Variable Specific Constant Elasticities of Substitution in Production*

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Abstract

In the empirical literature of the last 65 years the mean value production function has been called upon to give meaning to the wage regression coefficient in the demand for labor. Little effort has been made towards testing the hypothesis of a constant parameter among different labor types and of the goodness of fit obtained, when all intuition and a priori evidence points to unequal contribution of one type to another in the production process. In many studies aggregation to a single variable is not interesting. What stands in the way of an interpretation of statistical results with many variables is an economic model of production in which different factors are given different parameters and the statistical model is an equation system satisfying simultaneously the first order conditions of all variables together.

In this paper a modified production function is introduced in which each variable plays a distinct role different from the role of other variables through a production function $V_S(z)$ that assigns a variable specific constant negative power $r_\ell$ to variable $z_\ell$. The conditions under which this function is quasi-concave in inputs and quasi-convex in joint outputs are obtained. Slutsky-Schultz technical substitution coefficients are found in the inverse of the bordered Hessian and the pairwise Joan Robinson measure of substitution between variables $z_\ell$ and $z_\lambda$ is shown to be constant and equal to the harmonic mean of $\left(\frac{1}{1-r_\ell}, \frac{1}{1-r_\lambda}\right)$. Under the $V_S(z)$ return to scale is not constant but changes with the chosen bundle of variables hired in markets. When prices are changing short run profits may be negative. The entrepreneurial choice is not a single supply or factor demand but a bundle of variables, with the bundling dependent on the powers $r_\ell$ and the bundling itself to be estimated. Given a timeseries of a firm’s activity, the MLE estimator of the $V_S(z)$ is relatively simple and obtainable by iterated OLS.

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1 Introduction.

Closely following Hicks’ entrepreneurial decision making system, short run profit maximization with respect to the homegood price at given pre-determined homegood output is assumed, period after period. Under this assumption the entrepreneur constructs a timeseries $y_t$ of homegood outputs in which the comparative static coefficients of prices of inputs and of joint outputs relative to homegood output plays its role, period after period. The observed time series of outputs and inputs so generated are the data generated compatibly with an assumed operation model known to the entrepreneur. The comparative static coefficients of the operation model guide the manager in forming the sequence in the environment of random variations in prices and quantities.

The operation model is a generalization of the CES model. It starts with the assumption that different variables make unequal contributions. Equality is a testable hypothesis and appropriate testing tools exist. But a considerable number of pages is required to derive the comparative static coefficients from the inverse bordered Hessians and exogeneity of covariate variables is to be avoided. All these make this for a very long paper. Nevertheless, who can disagree with Samuelson in [?], page 378, reminding his readers that it is of considerable importance for many economic problems to know what is implied with respect to the inverse of bordered Hessians. To be sure in econometrics is meant a numerical inverse and not just an algebraic language of inequalities.

The operational model is expressed in a variable specific power function $y = VS(z)$ that is a modification of the standard one constant order mean value $y = CES(z)$. Given variables $z = (z_\ell)$ of intermediate or nonproduced inputs or joint outputs, the home output $y = VS(z)$ is the function

$$y = (q_1z_1^{r_1} + q_2z_2^{r_2} + \cdots + q_Lz_L^{r_L})^{\frac{1}{r_0}}, \ y > 0, \ z = (z_\ell > 0), \ r_\ell < 0, \ q_\ell \neq 0, \ \ell = 0, \ldots, L. \ (1)$$

If $q_\ell > 0$ the variable $z_\ell$ is an intermediate or nonproduced input and if $q_\ell < 0$ variable $z_\ell$ is a joint output. Each power $r_\ell$ is negative and $\sigma_\ell = \frac{1}{1 - r_\ell} < 1$ is the well known elasticity. The function (1) is not in Fuss, McFadden, Mundlak [?] and its properties have not received the attention of applied mathematicians.

Special cases of the $VS(z)$ with $r_\ell = r < 0$, $\ell = (1, \ldots, L)$ are the decreasing $DRS$, $r < r_0$, and increasing returns to scale $IRS$, $r_0 < r$. The constant return to scale $CES(z)$ with all powers equal to $r < 0$ or $\sigma_\ell = \sigma < 1$ is a limiting special member of the family.
The classification of the variables in inputs, both intermediate and nonproduced, joint outputs and the homegood is traditional, as in J.R. Hicks’s The Equilibrium of the Firm, Chapter VI and beyond in [?]. The marginal product of \( z_\ell \) is 
\[
\frac{\partial y}{\partial z_\ell} = \frac{q_{r_0} r_0^y z_\ell^{1-r_0}}{z^{r_0}}
\]
and the ratio of any 2 partials of different powers, changes with homgood output \( y \), i.e. (1) is not separable. Observe that a marginal product is never zero, either positive or negative for all \( z \) and the isoquants do not cut the axes.

Hessians and bordered Hessians are stated in the Appendix. The own 2\(^{nd} \) order derivatives are shown to be positive for joint outputs, negative for inputs if \( \sigma_{z_{z_\ell}} \frac{\partial y}{\partial y} \frac{\partial y}{\partial z_\ell} < 1 \) and cross partials are products of first derivatives. Following the Allen-Hicks law of diminishing marginal rate of substitution, the conditions are stated under which the \( VS(z) \) function is quasi-concave in inputs, quasi-convex in joint outputs. This law does not require the Hessian to have an inverse, but it must be shown that the nested principal minors of the bordered Hessian are not zero, these being conditions known to Hicks [?]. Arrow-Endhoven [?] derived quasi-concavity conditions and recently Topkis studied quasisupermodularity [?] without differentiability as the condition for comparative statics. The \( VS(z) \) matrix inverses are tractable enough to verify that the necessary and sufficient conditions for quasi-concavity-convexity are satisfied for all \( z \in Z \), where \( Z \) is a subset of \( z = (z_\ell > 0) \) is to be defined through a system of inequalities. For the \( VS(z) \) functions only the \( L \) principal minors of order \( L-1 \) and the determinant, not all principal minors of all orders, are to be different from zero.

Section 2 states the algebraic theory, subdivided into the conditions under which the \( VS(z) \) is concave-convex and quasi-concave-convex in Theorem 1. The sign patterns in the bordered Hessian inverse and in the Hessian inverse are shown when the power \( r_0 \) divides the joint output powers from the input powers. In its Subsection 2.4 the Return to Scale \( RL(z) \) function is defined. Short run competitive profits are positive under decreasing return to scale and negative in the increasing return to scale region. The curve \( \delta_L(z) = 1 \) separates the decreasing from the increasing return to scale regions.

In Section 3 three pictures illustrate graphically some shapes of the function \( VS(z) \). Figure 1 shows that the \( VS(z) \) function representing the model with one input and one joint output fails to be quasi-concave-convex. Corresponding to the model with two inputs and one joint output, the algebra is shown and in Figure 2 the geometry is shown of the joint output isoquant running into the quasi-concave-convex region, producing negative profits there, and continuing into the concave-convex region with positive profits.
Figure 3 draws pure production isoquants, each cut by the $\delta_2(z) = 1$ curve and the constant return to scale $RS_2(z) = 1$ curve. Allocations in the concave region where the return to scale is increasing yield negative profit. In Figure 4 with a joint output it is shown how a long run increasing return to scale operation can be converted to a concave-convex profitable short run process.

In Section 4 Firm $j$ produces the home good $y$ under the technology $y = VS(z^j)$ and prices $p_j, p'_j)$. The Entrepreneurial Behavior is characterized by profit maximization under continuous monitoring of the level of the $2L$ variables $(z, p)$. Decisions are choices of $L$ variables in which Firm $j$ is perceived to have profit interest given the other $L$ variables. These choices are recomputed as often and as much as is signaled by the profit function under changed data. Three managerial firm models are considered.

