractual choice) but also on the report delivered by the second firm (or the latter’s contractual choice), which is informative about the true cost of the first firm. Formally, the take-it-or-leave-it offer is a profile of allocations $\{q(\theta), t_s(\theta)\}, \forall \theta \in \Theta, \forall s \in N$, where $q(\theta)$ is the quantity an agent of type $\theta$ will produce and $t_s(\theta)$ the transfer he will receive in state $s$. Considering both the production cost and the opportunity cost borne by the agent, the profit that type $\theta$ obtains when reporting $\theta'$ in state $s$ includes two components. The first is the profit assigned to the mimicked type in that state. The second, which is independent of the realised signal instead, is the difference between the fake cost, reimbursed according to the report, and the true cost, incurred to produce the quantity recommended from the reported type. Therefore, the expected payoff of type $\theta$ announcing $\theta'$, which is written as

$$\mathbb{E}_s[\pi_s(\theta' | \theta)] = \sum_s \pi_s(\theta') p_s(\theta) + (\theta' - \theta) q(\theta') + K(\theta') - K(\theta), \quad (3)$$
blends together a lottery with possible outcomes the profits assigned to the pretended type in the different states and a fixed amount given by the difference in cost reimbursement. Of course, (3) reduces to \( \mathbb{E}_s [\pi_s (\theta)] = \sum_s \pi_s (\theta) p_s (\theta) \), i.e., the expected payoff of type \( \theta \) is just the expected value of the lottery, if \( \theta' = \theta \).

The principal’s programme The principal’s programme is formulated as follows:

\[
\begin{align*}
\text{Max}_{\{q(\theta); \pi(\theta); \forall \theta\}} & \quad \int \overline{\theta} \{ S (q (\theta)) - (\theta q (\theta) + K (\theta)) - \mathbb{E}_s [\pi_s (\theta)] \} \, dF (\theta) \\
\text{subject to} & \\
\mathbb{E}_s [\pi_s (\theta)] & \geq \mathbb{E}_s [\pi_s (\theta' | \theta)] , \; \forall \theta, \theta' \in \Theta , \quad (IC_\theta^{\theta'}) \\
\mathbb{E}_s [\pi_s (\theta)] & \geq 0 , \; \forall \theta \in \Theta , \quad (PC_\theta) \\
\pi_s (\theta) & \geq -L , \; \forall \theta \in \Theta , \; \forall s \in N . \quad (LL_\theta)
\end{align*}
\]

\((IC_\theta^{\theta'})\) is the incentive constraint whereby type \( \theta \) weakly prefers to tell the truth rather than claiming \( \theta' \neq \theta \). \((PC_\theta)\) is the ex-ante participation constraint which ensures that type \( \theta \) obtains a non-negative profit in expectation. \((LL_\theta)\) is the limited liability constraint which ensures that the highest deficit to which the agent is exposed does not exceed \( L > 0 \) in any state \( s \). This form of limited liability represents situations in which the principal would like to avoid the agent becoming so financially distressed that the activity must be interrupted, at least as long as the agent does not attempt to conceal information. For instance, this is common practice in regulated industries, where \( L \) could be interpreted as an indicator of financial viability, beyond which the regulated firm would go bankrupt.\(^4\)

The first-best allocation At the first-best allocation quantities and profits are such that \( S'(q^* (\theta)) = \theta \) and \( \mathbb{E}_s [\pi_s (\theta)] = 0 \) for all \( \theta \in \Theta \). Given the properties of the function \( S (\cdot) \), the first-best quantity is positive and unique for any given value of \( \theta \) and the function \( q^* (\cdot) \) is continuous for all values of \( \theta \).

3 Background: countervailing incentives without signals

Before plunging into our analysis, we find it useful to recall the main features of the optimal contract without informative signals, as drawn by MRC. This will enable us to provide a preliminary overview of what changes when the compensation to the agent can be conditioned on a signal correlated with type.

\(^4\)In the most extreme case of \( L = 0 \), limited liability constraints would be ex-post participation constraints, in fact. Focusing on that case would prevent us from investigating first-best implementation, which is beyond reach unless the agent can be exposed to some deficit ex post. A different form of limited liability, mirroring the principal’s inability to tax the agent rather than her commitment to preserve the agent’s financial viability, would require the transfer made to the agent being non-negative in any state of nature.
Absent informative signals, it is not possible to use lotteries and the profit targeted to type \( \theta \), denoted \( \pi(\theta) \), must be set to satisfy the incentive constraint

\[
\pi(\theta') - \pi(\theta) \geq (\theta - \theta') q(\theta) + K(\theta) - K(\theta') , \quad \forall \theta' \neq \theta , \tag{4}
\]

whereby \( \theta \) is not an attractive report to any type \( \theta' \). In addition, it must be the case that \( \pi(\theta) \geq 0 \) to ensure participation in the contract. Because the opportunity cost is decreasing, a type \( \theta' \) that pretends \( \theta \) faces two opposite effects. If \( \theta > \theta' \), then the lie yields a higher reimbursement on the variable cost \( ((\theta - \theta') q(\theta) > 0) \) but a lower reimbursement on the opportunity cost \( (K(\theta) - K(\theta') < 0) \). The converse is true if \( \theta < \theta' \). Thus, depending on the prevailing effect, type \( \theta' \) is eager either to overstate or to understate information vis-à-vis the principal. In particular, if there exists \( \hat{\theta} \in (\underline{\theta}, \overline{\theta}) \) defined as being such that:

\[
q^*(\hat{\theta}) + K'(\hat{\theta}) = 0 ,
\]

then one effect prevails on the other if \( \theta' < \hat{\theta} \), and vice versa if \( \theta' > \hat{\theta} \). Which exact effect dominates on each range of types depends, in turn, on the shape of the opportunity cost. Henceforth, we take \( \hat{\theta} \) to exist, indeed, unless differently stated for presentation of some specific results. The optimal contract displays the following features.

1. The output profile must be non-increasing in type. This is necessary to ensure that the benefit type \( \theta \) would obtain by claiming \( \theta' > \theta \) is lower than the penalty type \( \theta' \) would incur by claiming \( \theta < \theta' \), and hence that either such type can be induced to release information. Notice that this is standard in any screening problems without correlated information, provided that the incentives of type \( \theta \) and type \( \theta' \) to mimic each other only depend on the variable cost and are thus unrelated to the properties of the opportunity cost.

2. When the opportunity cost is little convex or concave \((K''(\cdot) \leq -(q^*(\cdot))')\), the principal distorts the output levels downwards for types below \( \hat{\theta} \) and upwards for types above \( \hat{\theta} \), trading off efficiency and rent-extraction purposes. Given requirement (1) and with the output levels distorted in opposite directions for types in the two different ranges, pooling is optimally induced for a range of intermediate types. However, countervailing incentives do not necessarily lead to pooling. Indeed, when the opportunity cost is sufficiently convex \((K''(\cdot) > -(q^*(\cdot))')\) types below \( \hat{\theta} \) are tempted to understate information and types above \( \hat{\theta} \) to overstate information, and the principal can afford offering a fully separating contract.

3. When the opportunity cost is little convex no rent is given up to the intermediate types which are recommended the same production. This is because countervailing incentives are strong and the potential benefits from private information are nearly constant across types. Then, output distortions suffice to ensure the agent’s participation without triggering lies. This is no longer true when \( K(\cdot) \) is concave. In that case, countervailing incentives are strong only for intermediate types and increasingly weaker for more extreme types. Whereas the former types can be induced to tell the truth at a low (or no) cost as they are turned between the temptation
to overstate and that to understate, higher information rents must be given up to the latter
types, which have clear incentives to lie in one direction. On the other hand, when \( K(\cdot) \) is
sufficiently convex, countervailing incentives are strong only for extreme types and increasingly
weaker for more interior types, which are now those that receive higher information rents.

A neat conclusion is to be drawn from the analysis of MRC. Screening is less costly when
countervailing incentives are strong. This is the case for a range of intermediate types when
the cost is concave or little convex, and for extreme types when the cost is sufficiently convex.
Particularly, whereas in the concave case countervailing incentives are sufficiently strong only
for one intermediate type, from which all surplus is then retained, in the little convex case the
range of such types is wider. Yet, the countervailing effect is weaker the more pronounced that
the convexity is. When the cost is sufficiently convex that effect is strong enough to extract all
surplus only for the two extreme types.

