Optimal taxation of motherhood*

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May 30, 2016

Abstract

Mothers earn substantially lower wages and incomes, and have a larger labor supply elasticity than women without children. In the presence of a progressive tax-transfer scheme, the discrete change in family status of becoming a mother can thus generate substantial fiscal externalities. We build an optimal taxation model that combines an intensive labor supply margin with the extensive child-bearing decision. The latter endogenously affects female productivity in terms of market remuneration as well as the labor supply elasticity. The resulting reduction in gross income reduces tax revenue under progressive income taxation, such that the decision for a child entails a fiscal externality. In addition, the higher labor supply sensitivity to tax changes increases the impact of the tax schedule as a policy instrument. Our analysis reveals the nature of this interaction and shows that this externality can be important.

JEL classification: H21, J13, J16, J22

Keywords: Optimal taxation, redistribution, child-bearing, family gap, motherhood wage penalty, gender gap, family size, multi-dimensional screening, delayed optimal control

\textsuperscript{*}Preliminary and incomplete. Please neither cite nor distribute.
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1 Introduction

Despite substantial convergence the gender pay gap between women and men remains substantial. Recent evidence points to the key role of child-bearing for the remaining gap. Using administrative data from Denmark, Kleven et al. (2015) find that about 80% of the wage income difference can be attributed to the decision to become a mother. This confirms earlier findings by Waldfogel (1998), Waldfogel (1997) and Joshi et al. (1999), among others, who show similar effects in other countries. This family gap appears to be considerably persistent over time, whereas the pure gender gap seems to have diminished largely, at least for Denmark, see Kleven et al. (2015). Moreover, the negative effect of the individual child-bearing decision on female gross incomes is due to lower wages and an increased labor supply elasticity. We analyze the consequences for optimal taxation, if child-bearing endogenously affects female wages and female labor supply elasticity. Our key innovation is to treat the child-bearing decision and its consequences for wages and labor supply as endogenous. This is in contrast with most of the optimal taxation literature that has addressed the optimal taxation of families and children for a given family structure and productivity, see Blundell and Shepard (2009) and Brewer et al. (2010).

We consider the labor supply and the fertility decision jointly, and we allow the discrete fertility margin to endogenously determine productivity and the labor supply elasticity. In line with the empirical evidence, we model the decision to have a child to reduce the productivity in terms of market compensation per hour worked as well as affecting the women’s labour supply elasticity. Our analysis illuminates how the decision to have a child bears a fiscal externality. The decision to have a child reduces gross wage income and thus, in the presence of an increasing tax schedule, tax payments to the government. Following the empirical evidence, our framework allows for a reduction of gross income due to a lower compensation per hour worked as well as for the effects that originate from the increase in the labor supply elasticity. Reducing the tax payment for mothers triggers marginal potential mothers into child-bearing. These additional mothers’ gross income is reduced via the productivity and the labor supply elasticity channel, reducing net tax payments. These fiscal costs are the only welfare costs to be considered since the child-bearing decision is taken optimally, such that the marginal mothers are indifferent.

Most countries have family policies in place which subsidize children. From an efficiency perspective, such subsidies can be justified by the positive externality that children generate in the presence of a public pay-as-you-go-pension scheme, where people individually take the rate of return of the system as given, but, which is actually determined by the growth rate of the total wage bill. Moreover, there may be further positive exter-
nalities from a younger society. A higher share of young people is likely to be conducive to innovation and entrepreneurship, which are activities predominantly carried out by younger individuals. Our analysis presents a countervailing force to these arguments for subsidizing children. The optimal policy should balance these classic positive externalities of children with the negative fiscal externalities that we identify in our analysis.

Our analysis additionally also provides an important contribution to the discussion on optimal child support policies. Frequently, this discussion focuses on the question whether such policies should take the form of cash benefits or whether benefits should be delivered in kind (child care facilities, parent-friendly-workplaces and working times etc.), and should change the perception of working mothers in society. These policy alternatives are usually regarded as substitutes. Our analysis reveals, however, that, to the extent that in kind policies or changes in the perception of working mothers positively reduce the fiscal externality, optimal cash benefits are complementary to these in kind policies.