(a) In the Hicks Model Firm $j$ profit maximizes by choosing $(p_j, z^j)$ given $(z_j, p'_j)$. The Hicks entrepreneur is a supply price $p_j$ setter for a given planned output $z_j$ under quasi-concave-convex technology. The differential equation system and the Jacobian impact coefficient matrix $\frac{\partial (p_j, z^j)}{\partial (z_j, p'_j)}$ guide the entrepreneur into implementing desired changes.

(b) In the Cost/Revenue Competitive Model the entrepreneur chooses $(z_j, z^j)$ at given $(p_j, p'_j)$. Under a concave convex technology the choice $z^j(z_j)$ is unique and so is $z_j$ through the $z_j = VS(z^j)(z_j)$ technology. The latter is a variable that generates the cost/revenue function and produces the subset $\frac{\partial z_j}{\partial p'_j}$ of supply joint output and factor demand price slopes. There is no information about how the supply price $p_j$ changes with output $z_j$ or with other prices and how inputs grow with expanding homegood $z_j$. The competitive firm is a cost/revenue minimizer at any home good output.

(c) In the Admissible Competitive Model the entrepreneur profit maximizes with respect to homegood output $z_j$ under competitive all prices given, which is a reversal of $z_j, p_j$ in the Hicks Model, as first named and shown in Samuelson [?]. Its comparative analytical results are derived indirectly from those of the Hicks Model. The latter imperfectly competitive model is central to the alternatives.

Section 3 prints and compares the comparative statics of all 3 models.
2 $VS(z)$ Saddle Function Conditions

2.1 Nonsingularity of the Hessians and bordered Hessians

In (A65) the Hessian is shown to be nonsingular in $Z$ if for all $z \in Z$

$$
\delta_L(z) = \sum_{\ell=1}^{L} \frac{\sigma_\ell z_\ell}{\sigma_0 y} \frac{\partial y}{\partial z_\ell} = \frac{\sum_{\ell=1}^{L} r_\ell \sigma_\ell q_\ell z_\ell^{r_\ell}}{\sum_{\ell=1}^{L} q_\ell z_\ell^{r_\ell}} = y^{-\rho_0} \sum_{\ell=1}^{L} \frac{1 - \sigma_\ell}{1 - \sigma_0} q_\ell z_\ell^{r_\ell} < 1
$$

(2)

and in (A68) the bordered Hessian is shown to be nonsingular in $Z$ if for all $z \in Z$

$$
0 < \delta_L(z) = \sum_{\ell=1}^{L} \frac{\sigma_\ell z_\ell}{\sigma_0 y} \frac{\partial y}{\partial z_\ell} = \frac{\sum_{\ell=1}^{L} r_\ell \sigma_\ell q_\ell z_\ell^{r_\ell}}{\sum_{\ell=1}^{L} q_\ell z_\ell^{r_\ell}} = y^{-\rho_0} \sum_{\ell=1}^{L} \frac{1 - \sigma_\ell}{1 - \sigma_0} q_\ell z_\ell^{r_\ell}.
$$

(3)

**Theorem 1** The $VS(z)$ is concave in inputs, convex in joint outputs in $Z$ if and only if

$$
0 < y(z), \quad \delta_\ell(z) = \sum_{\ell=1}^{r} \frac{\sigma_\ell z_\ell}{\sigma_0 y} \frac{\partial y(z)}{\partial z_\ell} < 1, \quad r = 1, \ldots, L, \text{ for all } z \in Z.
$$

(4)

The $VS(z)$ is quasi-concave in inputs, quasi-convex in joint outputs in $Z$ if and only if

$$
0 < y(z), \quad 0 < \delta_\ell(z) = \sum_{\ell=1}^{r} \frac{\sigma_\ell z_\ell}{\sigma_0 y} \frac{\partial y(z)}{\partial z_\ell}, \quad r = 1, \ldots, L, \text{ for all } z \in Z.
$$

(5)

**Proof.** Theorems 6 and 7 of the Appendix show that $(|H_r|, |B_r|)$, the principal minors of order $r$ in the Hessian and bordered Hessian, respectively, have the proportional form

$$
|H_r| = (1 - \delta_r(z)) \prod_{\ell=1}^{r} \frac{1}{\sigma_\ell z_\ell} (-\frac{\partial y}{\partial z_\ell}), \quad |B_r| = \sigma_0 y \delta_r(z) \prod_{\ell=1}^{r} \frac{1}{\sigma_\ell z_\ell} (-\frac{\partial y}{\partial z_\ell}), \quad r = 1, \ldots, L.
$$

(6)

The concave-convex Hessian conditions are satisfied if $(1 - \delta_1(z), 1 - \delta_2(z), \ldots, 1 - \delta_L(z))$ are positive numbers and the extended quasi-concave-convex Samuelson [?] and Arrow-Enthoven [?] conditions are satisfied if $(\delta_1(z), \delta_2(z), \ldots, \delta_L(z))$ are positive numbers. □

These conditions are valid for any ordering of the variables. Since the bordered Hessian inverse is known parametrically, econometrically, empirically, as stated in the Appendix, consideration of multiple orderings is avoided. The practical conclusion is summarized as follows.

**Corollary 1** The $VS(z)$ function $y(z) = (\sum_{\ell=1}^{L} q_\ell z_\ell^{r_\ell})^{1/\rho_0}$ is quasi-concave in inputs and quasi-convex in joint outputs for all $z \in Z$ if and only if

$$
0 < y(z), \quad 0 < \delta_{L,1}(z), \ldots, 0 < \delta_{L,r}(z), \ldots, 0 < \delta_{L,L}(z), \quad 0 < \delta_L(z), \quad r = 1, \ldots, L,
$$

(7)

for all $z \in Z$, where $\delta_{L,\ell}(z) \sum_{\ell=1, \neq \ell=1}^{L} \frac{\sigma_\ell z_\ell}{\sigma_0 y} \frac{\partial y(z)}{\partial z_\ell}, \quad \delta_L(z) = \sum_{\ell=1}^{L} \frac{\sigma_\ell z_\ell}{\sigma_0 y} \frac{\partial y(z)}{\partial z_\ell}$. 

5
Remark Conca-vity-convexity as characterized algebraically by \(0 < \delta_r < 1\) is a special case of quasi-concavity-convexity as characterized by \(0 < \delta_r\) in their principal minor of order \(r\). Geometrically the VS function \(y(z_1, z_2, \ldots)\) is concave-convex in the direction of \((z_1, z_2)\), (input, joint output) if the chord joining any two points of its graphs \(y(z_1, \ldots), y(\ldots, z_2, \ldots)\) lies everywhere on or below \(y(z_1, \ldots)\)-above \(y(\ldots, z_2, \ldots)\). Geometrically the VS function \(y(z_1, z_2, \ldots)\) is quasi-concave-convex if \((z_1 | y(z_1, \ldots) \geq c)\) and \((z_2 | y(\ldots, z_2, \ldots) \leq c)\) are both convex sets for each real number \(c\).

**Signs in the Hessian and bordered Hessian Inverses.**

i). Under (4) from \(\mathcal{H}_L^{-1}\) at (A68), the diagonal element \((\mathcal{H}_L^{-1})_{rr}\) is negative if \(z_r\) is an input and is positive if \(z_r\) is a joint output. All not diagonal elements are negative.

ii). Under (5) from \(\mathcal{B}_L^{-1}\) at (A71), the \((1,1)\) element has the sign of \(1 - \delta_L(z)\). The diagonal element \((\mathcal{B}_L^{-1})_{r+1,r+1}\) is negative if \(z_r\) is an input and is positive if \(z_r\) is a joint output, when the bordering is on top and to the left. All not diagonal elements are positive.