Turning to the settings with correlated information, the difference between reimbursed cost
and true cost, namely

\[ C(q(\theta'), \theta) - C(q(\theta'), \theta') = (\theta - \theta') q(\theta') + K(\theta) - K(\theta'), \]

is only one of the two components of the payoff that type \( \theta' \) obtains, if it claims \( \theta \). There is also a
second component. This is a lottery with expected value of \( \sum_s \pi_s(\theta) p_s(\theta') \) and represents the
instrument by means of which, unlike in the absence of informative signals, the principal can
attain the first-best outcome. When the cost increases for all types (the standard case in the
literature on correlated information), the lottery targeted to type \( \theta \) is intended to extract the
bonus in cost reimbursement that lower types could obtain by claiming \( \theta \), without yet attracting
lies from higher types. From Danau and Vinella [8], we know what the most effective lottery
looks like under limited liability when the agent’s cost is monotonic in type. However, we do
not know whether that lottery is most effective also when the cost is non-monotonic, provided
in this case the difference between pretended cost and true cost does not have the same sign
across types, and may not have the same sign across types above and across types below the
target type. To clarify this aspect, it is necessary to develop a specific analysis, along the lines
of Danau and Vinella [8]. Once the most effective lottery with non-monotonic cost is identified,
it will also emerge how the contract characterized by MRC is amended when the compensation
to the agent is conditioned on informative signals.

4 First-best implementation

To effect first best the principal must be able to recommend the first-best output level and
construct lotteries such that all surplus is retained from all types without exposing the agent to
excessive deficits \textit{ex post}. To investigate first-best implementation, we first explore the design
of incentive compatible lotteries when the first-best quantity is recommended. We then identify
conditions under which, given the agent’s level of liability, all surplus is extracted by means of
those lotteries when the opportunity cost is convex and when it is concave.

4.1 The lottery design

Take any triplet of types \( \{\theta^-, \theta, \theta^+\} \in \Theta \) such that \( \theta^- < \theta < \theta^+ \). Resting on the condition for full surplus extraction

\[
\sum_s \pi_s (\theta) p_s (\theta) = 0, \tag{5}
\]

the incentive constraints \( IC^-_{\theta^-} \) and \( IC^+_{\theta^+} \), whereby \( \theta \) is an attractive report neither to type \( \theta^- \) nor to type \( \theta^+ \), are conveniently reformulated as follows:

\[
\pi_n (\theta) \leq \frac{\theta - \theta^-}{-p_n(\theta)} q^*(\theta) + \frac{K(\theta^-) - K(\theta^-)}{\theta - \theta^-} - \sum_{s \neq 1,n} \pi_s (\theta) \frac{p_s(\theta^-) - p_1(\theta^-)}{p_1(\theta^-)} - \sum_{s \neq 1,n} \pi_s (\theta) \frac{p_s(\theta^+) - p_1(\theta^+)}{p_1(\theta^+)} \tag{6}
\]

\[
\pi_n (\theta) \geq \frac{\theta^+ - \theta}{-p_n(\theta)} q^*(\theta) + \frac{K(\theta^+) - K(\theta^-)}{\theta^+ - \theta} - \sum_{s \neq 1,n} \pi_s (\theta) \frac{p_s(\theta^+) - p_1(\theta^+)}{p_1(\theta^+)} - \sum_{s \neq 1,n} \pi_s (\theta) \frac{p_s(\theta^-) - p_1(\theta^-)}{p_1(\theta^-)} \tag{7}
\]

Under Assumption 1, the higher that \( \pi_n (\theta) \) is the more eager that type \( \theta^- \) is to claim \( \theta \). On the other hand, the lower that \( \pi_n (\theta) \) is the more eager that type \( \theta^+ \) is to claim \( \theta \). Therefore, types \( \theta^- \) and \( \theta^+ \) are both unwilling to claim \( \theta \) only if the value of \( \pi_n (\theta) \) is set neither too low nor too high. Taking the limit of the right-hand side of (6) and (7), respectively, as \( \theta^- \to \theta \) and \( \theta^+ \to \theta \), we see that the two conditions hold at once if and only if:

\[
\pi_n (\theta) = \frac{q^*(\theta) + K'(\theta) + \sum_{s \neq 1,n} \pi_s (\theta) p_s (\theta) \left( \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_s(\theta)}{p_s(\theta)} \right)}{-p_n(\theta) \left( \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right)} \tag{8}
\]

This is the state—\( n \) profit such that any incentives to mimic a neighboring type are eliminated. Using (8), (6) and (7) are respectively reformulated as

\[
q^*(\theta) + K'(\theta) \geq (\theta - \theta^-) \left( q^*(\theta) + \frac{K(\theta^-) - K(\theta^-)}{\theta^+ - \theta^-} \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_s(\theta)}{p_s(\theta)} \right) + \sum_{s \neq 1,n} \pi_s (\theta) p_s (\theta) \left( \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_s(\theta)}{p_s(\theta)} \right) \tag{9}
\]

and

\[
q^*(\theta) + K'(\theta) \leq (\theta^+ - \theta) \left( q^*(\theta) + \frac{K(\theta^+) - K(\theta^-)}{\theta^+ - \theta} \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_s(\theta)}{p_s(\theta)} \right) + \sum_{s \neq 1,n} \pi_s (\theta) p_s (\theta) \left( \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_s(\theta)}{p_s(\theta)} \right) \tag{10}
\]
We see that (9) and (10) hold jointly only if

\[
(\theta^+ - \theta) \frac{q^*(\theta)}{p_1(\theta)} + \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} - (\theta - \theta^-) \frac{q^*(\theta)}{p_1(\theta)} + \frac{K(\theta) - K(\theta^-)}{\theta^- - \theta} \leq \sum_{s \neq 1, n} \pi_s(\theta) p_s(\theta) \left( \frac{p_s(\theta^-)}{p_s(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)} \right) \frac{q^*(\theta)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)} \frac{p_1(\theta^-)}{p_1(\theta)} \frac{p_n(\theta^+)}{p_n(\theta^+)} \frac{p_1(\theta^+)}{p_1(\theta^+)} \right).
\]

In the limit, (11) reduces to (9) as \(\theta^- \to \theta\) and to (10) as \(\theta^+ \to \theta\). The incentive constraints can thus be replaced by the local incentive constraint (8), which must hold for all \(\theta\), together with (11), which must hold for all triplets \(\theta^-, \theta, \theta^+\). Accordingly, provided that no information rent is left to the agent, the principal’s programme is reformulated as follows:

\[
\max_{\{q(\theta), \pi(\theta) : \theta\} \geq \theta} \int_{\theta}^3 (S(q(\theta)) - C(q(\theta), \theta)) dF(\theta)
\]

subject to

\[
(5), (8), (11) \text{ and } (LL_0^n).
\]

We begin by looking at (8) and \((LL_0^n)\). Consider a type \(\theta\) such that \(q^*(\theta) + K'(\theta) > 0\). The easiest way of satisfying the two constraints in all states for this type is to set the profit in state \(n\) equal to the profit in all other states but 1. Accordingly, we consider the following lottery, to be denoted \(\pi^a(\theta)\):

\[
\pi_1(\theta) = (q^*(\theta) + K'(\theta)) \frac{1 - p_1(\theta)}{p_1(\theta)}
\]

\[
\pi_s(\theta) = (q^*(\theta) + K'(\theta)) \frac{p_1(\theta)}{-p_1(\theta)}, \quad \forall s \neq 1.
\]

We remark that \(\pi_1(\theta)\) is a reward, whereas \(\pi_s(\theta)\) is a punishment. This structure of the lottery is explained as follows. Types below \(\theta\) could exaggerate information to gain the difference between fake and true cost. First take \(\pi_s(\theta)\) as given for all signals but 1 and \(n\). The lower that \(\pi_n(\theta)\) is, the more the surplus that the principal can extract from such types through the lottery. Of course, given \(\pi_s(\theta), \forall s \neq 1, n\), lowering \(\pi_n(\theta)\) involves raising \(\pi_1(\theta)\) to ensure that type \(\theta\) is willing to participate in the contract. However, this does not prevent the principal from retaining all surplus from types below \(\theta\), provided that those types are more likely to draw signal \(n\) than signal 1. Therefore, \(\pi_n(\theta) < 0\) and \(\pi_1(\theta) > 0\). Further setting \(\pi_s(\theta) = \pi_n(\theta)\), \(\forall s \neq 1, n\), enables the principal to minimize the punishment to which type \(\theta\) is exposed.

Next consider a type \(\theta\) such that \(q^*(\theta) + K'(\theta) < 0\). The reasoning goes as before, except that now types above \(\theta\) gain in terms of cost reimbursement, if they understate information. Hence, an effective lottery is such that in state 1 type \(\theta\) is punished, in state \(n\) it is rewarded, and in all other states it receives the same profit as in state 1 so that \((LL_0^n)\) holds. This lottery,
to be denoted $\pi^\beta(\theta)$, is composed as follows:

$$
\pi_n(\theta) = (q^*(\theta) + K'(\theta)) \frac{1 - p_n(\theta)}{p'_n(\theta)} \quad (14)
$$

$$
\pi_s(\theta) = (q^*(\theta) + K'(\theta)) \frac{p_n(\theta)}{-p'_n(\theta)} \quad \forall s \neq n. \quad (15)
$$

**Lemma 1** (8) *is most likely to be satisfied by the lottery $\pi^\alpha(\theta)$ for all $\theta$ such that $q^*(\theta) + K'(\theta) < 0$ and the lottery $\pi^\beta(\theta)$ for all $\theta$ such that $q^*(\theta) + K'(\theta) > 0$.