2 Related literature

In the optimal income taxation literature the question of how family size should influence income tax schedule reaches back to Mirrless (1972) who, though, stopped his analysis before going into detail. Gelber and Weinzierl (2012) discuss the endogeneity of future income distribution since, to a large part, children’s ability depend on their parents’ ability. Cremer et al. (2003) analyze the utility-generating property of children in parents’ well-being and find that marginal tax rates should decrease with family size for low abilities. They also give an overview of actual child considerations in the tax regulations of European countries. However, they do not consider a productivity or wage effect of children. Piketty and Saez (2013) give a recent discussion on transfer and household income tax issues concerning children. However, they do not discuss the potential productivity loss due to child bearing and rearing. Guner at al. (2014) estimate parameters and the effective tax functions for U.S. households. Kurnaz (2016) connects a Mirrleesian optimal tax framework with a Beckerian home production process of (time-) costly children. In addition, families differ in their taste for children and make fertility decisions on the size of their family entailing an endogenous labor supply elasticity. In this study, child benefits are examined by considering child-dependend optimal tax schedules. However, Kurnaz (2016) differs in two important aspects from our work.

First, Kurnaz (2016) considers families that are a joined entity of the spouses having joined parameters such as a common labor supply elasticity. There are related studies about optimal taxation of couples, treating unitary and collective family decision mak-
ing, see Kleven et al. (2009) and Immervoll et al. (2011). The former authors study the optimal taxation of couples, where a cooperative household decides on its intensive labor supply and on the decision of whether to become a two-income earner household or not. They model directly the extensive decision of the spouse. The latter study examines the extensive decision of both spouses unitary and collective, respectively, in a two-step double-extensive approach using sufficient statistics. Their basic framework is then used to analyze the transfer schemes under different reforms for several European countries. Both studies neglect the intensive decisions of the secondary earner. Moreover, both studies do not consider the child-bearing decision. Given the fact that the share of stepfamilies increased substantially and long-lasting families got more and more uncommon in developed countries, a distinctive perspective on women is required. This is underpinned by the fact that most countries nowadays tax spouses separately. Especially when considering the fertility decision of women, neither a household bargaining process nor a full family comprising view may be adequate anymore for a substantially amount of women. Our study focus on women as independent entities with a fertility and a labor supply choice and, hence, we contribute to this relevant perspective from a redistributive taxation view. In this sense, our study is also related to the literature on lone mothers, see, among others, Blundell et al. (2009) who empirically analyze the tax and transfer scheme to lone mothers in Germany and the UK using an optimal taxation framework. In this study, the authors find substantially differences in the extensive as well as in the intensive labour decision depending on whether the lone mother has a child of pre-school age or not. The likely but advanced study of Blundell and Shephard (2011) underpinnnes clearly the results of the previous study.

Second and most relevant, Kurnaz (2016) neglects the important productivity and wage effect, respectively, that women are faced when they gave birth to a child. This is a relevant topic even outside the economic literature, see for example Budig and England (2001) and, for a more recent overview, Grimshaw and Rubery (2015). As the fertility decision affects labor force relevant aspects, namely the wage and the labor supply elasticity, we simultaneously analyze the, perhaps, two most influencing decisions in a woman’s life, to wit the decisions of becoming a mother and of the extend, if at all, participating in the labor force. The fiscal externality resulting from the productivity decrease creates an opposite effect on marginal tax rate compared to the pure elasticity effect considered by Kurnaz (2016). Higher marginal tax rates on mothers ensures efficiency and, hence, counteracts the elasticity channel that lowers tax rates. In our model, we take account also of this productivity effect and examine the interaction of these counteracting effects. To our best knowledge, the analysis of tax related child allowance in an optimal taxation frame-
work that allows for endogenous motherhood productivity changes has been neglected so far, despite the fact of frequent consideration of children in real-life tax regulations.

Kleven et al. (2015) show for Denmark that the persistent gender income gap is mostly driven by women’s decision to become a mother. They find both, a permanent decrease in productivity together with a permanent change in labor supply elasticity of a women just in the moment she gives birth to a child. This is in line with other studies, see, among others, Blundell et al. (1993) who investigate the labor supply of women during their lifecycle find higher labor supply elasticities of child-rich women compared to childless women. Davies and Pierre (2005) study the family gap across European countries and find significant wage penalties for several of them, among others, for Germany, Denmark, and the UK. Wilde et al. (2010) give recent evidence of a family gap in the US. In addition, they provide estimates for different skill levels.