The Samuelson [-Arrow-Enthoven extended quasi-concave conditions require that the bordered principal minor \(|\mathcal{B}_r|\) of order \(r\) has the sign opposite to the sign of \(|\mathcal{B}_{r-1}|\) if \(z_r\) is an input and the same sign if \(z_r\) is a joint output.

### 2.2 Power Separation \(\left\{ \frac{r}{r_0} q_{\ell} \leq q_{\ell}, \forall \ell \right\}\)

If the VS(z) powers \(r_\ell\) satisfy the inequalities \(\left\{ \frac{r}{r_0} q_{\ell} \leq q_{\ell}, \forall \ell \right\}\) they are weakly separated by \(r_0\) by having a joint output power \(\left\{ r_\ell \leq r_0, \forall \ell \right\}\) and an input power \(\left\{ r_0 \geq r_\ell, \forall \ell \right\}\). Equivalently \(\left\{ \frac{r}{r_0} q_{\ell} \leq q_{\ell}, \forall \ell \right\}\) implies a joint output elasticity \(\{\sigma_\ell = \frac{1}{1-r_\ell} \leq \sigma_0 = \frac{1}{1-r_0}, \forall \ell\}\) and an input elasticity \(\{\sigma_\ell = \frac{1}{1-r_\ell} \geq \sigma_0 = \frac{1}{1-r_0}, \forall \ell\}\).

To have strict separation at least one inequality holds strictly. From (2) under power separation \(\delta_L(z) = y^{-r_0} \sum_{\ell=1}^{L} \frac{1-\sigma_\ell}{1-\sigma_0} q_{\ell} z_{\ell}^{r_\ell} \leq y^{-r_0} \sum_{\ell=1}^{L} q_{\ell} z_{\ell}^{r_\ell} = 1\), and under inverse power separation \(\left\{ \frac{r}{r_0} q_{\ell} \leq q_{\ell}, \forall \ell \right\}\) we have \(\left\{ \frac{r}{r_0} q_{\ell} \geq q_{\ell}, \forall \ell \right\}\), \(\delta_L(z) \geq 1\).

### 2.3 Signs bordered Hessian and Hessian inverse.

Under strict power separation \(\left\{ \frac{r}{r_0} q_{\ell} \leq q_{\ell}, \forall \ell \right\}\), the VS(z) is quasi-concave in inputs and quasi-convex in joint outputs under inequalities (3) with a negative definite Hessian in inputs and a positive definite Hessian in the joint outputs. With \(z_I\) inputs and \(z_J\) joint
whereas concavity-convexity holds if \( \sigma < \delta \) and quasi-concavity-convexity if \( 0 \leq \sigma \leq \delta \).

The curve \( \delta_L(z) = 1 \) draws the border where quasi concave-convex regions are separated from concave-convex regions.

### 2.4 Return to Scale \( RS_L(z) \) and Return to Variable \( z_\ell \)

In the \( VS(z) \) the return to scale \( \frac{1}{y} \frac{\partial y(tz)}{\partial t} \bigg|_{t=1} \) is not a constant but the function

\[
0 < y, \quad RS_L(z) = \frac{1}{y} \sum_{\ell=1}^{L} z_\ell \frac{\partial y}{\partial z_\ell} = y^{-r_0} \sum_{\ell=1}^{L} \frac{r_\ell}{r_0} q_\ell z_\ell^r = y^{-r_0} \sum_{\ell=1}^{L} \sigma_0 \frac{1 - \sigma_\ell}{\sigma_\ell - \sigma_0} q_\ell z_\ell^r, (8)
\]

whereas concavity-convexity holds if

\[
0 < y, \quad 0 < \delta_L(z) = \sum_{\ell=1}^{L} \sigma_\ell z_\ell \frac{\partial y}{\partial z_\ell} = \sum_{\ell=1}^{L} \frac{r_\ell}{r_0} \frac{\sigma_\ell}{\sigma_0} q_\ell z_\ell^r = y^{-r_0} \sum_{\ell=1}^{L} \frac{1 - \sigma_\ell}{1 - \sigma_0} q_\ell z_\ell^r < 1
\]

and quasi-concavity-convexity if \( 0 < \delta_L(z) \), where \( y^{-r_0} = \sum_{\ell=1}^{L} q_\ell z_\ell^r \).

\( RS_L(z) = \frac{\sigma_0}{\sigma_\ell} \delta_L(z) \) for all \( z \) if \( \sigma = \sigma_\ell \) is a constant for all \( \ell \), which is the model with \( \sigma = \sigma_\ell \neq \sigma_0 \), having decreasing return if \( \sigma_0 < \sigma \) and increasing returns if \( \sigma_0 > \sigma \). In the \( y = CES(z) \) case with all elasticities the same, \( RS_L(z) = \delta_L(z) = 1 \) for all \( z \).

With no joint outputs and strictly separability, having \( \sigma_0 < \sigma_\ell \) and all \( q_\ell > 0 \), the return to scale \( RS_L(z) < \delta_L(z) \) and in the strictly inverse separable model \( RS_L(z) > \delta_L(z) \).

With joint outputs and with or without power separation the difference between \( RS_L(z) \) and \( \delta_L(z) \) is not predictable and can change sign with \( z \).

In each period the Hicks manager selects inputs and outputs \( z \) that technically and optimally produce the planned output \( y = VS(z) \). This strategy results in losses if
the return to scale is increasing and prices are $p_z = p_z^\frac{\partial y}{\partial z}$. Hicks ignores this problem and assumes decreasing return to scale. This is unrealistic, periodic losses with $1 < RS_L(z)$ do occur. One reason is that some inputs are lumpy and the adjustment to a profitable level takes several periods. Consider a partition $z = (z'_I, z''_I)'$, where $z_I$ are $\Lambda \leq L$ short run flexible inputs or joint outputs. The variables $z_{II}$ are the up to period $t$ accumulated capital investments including durable machinery not adjustable instantaneously. At prices $p_I = p_z^\frac{\partial y(z)}{\partial z_I}$, payments to flexible inputs plus sale of joint outputs imply current positive cash flow $C = py(z)(1 - \frac{1}{y(z)}z_I^I\frac{\partial y}{\partial z_I})$ if $RS_A(z_I) < 1$. If no $z_{II}$ expenses are incurred now, $C$ is added to the $z_{II}$ investment pool of the next period. With $z_{II}$ lumpy inputs, $RS_A(z) < RS_L(z)$ and with $z_{II}$ bulky durable inputs, the entrepreneur can choose successive allocations that are profitable in each period, especially if the firm is planning for longer run expansion of the operations. In some period cash assets may be accumulated that are added to the financing pool of the capital assets installed in some other. Over time planned $z$ differ if the variable adjustment rates differ.