Unlike the local incentive constraint (8), the global incentive constraint (11) is most likely to hold when both the profit assigned in state 1 and the profit assigned in state $n$, rather than only one of them, are set to differ from the profit assigned in all other states. This is because the principal can take better advantage of the correlation between signal and type, if she saturates $(LL_s^\alpha), \forall s \neq 1, n$, and then uses the profit in state $n$ to satisfy (8) and adjusts the profit in state 1 to retain all surplus. For any type $\theta$ this lottery, to be denoted $\pi^\gamma(\theta)$, looks as follows:

$$
\pi_1(\theta) = \frac{q^*(\theta) + K'(\theta) - L \frac{p'_1(\theta)}{p_1(\theta)}}{p_1(\theta) \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)}} - L \quad (16)
$$

$$
\pi_n(\theta) = \frac{L \frac{p'_1(\theta)}{p_1(\theta)} - (q^*(\theta) + K'(\theta))}{p_n(\theta) \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)}} - L \quad (17)
$$

$$
\pi_s(\theta) = -L, \forall s \neq 1, n. \quad (18)
$$

**Lemma 2** (11) *is most likely to be satisfied by the lottery $\pi^\gamma(\theta), \forall \theta$.

Noticeably, the profile of profits (16) - (18) looks the same regardless of whether $q^*(\theta) + K'(\theta)$ is positive or negative. Nonetheless, $\pi_1(\theta)$ is higher in the former case than in the latter, whereas the converse is true for $\pi_n(\theta)$.

The contractual offer such that all surplus is retained from the agent will, of course, differ depending on which one, between the pair of lotteries $\{\pi^\alpha(\cdot), \pi^\beta(\cdot)\}$ and the lottery $\pi^\gamma(\cdot)$, is more appropriate.

To see how exactly the lottery is chosen and how the non-monotonicity of the cost affects that choice, suppose first that the cost is monotonic instead, such that $q^*(\theta) + K'(\theta) > 0 \forall \theta$. Then, as Danau and Vinella [8] show and as also emerges from our previous lemmas, the principal has to select one between the lotteries $\pi^\alpha(\cdot)$ and $\pi^\gamma(\cdot)$ for all types. To understand what drives the choice, it is useful to have a look at the way in which the use of lotteries helps the principal extract surplus. By announcing $\theta$, type $\theta^-$ gains an amount of $(\theta - \theta^-) q^*(\theta) + K(\theta) - K(\theta^-)$ in terms of cost reimbursement but it also incurs a penalty equal to the expected...
value of the lottery, which is negative:

\[
\sum_s \pi_s (\theta) p_s (\theta^-) = \pi_n (\theta) \sum_{s \neq 1} p_s (\theta) \left( \frac{p_s (\theta^-)}{p_s (\theta)} - \frac{p_1 (\theta^-)}{p_1 (\theta)} \right) \\
= \pi_n (\theta) \frac{p_1 (\theta) - p_1 (\theta^-)}{p_1 (\theta)}.
\]

On the other hand, by announcing \( \theta \), type \( \theta^+ \) loses an amount of \( (\theta^+ - \theta) q^* (\theta) + K (\theta^+) - K (\theta) \) in terms of cost reimbursement but it also obtains a gain equal to the expected value of the lottery, which is now positive:

\[
\sum_s \pi_s (\theta) p_s (\theta^+) = \pi_n (\theta) \sum_{s \neq 1} p_s (\theta) \left( \frac{p_s (\theta^+)}{p_s (\theta)} - \frac{p_1 (\theta^+)}{p_1 (\theta)} \right) \\
= \pi_n (\theta) \frac{p_1 (\theta) - p_1 (\theta^+)}{p_1 (\theta)}.
\]

With \( p_1 (\cdot) \) being concave, the principal can use \( \pi^\alpha (\cdot) \), which is most likely to satisfy the limited liability constraints, if the ratio between the penalty incurred by type \( \theta^- \) and the gain obtained by type \( \theta^+ \) on the lottery is greater than the ratio between the gain obtained by type \( \theta^- \) and the penalty incurred by type \( \theta^+ \) in terms of cost reimbursement. This is the case when following condition is satisfied:

\[
\frac{q^* (\theta)}{q^* (\theta)} + \frac{K (\theta) - K (\theta^-)}{\theta - \theta^-} \leq \frac{p_1 (\theta) - p_1 (\theta^-)}{\theta - \theta^-}.
\]

(19)

Under (19), there exists a negative value of \( \pi_n (\theta) \) such that neither type \( \theta^- \) nor type \( \theta^+ \) finds it convenient to announce \( \theta \). When (19) does not hold but the limited liability constraints are not saturated, rather than insisting on \( \pi^\alpha (\cdot) \), the principal should rely on the lottery \( \pi^\gamma (\cdot) \) to inflict higher punishments, and hence to relax the conflict between upward and downward incentive constraints to the utmost.

Let us now turn to the case in which the cost is non-monotonic in type. By looking at this case, it will become evident that the lottery choice is not the only contractual aspect to which the shape of the cost is relevant. The use of correlated information in agency problems with countervailing incentives has two effects on contractual design.

First, as regards types with decreasing cost \( (q^* (\cdot) + K' (\cdot) < 0) \), the principal has to choose a lottery between \( \pi^\beta (\cdot) \) and \( \pi^\gamma (\cdot) \), rather than choosing between \( \pi^\alpha (\cdot) \) and \( \pi^\gamma (\cdot) \), because those types obtain a bonus in terms of cost reimbursement if they understate (rather than overstating) information. Thus, the condition under which upward and downward incentive
constraints are jointly satisfied for any triplet \( \{ \theta^-, \theta, \theta^+ \} \) is specified as follows:

\[
\frac{p_\alpha(\theta)-p_\omega(\theta^-)}{\theta-\theta^-} \leq q^*(\theta) + \frac{K(\theta)-K(\theta^-)}{\theta-\theta^-} \quad \frac{p_\alpha(\theta^+)-p_\omega(\theta^+)}{\theta^+-\theta} \leq q^*(\theta) + \frac{K(\theta^+)-K(\theta)}{\theta^+-\theta}.
\]  

(20)

When (20) holds, there exists a negative value of \( \pi_1(\theta) \) such that \( \theta \) is an attractive report neither to type \( \theta^- \) nor to type \( \theta^+ \). Once again, when (20) fails to hold and provided that the limited liability constraints are not saturated, the principal should rather rely on the lottery \( \pi^+ (\cdot) \). Overall, the choice is between \( \pi^\alpha (\cdot) \) and \( \pi^\gamma (\cdot) \) for types such that \( q^*(\cdot) + K'(\cdot) > 0 \) and between \( \pi^\beta (\cdot) \) and \( \pi^\gamma (\cdot) \) for types such that \( q^*(\cdot) + K'(\cdot) < 0 \). As shown by MRC, it is the shape of \( K(\cdot) \) what determines the ranges of types with increasing and decreasing cost. In our case, the shape of \( K(\cdot) \) determines whether the lottery \( \pi^\alpha (\cdot) \) or \( \pi^\beta (\cdot) \) should be used for the types in the two ranges, when (19) and (20) are not an issue.

Second, whereas with a monotonic cost all types below (above) \( \theta \) lose in terms of cost reimbursement and gain in terms of lottery, with a non-monotonic cost this may or may not be the case. To clarify this aspect, it is useful to define \( h(\theta) \) such that:

\[
q^*(\theta) + \frac{K(\theta) - K(h(\theta))}{\theta - h(\theta)} = 0.
\]

(21)

That is, \( h(\theta) \) is the type which neither gains nor loses in terms of cost reimbursement, if it mimics type \( \theta \). Obviously, such a type does not exist with a monotonic cost. First take \( \theta \) such that \( q^*(\theta) + K'(\theta) < 0 \). By pretending \( \theta \), type \( \theta^+ \) incurs a penalty in cost reimbursement if \( \theta^+ < h(\theta) \) but not otherwise. Hence, if \( \theta^+ > h(\theta) \), then reporting \( \theta \) would be beneficial to type \( \theta^+ \) in terms of both cost reimbursement and lottery. Next take \( \theta \) such that \( q^*(\theta) + K'(\theta) > 0 \). By pretending \( \theta \), type \( \theta^+ \) would lose on the lottery; in addition, it would also lose on the cost reimbursement if \( h(\theta) < \theta^+ \). To ascertain whether all types below (above) \( \theta \) get a bonus by pretending \( \theta \), or they all lose, it is necessary to assess how \( h(\theta) \) compares with \( \theta \). Again, this is related to the shape of the cost, in turn.

