Dynamic Tax Evasion with Habit Formation

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Abstract

Although tax evasion and auditing are dynamic processes, they have been approached in a dynamic framework only recently. In this paper we argue that the decision to evade taxes is dynamically embedded with consumption decisions, which in turn are driven by consumption habits. The model is cast in a dynamic context with an infinite horizon. Consumer’s utility is a strictly increasing and concave in the (positive) difference between current consumption and a habit level given by a weighted mean of past consumptions. Our paper makes several contributions to the existing literature on tax evasion: 1) habit formation has a dampening effect on tax evasion; 2) as the representative consumer grows older, the gap between habit and consumption decreases and his/her tax evasion decreases. This has important implications for fiscal policies aimed at reducing tax evasion. 3) The effect of an increase in tax evasion depends on the ratio of habit to capital, i.e. the presence of the Yitzhaki [1974] paradox depends on such ratio; 4) we show that in the long run the ratio increases, i.e. over time this effect from positive becomes negative.

JEL Classification: H26, G11
1 Introduction

Economic decisions at any level involve inter-temporal optimisation. One of the most relevant examples is the dynamic optimisation problem for economic agents who want to maximise the inter-temporal utility of their consumption. The inter-temporal nature of such a decision implies an optimal behaviour which excludes extreme changes in consumption over time, and thus implies a so-called “consumption smoothing”. Furthermore, since a consumer is prone to adapt to his/her past consumption level, also a certain degree of habit formation may exist. The literature has long recognised this problem and in fact several models have been proposed that incorporate habit formation in consumption decisions (Fuhrer, 2000, Chetty and Szeidl, 2004, 2014), asset allocation (Constantinides, 1990, Otrok et al., 2002), investment labour supply and tax design (Koehne and Kuhn, 2015). Behavioural economics and neuroscience are emphasising the role of habit formation and recently also the literature on public finance has started studying habit formation, especially with reference to optimal tax design. Surprisingly, tax evasion has not received the same attention in spite of its relevant policy implications. Tax evasion is in fact one of the least desired consequences of government intervention in the economy. The most recent estimates (Feige and Cebula, 2011) show that intentional under-reporting of income is about 18-19% of the total reported income in the US, leading to a tax gap of about 500 billion dollars. Schneider [2011] and Buehn and Schneider [2007] show that the shadow economy, a good proxy for tax evasion, is rising both in OECD countries (from 13.2% in 1990 to 16.7% in 2010) and in transition economies.

Although tax evasion and auditing are dynamic processes (Allingham and Sandmo, 1972, Engel and Hines, 1999), they have been approached in a dynamic framework only recently (Wen-Zhung and Yang 2001, Dzhumashev and Gahramanov 2011, Levaggi and Menoncin 2012, 2013, Bernasconi et al. 2012, Niepelt 2005). The incentives to tax evasion are dynamically embedded with consumption decision and are driven by consumption habit. In particular, tax evasion may be interpreted as a tool that is used by individuals to reach and maintain their optimal consumption path. Given the importance of habit in consumption decision, it seems natural to embed it into a dynamic model of tax evasion. A strand of literature has somehow recognised the importance of a reference point around which consumers decisions revolve. In our model we argue that consumption habit formation plays such a role.

We model a representative individual that has to decide how much of his/her income to consume and to evade in a dynamic context with an infinite horizon, given a dynamic capital accumulation. His/her utility is a strictly increasing and concave in the (positive) difference between current consumption and a minimum level of consumption given by a weighted mean of past consumptions.

This paper makes several contributions to the existing literature on tax evasion: first of all we are able to study the effects of tax evasion through time, not only in at a specific point. In this framework the policy implications of our model are richer. The evolution of tax evasion and its response through time can better inform policymakers on more targeted fiscal policies.
2 Habit in consumption decisions

Most choices require decision-makers to trade-off costs and benefits at different points in time; decisions about consumption and savings are certainly inter-temporal and consequently also decisions about tax evasion are set in a dynamic environment.