Decreasing return $0 < RS_L(z) = y^{-\tau_0} \sum_{\ell=1}^L \frac{\sigma_0}{\sigma_0 - \tau_0} q_{\ell} z_{\ell}^\tau < 1$ and the concavity-convexity condition $0 < \delta_L(z) = y^{-\tau_0} \frac{1}{1-\tau_0} q_{\ell} z_{\ell}^\tau$ are closely related. Both $RS_L(z)$ and $\delta_L(z)$ are sums. The share of variable $z_\ell$ in the competitive income distribution

$$\frac{p_\ell z_\ell}{\sum_{\ell=1}^L p_\ell z_\ell} = \frac{z_\ell^I p_\ell^I}{\sum_{\ell=1}^L z_\ell^I p_\ell^I} = \frac{\frac{\sigma_0}{\sigma_0 - \tau_0} q_{\ell} z_{\ell}^\tau}{\sum_{\ell=1}^L \frac{\sigma_0}{\sigma_0 - \tau_0} q_{\ell} z_{\ell}^\tau}$$

is the normalized $\ell - \text{th}$ term in $RS_L(z)$ and the $\ell - \text{th}$ term in $\delta_L(z)$ is the $\ell - \text{th}$ share in the nonsingularity measure of the bordered Hessian. The latter is equal to the income share of $z_\ell$ if $\sigma_\ell = \sigma_0$ and is larger if $\sigma_\ell < \sigma_0$. In general strict $\{\frac{\tau_{\ell}}{\tau_0} q_{\ell} \leq q_{\ell}, \forall \ell\}$ implies $RS_L(z) < \delta_L(z) < 1$.

The competitive profit or loss is

$$py - \sum_{\ell=1}^L p_\ell z_\ell = py(1 - RS_L(z)) = py^{1-\tau_0} \sum_{\ell=1}^L \frac{\sigma_\ell - \sigma_0}{\sigma_\ell (1 - \sigma_0)} q_{\ell} z_{\ell}^\tau.$$

Variable $z_\ell$ makes a positive contribution to profit if its $(\sigma_\ell - \sigma_0) q_{\ell} > 0$, such as an input with its $\sigma_0 < \sigma_\ell$ or a joint output with $\sigma_\ell < \sigma_0$. Inputs with low elasticity of substitution $\sigma_\ell < \sigma_0$ contribute negatively.

The curve $RS_L(z) = 1$ draws the boundary between the production plans with a positive profit and those with a negative profit. The $\ell - \text{th}$ share in the income distri-
bution is \( z_{\ell} \frac{\partial y}{\partial z_{\ell}} / \sum_{\ell=1}^{L} z_{\ell} \frac{\partial y}{\partial z_{\ell}} \). The firm’s bylaws specify how periodic profits are disbursed or reinvested. Under power separation \( \{ \frac{r_{\ell}}{r_{0}} q_{\ell} \leq q_{\ell}, \forall \ell \} \), the return to scale is diminishing. Letting \( \delta_{L} z_{\ell} = \frac{\sum_{\ell=1}^{L} \frac{1-\sigma_{\ell}}{\sigma_{\ell}} q_{\ell} z_{\ell}^{r_{\ell}}}{\sum_{\ell=1}^{L} q_{\ell} z_{\ell}^{r_{\ell}}} \leq 1 \) and the curve \( RS_{L}(z) = 1 \) disappears towards infinity.

The curve \( \delta_{L}(z) = 1 \) divides the concavity-convex production plans and those that are quasi-concave-convex. Under power separation \( \{ \frac{r_{\ell}}{r_{0}} q_{\ell} \leq q_{\ell}, \forall \ell \} \), the nonsingularity measure is \( \delta_{L}(z) = \frac{\sum_{\ell=1}^{L} \frac{1-\sigma_{\ell}}{\sigma_{\ell}} q_{\ell} z_{\ell}^{r_{\ell}}}{\sum_{\ell=1}^{L} q_{\ell} z_{\ell}^{r_{\ell}}} \leq 1 \) and the curve \( \delta_{L}(z) = 1 \) disappears towards infinity making the transition between concave-convex and quasi-concave-convex invisible. Standard 2-dimensional geometric graphs may be drawn in three examples.

**EXAMPLE I.** \( y(z) = (z_{1}^{-1} - z_{2}^{-3})^{\frac{1}{2}} \)

In this simple model of 1 input and 1 joint output, elasticities are \( (\sigma_{0}, \sigma_{1}, \sigma_{2}) = (\frac{1}{3}, \frac{1}{2}, \frac{1}{4}) \), homegood output \( y \) satisfies \( y^{-2} = z_{1}^{-1} - z_{2}^{-3} \), partials are \( \frac{\partial y}{\partial z_{1}} = \frac{1}{2} z_{1}^{-2} y^{3}, \frac{\partial y}{\partial z_{2}} = -\frac{3}{2} z_{2}^{-4} y^{3} \).

From Corollary 1 find \( Z \) such that \( z \in Z \) implies \( 0 < y(z) \) if \( 0 < z_{1}^{-1} - z_{2}^{-3} \) and

\[
0 < \delta_{2,1}(z) = \frac{-9}{8} z_{2}^{-3}, \quad 0 < \delta_{2,2} = \frac{3}{4} z_{1}^{-1} - z_{2}^{-3}, \quad 0 < \delta_{2}(z) = \sum_{\ell=1}^{2} \sigma_{\ell} z_{\ell} \frac{\partial y}{\partial y} \frac{\partial y}{\partial z_{\ell}} = \frac{3}{4} z_{1}^{-1} - \frac{9}{8} z_{2}^{-3}.
\]

\( Z \) is empty since \( \delta_{2,1} \) is negative for all \( z \). The VS(z) function is not quasi-concave-convex.

If \( z_{2} > (\frac{3}{2} z_{1})^{\frac{1}{3}} \) both homegood output \( y(z) \) and \( \delta_{2}(z) \) are positive, the bordered Hessian is nonsingular. Furthermore both \( \delta_{2}(z) \) and \( RS_{2}(z) = \frac{z_{1}^{-1} - z_{2}^{-3}}{z_{1}^{-1} - z_{2}^{-3}} \) are less than 1 and the return to scale is diminishing. Letting \( f_{\ell} = \frac{\partial y(z)}{\partial z_{\ell}} \) the bordered Hessian inverse \( B_{2}^{-1} \) is

\[
\begin{pmatrix}
0 & f_{1} & f_{2} \\
\frac{2}{y} f_{1} f_{2} & \frac{2}{y} f_{1} f_{2} & \frac{2}{y} f_{1} f_{2} \\
\frac{2}{y} f_{1} f_{2} & \frac{2}{y} f_{1} f_{2} & \frac{2}{y} f_{1} f_{2}
\end{pmatrix}^{-1} = \frac{1}{\delta_{2}(z)} \begin{pmatrix}
\frac{3(1-\delta_{2}(z))}{y} & \frac{3 z_{1}}{2 y} & \frac{3 z_{2}}{4 y} \\
\frac{3 z_{1}}{2 y} & \frac{3 z_{1}}{2 y} & \frac{3 z_{1} z_{2}}{8 y} \\
\frac{3 z_{2}}{4 y} & \frac{3 z_{1} z_{2}}{8 y} & \frac{3 z_{2} z_{2}}{8 y}
\end{pmatrix}
\]

As shown below each entry has the sign of a regular concave-convex function except the 2nd diagonal element \( \frac{3 z_{1}}{4 f_{1}} \delta_{2,1}(z) \delta_{2}(z) \) is positive instead of negative. Its comparative static interpretation is that \( \frac{\partial y}{\partial p_{1}} > 0 \), which is a source of instability. Clearly \( \delta_{2,1} \) is negative in any VS(z) function with 1 input and 1 joint output and none can be quasi-concave-convex.

Two graphs are drawn that provide geometric descriptions of Example I.
2.4.1 Fig. 1a. Homegood $y$-Isoquant Map $y(z) = (z_1^{-1} - z_2^{-3})^{\frac{1}{2}}$ in $(z_1, z_2|y(z))$

The $y$–isoquants do not intersect all converging to a different point of the open origin $(0,0)$ and to $(y^2,\infty)$ at their vertical end. The strip along the $y$–isoquant from the open origin to $\frac{2}{3}$ is not in the nonsingular $\delta_2(z) > 0$ region. Clearly the slope of the $y$-curves will determine the desirable allocation including the optimal input $z_1$, but $\frac{\partial z_1}{\partial y}$ is positive, not negative. The $y$–isoquants have a third degree polynomial appearance reflecting the concavity to convexity transition.