**Lemma 3** If either \( K''(\cdot) < 0 \) or \( K''(\cdot) \geq - (q^*(\cdot))' \), then \( h(\theta) > \theta \) if and only if \( \theta < \tilde{\theta} \). If \( 0 < K''(\cdot) < - (q^*(\cdot))' \), then \( h(\theta) < \theta \) if and only if \( \theta < \tilde{\theta} \).

The various cases identified by the lemma are better understood by looking at the graphs in Figure 4.1. In each graph, the thick line represents \( q^*(\theta) + K'(\theta) \) as a function of \( \theta \in \Theta \). Each of the two dashed lines represents \( q^*(\theta) + \frac{K(\theta)-K(x)}{\theta-x} \) as a function of \( x \in \Theta \) for some given value of \( \theta \), taken to be \( \theta_1 < \tilde{\theta} \) for the upper line and \( \theta_2 > \tilde{\theta} \) for the other. These are the values of \( \theta \) at which the thick line and each of the two dashed lines cross. The values of \( x \) at which the dashed lines cross the horizontal axis are \( h(\theta_1) \) and \( h(\theta_2) \). Graph (i) shows that when \( K(\cdot) \) is concave, by reporting \( \theta_1 < \tilde{\theta} \), any type \( x < \theta_1 \) and any type \( x \in (\theta_1, h(\theta_1)) \) gets a prize in cost reimbursement because \( q^*(\theta_1) + \frac{K(\theta_1)-K(x)}{\theta_1-x} > 0 \) for all types in those ranges; by contrast, any type \( x > h(\theta_1) \) faces a penalty in cost reimbursement because \( q^*(\theta_1) + \frac{K(\theta_1)-K(x)}{\theta_1-x} < 0 \). By
reporting $\theta_2 > \hat{\theta}$, any type $x < h(\theta_2)$ obtains a prize because $q^*(\theta_2) + \frac{K(\theta_2)-K(x)}{\theta_2-x} > 0$, whereas any type $x > h(\theta_2)$, whether below or above $\theta_2$, faces a penalty because $q^*(\theta_2) + \frac{K(\theta_2)-K(x)}{\theta_2-x} < 0$. Graphs (ii) and (iii) are interpreted in a similar manner, mutatis mutandis.

Taking the two effects together, it is not surprising that the shape of $K(\cdot)$ determines the principal’s contractual attainments, which we now turn to present.

### 4.2 $K(\cdot)$ convex

**Lemma 4** If $K''(\cdot) > 0$, then (24) is satisfied and full surplus extraction is most likely to be attained through the lottery $\pi^\alpha(\theta)$ for all $\theta$ such that $q^*(\theta) + K'(\theta) > 0$ and the lottery $\pi^\beta(\theta)$ for all $\theta$ such that $q^*(\theta) + K'(\theta) < 0$.

According to this lemma, it is not an issue to attain global incentive compatibility when the opportunity cost is convex, regardless of the exact degree of convexity. When $K(\cdot)$ is little convex, the total cost changes little as type changes (the thick line in graph (ii) in Figure 4.1), entailing that there is little to gain, in terms of cost reimbursement, from a lie. Thus, (24) can be satisfied with the little risky lotteries $\pi^\alpha(\cdot)$ and $\pi^\beta(\cdot)$. When $K(\cdot)$ is sufficiently convex, the total cost varies more with type (the thick line in graph (iii) of Figure 4.1). This entails that countervailing incentives are weaker and a lie is potentially more convenient in terms of cost reimbursement. However, it is easier for the principal to induce information release at no cost because the incentives to lies are opposite to those displayed with other shapes of $K(\cdot)$, and hence the second effect previously explained is not at work. To illustrate, take type $\theta^+ < h(\theta)$ and suppose that $\theta < \hat{\theta}$ so that $q^*(\theta) + K'(\theta) < 0$. Then, $\pi^\beta(\theta)$ is used. With $K(\cdot)$ being sufficiently convex, understatement yields a double penalty to type $\theta^+$, in terms of both lottery and cost reimbursement, and is thus easily prevented. This all compares interestingly with settings without correlated information. In the latter, the principal finds it more costly to elicit information when the agent’s opportunity cost is sufficiently convex and
countervailing incentives are weak. In our framework, weak countervailing incentives associated with a sufficiently convex opportunity cost facilitates the use of lotteries to retain surplus and it suffices that local incentive constraints be satisfied. The following result is obtained.

**Proposition 1**

Suppose that $K'' (\cdot) \geq 0$. Then, first best is implemented if and only if:

\[
(q^* (\theta) + K' (\theta)) \frac{p_\theta(\theta)}{p_\theta'(\theta)} \leq L, \ \forall \theta \text{ such that } q^* (\theta) + K' (\theta) > 0 \quad (22)
\]

\[
(q^* (\theta) + K' (\theta)) \frac{p_\theta(\theta)}{p_\theta'(\theta)} \leq L, \ \forall \theta \text{ such that } q^* (\theta) + K' (\theta) < 0. \quad (23)
\]

This result compares with Proposition 2 in Gary-Bobo and Spiegel [11]. They assume that the agent’s cost is increasing and convex for all types. In that case, in spite of the agent being protected by limited liability, first best is at hand provided that local incentive constraints hold. When the cost is non-monotonic but still convex in type there is only one novel aspect to that finding: local incentive compatibility requires targeting a different lottery to the types with decreasing cost.

### 4.3 $K (\cdot)$ concave

When the opportunity cost is concave, the lottery that is most likely to extract surplus under limited liability depends on the exact degree of concavity. This is at odds with settings without correlated information (MRC) but in line with settings with correlated information and monotonic cost (Danau and Vinella [8]). Due to the properties of the conditional probabilities of signals 1 and $n$, the lotteries $\pi^\alpha (\cdot)$ and $\pi^\beta (\cdot)$ may still be effective when the concavity of $K (\cdot)$ is little pronounced, provided that (19) and (20) are satisfied. However, this is not true otherwise. Besides, even when $K (\cdot)$ is little concave full surplus extraction with those lotteries cannot be taken for granted. Indeed, whether or not that outcome is achieved will depend on how the effects evidenced by Lemma 3 are at work. Take $\theta < \hat{\theta}$, for instance. Then, $h (\theta) > \hat{\theta}$ and by using $\pi^\alpha (\cdot)$ the principal will succeed in discouraging types in $(\theta, h (\theta))$ from understating information because, following that lie, those types would lose more on cost reimbursement than could they gain on the lottery. However, types above $h (\theta)$ will find it convenient to claim $\theta$ as they would gain on both components of their payoffs (recall that, on the opposite, types above $h (\theta)$ incur a double penalty when $K (\cdot)$ is sufficiently convex). Noticeably, replacing $\pi^\alpha (\cdot)$ with $\pi^\beta (\cdot)$ would not be a solution because $\pi^\beta (\cdot)$ would then attract lies from types in $(\theta, h (\theta))$. The principal does not extract all surplus unless the lottery $\pi^\gamma (\cdot)$ is adopted. In the next lemma we present the best lottery to be targeted to different ranges of types, according to whether or not there exists a range of types $(\theta, h (\theta))$ if $\theta < \hat{\theta}$, and a range of types $(h (\theta), \theta)$ if $\theta > \hat{\theta}$.

**Lemma 5**

Suppose that $K'' (\cdot) < 0$.

(i) $\theta < \hat{\theta}$ and $h (\theta) \notin (\theta, \bar{\theta})$: The lottery that is most likely to extract all surplus is $\pi^\alpha (\cdot)$ when (19) holds, and $\pi^\gamma (\cdot)$ otherwise.
(ii) $\theta > \hat{\theta}$ and $h(\theta) \notin (\hat{\theta}, \overline{\theta})$: The lottery that is most likely to extract all surplus is $\pi^\alpha(\cdot)$ when (20) holds, and $\pi^\gamma(\cdot)$ otherwise.

(iii) $h(\theta) \in (\hat{\theta}, \overline{\theta})$: The lottery that is most likely to extract all surplus is $\pi^\gamma(\cdot)$.

In substance, $\pi^\gamma(\cdot)$ is more effective than a lottery yielding the same punishment in all states but one not only when $K(\cdot)$ is too concave to have (19) or (20) satisfied, but also when non-monotonicity of the total cost makes global incentive compatibility difficult to attain (formally, when $h(\theta) \in (\hat{\theta}, \overline{\theta})$). We now derive conditions under which first best is implemented, according to the optimal lotteries to be targeted to different ranges of types for different degrees of concavity of $K(\cdot)$.