The literature has long recognised the inter-temporal nature of consumption decisions (Deaton 1992, Jappelli and Pistaferri 2006) and the existence of frictions in the adjustments to sudden income changes. In this article we argue that a similar process can be applied to tax evasion through consumption decisions. Tax evasion is an extra source of income (through a reduced tax bill if evasion is not detected) and of risk (through an increased tax bill if evasion is detected). Both aspects affect inter-temporal consumption decisions, and hence the decision of how much income to conceal to the tax authority. In our model we incorporate habit formation in consumers decision by assuming that consumers form habits in their consumption patterns. Consumers derive utility from consumption in excess of a “subsistence” level that has two components:

- a fixed element that represents his/her consumption target. In the traditional literature on consumption this level coincides with physical subsistence;
- an adjustment through time which measures how quickly consumers adapt to past consumption levels.

Our habit formation differential equation is similar to the one presented in Constantinides [1990] and allows, through an appropriate choice of the parameters, to model several degrees of persistence in habit formation. In this respect, the model we propose is more general than that in Koehne and Kuhn [2015].

3 The model

Over an infinite horizon, we model the behaviour of a representative consumer who receives income from capital that is (deterministically) accumulated over time. In each period the consumer has to decide how much to consume and to evade in order to maximise the expected inter-temporal utility which contains habit.

3.1 Accumulation of capital

We model an economy in the period $[t_0, \infty[$, where total income $y_t$ is produced by using a linear production function of accumulated capital $k_t$:

$$y_t = Ak_t,$$  \hspace{1cm} (1)

\footnote{The literature does not in fact agree on the degree of persistence. See Kueng and Yakovlev [2014]; Führer [2000] and references therein}
where $A$ measures total factor productivity.

Without taxation, the dynamic equation of capital accumulation is described by the following differential equation:

$$dk_t = (y_t - c_t) \, dt,$$

(2)

where the expected increment in capital is given by income $y_t$ net of consumption $c_t$. Like in Dzhumashev and Gahramanov [2011], Wen-Zhang and Yang [2001], capital is non stochastic by itself, but it becomes stochastic because of the uncertainty of the auditing process.

The government levies a proportional tax $0 \leq \tau \leq 1$ on income. Without evasion, the net change in capital becomes

$$dk_t = ((1 - \tau_t) y_t - c_t) \, dt.$$

(3)

The agent may hide a proportion $e_t \in [0, 1]$ of his/her income $y_t$. If evasion is detected, a fine $\eta(\tau)$ must be paid (where $\eta(\tau)$ is a non-decreasing function of $\tau$). Thus, the total evasion fine is

$$\eta(\tau) e_t y_t.$$

(4)

The specification of $\eta(\tau)$ allows us to consider several fine regimes:

- $\eta(\tau) = \eta_0$: the fine is on evaded income as in Allingham and Sandmo [1972];
- $\eta(\tau) = \eta_1 \tau$: the fine is on evaded tax as in Yitzhaki [1974];
- $\eta(\tau) = \eta_0 + \eta_1 \tau$: the fine is a combination of these two previous cases.

Evasion introduces risk in equation (3) since the fine may or may not be paid. If the proportion $e_t$ of income is evaded, the amount of tax $\tau e_t y_t$ remains unpaid and the expected change in capital becomes

$$E_t [dk_t] = ((1 - \tau + \tau e_t) y_t - c_t) \, dt,$$

(5)

where $E_t [\cdot]$ is the expected value operator conditional on the information set at time $t$.

As in Levaggi and Menoncin [2012, 2013], Bernasconi et al. [2015], we model auditing as a Poisson jump process $d\Pi_t$ with expected value and variance given by

$$E_t [d\Pi_t] = \lambda_t dt,$$

$$V_t [d\Pi_t] = \lambda_t dt,$$

(6)

(7)

where $V_t [\cdot]$ is the variance operator and the function $\lambda_t \in [0, \infty[$ (the “intensity” of the process) determines the frequency of audits within a time interval. When

\footnote{This process can be thought of as the limit of a binomial model whose value is 1 with probability $\lambda dt$ and 0 otherwise.}
\( \lambda_t = 0 \), the probability of being caught is zero and when \( \lambda_t \) tends towards infinity, the probability of being caught tends towards 1.