The structure of our approach owes much to Kessing et al. (2015), who combine the intensive labor supply decision with an extensive, productivity-increasing migration margin, and to Kleven et al. (2009). Hungerbühler et al. (2006) model income taxation with labor market frictions in a job matching framework. Their setting may act as another way of modeling the labor decision of women since, in a framework of random offers of better jobs, women tend to less decide to change job in comparison to men.

Our analysis of the potential benefits of differentiated taxation relates to the increased interest in tagging in the design of tax-transfer-schemes. The idea that the government’s information problem can be relaxed by using additional observable characteristics (“tags”) that are correlated with the individual productivity goes back to Akerlof (1978) and has recently been discussed intensively in the optimal taxation literature, see Immonen et al. (1998), Weinzierl (2011), Mankiw and Weinzierl (2010), Boadway and Pestieau (2005), Cremer et al. (2010) and Best and Kleven (2013). We add to this literature in several ways. First, we consider the decision of having a child as a potential tag. Secondly, we explicitly study a tag that is endogenous and can be adjusted by individuals subject to some cost. Moreover, changing the tag directly affects productivity.

3 The framework

We consider the optimal taxation of women who may or may not have children. We assume that there are originally two sources of heterogeneity across women: innate productivity $n$ and the utility derived from having a child $q$. These original individual characteristics are distributed over $[n_{\min}, n_{\max}] \times [0, +\infty)$, and the government can neither observe productivity nor the benefits from having children. For each woman, there are two possible
states of the world $i \in [0, 1]$, which depend endogenously on each woman’s choice. State 0 (1) denotes a woman who is childless (a mother). Total population of women is normalized to one and consists of the two subgroups according to the two states of the world. It is straightforward to extend our framework to allow for different number of children, but, for clarity of exposition, we treat all mothers alike. Next our framework incorporates the empirical features that childbearing negatively affects productivity as measured by market compensation and positively affects women’s labor supply elasticity. To model endogenous productivity we assume that a woman’s actual or realized productivity $n_i$ is a function of her innate productivity and her decision $n_i = \omega(n, i) = \omega_i(n)$, where $\omega_i$ is strictly increasing in $n$. We normalize $n_0 = \omega_0(n) = n$. Accordingly, the function $n_i = \omega_1(n) = \omega(n)$ indicates the transformation of productivity of all women who decide to have a child. Without loss of generality and according to empirical findings, too, we assume that childless women are more productive than mothers, so that $\omega(n) < n$. The lag function $\kappa(n) \equiv w(n) - n$ describes the decrease in ability from having a child. In-nate productivity is distributed according to the unconditional probability distribution $f(n)$ on $[n_{\min}, n_{\max}]$. With the exception of the individual child-bearing decision, we treat wages as exogenous and independent of individual labor supply and aggregate decisions of having children, as in most of the optimal taxation literature.

We also endogenize the individual labor supply elasticity as a function of the family status. Thus, following Diamond (1998), we use preferences that are separable in consumption and labor, but also depend on the decision of having a child,

$$u(c, z, l) = c_i - n_i h_i \left( \frac{z_i}{n_i} \right) + q l, \tag{1}$$

where $l$ is an indicator variable that takes the value of 1 in case of having a child. The function $h_i(\cdot)$ is increasing, convex and twice-differentiable. It is normalized such that $h_i'(1) = 1$ and $h_i(0) = 0$. Note that we allow this function to depend on the child-bearing decision. This allows us to endogenize the labor supply elasticity. The other variables have standard interpretations. Consumption $c_i$ equals gross income $z_i$ minus taxes $T_i$, which itself depend on gross income, $c_i = z_i - T_i(z_i)$. The last term in (1) measures the money-metric utility women derive from being a mother.