Fig. 1b. Input $z_1$ – isoquant Map when $y(z) = (z_1^{-1} - z_2^{-3})^{\frac{1}{2}} \in (y, z_2|z_1)$

With a given input $z_1$ the substitutions between the homegood $y$ and the joint output $z_2$ are along a smooth diminishing marginal rate of substitution curve with $z_2$ reaching the amount $z_1^{\frac{1}{4}}$ at the horizontal end and $y$ reaching the amount $z_1^{\frac{1}{2}}$ at the vertical end.
EXAMPLE II. \( y(z) = (z_1^{-1} + z_2^{-1.5} - z_3^{-1.25})^{1/25} \)

In this model of 2 inputs and 1 joint output, \((r_0, r_1, r_2, r_3) = (-1.25, -1, -1.5, -1.25)\) are not separated and elasticities are \( (\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (\frac{4}{9}, \frac{1}{2}, \frac{2}{5}, \frac{2}{5}) \). In determining quasi-concavity-convexity with positive \( y^{-1.25} = z_1^{-1} + z_2^{-1.5} - z_3^{-1.25} \), the partials are

\[
\begin{align*}
    f_1 &= \frac{\partial y}{\partial z_1} = 4 \frac{1}{5} z_1^{-2} y \frac{2}{5}, \\
    f_2 &= \frac{\partial y}{\partial z_2} = 6 \frac{1}{5} z_2^{-2.5} y \frac{2}{5}, \\
    f_3 &= \frac{\partial y}{\partial z_3} = -z_3^{-2.25} y \frac{2}{5}.
\end{align*}
\]

From (7) the principal minors that must be positive are

\[
\delta_3(z) = \frac{\frac{9}{10} z_1^{-1} + \frac{27}{25} z_2^{-1.5} - z_3^{-1.25}}{y^{-1.25}},
\]

\[
\delta_{3,1} = \frac{27}{25} z_2^{-1.5} - z_3^{-1.25} > 0, \quad \delta_{3,2} = \frac{9}{10} z_1^{-1} - z_3^{-1.25} > 0, \quad \delta_{3,3} = \frac{9}{10} z_1^{-1} + \frac{27}{25} z_2^{-1.5} > 0.
\]

The inequalities corresponding to \((y(z), \delta_{3,1}(z), \delta_{3,2}(z), \delta_{3,3}(z), \delta_{3}(z))\) positive are

\[
z_3 > \text{Max}\{ (z_1^{-1} + z_2^{-1.5})^{1/25}, (\frac{25}{27})^{1/25} z_2^{1.25}, (\frac{10}{9})^{1/25}, 0, (\frac{9}{10} z_1^{-1} + \frac{27}{25} z_2^{-1.5})^{1/25} \},
\]

where the highest boundary curves intersect also with the inequalities

\[
0 < R_{3} R_{3}(z) = \frac{\frac{9}{10} z_1^{-1} + \frac{27}{25} z_2^{-1.5} - z_3^{-1.25}}{y^{-1.25}} \quad \text{if} \quad (\frac{8}{10} z_1^{-1} + \frac{6}{5} z_2^{-1.5})^{1/25} < z_3 \quad \text{and} \quad 1 < R_{3}(z) \quad \text{if} \quad z_1 > z_2^{1.5}.
\]

\[
0 < \delta_{3}(z) = \frac{9}{10} z_1^{-1} + \frac{27}{25} z_2^{-1.5} - z_3^{-1.25} \quad \text{if} \quad (\frac{9}{10} z_1^{-1} + \frac{27}{25} z_2^{-1.5})^{1/25} < z_3 \quad \text{and} \quad 1 < \delta_{3}(z) \quad \text{if} \quad z_1 > \frac{5}{4} z_2^{1.5}.
\]

From the Appendix the bordered Hessian and its inverse \( B_{3}^{-1} \) are

\[
B_{3}^{-1} = \begin{pmatrix}
0 & f_1 & f_2 & f_3 \\
-\frac{2f_1}{z_1} (1 - \frac{5z_1}{8y} f_1) & \frac{5}{4y} f_1 f_2 & \frac{5}{4y} f_1 f_3 \\
\frac{5}{4y} f_2 f_1 & -\frac{5f_2}{2z_2} (1 - \frac{z_2}{2y} f_2) & \frac{5}{4y} f_2 f_3 \\
\frac{5}{4y} f_3 f_1 & \frac{5}{4y} f_3 f_2 & -\frac{9f_3}{4z_3} (1 - \frac{z_3}{9y} f_3)
\end{pmatrix}^{-1} = \frac{1}{\delta_3(z)} \begin{pmatrix}
\frac{9(1-\delta_3(z))}{5y} & \frac{9z_1}{8y} & \frac{9z_2}{10y} & \frac{z_3}{y} \\
\frac{9z_1}{8y} & \frac{9z_1 z_2}{20y} & \frac{9z_2 z_3}{20y} & \frac{z_1 z_3}{2y} \\
\frac{9z_2}{10y} & \frac{9z_1 z_2}{20y} & \frac{9z_2 z_3}{20y} & \frac{z_1 z_3}{2y} \\
\frac{z_3}{y} & \frac{z_1 z_3}{2y} & \frac{z_1 z_3}{2y} & \frac{4z_1 z_2 z_3}{9f_3}
\end{pmatrix}.
\]

In \( B_{3}^{-1} \) under decreasing return the \((1,1)\) element is positive. Under the inequalities above the diagonal element is negative in an input row and positive in a joint output row. All off-diagonal entries are positive.
2.4.2 Fig. 2. Quasi-Concave-Convex Inequalities $y(z) = (z_1^{-1} + z_2^{-1.5} - z_3^{-1.25})^{-1}$ in $(z_1, z_3|z_2 = 10, 50)$.

In Fig.2a eight curves are drawn. The curves portray the quasi-concave-convex conditions of the 3-dimensional function in two dimensions by fixing $z_2 = 10$. The 3 lowest correspond to the conditions $(y(z), \delta_3(z), RS_3(z))$ positive. These 3 curves run very closely together for the chosen elasticities, but they are different. Allocations above these imply positive homegood output, nonsingular bordered Hessian and positive return to scale. The next 2 curves translate graphically the conditions that $(\delta_{3,1}(z), \delta_{3,2}(z))$ are positive for $z$ vertically above the curves that are in the $VS(z|z_2 = 10)$ quasi-concave-convex region. All conditions are conditional on the second input $z_2 = 10$. The allocations $z' = (z_1, z_2 = 10, z_3)$ above $(\delta_{3,1} = 0, \delta_{3,2} = 0)$ are rational except profits are negative if $z_1 > 10^{1.5} = 31.62$. The curves $\delta_{3,1}(z|z_2 = 10), \delta_{3,2}(z|z_2 = 10)$ intersect at $z_1 = 28.67$.

Three homegood output isoquants are shown and other that for any level $y(z) \leq 10^{1.2} = 15.848$ could have been drawn. Each isoquant reaches a vertical asymptote $(z_1, z_3) = (y^{-1.25} - 10^{-1.5})^{-1}$, $\infty$) where $z_1 = (9.39, 40.63, 188.54)$ at $y = (6, 10, 14)$ respectively.
EXAMPLE III.