Finally, the stochastic process of capital accumulation can be written as

\[
dk_t = ((1 - \tau + \tau e_t) y_t - c_t) dt - \eta(t) e_t y_t d\Pi_t,
\]

from which,

\[
\mathbb{E}_t [dk_t] = ((1 - \tau + (\tau - \eta(t) \lambda_t) e_t) y_t - c_t) dt,
\]

\[
\mathbb{V}_t [dk_t] = \eta(t)^2 e_t^2 y_t^2 \lambda_t dt.
\]

### 3.2 Consumer’s preferences

Consumer’s preferences are represented by an Hyperbolic Absolute Risk Aversion utility function with a minimum level of consumption \( (h_t) \) coinciding with a weighted mean of the past consumptions. The inter-temporal utility function is

\[
U(c_t) = \frac{(c_t - h_t)^{1-\delta}}{1-\delta},
\]

where \( h_t \) is the solution to the following differential equation

\[
dh_t = (\alpha c_t - \beta h_t) dt,
\]

with exogenous initial value of \( h_t \) given by \( h_0 \). The (unique) solution to (11) is

\[
h_t = h_0 e^{-\beta t} + \alpha \int_0^t c_s e^{-\beta(t-s)} ds.
\]

The parameter \( \delta \) measures the risk aversion and we assume it is higher than 1. When taking the limit for \( \delta \to 1 \), then result of a logarithm utility is obtained.\(^3\)

The parameter \( \beta \) measures “time preference”: the higher \( \beta \) the lower the weight of the past consumption on the habit. In particular:

- \( \beta = 0 \): time does not affect the relevance of past consumptions which are all taken into account with the same weight, independently of the time they were available;
- \( \beta \to \infty \): there is no habit formation and the utility function is traced back to the Constant Relative Risk Aversion case (with a subsistence consumption level equal to zero).

\(^3\)In fact, we can think of the following transformation of the utility:

\[
V(c_t) = \left(\frac{(c_t - h_t)^{1-\delta} - 1}{1-\delta}\right),
\]

which does not alter the consumer’s preferences, and whose limit is

\[
\lim_{\delta \to 1} \left(\frac{(c_t - h_t)^{1-\delta} - 1}{1-\delta}\right) = \lim_{\delta \to 1} \frac{-(c_t - h_t)^{1-\delta} \ln c_t}{1} = \ln c_t.
\]
The parameter $\alpha$ measures the percentage of past consumptions which contributes to create the minimum consumption level $h_t$ and it could be interpreted as the speed at which past consumption becomes an habit. Some particular cases are worthy highlighting:

- $\alpha = 0$: past consumption does not affect the subsistence level, which just depends on time (through the parameter $\beta$);
- $\beta = \alpha = 0$: $h_0$ can be interpreted as a subsistence consumption level as in Levaggi and Menoncin [2013];

The habit $h_t$ is a positive function of $\alpha$ and a negative function of $\beta$.

## 4 Results

The consumer optimises his/her inter-temporal utility solving the following problem:

$$
\max_{c_t, e_t} \mathbb{E}_0 \left[ \int_0^\infty \frac{(c_s - h_s)^{1-\delta}}{1-\delta} e^{-\rho s} ds \right],
$$

where $\rho > 0$ is a subjective discount factor. The state variables $k_t$ and $h_t$ solve (8) and (11), respectively.

The solution to Problem (12) is shown in the following proposition.

**Proposition 1.** The optimal consumption and evasion which solve Problem (12), given the capital accumulation (8) and the habit formation (11), are

$$
e^*_t = \frac{1}{\eta A} \left( 1 - \frac{h_t}{k_t (1 - \tau) A + \beta - \alpha} \right) \left( 1 - \left( \frac{\lambda \eta}{\tau} \right)^{\frac{1}{\delta}} \right),
$$

$$
c^*_t = h_t + \frac{k_t ((1 - \tau) A + \beta - \alpha)}{(1 - \tau) A + \beta} \left( \frac{\rho + \lambda}{\delta} + \frac{\delta - 1}{\delta} \left( \frac{\tau}{\eta} + (1 - \tau) A \right) - \frac{\tau}{\eta} \left( \frac{\lambda \eta}{\tau} \right)^{\frac{1}{\delta}} \right).
$$