**Proposition 2** Take $K''(\cdot) < 0$. First best is implemented if and only if either:

(i) (19) and (22) hold for $\theta \leq \hat{\theta}$, (20) and (23) hold for $\theta \geq \hat{\theta}$, and $h(\theta) \notin (\hat{\theta}, \overline{\theta})$;

or

(ii) the following condition holds for all triplets $\{\theta^-, \theta, \theta^+\}$:

$$
\begin{align*}
\frac{q^+(\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} p_n(\theta^-)}{p_n(\theta^-)} & - \frac{q^+(\theta) + \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} p_n(\theta)}{p_n(\theta)} \\
\leq L \sum_{s \neq 1, n} p_s(\theta) \left( \frac{p_1(\theta^-) - p_2(\theta^-)}{p_1(\theta)} \right) - \left( \frac{p_1(\theta^+) - p_2(\theta^+)}{p_1(\theta)} \right)
\end{align*}
$$

(24)

**Corollary 1** Suppose that $K''(\cdot) < 0$ and that $q^+(\theta) + K'(\theta) > 0, \forall \theta$. First best is implemented if and only if either (19) and (22) are jointly satisfied or (24) holds.

When the cost increases with type on the whole support the principal should adopt either the lottery $\pi^\alpha(\cdot)$, which is most likely to attain local incentive compatibility, or the lottery $\pi^\gamma(\cdot)$, which is most likely to avoid conflicts between upward and downward incentive constraints. The former is preferable with a mild concavity of the cost, the latter with a more pronounced concavity. The choice between $\pi^\alpha(\cdot)$ and $\pi^\gamma(\cdot)$ explains the alternative conditions required in Corollary 1. The result in Proposition 1 is similar in this respect. Actually, the choice between the lotteries $\pi^\alpha(\cdot)$ and $\pi^\beta(\cdot)$, on one side, and that of $\pi^\gamma(\cdot)$, on the other, depend on the degree of concavity of the opportunity cost. This explains why in Proposition 1 there are two pairs of relevant conditions, namely (22) and (19) for $\theta < \hat{\theta}$ and (23) and (20) for $\theta > \hat{\theta}$.

There are nonetheless two essential differences between Proposition 1 and Corollary 1. In the framework of Proposition 1, the lottery $\pi^\alpha(\cdot)$ is more likely to be preferable when the restrictions imposed by the non-monotonicity of the cost are important (formally, when $h(\theta) \in (\hat{\theta}, \overline{\theta})$), an issue which is of course absent in the context of Corollary 1. Less evident is
that if the lottery \( \pi^\gamma(\cdot) \) is adopted, then the relevant condition (24) is not equally tight when the cost is monotonic and when it is not. As stated in the next corollary, for some intermediate degree of concavity of \( K(\cdot) \), the principal attains the first-best outcome through the lottery \( \pi^\gamma(\cdot) \) when the cost increases for all types ((24) holds) but this is not necessarily the case otherwise.

**Corollary 2** Take \( K''(\cdot) < 0 \).

(i) Suppose that \( q^*(\theta) + K'(\theta) > 0 \ \forall \theta \) and that \( K(\cdot) \) is "sufficiently little concave" to have

\[
\frac{q^*(\theta) + K'(x)}{q^*(\theta) + K(\theta) - K(x)} \geq \frac{p'_1(x) - p'_n(x)}{p(\theta) - p_2(x)} - \frac{q^*(\theta) + K'(x)}{q^*(\theta) + K(\theta) - K(x)} \leq \frac{p'_1(x) - p'_n(x)}{p(\theta) - p_2(x)} - \frac{q^*(\theta) + K'(x)}{q^*(\theta) + K(\theta) - K(x)}
\]

(25)

Then, (24) holds for all triplets \( \{\theta^-, \theta^+\} \).

(ii) Suppose that \( q^*(\theta) + K'(\theta) > 0 \ \forall \theta < \hat{\theta} \) and \( q^*(\theta) + K'(\theta) < 0 \ \forall \theta > \hat{\theta} \) and that \( K(\cdot) \) is "sufficiently little concave" that

1. (25) holds if either \( x > \theta \) and \( x > h(\theta) \) or \( x < \theta \) and \( x < h(\theta) \);

2. (26) holds if \( h(\theta) < x < \theta \) or \( \theta < x < h(\theta) \).

Then, (24) holds for all triplets \( \{\theta^-, \theta^+\} \) if and only if

\[
\frac{q^*(\theta) + K'(\theta)}{p'_1(\theta) - p'_n(\theta)} \leq L \sum_{s \neq 1, n} p_s(\theta) \left( \frac{p'_1(x) - p'_n(x)}{p_1(\theta) - p_n(\theta)} - \frac{p_1(\theta) - p_n(\theta)}{p_1(\theta) - p_n(\theta)} \right), \text{ if } \theta < \hat{\theta}
\]

(27)

\[
\frac{q^*(\theta) + K'(\theta)}{p'_1(\theta) - p'_n(\theta)} \geq L \sum_{s \neq 1, n} p_s(\theta) \left( \frac{p'_1(x) - p'_n(x)}{p_1(\theta) - p_n(\theta)} - \frac{p_1(\theta) - p_n(\theta)}{p_1(\theta) - p_n(\theta)} \right), \text{ if } \theta > \hat{\theta}
\]

(28)

The bonus in cost reimbursement, which the agent can appropriate by pretending a type that is "distant" from the true type, is lower the less concave that the opportunity cost is. Thus, as long as the concavity of \( K(\cdot) \) is sufficiently mild, correlated information is a powerful tool to make such a lie unprofitable. That is, lotteries can be used to extract the associated benefit, without yet attracting lies from other types which could rather gain in terms of lottery.

Actually, this is true both when the total cost is monotonic - case (i) in Corollary 2 - and when it is not - case (ii) - but not to the same extent. Whereas in the former case it is not complicated to ensure that upward and downward incentive constraints are jointly satisfied, difficulties can arise in the latter case instead. To see this, first consider \( \theta < \hat{\theta} \) and suppose that the principal designs a lottery for type \( \theta \) such that no type \( \theta^- \) obtains any benefit if it announces \( \theta \). Then, one cannot take for granted that such a lottery will also be unattractive to type \( \theta^+ \). For instance, if \( \theta^+ > \hat{\theta} \), then by reporting \( \theta \) type \( \theta^+ \) might gain on both the lottery and the cost reimbursement,
provided that the total cost decreases for types in $(\hat{\theta}, \theta^+)$, in a similar fashion, when $\theta > \hat{\theta}$ a lottery designed for type $\theta$ in such a way that it is unattractive to type $\theta^+$, may end up attracting some type $\theta^- < \hat{\theta} < \theta$. As a result, in case (ii) the first-best outcome is not at reach unless the agent can be exposed to higher deficits than in case (i). Indeed, the agent must have a sufficiently deep pocket for the relevant condition between (27) and (28) to be satisfied, and it turns out that those two conditions are tighter than their counterparts with a monotonic cost, namely (25) and (26). The next corollary is an implication of Lemma (8) and (11). When $K(\cdot)$ is sufficiently concave to have $\pi^a(\cdot)$ and $\pi^\beta(\cdot)$ ineffective at implementing first best, the principal should opt for $\pi^\tau(\cdot)$. Yet, this lottery does not make a better job unless the agent’s liability is high enough.

**Corollary 3** (27) implies (22) and (28) implies (23).

**References**


A Derivation of lotteries

Lottery \( \pi^a(\cdot) \)

Setting \( \pi_n(\theta) = \pi_s(\theta) \) in (6) and (7) and rearranging, we obtain:

\[
\pi_n(\theta) \leq \frac{-(\theta - \theta^-)}{p_1(\theta) - p_1(\theta^-)} \left( q^*(\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} \right) \tag{29}
\]

\[
\pi_n(\theta) \geq \frac{-(\theta^+ - \theta)}{p_1(\theta^+) - p_1(\theta)} \left( q^*(\theta) + \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} \right) \tag{30}
\]

As \( \theta^- \to \theta \) and \( \theta^+ \to \theta \) these conditions are jointly satisfied if and only if \( \pi_n(\theta) \) is expressed as in (8). (13) is obtained because as \( \pi_n(\theta) = \pi_s(\theta), \forall s \neq 1, n \). Using this in \( \sum_s \pi_s(\theta) p_s(\theta) = 0 \), we find (12).
Lottery \( \pi^3 (\cdot) \)

From \( \sum_{s} \pi_s (\theta) p_s (\theta) = 0 \) we have \( \pi_n (\theta) = - \sum_{s \neq n} \pi_s (\theta) p_s (\theta) / p_n (\theta) \), and hence we can formulate \( IC_{\theta^-} \) and \( IC_{\theta^+} \) as follows:

\[
\pi_1 (\theta) \geq \frac{(\theta - \theta^-) \left( q^* (\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} \right) - \sum_{s \neq 1, n} \pi_s (\theta) p_s (\theta) \left( \frac{p_n (\theta^-)}{p_n (\theta)} - \frac{p_s (\theta^-)}{p_s (\theta)} \right)}{p_1 (\theta) \left( \frac{p_n (\theta^-)}{p_n (\theta)} - \frac{p_1 (\theta^-)}{p_1 (\theta)} \right)} \tag{31}
\]

\[
\pi_1 (\theta) \leq \frac{(\theta^+ - \theta) \left( q^* (\theta) + \frac{K(\theta) - K(\theta^+)}{\theta - \theta^+} \right) + \sum_{s \neq 1, n} \pi_s (\theta) p_s (\theta) \left( \frac{p_n (\theta^+)}{p_n (\theta)} - \frac{p_s (\theta^+)}{p_s (\theta)} \right)}{p_1 (\theta) \left( \frac{p_1 (\theta^+)}{p_1 (\theta)} - \frac{p_n (\theta^+)}{p_n (\theta)} \right)} \tag{32}
\]