Each woman chooses $l$ and $z_i$ to maximize (1) for a given tax schedule, i.e. she decides whether to have a child or not and determines her gross earnings, given that her desire to have a child is $i$. The first order condition for gross earnings is

$$h_i' \left( \frac{z_i}{n_i} \right) = 1 - \tau_i(z_i), \tag{2}$$
where \( \tau_i \) is the marginal tax rate. Accordingly, \( n_i \) can be interpreted as potential income, given that women facing a marginal tax rate of zero would realize this level of gross earnings. The elasticity of gross earnings with respect to net-of-tax-rate \( \varepsilon_i \) as a function of gross earnings and the motherhood status \( i \) is defined as

\[
\varepsilon_i \equiv \frac{1 - \tau_i}{z_i} \frac{\partial z_i}{\partial (1 - \tau_i)} = \frac{n_i h'_i \left( \frac{z_i}{n_i} \right)}{z_i h''_i \left( \frac{z_i}{n_i} \right)}.
\]

Furthermore, we require the following assumption.

**Assumption** The function \( x \to \frac{1-h'_i(x)}{x h''_i(x)} \) is decreasing for all \( i \).

Consider now the decision to have a child. We denote by \( p(q|n) \) the density of \( q \) conditional on \( n \), and by \( P(q|n) \) the cumulated distribution of \( q \) conditional on \( n \). Conditional on the desire to have a child \( i \), the individuals’ choice of gross earnings is determined by (2), which allows to define indirect utility conditional on the decision to have a child and net of the benefits from children

\[
V_i(n_i) = z_i - T_i(z_i) - n_i h_i \left( \frac{z_i}{n_i} \right).
\]

Women will not have a child, whenever their benefits from motherhood are lower than the net loss from having a child, such that \( q \equiv V_0(n_0) - V_1(n_1) \) is the critical level of benefit from motherhood that determines the actual number of childless women for any innate productivity level.

### 3.1 The government’s optimal tax problem

The government wants to maximize the social welfare function

\[
\int_{n_{\text{min}}}^{n_{\text{max}}} \int_{0}^{+\infty} \Psi \left( V_i(n) + q^b l - q^d (1 - l) \right) p(q, n) f(n)dqdn,
\]

where \( \Psi(.) \) is a concave and increasing transformation of individual utilities. Denoting by \( E \) the exogenous expenditure requirements, it needs to respect the budget constraint

\[
\int_{n_{\text{min}}}^{n_{\text{max}}} \int_{0}^{+\infty} T_i(z_i) p(q, n) f(n)dqdn \geq E.
\]
Moreover, the government’s tax schedule needs to be incentive compatible for individuals with and without children. This implies

\[ \hat{V}(n) = \left[ -h_i \left( \frac{z_i}{n_i} \right) + \frac{z_i}{n_i} h'_i \left( \frac{z_i}{n_i} \right) \right] \omega'_i(n) \geq 0, \]

where the dot above a variable denotes its derivative with respect to \( n \). We show in the appendix that a path for \( z_0 \) and \( z_1 \) can be truthfully implemented by the government using a non-linear tax schedule.

Let \( \lambda > 0 \) be the multiplier associated with the budget constraint (4). The government’s redistributive tastes may be represented by child-dependent social marginal welfare weights. In terms of income, our welfare weights will take the form of

\[
g_0(z) = \frac{\Psi'(V_0(z)) P(q|z)}{\lambda P(q|z)} = \frac{\Psi'(V_0(z))}{\lambda}
\]

\[
g_1(z) = \frac{\int_0^\infty \Psi'(V_1(z) + q^b) p(q|z) dq}{\lambda (1 - P(q|z))}
\]

for the benefit from children model. Note that this implies that individuals with children should receive a lower marginal welfare weight, since they realize the benefits from having children and our thus considered better off. However, the government may additionally take into account that the individual will use part of her income for children, such that the disposable income for herself is reduced. ’s objective the individuals net consumption may be more

\[ \text{4 Optimal differentiated taxation} \]

Given that most countries condition their tax-transfer schedule, we focus on the case where the tax-transfer-schedule is conditioned on family status. Moreover, for clarity of exposition we focus on the simple binary choice between children and no children. Our framework can be extended to allow for more than two potential family status possibilities. The optimal taxation problem in this context amounts to a tagging problem. However, the tag here is endogenous and causal for affecting a woman’s productivity and labor supply elasticity. Our analysis employs the perturbation approach introduced by Picketty (1999) and Saez (2001). We derive our results rigorously in the Appendix. motherhood productivity differences but no further pregnancies, this setting would correspond to the analysis of an optimal tax scheme with tagging on the motherhood status. However, we continue to assume that giving birth to a first child is possible, that productivity is motherhood-dependent, and that individuals are heterogenous with respect to their
child benefits, which are unobservable by the government. We employ the perturbation approach and delegate the formal proofs to the Appendix.