**Proof.** See Appendix A.

Equation (13) measures the fraction of income that is concealed from the tax authority, which is a negative function of $\alpha$ and a positive function of $\beta$. In other words, habits in consumption decisions dampens tax evasion. Other things being equal the speed at which consumers adjusts their habit makes them less prone to evade. This result is also confirmed by the result for $\beta \to \infty$, when the utility exhibits constant risk aversion and the consumer increases tax evasion. The other interesting observation is that tax evasion negatively depends on the ratio $\frac{h_t}{k_t}$. Other things being equal, the higher the ratio, i.e. the closer the level of habit to income, the lower the tax evasion. This result is rather intuitive: tax evasion makes income more volatile and the consumer is allowed to take this
risk only if his income is sufficiently higher than his habite. It is also interesting to note that if habit formation simply depends on a fixed subsistence level \((\alpha = \beta = 0)\), Equation (13) becomes

\[
e^*_t = \frac{1}{\eta A} \left( 1 - \frac{h_0}{k_t} \frac{1}{(1 - \tau) A} \right) \left( 1 - \frac{\lambda \eta}{\tau} \right),
\]

and the optimal percentage of concealed yield increases over time.

In our model, as in any habit formation framework, preferences are non time separable, i.e. decisions are not simply taken by considering present values of the state variables, but also their whole history. This implies that solutions are highly complex and it is difficult to determine the effects of habit and fiscal parameters on the optimal path. In spite of this general difficulty, a conclusion for the relationship between tax rate and tax evasion is possible.

**Proposition 2.** The elasticity of tax evasion to a change in the tax rate changes depends on the relative importance of the ratio \(\frac{k_t}{h_t}\):

\[
\frac{\partial e^*_t}{\partial \tau} \frac{\tau}{e^*_t} \gtrless 0 \iff \frac{k_t}{h_t} \gtrless \frac{A}{(1 - \tau) A + \beta - \alpha} \left( \frac{\eta(\tau)}{\tau} \right) - \frac{1}{\delta} \left( \frac{\lambda \eta(\tau)}{\tau} \right) \left( 1 - \frac{\partial \eta(\tau)}{\partial \tau} \frac{\tau}{\eta(\tau)} \right) + \frac{1}{(1 - \tau) A + \beta - \alpha} \frac{A}{1 - \left( \frac{\lambda \eta(\tau)}{\tau} \right)}.
\]

**Proof.** The derivative of (13) with respect to \(\tau\) is

\[
\frac{\partial e^*_t}{\partial \tau} \frac{\tau}{e^*_t} = -\frac{\partial \eta(\tau)}{\partial \tau} \frac{\tau}{\eta(\tau)} + \frac{1}{\delta} \left( \frac{\lambda \eta(\tau)}{\tau} \right)^{\frac{1}{2}} \left( 1 - \frac{\partial \eta(\tau)}{\partial \tau} \frac{\tau}{\eta(\tau)} \right) + \frac{1 - \frac{h_0}{k_t} \frac{1}{(1 - \tau) A + \beta - \alpha}}{(1 - \tau) A + \beta - \alpha}.
\]

Now, since \(1 - \frac{h_0}{k_t} \frac{1}{(1 - \tau) A + \beta - \alpha} > 0\), then we can write that \(\frac{\partial e^*_t}{\partial \tau} \frac{\tau}{e^*_t} \gtrless 0\) if and only if

\[
\left( 1 + \frac{1}{\delta} \left( \frac{\lambda \eta(\tau)}{\tau} \right)^{\frac{1}{2}} \left( 1 - \frac{\partial \eta(\tau)}{\partial \tau} \frac{\tau}{\eta(\tau)} \right) \right) \geq \frac{A}{h_t} \left( \frac{\eta(\tau)}{\tau} \right) - \frac{1}{\delta} \left( \frac{\lambda \eta(\tau)}{\tau} \right) \left( 1 - \frac{\partial \eta(\tau)}{\partial \tau} \frac{\tau}{\eta(\tau)} \right) (1 - \tau) A + \beta - \alpha.
\]