Setting \( \pi_1 (\theta) = \pi_s (\theta) \) in these inequalities yields:

\[
\pi_1 (\theta) \geq \frac{(\theta - \theta^-) p_n (\theta)}{p_n (\theta^-) - p_n (\theta)} \left( q^* (\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} \right) \tag{33}
\]

\[
\pi_1 (\theta) \leq \frac{(\theta^+ - \theta) p_n (\theta)}{p_n (\theta) - p_n (\theta^+)} \left( q^* (\theta) + \frac{K(\theta) - K(\theta^+)}{\theta - \theta^+} \right). \tag{34}
\]

As \( \theta^- \to \theta \) and \( \theta^+ \to \theta \) these conditions are jointly satisfied if and only if \( \pi_1 (\theta) \) is given by:

\[
\pi_1 (\theta) = (q^* (\theta) + K' (\theta)) \frac{p_n (\theta)}{-p_n' (\theta)}.
\]

Because \( \pi_1 (\theta) = \pi_s (\theta), \forall s \neq 1, n \), (15) is obtained. Using this in \( \pi_n (\theta) = - \sum_{s \neq n} \pi_s (\theta) p_s (\theta) / p_n (\theta) \) we derive (14).

Lottery \( \pi^7 (\cdot) \)

Setting \( \pi_s (\theta) = -L \) in (8) and rearranging, we obtain (17). From \( \sum_{s} \pi_s (\theta) p_s (\theta) = 0 \) we have \( \pi_1 (\theta) = - \sum_{s \neq 1} \pi_s (\theta) p_s (\theta) / p_1 (\theta) \). In this expression we use (17), together with \( \pi_s (\theta) = -L, \forall s \neq 1, n \), to derive (16).

**B Proof of Lemma 3**

Here and in subsequent proofs, we will be based on the equivalence between \( K'' (\cdot) < 0 \) and the following:

\[
K' (x) > \frac{K (\theta) - K (x)}{\theta - x} > K' (\theta), \forall x < \theta \tag{35}
\]

\[
K' (x) < \frac{K (x) - K (\theta)}{x - \theta} < K' (\theta), \forall x > \theta. \tag{36}
\]
(I) \( \theta < \hat{\theta} \). Using (35) and (36), we deduce the following:

\[
K''(\cdot) < 0 \iff \frac{K(h(\theta)) - K(\theta)}{h(\theta) - \theta} > K'(\theta) \text{ if } h(\theta) < \theta
\]
\[
K''(\cdot) < 0 \iff \frac{K(h(\theta)) - K(\theta)}{h(\theta) - \theta} < K'(\theta) \text{ if } h(\theta) > \theta.
\]

By the definition of \( h(\theta) \), namely \( q(\theta) + \frac{K(h(\theta)) - K(\theta)}{h(\theta) - \theta} = 0 \), we further deduce the following:

\[
K''(\cdot) < 0 \iff q^*(\theta) + K'(\theta) < 0 \text{ if } h(\theta) < \theta \tag{37}
\]
\[
K''(\cdot) < 0 \iff q^*(\theta) + K'(\theta) > 0 \text{ if } h(\theta) > \theta. \tag{38}
\]

Recall that for \( \theta < \hat{\theta} \) it is \( q^*(\theta) + K'(\theta) < 0 \) if and only if \( K''(\cdot) \geq -(q^*(\theta))' \). We use this in (37) and (38) to deduce the following:

- if \( K''(\cdot) < 0 \), then \( q^*(\theta) + K'(\theta) > 0 \), and hence \( h(\theta) > \theta \);
- if \( 0 \leq K''(\cdot) < -(q^*(\cdot))' \), then \( q^*(\theta) + K'(\theta) > 0 \), and hence \( h(\theta) < \theta \);
- if \( K''(\cdot) > -(q^*(\cdot))' \), then \( q^*(\theta) + K'(\theta) < 0 \), and hence \( h(\theta) > \theta \).

(II) \( \theta > \hat{\theta} \). Recall that for \( \theta > \hat{\theta} \) it is \( q^*(\theta) + K'(\theta) < 0 \) if and only if \( K''(\cdot) \leq -(q^*(\theta))' \). We use this in (37) and (38) to deduce the following:

- if \( K''(\cdot) < 0 \), then \( q^*(\theta) + K'(\theta) < 0 \), and hence \( h(\theta) < \theta \);
- if \( 0 \leq K''(\cdot) < -(q^*(\cdot))' \), then \( q^*(\theta) + K'(\theta) < 0 \), and hence \( h(\theta) > \theta \);
- if \( K''(\cdot) \geq -(q^*(\cdot))' \), then \( q^*(\theta) + K'(\theta) > 0 \), and hence \( h(\theta) < \theta \).

C Proof of Lemma 4

(I) \( 0 \leq K''(\cdot) < -(q^*(\cdot))' \)

(I.1) \( \theta < \hat{\theta} \), in which case \( q^*(\theta) + K'(\theta) > 0 \). Recall that setting \( \pi_n(\theta) = \pi_s(\theta) \), \((IC^{\theta_-}_{\theta^-})\) and \((IC^{\theta_+}_{\theta^+})\) are rewritten as (29) and (30). \( \exists \pi_n(\theta) \) such that both conditions hold if and only if:

\[
\frac{\theta - \theta^-}{p_1(\theta) - p_1(\theta^-)} \left( q^*(\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} \right) \leq \frac{\theta^+ - \theta}{p_1(\theta^+) - p_1(\theta)} \left( q^*(\theta) + \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} \right). \tag{39}
\]

From (36), we have \( \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} > K'(\theta) \). Because \( q^*(\theta) + K'(\theta) > 0 \), we also have \( q^*(\theta) + \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} > 0 \) so that (39) is rewritten as (19). Because \( p_1^r(\cdot) < 0 \) under (2), the right-hand side of (19) is above one. Because \( K''(\cdot) > 0 \) implies \( \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} > \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} \), the left-hand side of (19) is below one. Hence, in this case, (19) is satisfied.
\( (I.2) \) \( \theta > \hat{\theta} \), in which case \( q^*(\theta) + K'(\theta) < 0 \). Recall that setting \( \pi_1(\theta) = \pi_s(\theta) \), \((IC_{\theta^-})\)
and \((IC_{\theta^+})\) are rewritten as (33) and (34). \( \exists \pi_1(\theta) \) such that both conditions hold if and only if:

\[
\frac{\theta - \theta^-}{p_n(\theta^-) - p_n(\theta)} \left( q^*(\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} \right) \leq \frac{\theta^+ - \theta}{p_n(\theta) - p_n(\theta^+)} \left( q^*(\theta) + \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} \right) .
\]

(40)

From (35) and (36), we have \( \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} > K'(\theta) > \frac{K(\theta^-) - K(\theta^-)}{\theta - \theta^-} \). Because \( q^*(\theta) + K'(\theta) < 0 \), we also have \( q^*(\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} < 0 \). (40) is satisfied when \( q^*(\theta) + \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} > 0 \Leftrightarrow \theta^+ > h(\theta) \), which is true because \( h(\theta) < \hat{\theta} \) and \( \theta < \theta^+ \).

(II) \( K''(\cdot) \geq -\left( q^*(\cdot) \right) \)

\( (II.1) \) \( \theta < \hat{\theta} \), in which case \( q^*(\theta) + K'(\theta) < 0 \). Recall from Lemma 3 that \( \theta < h(\theta) \). If \( \theta^+ < h(\theta) \), then the right-hand side of (40) is negative. The left-hand side of (40) is negative as well. Hence, (40) is rewritten as (20). Because \( p''_p(\theta) < 0 \) under (2), the left-hand side of (20) is below 1. Moreover, because \( K''(\cdot) > 0 \), the right-hand side of (20) is above 1. Hence, (20) is satisfied.

\( (II.2) \) \( \theta > \hat{\theta} \), in which case \( q^*(\theta) + K'(\theta) > 0 \). Because \( q^*(\theta) + \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} > 0 \), (39) is rewritten as (19). Because \( p''_p(\theta) < 0 \) under (2), the right-hand side of (19) is higher than 1. Because \( K''(\cdot) > 0 \), the left-hand side of (19) is lower than 1. Hence, (19) holds.

**D Proof of Proposition 1**

For \( \theta < \hat{\theta} \) the optimal lottery is \( \pi^a(\theta) \) (Lemma 4). \((LL_{\theta^o})\) is satisfied for all types if and only if \( \pi_s(\theta) \geq -L, \forall \theta, \forall s \neq 1 \). Using (13), this inequality is rewritten as (22).