We use the endogenously realized distribution of gross incomes in both subgroups denoted by \( v_i(z_i) \), and we denote by \( k \) the endogenously defined, strictly increasing function that maps gross income of childless women to the gross income this female would earn if she gives birth to a child, given her innate productivity and the respective tax treatment, i.e. \( z_1 = k(z_0) \).\(^1\)

We first study the optimal tax schedule for the higher productive, but motherless women. Consider an increase of taxes for all childless women above gross income \( z_0 \). The increase is engineered through an increase in the marginal tax rate \( d\tau_0 \) in the small band \((z_0, z_0 + d z_0)\), such that all individuals with gross earnings above \( z_0 \) increase their tax payments by \( dz_0 d\tau_0 \). This generates three effects.

**Revenue effect:** All childless taxpayers pay additional taxes of \( dz_0 d\tau_0 \). The net welfare effect of this tax payment for an women with gross earnings \( z'_0 \) is given by \( dz_0 d\tau_0 (1 - g_0(z'_0)) \) and the total effect is

\[
R_0 = dz_0 d\tau_0 \int_{z_0}^{\infty} \left[1 - g_0(z'_0)\right] v_0(z'_0) s_0(z'_0) dz'_0,
\]

with \( s_0(z'_0) \equiv 1 - P(\bar{q} \mid z'_0) \) and accordingly \( s_1(z'_1) \equiv P(q \mid z'_1) \).

**Behavioral effect:** Women in the band \((z_0, z_0 + d z_0)\) will change their labor supply in response to the increase in the marginal tax rate. Given that \( \epsilon_i \equiv \frac{1 - \tau_0}{\tau_i} \frac{dz_i}{d(1 - \tau_i)} \), each individual in the band will reduce its income by \(-d\tau_0 \epsilon_0 \frac{z_0}{1 - \tau_0} \). There are approximately \( dz_0 v_0(z_0) s_0(z_0) \) of these individuals, such that the total effect on tax revenue is

\[
B_0 = -d\tau_0 dz_0 \epsilon_0 \tau_0 \frac{z_0 v_0(z_0) s_0(z_0)}{1 - \tau_0}.
\]

**Child effect:** An increase in taxes for all individuals above gross income \( z_0 \) affects the pregnancy decision of childless women with gross income above this level. At any income level \( z \geq z_0 \) women whose benefit of children is between \( \bar{q} \) and \( q - dz_0 d\tau_0 \) will now decide to give birth to their first child. There are \( p(\bar{q} \mid z_0) v_0(z_0) dz_0 d\tau_0 \) affected individuals with a resulting tax effect of \( T_1(k(z_0)) - T_0(z_0) \) for each of them. The total effect is thus

\[
C_0 = dz_0 d\tau_0 \int_{z_0}^{\infty} \left[T_1(k(z'_0)) - T_0(z'_0)\right] p(\bar{q} \mid z'_0) v_0(z'_0) dz'_0.
\]

In the optimum, these effects should cancel out such that optimal marginal tax rates can

\(^1\)In terms of our previous formulation, a female of ability \( n \) receives gross income \( z_0 = z(n) \) without a child and gross income \( z_1 = z(\omega(n)) \) as a mother, where this notation abstracts from the fact that the gross income also depends on the tax schedule.
be characterized by

\[
\frac{\tau_0}{1 - \tau_0} = \frac{1}{\varepsilon_0 z_0 v_0(z_0) s_0(z_0)} \times \int_{z_0}^{+\infty} \left\{ \left[ 1 - g_0(z_0') \right] s_0(z_0') + \left[ T_1(k(z_0')) - T_0(z_0') \right] p\left( \tilde{q} \mid z_0' \right) \right\} v_0(z_0') dz_0' (6a)
\]