Given the for \(\eta(\tau) = \eta_0 + \eta_1 \tau\), we know that

\[
1 - \frac{\partial \eta(\tau)}{\partial \tau} \frac{\tau}{\eta(\tau)} = \frac{\eta_0}{\eta_0 + \eta_1 \tau} > 0,
\]
and so we can conclude that \( \frac{\partial e^*_t \tau}{\partial \tau} \geq 0 \) if and only if

\[
k_t \geq \frac{h_t}{h_t} \iff A \left( \frac{\partial \eta(\tau)}{\partial \tau} \eta(\tau) - \frac{1}{2} \left( \frac{\lambda \eta(\tau)}{1-\left(\frac{\lambda \eta(\tau)}{1} \right)} \right)^\frac{1}{2} \left( 1 - \frac{\partial \eta(\tau)}{\partial \tau} \eta(\tau) \right) \right) \left( (1-\tau) A + \beta - \alpha \right)
\]

\[
\left( (1-\tau) A + \beta - \alpha \right)^2 \left( 1 + \frac{1}{2} \left( \frac{\lambda \eta(\tau)}{1-\left(\frac{\lambda \eta(\tau)}{1} \right)} \right)^\frac{1}{2} \left( 1 - \frac{\partial \eta(\tau)}{\partial \tau} \eta(\tau) \right) \right)
\]

We highlight some particular cases:

- without habit formation (i.e. \( h_t = 0 \)): the ratio \( \frac{k_t}{h_t} \) explodes and from (16) we obtain that the optimal evasion positively depends on the tax rate: \( \frac{\partial e^*_t \tau}{\partial \tau} \geq 0 \);

- with \( \eta(\tau) = \eta_0 \) (and so \( \frac{\partial \eta(\tau)}{\partial \tau} = 0 \)), the condition in Proposition (2) becomes

\[
\frac{\partial e^*_t \tau}{\partial \tau} \geq 0 \iff k_i \geq \frac{1}{(1-\tau) A + \beta - \alpha} + \frac{1}{2} \left( \frac{\lambda \eta(\tau)}{1-\left(\frac{\lambda \eta(\tau)}{1} \right)} \right)^\frac{1}{2} \left( 1 - \frac{\partial \eta(\tau)}{\partial \tau} \eta(\tau) \right)
\]

- with constant habit (i.e. \( \beta = \alpha = 0 \) and \( h_t = h_0 \)) we have

\[
\frac{\partial e^*_t \tau}{\partial \tau} \geq 0 \iff k_i \geq \frac{1}{(1-\tau) A} \left( \frac{\partial \eta(\tau)}{\partial \tau} \eta(\tau) - \frac{1}{2} \left( \frac{\lambda \eta(\tau)}{1-\left(\frac{\lambda \eta(\tau)}{1} \right)} \right)^\frac{1}{2} \left( 1 - \frac{\partial \eta(\tau)}{\partial \tau} \eta(\tau) \right) \right)
\]

\[
(1-\tau) A \left( 1 + \frac{1}{2} \left( \frac{\lambda \eta(\tau)}{1-\left(\frac{\lambda \eta(\tau)}{1} \right)} \right)^\frac{1}{2} \left( 1 - \frac{\partial \eta(\tau)}{\partial \tau} \eta(\tau) \right) \right)
\]

5 Simulations

In this section we propose a simulation exercise that allows to study consumption, tax evasion and income path of our representative consumer. The evolution through time of decisions is in fact one of the distinguishing characteristics of habit models.

We start by presenting a basic scenario where the parameters have been initialised as follows: \( A=0.8; \tau=0.4; \lambda=0.25; \rho=0.05; \alpha=1/3; \beta=0.05; \delta=2.5; T=35; N=100; dt=1/250; k_0=100; h_0=10. \)

The solution is presented in figure 1.

In the long run, consumption coincides with habit, something that in our model can be interpreted as a steady state. Evasion decreases over time and reaches about zero in our 35 lifetime span. In this respect, our model predicts
Figure 1: Base scenario
that people evade less (in percentage of their income) as they grow older, a result that is in line with the literature on risk taking. However, it should be noticed that the policy implications of our model are different from the standard literature. In the risk taking literature, older people risk less because they become less risk averse. In our model their risk aversion parameter does not change; they simply face a higher risk since their consumption is close to their habit, which is an increasing proportion of their income. Since the risk aversion parameter does not change over time, tax audits cannot be age targeted; in particular tax authorities should not reduce the number of controls as age increases because if this behaviour can be observed by the individuals, they will keep on evading as they did when they were younger.

The second interesting result is that increases over time. This means that, as shown in proposition 2 older people are more likely to react with a further decrease in evasion to an increase in the tax rate.

Let us now examine the effects of habit on consumption and tax evasion. As shown in figure 2, a decrease in from to increases tax evasion and changes the variability of all the lifetime decision. It is interesting to note that in this case both consumption and habit as percent of income seems to slightly decrease towards the end of the period considered.