Analogously, for \( \theta > \hat{\theta} \) the optimal lottery is \( \pi^\beta(\theta) \) (Lemma 4). \((LL_{\theta^o})\) is satisfied for all types if and only if \( \pi_s(\theta) \geq -L, \forall \theta, \forall s \neq n \). Using (15), this inequality is rewritten as (23).

**E Proof of Lemma 5**

\( (I) \) \( \theta < \hat{\theta} \). In this case, \( q^*(\theta) + K'(\theta) > 0 \) and the choice is between \( \pi^a(\theta) \) and \( \pi^\gamma(\theta) \) (Lemma 1 and 2). Because \( K''(\cdot) < 0 \) it follows from (35) that \( \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} > K'(\theta) \), and from (36) that \( K'(\theta) > \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta} \). Moreover, because \( q^*(\theta) + K'(\theta) > 0 \), we have \( q^*(\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} > 0 \). If \( \theta^+ < h(\theta) \), then:

\[
q^*(\theta) + \frac{K(\theta) - K(h(\theta))}{\theta - h(\theta)} = 0 < q^*(\theta) + \frac{K(\theta) - K(\theta^+)}{\theta - \theta^+}
\]

and (39) is rewritten as (19). Both the left-hand side and the right-hand side of (19) are positive so that (19) is not implied by the assumptions of the model and is to be verified. If (19) is satisfied, and hence (11) is satisfied with \( \pi^a(\theta) \), then \( \pi^a(\theta) \) is optimal (Lemma 1 and 2). If \( \theta^+ \geq h(\theta) \), then with analogous reasoning we deduce that \( q^*(\theta) + \frac{K(\theta) - K(\theta^+)}{\theta - \theta^+} < 0 \) so that
the right-hand side of (39) is negative. Because the left-hand side is positive, (39) is violated. Hence, (11) is not satisfied with $\pi^a(\theta)$ and $\pi^r(\theta)$ is optimal.

$(II)$ $\theta > \hat{\theta}$. In this case, $q^*(\theta) + K'(\theta) < 0$ and the choice is between $\pi^b(\theta)$ and $\pi^r(\theta)$ (Lemma 1 and 2). Because $K''(\cdot) < 0$, it follows from (36) that $\frac{K(\theta^+)-K(\theta)}{\theta^+-\theta} < K'(\theta)$, and from (35) that $K'(\theta) < \frac{K(\theta)-K(\theta^-)}{\theta-\theta^-}$. Moreover, because $q^*(\theta) + K'(\theta) < 0$, we have $q^*(\theta) + \frac{K(\theta^+)-K(\theta)}{\theta^+-\theta} < 0$. If $\theta^- > h(\theta)$, then:

$$q^*(\theta) + \frac{K(\theta)-K(\theta^-)}{\theta-\theta^-} < q^*(\theta) + \frac{K(\theta)-K(h(\theta))}{\theta-h(\theta)} = 0$$

and (40) is rewritten as (20). With $p''_s(\cdot) < 0$, the left-hand side of (20) is below 1. Because $\frac{K(\theta)-K(\theta^-)}{\theta-\theta^-} > \frac{K(\theta^+)-K(\theta)}{\theta^+-\theta}$, the right-hand side of (20) is below 1 as well. Hence, we cannot conclude that (20) is satisfied and it must be verified. If (20) is satisfied, and hence (11) is satisfied with $\pi^b(\theta)$, then $\pi^b(\theta)$ is optimal (Lemma 1 and 2). If $\theta^- \leq h(\theta)$, then with analogous reasoning we deduce that $q^*(\theta) + \frac{K(\theta^+)-K(\theta^-)}{\theta^+-\theta^-} > 0$, involving that the left-hand side of (40) is positive and the condition is violated. Hence, (11) is not satisfied with $\pi^b(\theta)$ and $\pi^r(\theta)$ is optimal.

**F Proof of Proposition 2**

Derivation of (24)

Under (1), the difference in the brackets multiplied by $\pi_s(\theta)p_s(\theta)$ in (11) is negative for all triplets $\{\theta^-,\theta,\theta^+\}$. Hence, (11) is weakest when $\pi_s(\theta) = -L, \forall s \neq 1, n$. Replacing these values, (11) is reformulated as (24).

Limited liability satisfied

We are left with verifying $(LL^b_\theta)$ and $(LL^a_\theta)$.

$(I)$ $\theta < \hat{\theta}$. From (17), we see that $(LL^a_\theta)$ holds because $\frac{\nu'_1(\theta)}{p_1(\theta)} - \frac{\nu'_n(\theta)}{p_n(\theta)} > 0$ (by (1)) and $q^*(\theta) + K'(\theta) \leq Lp'_1(\theta)/p_1(\theta)$ (by (22)). From (16), we see that $(LL^b_\theta)$ holds because $\frac{\nu'_1(\theta)}{p_1(\theta)} - \frac{\nu'_n(\theta)}{p_n(\theta)} > 0$, $p'_n(\theta) < 0$ (by (1)) and $q^*(\theta) + K'(\theta) > 0$.

$(II)$ $\theta > \hat{\theta}$. From (16), we see that $(LL^b_\theta)$ holds because $p'_n(\theta) < 0$ (by (1)) and $q^*(\theta) + K'(\theta) \geq Lp'_n(\theta)/p_n(\theta)$ (by (23)). From (17), we see that $(LL^a_\theta)$ holds because $q^*(\theta) + K'(\theta) < 0$. Overall, $(LL^b_\theta)$ and $(LL^a_\theta)$ are satisfied $\forall \theta$.

**G Proof of Corollary 2**

Define the functions:

$$\varphi(x) \equiv \frac{q^*(\theta) + \frac{K(\theta)-K(x)}{\theta-x}}{\frac{p_1(\theta)-p_1(x)}{(\theta-x)p_1(\theta)} - \frac{p_n(\theta)-p_n(x)}{(\theta-x)p_n(\theta)}}$$
and \( g(\theta) \) such that \( q^*(\theta) + K'(g(\theta)) = 0 \). We will identify the conditions under which \( \varphi'(x) \geq 0 \) for \( x = \theta^- \) and for \( x = \theta^+ \). We have \( \varphi'(x) \geq 0 \) if and only if:

\[
(q^*(\theta) + K'(x)) \left( \frac{p_1(\theta) - p_1(x)}{p_1(\theta)} - \frac{p_n(\theta) - p_n(x)}{p_n(\theta)} \right) \leq (\theta - x) \left( q^*(\theta) + \frac{K(\theta) - K(x)}{\theta - x} \right) \left( \frac{p_1'(x)}{p_1(\theta)} - \frac{p_n'(x)}{p_n(\theta)} \right). \tag{41}
\]

Under (1), \( \frac{p_1(\theta) - p_1(x)}{p_1(\theta)} - \frac{p_n(\theta) - p_n(x)}{p_n(\theta)} > 0 \) if and only if \( x < \theta \). Hence, (41) is equivalent to the following pair of conditions:

\[
q^*(\theta) + K'(x) \leq (q^*(\theta) + \frac{K(\theta) - K(x)}{\theta - x}) \left( \frac{p_1'(x)}{p_1(\theta)} - \frac{p_n'(x)}{p_n(\theta)} \right), \quad \text{if } x < \theta \tag{42}
\]

\[
q^*(\theta) + K'(x) \geq (q^*(\theta) + \frac{K(\theta) - K(x)}{\theta - x}) \left( \frac{p_1'(x)}{p_1(\theta)} - \frac{p_n'(x)}{p_n(\theta)} \right), \quad \text{if } x > \theta. \tag{43}
\]

(I) either \( \hat{\theta} \) or \( \bar{\theta} > \bar{\theta} \). In this case, \( q^*(\theta) + K'(x) > 0 \) and \( q^*(\theta) + \frac{K(\theta) - K(x)}{\theta - x} > 0 \), \( \forall x, \forall \theta \). For \( x < \theta \) (41) is equivalent to (42), which is reformulated as (25); for \( x > \theta \) (41) is equivalent to (43), which is reformulated as (26). If \( K(\cdot) \) is sufficiently little concave that (25) and (26) hold, then the left-hand side of (24) increases with \( \theta^- \) and decreases with \( \theta^+ \). As the right-hand side of (24) decreases with \( \theta^- \) and increases with \( \theta^+ \) (by (1)), it follows that (24) is tightest as \( \theta^- \to \theta \) and \( \theta^+ \to \theta \), in which case it is satisfied. Hence, (24) is satisfied any triplet \( \{\theta^-, \theta, \theta^+\} \).