We turn now to the optimal tax schedule for mothers. We consider a small increase in taxes by \( dz_1 d\tau_1 \) for all mothers above \( z_1 \). However, since there is also a child effect for women that would have a child if the tax change does not come into account, the tax increase generates again three effects, which must balance out along the optimal tax schedule, such that

\[
\frac{\tau_1}{1 - \tau_1} = \frac{1}{\varepsilon_1 z_1 v_1(z_1) s_1(z_1)} \times \int_{z_1}^{+\infty} \left\{ \left[ 1 - g_1(z_1') \right] s_1(z_1') - \left[ T_1(k(z_1')) - T_0(k^{-1}(z_1')) \right] p\left( \tilde{q} \mid z_1' \right) \right\} v_1(z_1') dz_1'.
\] (6b)

Both optimal tax schedules are derived rigorously in the Appendix. The optimal tax formulae not only differ by the different average welfare weights and the respective productivity distribution above the gross income level for which taxes are increased, but they also take the fiscal externality from the effect on motherhood into account. We have the following result.

**Proposition 1** For all levels of innate productivity and the corresponding gross incomes the marginal tax rate for mothers \( \tau_1 \) is increasing in the difference in total tax liability between women with and without children, and the marginal tax rate for childless women \( \tau_0 \) is decreasing in this difference in total tax liability.

**Proof.** The result follows directly from (6a) and (6b).

Intuitively, the larger the potential fiscal loss are from having a child instead of higher income and higher labor supply of childless women, the more the government distorts labor supply of potential mothers and the less it distorts labor supply for potential childless women. This indicates that the marginal tax rates are used to suppress childbearing. Differences in the demogrant may be used instead to target redistribution by using a child as a productivity tag.

### 5 Concluding remarks

In the presence of a progressive tax-transfer scheme and given the substantially lower wages and larger labor supply elasticity, the discrete choice of becoming a mother can
thus generate substantial fiscal externalities. Our optimal taxation model combines an intensive labor supply margin with the extensive child-bearing decision. The latter endogenously affects female productivity in terms of market remuneration as well as the labor supply elasticity. The resulting reduction in gross income reduces tax revenue under progressive income taxation, such that the decision for a child entails a fiscal externality. Our analysis reveals the nature of this externality and shows that they can be important since marginal tax rates are higher for potential mothers. Basically, the framework could be expanded to a world of uniform taxation. In such a setting the delayed optimal control technique used in Kessing et al. (2015) applies. Furthermore, simulations are required to indicate the quantitative importance of the result. Additionally, future research should contrast the results with in-kind benefits.
6 Appendix A: Formal derivation of the optimal tax formulae

We now show that the optimal tax formulae (6a) and (6b).

6.1 Child-dependent differentiated taxation

The government maximizes

\[
W = \int_{n_{\text{min}}}^{n_{\text{max}}} \left[ \int_{0}^{\hat{q}} \Psi \left( V_0(n) - q^d \right) p(q|n) dq + \int_{\hat{q}}^{+\infty} \Psi \left( V_1(\omega(n)) + q^b \right) p(q|n) dq \right] f(n) dn,
\]

where \( q = q^b + q^d \), and either \( q^b = 0 \) or \( q^d = 0 \). The first term in this expression stands for the social welfare from the childless women, the second term stands for that of the population of mothers.

The maximization is subject to

\[
\text{and the corresponding incentive compatibility constraints.}
\]

Let the Hamiltonian be \( H(z_0, z_1, V_0, V_1, \lambda, \mu_0, \mu_1, n) \). The necessary conditions are

1. There exist absolutely continuous multipliers \( \mu_0(n), \mu_1(n) \) such that on \( (n_{\text{min}}, n_{\text{max}}) \)
   \[
   \dot{\mu}_1(n) = -\frac{\partial H(n)}{\partial \dot{\omega}_1(n)} \quad \text{and} \quad \dot{\mu}_0(n) = -\frac{\partial H(n)}{\partial \dot{\omega}_0(n)} \quad \text{almost everywhere with} \quad \mu_1(n_{\text{min}}) = \mu_1(n_{\text{max}}) = 0.
   \]

2. We have \( H(z_i(n), V_i, \lambda, \mu_i, n) > H(z_i, V_i, \lambda, \mu_i, n) \) almost everywhere in \( n \) for all \( z \). The first order conditions are
   \[
   \frac{\partial H}{\partial z_0} = 0, \quad \frac{\partial H}{\partial z_1} = 0.
   \]