The empirical literature on habit does not agree on its persistence. Dynan [2000], Fuhrer [2000] and aruge that there is a very limited persistence while Kueng and Yakovlev [2014] show that persistence is very high. In our framework it is possible to take into account very short living habit formation by assuming a high value for (in the simulations we use ). Accordingly, a high value for is needed in order to prevent a too low level of habit (we take ). The results are presented in figure 3.

When habits are short lived, evasion is higher, as one might expect. As shown in the previous section, habit formation is a barrier to tax evasion: in order to keep a determined lifestyle the consumer has to reduce the risk in income falls deriving from auditing. The path through time is also quite interesting: tax evasion decreases, but much less than in the base scenario and the variance in considerably higher.

Let us now study the effects of an increase in the tax rate that shoots up from 40% to 50%.

The results are presented in figure 4. An increase in the tax rate produces a reduction in tax evasion (as percentage of income), hence our results are in line with the Yitzhaki [1974] hypothesis. However, as shown in 2, the short run effect could also be positive: in fact it depends on the ratio .

6 Conclusions

Tax evasion decisions have important intertemporal dimensions that the traditional literature has been ignoring until the recent past. In this paper we use a dynamic optimisation model to show the effects of intertemporal substitution
Figure 2: Decrease in $\alpha$ to 0.1
Figure 3: Short persistence of habit. $\alpha = 0.8 \beta = 1$
Figure 4: Increase in the tax rate from 40% to 50%
effects on the decision to evade. We introduce habit formation in consumption and we show that under this hypothesis tax evasion and consumption evolves through time following specific paths that characterize their evolution. In this way, we add a further dimension to the study of tax evasion, i.e. it evolution through time.

Our model shows that risk taking is not the only determinant of the propensity of consumers to evade. In fact, consumers through their life cycle, even when fiscal parameters and risk aversion do not change, tend to reduce their level of tax evasion. This is because of habit in consumption: in the long run the consumer wants to keep his standard of living and it is less open to risk an income loss deriving from fines due for tax evasion.

The standard comparative static of the model shows that the response of tax evasion to an increase in the tax rate is ambiguous; in fact the response depends on the ratio \( \frac{k}{h} \) which measures how far is habit from capital, i.e. from a measure of income. The evolution of the variables of interest through time has been presented using a numerical simulation; we show that auditing and fines reduces tax evasion, but not in the same way. Audits allows to reduce the variance in consumption and income: other thing being equal it is better to induce people to reduce their level of tax evasion with controls rather than fines. The intuitive explanation for this result is that higher fines produces a more pronounced income loss by consumers that are found cheating. On the other hand, auditing makes cheating less profitable without having to pay a higher fine.

**References**


### A Proof of Proposition 1

Given Problem (12), the value function $J_t(h_t, k_t)$ which solves it can be defined as

$$J_t(h_t, k_t)e^{-\rho t} = \max_{c_t, e_t} \mathbb{E}_t \left[ \int_t^\infty \frac{(c_s - h_s)^{1-\delta}}{1-\delta} e^{-\rho s} ds \right],$$

and $J_t(h_t, k_t)$ must solve the following Hamilton-Jacobi-Bellman equation:

$$0 = \frac{\partial J_t}{\partial t} - \rho J_t + \max_{c_t} \left[ \frac{(c_t - h_t)^{1-\delta}}{1-\delta} + \frac{\partial J_t}{\partial h_t} (\alpha c_t - \beta h_t) - \frac{\partial J_t}{\partial k_t} c_t \right] \tag{17}$$

$$+ \max_{e_t} \left[ (J_t(h_t, k_t - \eta e_t A_k) - J_t) \lambda + \frac{\partial J_t}{\partial k_t} (1 - \tau + \tau e_t) A_k \right].$$

The first order conditions on $c_t$ and $e_t$ are:

$$c_t^* = h_t + \left( \frac{\partial J_t}{\partial k_t} - \frac{\partial J_t}{\partial h_t} \right)^{-\frac{1}{2}},$$

$$\frac{\partial J_t(h_t, k_t - \eta e_t^* A_k)}{\partial e_t^*} \lambda + \frac{\partial J_t}{\partial k_t} \tau A_k = 0.$$

In this case the so-called “guess function” is

$$J_t = F_t \frac{(k_t - h_t H_t)^{1-\delta}}{1-\delta},$$

where the functions $F_t$ and $H_t$ must be found in order to solve the HJB equation. The two transversality conditions are

$$\lim_{t \to \infty} F_t = 0,$$
\lim_{t \to \infty} H_t = 0.