(II) \( \hat{\theta} \in (\theta, \bar{\theta}) \). Using \( \frac{d}{dx} \left( \frac{K(x) - K(\theta)}{x - \theta} \right) < 0 \) (which follows from \( K''(\cdot) < 0 \)) together with the definition of \( h(\theta) \), we see that \( q^*(\theta) + \frac{K(\theta) - K(x)}{\theta - x} > 0 \) if and only if \( x < h(\theta) \). (41) is rewritten as (25), if either \( x < \theta \) and \( x < h(\theta) \) or \( x > \theta \) and \( x > h(\theta) \); it is rewritten as (26), if either \( x < \theta \) and \( x > h(\theta) \) or \( x > \theta \) and \( x < h(\theta) \).

We now verify the pairs \( \{\theta, x\} \) for which the associated condition (25) or (26) is violated. Because \( K''(\cdot) < 0 \), \( \bar{\theta} < g(\theta) < h(\theta) \) \( \forall \theta < \hat{\theta} \) and \( h(\theta) < g(\theta) < \bar{\theta} \forall \theta > \hat{\theta} \).

(II.1) If \( \theta < \hat{\theta} \) together with \( g(\theta) < x < h(\theta) \), then the associated condition (26) is violated and \( \varphi'(x) < 0 \) for any degree of concavity of \( K(\cdot) \). If \( x \notin [g(\theta), h(\theta)] \), then the associated condition (25) or (26) can be satisfied. Because \( \theta^- < \theta < g(\theta) \), only \( \theta^+ \) can belong to \( [g(\theta), h(\theta)] \). Hence, for any \( \theta^- < \theta, \varphi'(\theta^-) > 0 \) if the associated condition (25) or (26) is satisfied so that (24) is tightest as \( \theta^- \to \theta \). Therefore, (24) is to be verified for \( \theta^- \to \theta \) and some \( \theta^+ > \theta \) (see below).

(II.2) If \( \theta > \hat{\theta} \) together with \( h(\theta) < x < g(\theta) \), then the associated condition (26) is violated. Again, \( \varphi'(x) < 0 \) for any degree of concavity of \( K(\cdot) \). If \( x \notin [h(\theta), g(\theta)] \), then the associated condition (25) or (26) can be satisfied. Because \( \theta^+ > \theta > g(\theta) \), only \( \theta^- \) can belong to \( [h(\theta), g(\theta)] \). Hence, (24) is satisfied for any \( \theta^- < \theta \) if it is satisfied for \( \theta^+ \to \theta \). Therefore, (24) is to be verified for \( \theta^+ \to \theta \) and some \( \theta^- < \theta \) (see below).

Verify (24) on the range \([g(\theta), h(\theta)]\) when \( \theta < \hat{\theta} \) and \([h(\theta), g(\theta)]\) when \( \theta > \hat{\theta} \)

(I) \( \theta < \hat{\theta} \) and \( \theta^- \to \theta \).

\( \varphi'(\theta^+) > 0 \) if \( \theta^+ < g(\theta) \); \( \varphi'(\theta^+) < 0 \) if \( \theta^+ \in [g(\theta), h(\theta)] \); \( \varphi'(\theta^+) > 0 \) if \( \theta^+ > h(\theta) \). Hence, (24) is tighter at \( \theta \) than at \( g(\theta) \); it is tighter at \( h(\theta) \) than at \( g(\theta) \); it is tighter at \( h(\theta) \) than at \( \bar{\theta} \). To verify if (24) is tightest as \( \theta^+ \to \theta \) we need to recall the definition of \( \varphi(x) \) and
check whether:

$$\varphi(\theta) < \varphi(h(\theta)).$$

(44) is equivalent to:

$$\frac{q^*(\theta) + K'(\theta)}{\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)}} < \frac{q^*(\theta) + \frac{K(\theta) - K(h(\theta))}{\theta - h(\theta)}}{\frac{p_1(\theta) - p_1(h(\theta))}{(\theta - h(\theta))p_1(\theta)} - \frac{p_n(\theta) - p_n(h(\theta))}{(\theta - h(\theta))p_n(\theta)}} = 0.$$

This is impossible because the left-hand side is positive. Hence, the left-hand side of (24) is highest for $$\theta^+ = h(\theta)$$, in which case $$\varphi(\theta^+) = 0$$. Because the right-hand side of (24) is lowest for $$\theta^+ \to \theta$$, we need to compare (24) for $$\theta^+ = h(\theta)$$ and for $$\theta^+ \to \theta$$. For $$\theta^+ = h(\theta)$$ and $$\theta^+ \to \theta$$ (24) is rewritten as (27). For $$\theta^+ \to \theta$$ (24) reduces to $$0 \leq 0$$, and hence it is satisfied as an identity. Therefore, (24) holds for any triplet $$\{\theta^-, \theta, \theta^+\}$$ if and only if (27) is satisfied.

(II) $$\theta > \hat{\theta}$$ and $$\theta^+ \to \theta$$.

$$\varphi'(\theta^-) > 0$$ if $$\theta^- > g(\theta)$$; $$\varphi'(\theta^-) < 0$$ if $$\theta^- \in [h(\theta), g(\theta)]$$; $$\varphi'(\theta^-) > 0$$ if $$\theta^- < h(\theta)$$.

Hence, (24) is tighter at $$\theta$$ than at $$g(\theta)$$; it is tighter at $$h(\theta)$$ than at $$g(\theta)$$; it is tighter at $$h(\theta)$$ than at $$\hat{\theta}$$. To verify if (24) is tightest as $$\theta^- \to \theta$$ we need to check if:

$$\varphi(\theta) > \varphi(h(\theta))$$

(45) is equivalent to:

$$\frac{q^*(\theta) + K'(\theta)}{\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)}} > \frac{q^*(\theta) + \frac{K(\theta) - K(h(\theta))}{\theta - h(\theta)}}{\frac{p_1(\theta) - p_1(h(\theta))}{(\theta - h(\theta))p_1(\theta)} - \frac{p_n(\theta) - p_n(h(\theta))}{(\theta - h(\theta))p_n(\theta)}} = 0,$$

which is not true because the left-hand side is negative. As in (1), we need to verify (24) for $$\theta^- = h(\theta)$$. Provided that $$\theta^+ \to \theta$$, (24) is rewritten as the converse of (27), namely as (28).

H Proof of Corollary 3

(27) implies (22)

Multiply both sides of (27) by the positive difference $$\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)}$$ and rearrange to obtain

$$\begin{align*}
q^*(\theta) + K'(\theta) & \leq L \left\{ \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_1(\theta)}{p_1(\theta)}p_n(\theta) + p'_n(\theta) - \frac{p'_n(\theta)}{p_n(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \left( 1 - p_n(\theta) \right) \frac{p_1(h(\theta))}{p_1(\theta)} - 1 + p_n(h(\theta)) \right\}.
\end{align*}$$

As the right-hand side is lower than $$Lp'_1(\theta)/p_1(\theta)$$, this is tighter than (22) if and only if

$$\begin{align*}
& \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_1(\theta)}{p_1(\theta)}p_n(\theta) + p'_n(\theta) - \frac{p'_n(\theta)}{p_n(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \left( 1 - p_n(\theta) \right) \frac{p_1(h(\theta))}{p_1(\theta)} - 1 + p_n(h(\theta)) \right\} < p_n(\theta) \left( \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right).
& \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_1(\theta)}{p_1(\theta)}p_n(\theta) + p'_n(\theta) - \frac{p'_n(\theta)}{p_n(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \left( 1 - p_n(\theta) \right) \frac{p_1(h(\theta))}{p_1(\theta)} - 1 + p_n(h(\theta)) \right\} < p_n(\theta) \left( \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right).
\end{align*}$$

Multiplying both sides by $$\left( \frac{p_1(h(\theta))}{p_1(\theta)} - \frac{p_n(h(\theta))}{p_n(\theta)} \right) \left( \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right) > 0$$ and rearranging, we obtain $$\left[ (p_1(h(\theta)) - p_1(\theta))/p_1(\theta) \right] > 0$$, which is true by (1).
(28) implies (23)

Multiply both sides of (28) by the positive difference $\frac{p'_n(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)}$ and rearrange to obtain

$$q^*(\theta) + K' \cdot (\theta) \geq L \left\{ (1 - p_n(\theta)) \frac{p'_1(\theta)}{p_1(\theta)} + p'_n(\theta) - \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \left[ (1 - p_n(\theta)) \frac{p_1(h(\theta))}{p_1(\theta)} - 1 + p_n(h(\theta)) \right] \right\}.$$

As the right-hand side is higher than $L p'_n(\theta)/p_n(\theta)$, this is tighter than (23) if and only if

$$- \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} + \left[ (1 - p_n(\theta)) \frac{p_1(h(\theta))}{p_1(\theta)} - 1 + p_n(h(\theta)) \right] < (1 - p_n(\theta)) \left( \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right).$$

Multiplying both sides by \left[ \left( \frac{p_1(h(\theta))}{p_1(\theta)} - \frac{p_1(h(\theta))}{p_1(\theta)} \right) / \left( \frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right) \right] > 0 and rearranging, we obtain $[(p_n(h(\theta)) - p_n(\theta))/p_n(\theta)] > 0$, which is true by (1).