2.4.3 Fig.3 Production with Concave and Quasi-concave Regions.

With the production function \( y = (z_1^{-4} + z_2^{-6})^{\frac{1}{5}} \) the powers are not separable. The homegood \( y \) is positive in the positive quadrant, the quasi-concave conditions (7)

\[
0 < \delta_2(z) = \frac{24 z_1^{-4} + 36 z_2^{-6}}{z_1^{-4} + z_2^{-6}}, \quad 0 < \delta_{2,1} = \frac{36 z_2^{-6}}{z_1^{-4} + z_2^{-6}}, \quad 0 < \delta_{2,2} = \frac{24 z_1^{-4}}{z_1^{-4} + z_2^{-6}},
\]

are satisfied everywhere and the concavity conditions

\[
\delta_2(z) < 1, \quad \delta_{2,1} < 1, \quad \delta_{2,2} < 1 \quad \text{hold if } \left(\frac{5}{7}\right)^{\frac{1}{5}} z_1^{\frac{2}{3}} < z_2.
\]

The return to scale \( RS_2 = \frac{\frac{4 z_1^{-4} + 6 z_2^{-6}}{z_1^{-4} + z_2^{-6}}}{\frac{4 z_1^{-4}}{z_1^{-4} + z_2^{-6}}, \frac{6 z_2^{-6}}{z_1^{-4} + z_2^{-6}}} \) is less than 1 if \( z_1^{\frac{2}{3}} < z_2 \) and the factor returns are \( \left(\frac{\frac{4 z_1^{-4} + 6 z_2^{-6}}{z_1^{-4} + z_2^{-6}}}{\frac{4 z_1^{-4}}{z_1^{-4} + z_2^{-6}}, \frac{6 z_2^{-6}}{z_1^{-4} + z_2^{-6}}} \right) \). Factor 1 return exceeds Factor 2 return where \( \left(\frac{5}{7}\right)^{\frac{1}{5}} z_1^{\frac{2}{3}} < z_2 \).

\[\text{Input } z_2 \quad \text{Quasiconcave} \quad \text{Concave} \quad \text{Input } z_1 \]

\[\delta_2(z) = 1 \quad RS_2(z) = 1 \quad y = 7 \quad y = 14 \]

Profit is positive above the curve \( RS_2(z) = 1 \) and \( (z_1)^{\frac{2}{3}} < z_2 \). Concavity with \( \delta_2(z) < 1 \) does not imply \( RS_2(z) < 1 \) and positive profit. In the narrow band \( \left(\frac{5}{7}\right)^{\frac{1}{5}} z_1^{\frac{2}{3}} < z_2 < z_1^{\frac{2}{3}} \) of Fig. 3, \( RS_2(z) > 1, \delta_2(z) < 1 \) and profit is negative.

The isoquants do not cut the axes and are smooth without kinks. Above the curve \( \left(\frac{5}{7}\right)^{\frac{1}{5}} z_1^{\frac{2}{3}} \) the return to Factor 1 is larger than the return to Factor 2, as can be inferred from the tilt in the isoquant. If the optimal production plan is a chosen output \( y \) where the tangent of its \( y \)-isoquant is equal to the ratio of the input prices, the return to scale could be either decreasing or increasing. Any changes in either the output or in the factor input prices causes changes in the practiced technology, concave or quasiconcave.
2.5 Fig. 4 From Short-run to Long-run and back

Let \( y = VSP(z_I, \bar{z}_{II}) \) with \((L_1 \times 1)\) flexible \(z_I\) and \((\ell - th) \times 1)\) fixed variables \(\bar{z}_{II}\). The short run inverse Hessian \(\frac{\partial^2 y}{\partial z_i \partial z_j}\) and the short run bordered Hessian inverse are obtained from the inverses of the full matrices by retaining the \(\ell - th\), \(z_I \in z_I\) row and column, replacing \((\delta_L(z), \delta_{L,i}(z))\) in these rows and columns by

\[
(\delta_I, \delta_{I,i}) = \left(\sum_{\ell \in I} \sigma_{Iz_I} \frac{\partial y}{\partial z_I}, \sum_{\ell \in I, \neq i} \sigma_{Iz_I} \frac{\partial y}{\partial z_I}\right)
\]

\[
= \left(y^{-r_0} \sum_{\ell \in I} \frac{1 - \sigma_I}{1 - \sigma_0} q_{\ell} z_{r_\ell}^I, y^{-r_0} \sum_{\ell \in I, \neq i} \frac{1 - \sigma_I}{1 - \sigma_0} q_{\ell} z_{r_\ell}^I\right), i \in I.
\]

With these replacements in (2) and in (3), valid conditions are obtained in Theorem 1 under which the function \( y = VSP(z_I, \bar{z}_{II}) \) is a saddle function, quasi-concave-convex in the variable inputs and joint outputs, keeping all other fixed.

From Appendix 1, the inverse of the bordered principal submatrix of \( y = SCE(z_I, \bar{z}_{II}) \)

\[
\left(\begin{array}{cc}
0 & \frac{\partial y}{\partial z_I} \\
\frac{\partial y}{\partial z_I} & \frac{\partial^2 y}{\partial z_I \partial z_I}
\end{array}\right)^{-1} = \frac{1}{\delta_I} \left(\begin{array}{cccc}
\frac{1 - \delta_I}{\sigma_0} & -\frac{\sigma_{Iz_I}}{\sigma_0} & \cdots & -\frac{\sigma_{Iz_I}}{\sigma_0} \\
-\frac{\sigma_{Iz_I}}{\sigma_0} & \frac{1 - \delta_I}{\sigma_0} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{\sigma_{Iz_I}}{\sigma_0} & \cdots & \cdots & -\frac{\sigma_{Iz_I}}{\sigma_0}
\end{array}\right).
\]

Long run and short run coefficients have the same sign. From (6), the non-diagonal short run coefficient equals \(\delta_{L}(z)\) times the long run coefficient. The short run \(j - th\) diagonal coefficient equals \(\delta_{L}(z)\) \(\frac{\delta_{L,j}(z)}{\delta_{I,j}}\) times the long run coefficient. The ratio of the short run to the long run \((1, 1)\) coefficient under concavity is \(\frac{1 - \delta_I}{\delta_I} \frac{\delta_{L}(z)}{1 - \delta_{L}(z)}\), which is more than one if long run \(\delta_{L}(z)\) is larger than short run \(\delta_I\).

In Fig. 3 the long run \(y(z) = (z_I^{-2} + z_{I1}^{-1} + z_{I2}^{-1} + z_{I3}^{-1} - z_{I4}^{-1})^{1/4}\) displays inverse power separation. With \((\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3})\) the nonsingularity and return to scale ratios are

\[
\delta_4(z) = y^{-r_0} \left(\frac{4}{3} z_I^{-2} + z_{I1}^{-1} - z_{I2}^{-1} - z_{I3}^{-1} - z_{I4}^{-1}\right), \quad RS_4(z) = y^{-r_0} (2z_I^{-2} + z_{I1}^{-1} - z_{I2}^{-1} - z_{I3}^{-1} - z_{I4}^{-1})
\]

implying \(1 < \delta_4(z)\) and increasing return to scale. The saddle inequalities (3) are

\[
z_{I4}^{-4} < \text{Min}\{z_{I1}^{-1} + z_{I2}^{-1} + z_{I3}^{-1}, \frac{4}{3} z_{I1}^{-2} + z_{I2}^{-1} + z_{I3}^{-1}, z_{I1}^{-1} + z_{I2}^{-1} + z_{I3}^{-1}, \frac{4}{3} z_{I1}^{-2} + z_{I2}^{-1} + z_{I3}^{-1}\}.
\]
The long run technology is quasi-concave-convex where $z_4^{-4} < \min\{2z_2^{-1}, \frac{4}{3}z_1^{-2} + z_2^{-1}\}$ and $z_2 = z_3$. It is not operable under perfect competition in the long run.