Uniqueness of \( z_0 \) and \( z_1 \) that solve the equations above can be established in the similar way to Kleven et al. (2009), using the assumption that \( \varphi(x) = (1 - h'(x))/xh''(x) \) is decreasing in \( x \). Indeed, the FOCs can be rewritten as

\[
\frac{\mu_0(n)}{n} \left( \frac{z_0}{n} \right) + \lambda \left( 1 - h' \left( \frac{z_0}{n} \right) \right) P(q|n) f(n) = 0,
\]

\[
\varphi \left( \frac{z_0}{n} \right) = -\frac{\mu_0(n)}{\lambda n f(n) P(q|n)}
\]

for childless women and

\[
\frac{\mu_1(n)}{\omega(n)} \left( \frac{z_1}{\omega(n)} \right) + \lambda \left( 1 - h' \left( \frac{z_1}{\omega(n)} \right) \right) \left( 1 - P(q|n) \right) f(n) = 0
\]

\[
\varphi \left( \frac{z_1}{\omega(n)} \right) = -\frac{\mu_1(n)}{\lambda \omega(n) f(n) \left( 1 - P(q|n) \right)}
\]
for mothers. In both cases, LHS is decreasing in $z_0/n$ ($z_1/\omega(n)$) whereas RHS is constant, which implies that $z_i(n)$ is a unique solution and a global maximum indeed. Continuity can be then established in a way similar to Kleven et al (2009).

The conditions for $\mu_i(n)$ imply

$$-\mu_0(n) = f(n) \int_0^q \Psi' \left(V_0(n) - q^d\right) p(q|n) dq - \lambda P(q|n) + \lambda (T_0 - T_1) p(q|n),$$

and

$$-\mu_1(n) = f(n) \int_q^{+\infty} \Psi' \left(V_1(\omega(n)) + q^b\right) p(q|n) dq - \lambda (1 - P(q|n)) + \lambda (T_0 - T_1) p(q|n),$$

Integrating this, we have

$$-\frac{\mu_0(n)}{\lambda} = \int_n^{n_{\text{max}}} \left[-\frac{1}{\lambda} \int_0^q \Psi' \left(V_0(n) - q^d\right) p(q|n') dq + (1 - P(q|n')) - (T_0 - T_1) p(q|n')\right] f(n') dn',$$

Analogously, for mothers we get

$$-\frac{\mu_1(n)}{\lambda} = \int_n^{n_{\text{max}}} \left[-\frac{1}{\lambda} \int_q^{+\infty} \Psi' \left(V_1(\omega(n')) + q^b\right) p(q|n') dq + P(q|n') - (T_1 - T_0) p(q|n')\right] f(n') dn'.$$

Defining by $g_0(n)$ the average marginal social welfare weight of childless women with inborn ability $n$, by $g_1(n)$ the average marginal social welfare weight of mothers with inborn ability $n$, we have

$$g_0(n) = \frac{\int_0^q \Psi' \left(V_0(n) - q^d\right) p(q|n) dq}{\lambda (1 - P(q|n))},$$

$$g_1(n) = \frac{\int_q^{+\infty} \Psi' \left(V_1(\omega(n)) + q^b\right) p(q|n) dq}{\lambda P(q|n)}.$$

Using these, we can rewrite the optimality conditions as

$$-\frac{\mu_0(n)}{\lambda} = \int_n^{n_{\text{max}}} \left[(1 - g_0(n')) (1 - P(q|n')) - (T_0 - T_1) p(q|n')\right] f(n') dn',$$

$$-\frac{\mu_1(n)}{\lambda} = \int_n^{n_{\text{max}}} \left[(1 - g_1(n')) P(q|n') - (T_1 - T_0) p(q|n')\right] f(n') dn'.$$

Inserting into the FOCs, we get

$$\frac{1}{n f(n) \varepsilon(n)} \int_n^{n_{\text{max}}} \left[(1 - g_0(n')) (1 - P(q|n')) - (T_0 - T_1) p(q|n')\right] f(n') dn' = \frac{\tau_0}{1 - \tau_0},$$
\[
\frac{1}{\omega(n)f(n)\varepsilon_1 P(q|n)} \int_{n}^{n_{\text{max}}} [(1 - g_1(n')) P(q|n') - (T_1 - T_0) p(q|n')] f(n') dn' = \frac{\tau_1}{1 - \tau_1}
\]

for the marginal rates of childless women and mothers, respectively.

References