Given the guess function, the optimal consumption and evasion are

\[ c_t^* = \frac{k_t - h_t H_t}{\eta A k_t} \left( 1 - \left( \frac{\lambda \eta}{\tau} \right)^{\frac{1}{\delta}} \right), \]
\[ c_t^* = h_t + \frac{k_t - h_t H_t}{F_t} (1 + \alpha H_t)^{-\frac{1}{\delta}}. \]

After plugging these values into the HJB, it becomes

\[ 0 = \delta F_t^{\delta-1} \frac{(k_t - h_t H_t)^{1-\delta}}{1-\delta} \frac{\partial F_t}{\partial t} - h_t F_t^{\delta} (k_t - h_t H_t)^{-\delta} \frac{\partial H_t}{\partial t} - \rho F_t^{\delta} (k_t - h_t H_t)^{1-\delta} \]
\[ + \frac{1}{1-\delta} H_t (1 + \alpha H_t)^{-\frac{1}{\delta}} F_t^{\delta} (k_t - h_t H_t)^{-\delta} (\alpha h_t - \beta h_t) \]
\[ -\alpha H_t F_t^{\delta-1} (k_t - h_t H_t)^{1-\delta} (1 + \alpha H_t)^{-\frac{1}{\delta}} - F_t^{\delta} (k_t - h_t H_t)^{-\delta} h_t \]
\[ - F_t^{\delta-1} (k_t - h_t H_t)^{1-\delta} (1 + \alpha H_t)^{-\frac{1}{\delta}} + F_t^{\delta} (k_t - h_t H_t)^{-\delta} (1 - \tau) A k_t \]
\[ + F_t^{\delta} (k_t - h_t H_t)^{1-\delta} \left( \left( \frac{\lambda \eta}{\tau} \right)^{\frac{1-\delta}{\delta}} - 1 \right) \lambda + F_t^{\delta} (k_t - h_t H_t)^{1-\delta} \frac{\tau}{\eta} \left( 1 - \left( \frac{\lambda \eta}{\tau} \right)^{\frac{1}{\delta}} \right), \]

which can be simplified as two differential equations (one in \( F_t \) and one in \( H_t \)) as follows

\[ 0 = \frac{\partial F_t}{\partial t} - F_t \left( \frac{\rho + \lambda}{\delta} - \frac{1-\delta}{\delta} \left( \frac{\tau}{\eta} + (1 - \tau) A \right) - \frac{\tau}{\eta} \left( \frac{\lambda \eta}{\tau} \right)^{\frac{1}{\delta}} \right) + (1 + \alpha H_t)^{1-\frac{1}{\delta}}, \]
\[ 0 = \frac{\partial H_t}{\partial t} - H_t ((1 - \tau) A + \beta - \alpha) + 1. \]

Given the transversality conditions, the solutions of these two differential equations are:

\[ H_t = \int_t^{\infty} e^{\int_u^t ((1 - \tau) A + \beta - \alpha) du} ds = \frac{1}{(1 - \tau) A + \beta - \alpha}. \]
\[ F_t = \int_t^{\infty} (1 + \alpha H_s)^{1-\frac{1}{\delta}} e^{-\int_u^t \left( \frac{\rho + \lambda}{\delta} - \frac{1-\delta}{\delta} \left( \frac{\tau}{\eta} + (1 - \tau) A \right) - \frac{\tau}{\eta} \left( \frac{\lambda \eta}{\tau} \right)^{\frac{1}{\delta}} \right) du} ds \]
\[ = \frac{(1 - \tau) A + \beta - \frac{1-\delta}{\delta} \left( \frac{\tau}{\eta} + (1 - \tau) A \right) - \frac{\tau}{\eta} \left( \frac{\lambda \eta}{\tau} \right)^{\frac{1}{\delta}}}{(1 - \tau) A + \beta - \alpha}. \]

If \( F_t \) and \( H_t \) are substituted into the first order conditions, the result of Proposition is obtained.