If the firm acquires $\bar{z}_1$ units of $z_1$ as a fixed amount, the short run homegood function $y_s(z) = (\bar{z}_1^{-2} + z_1^{-1} + z_3^{-1} - z_4^{-1})^{\frac{1}{4}}$ satisfies $0 < \delta_3(z) = y_s(z_2^{-1} + z_3^{-1} - z_4^{-1}) = RS_3(z) < 1.$

The conditions (3)

$$y_s > 0, \quad y_s(z_2^{-1} + z_3^{-1} - z_4^{-1}) \geq 0, \quad y_s(z_3^{-1} - z_4^{-1}) \geq 0, \quad y_s(z_2^{-1} - z_4^{-1}) \geq 0$$

hold if $z_4 \geq (\bar{z}_1^{-2} + z_2^{-1} + z_3^{-1})^{-1}$ where the short run $y_s(z)$ is a concave-convex function in the variables $(z_2, z_3, z_4)$.

In Fig. 4. $y_s(z) = (\bar{z}_1^{-2} + z_2^{-1} + z_3^{-1} - z_4^{-1})^{\frac{1}{4}}$ with $(\bar{z}_1 = 5, z_2 = z_3, z_4)$ is drawn for levels of output ($2, 4, 10$).

The long run increasing return to scale technology $y(z) = (z_1^{-2} + z_2^{-1} + z_3^{-1} - z_4^{-1})^{\frac{1}{4}}$ is transformed into a concave-convex profit technology $y(z) = (\bar{z}_1^{-2} + z_2^{-1} + z_3^{-1} - z_4^{-1})^{\frac{1}{4}}$ which may compensate for the cost incurred in acquiring the fixed amount $\bar{z}_1$. Any other variable could be fixed in the short run with the same effect on the return to scale.
3 Appendix.

3.1 The \( y = \left( \sum_{\ell=1}^{L} q_{\ell} z_{\ell}^{r_{\ell}} \right)^{1/r_0} \) Derivatives

In the positive orthant \( \mathcal{P} \), the 1st and 2nd order partials of \( SCE(z) \) are

\[
\frac{\partial y}{\partial z_{i1}} = \frac{q_{i1} r_{i1} y^{1-r_{i0}}}{r_{i0} z_{i1}^{1-r_{i1}}}, \quad \frac{\partial y}{\partial z_{i2}} = \frac{q_{i2} r_{i2} y^{1-r_{i0}}}{r_{i0} z_{i2}^{1-r_{i2}}},
\]

\[
\frac{\partial^2 y}{\partial z_{i1}^2} = \frac{q_{i1} r_{i1}}{r_{i0}} \left( \frac{1}{\sigma_0} \frac{y^{-r_{i0}}}{z_{i1}^{1-r_{i1}}} - \frac{1}{\sigma_1} \frac{y^{1-r_{i0}}}{z_{i1}^{2-r_{i1}}} \right), \quad \frac{\partial^2 y}{\partial z_{i1} \partial z_{i2}} = \frac{q_{i2} r_{i2}}{r_{i0}} \left( \frac{1}{\sigma_0} \frac{y^{-r_{i0}}}{z_{i2}^{1-r_{i2}}} \right), \quad \ell_1 \neq \ell_2,
\]

i.e. \( \frac{\partial^2 y}{\partial z_{i1} \partial z_{i2}} = \text{diag}\{-\frac{1}{\sigma_{i1} z_{i1}}, \ldots, -\frac{1}{\sigma_{iL} z_{iL}}\} + \frac{1}{\sigma_{0y}} \frac{\partial y}{\partial z_{i1}} \frac{\partial y}{\partial z_{i2}}. \)

For all \( z \in \mathcal{P} \) the 1st order derivative is never zero, positive for an input and negative for a joint output. The second order derivative \( \frac{\partial^2 y}{\partial z_{i1}^2} \) of a joint output is always positive and for an input is negative if and only if the return to \( z_{i\ell} \), i.e., the \( \ell - \text{th} \) output elasticity \( z_{i\ell} \frac{\partial y}{\partial z_{i\ell}} < \frac{\partial y}{\partial z_{i\ell}} \). (64) is continously differentiable in the region \( \{z \in \mathcal{P}, y > 0\} \). If both variables \( (z_{i1}, z_{i\ell}) \) are inputs or joint outputs, the cross derivative \( \frac{\partial^2 y(z)}{\partial z_{i1} \partial z_{i\ell}} \) is positive and if one variable is an input and the other a joint output, the cross derivative is negative. Hicks-Allen gave this second order cross partial a name, calling them substitutes if positive and complements if negative. They have a physical meaning in a production function context and also agrees with the Slutsky labels between positive and negative.

In the classical days concavity of the production function is a standard assumption familiar from the established conditions under which a maximum exists. These conditions are to be extended here to a function concave in inputs and convex in joint outputs. Under what conditions is the \( VS(z) \) concave in inputs and convex in joint outputs, a saddle function, supporting a minimum in one direction and a maximum in another?

The derivatives of \( y(t) = \left( \sum_{\ell=1}^{L} q_{\ell} (a_{\ell t} z_{\ell t})^{r_{\ell}} \right)^{1/r_0} \), when \( \ell_1 \neq \ell_2 \), are

\[
\begin{pmatrix}
\frac{\partial y(t)}{\partial z_{i1 t}} = a_{i1 t}^{r_{i1}} q_{i1} t^{r_{i1}-1} y(t)^{1-r_{i0}}, \\
\frac{\partial^2 y(t)}{\partial z_{i1 t}^2} = a_{i1 t}^{r_{i1}} q_{i1} t^{r_{i1}-1} (\frac{1}{\sigma_0} \frac{y(t)^{-r_{i0}}}{z_{i1 t}^{1-r_{i1}}} - \frac{1}{\sigma_1} \frac{y(t)^{1-r_{i0}}}{z_{i1 t}^{2-r_{i1}}}'), \\
\frac{\partial y(t)}{\partial z_{i2 t}} = a_{i2 t}^{r_{i2}} q_{i2} t^{r_{i2}-1} y(t)^{1-r_{i0}}, \\
\frac{\partial^2 y(t)}{\partial z_{i2 t}^2} = a_{i2 t}^{r_{i2}} q_{i2} t^{r_{i2}-1} (\frac{1}{\sigma_0} \frac{y(t)^{-r_{i0}}}{z_{i2 t}^{1-r_{i2}}}'),
\end{pmatrix}
\]

and the 2nd derivatives stand to the 1st derivatives as in (64). With \( y(t) \) replacing \( y \) the inverse of the Hessian and of the bordered Hessian are the same.
3.2 Hessian and Hessian Inverse

With $y = \left(\sum_{\ell=1}^{L} q_{\ell} z_{\ell}^{r_{\ell}}\right)^{1/\sigma}$, from (64) the matrix of 2\textsuperscript{nd} order partials $\frac{\partial^{2} y}{\partial \sigma \partial z}$ is

$$
H_{L} = \left(\begin{array}{cccc}
-\frac{1}{\sigma_{y1}} \frac{\partial y}{\partial z_{1}} (1 - \sigma_{z_{1}} \frac{\partial y}{\partial z_{1}}) & 1 \frac{\partial y}{\partial \sigma_{01}} & \ldots & 1 \frac{\partial y}{\partial \sigma_{0L}} \\
1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial z_{2}} & -\frac{1}{\sigma_{z_{2}}} \frac{\partial y}{\partial z_{2}} (1 - \sigma_{z_{2}} \frac{\partial y}{\partial z_{2}}) & \ldots & 1 \frac{\partial y}{\sigma_{y2}} \frac{\partial y}{\partial \sigma_{02}} \\
\ldots & \ldots & \ldots & \ldots \\
1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial z_{L}} & 1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial \sigma_{0L}} & \ldots & -\frac{1}{\sigma_{yL}} \frac{\partial y}{\partial \sigma_{0L}} (1 - \sigma_{z_{L}} \frac{\partial y}{\partial \sigma_{0L}})
\end{array}\right)
$$

and its inverse

$$
H_{L}^{-1} = \frac{1}{1 - \delta_{L}(z)} \left(\begin{array}{cccc}
-\frac{\sigma_{z_{1}}}{\sigma_{y1}} (1 - \delta_{L,1}(z)) & -\sigma_{1} z_{1} \frac{\sigma_{z_{2}}}{\sigma_{0y}} & \ldots & -\sigma_{1} z_{1} \frac{\sigma_{z_{L}}}{\sigma_{0y}} \\
-\sigma_{2} z_{2} \frac{\sigma_{z_{1}}}{\sigma_{0y}} & -\frac{\sigma_{z_{2}}}{\sigma_{y2}} (1 - \delta_{L,2}(z)) & \ldots & -\sigma_{2} z_{2} \frac{\sigma_{z_{L}}}{\sigma_{0y}} \\
\ldots & \ldots & \ldots & \ldots \\
-\sigma_{L} z_{L} \frac{\sigma_{z_{1}}}{\sigma_{0y}} & -\sigma_{L} z_{L} \frac{\sigma_{z_{2}}}{\sigma_{0y}} & \ldots & -\sigma_{L} z_{L} \frac{\sigma_{z_{L}}}{\sigma_{0y}} (1 - \delta_{L,L}(z))
\end{array}\right)
$$

(11)

where

$$
\delta_{L,\lambda} = \delta_{L,\lambda}(z) = \sum_{\ell=1, \neq \lambda}^{L} \frac{\sigma_{z_{\ell}}}{\sigma_{0y}} \frac{\partial y}{\partial z_{\ell}} = y^{-\tau_{0}} \sum_{\ell=1, \neq \lambda}^{L} \frac{1}{1 - \sigma_{0} q_{\ell} z_{\ell}^{r_{\ell}}} , \lambda \in (1, \ldots, L).
$$

$$
0 < \delta_{L} = \delta_{L}(z) = \sum_{\ell=1}^{L} \frac{\sigma_{z_{\ell}}}{\sigma_{0y}} \frac{\partial y}{\partial z_{\ell}} = \frac{\sum_{\ell=1}^{L} r_{\ell} \sigma_{0y} q_{\ell} z_{\ell}^{r_{\ell}}}{\sum_{\ell=1}^{L} q_{\ell} z_{\ell}^{r_{\ell}}} = y^{-\tau_{0}} \sum_{\ell=1}^{L} \frac{1 - \sigma_{\ell}}{1 - \sigma_{0}} q_{\ell} z_{\ell}^{r_{\ell}} \neq 1.
$$

Proof. With $(H_{L})_{\ell}$ the $\ell$-th row of $H_{L}$ and $(H_{L}^{-1})^{\mu}$ the $\mu$-th column of $H_{L}^{-1}$, write

$$(H_{L})_{\ell} = \frac{\partial y}{\partial z_{\ell}} \left(\begin{array}{cccc}
1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial z_{1}} & \ldots & 1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial z_{\ell-1}} & -1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial z_{\ell}} (1 - \sigma_{z_{\ell}} \frac{\partial y}{\partial z_{\ell}}) \\
1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial z_{1}} & \ldots & 1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial z_{\ell}} & 1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial \sigma_{0y}} \\
\ldots & \ldots & \ldots & \ldots \\
1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial \sigma_{0y}} & \ldots & 1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial \sigma_{0y}} & 1 \frac{\partial y}{\sigma_{0y}} \frac{\partial y}{\partial \sigma_{0y}}
\end{array}\right)
$$

$$
(H_{L}^{-1})^{\mu} = -\epsilon \left(\begin{array}{cccc}
\frac{\sigma_{z_{1}}}{\sigma_{0y}} \frac{\partial y}{\sigma_{0y}} & \ldots & \frac{\sigma_{z_{\mu-1}}}{\sigma_{0y}} \frac{\partial y}{\sigma_{0y}} & \frac{\sigma_{z_{\mu}}}{\sigma_{0y}} (1 - \delta_{L,\mu}(z)) \\
\frac{\sigma_{z_{1}}}{\sigma_{0y}} \frac{\partial y}{\sigma_{0y}} & \ldots & \frac{\sigma_{z_{\mu-1}}}{\sigma_{0y}} \frac{\partial y}{\sigma_{0y}} & \frac{\sigma_{z_{\mu}}}{\sigma_{0y}} (1 - \delta_{L,\mu}(z)) \\
\ldots & \ldots & \ldots & \ldots \\
\frac{\sigma_{z_{1}}}{\sigma_{0y}} \frac{\partial y}{\sigma_{0y}} & \ldots & \frac{\sigma_{z_{\mu-1}}}{\sigma_{0y}} \frac{\partial y}{\sigma_{0y}} & \frac{\sigma_{z_{\mu}}}{\sigma_{0y}} (1 - \delta_{L,\mu}(z))
\end{array}\right)
$$

where $\epsilon = \frac{1}{1 - \delta_{L}(z)}$. When $\ell > \mu$, letting $h_{i} = \frac{\sigma_{z_{i}}}{\sigma_{0y}} \frac{\partial y}{\partial z_{i}}$, the product

$$(H_{L})_{\ell}(H_{L}^{-1})^{\mu} = \epsilon \frac{\partial y}{\partial z_{\ell}} \frac{\partial y}{\sigma_{0y}} \sum_{i=1}^{\mu-1} -h_{i} + \sum_{i=\mu+1}^{\ell-1} -h_{i} + \sum_{i=\ell+1}^{L} -h_{i} - (1 - \delta_{L,\mu}(z)) + (1 - h_{\ell})
$$

$$
= \epsilon \frac{\partial y}{\partial z_{\ell}} \frac{\partial y}{\sigma_{0y}} (-\delta_{L,\ell,\mu} + \delta_{L,\mu}(z) - h_{\ell}) = 0
$$
and
\[(H_L)_{\ell}(H_L^{-1})^\ell = \frac{1}{1 - \delta_L(z)} \left( -h_\ell \delta_{L,\ell} + (1-h_\ell)(1 - \delta_L(z)) \right) = \frac{1 - \delta_L(z)}{1 - \delta_L(z)} = 1. \]

\[H_L^{-1}\] at (65) is the general form of a nonsingular Hessian of any VS(z), over the whole range of \(0 < \delta_L(z) \neq 1\). But for a concave-convex VS(z) the diagonal element in an input row must be negative and in a joint output row must be positive. Therefore, \(y = VS(z)\) is concave-convex if
\[y > 0, \ 0 < \delta_L(z) < 1, \ 0 < \delta_L,\lambda(z) < 1, \ \lambda = 1, \ldots, L. \tag{12}\]
The off-diagonal elements of a nonsingular VS(z) Hessian inverse are all negative. When \(\delta(z) > 1\) the Hessian is no longer concave-convex but the VS(z) may still be quasi-concave-convex. In fact all off-diagonal coefficients may have changed sign under increasing returns to scale. The main conclusion is that the VS(z) can represent both decreasing at some action plan \(z\) and increasing returns to scale at another.

**Theorem 2** In the VS \(\sum_{\ell=1}^{L} q_\ell z_\ell^{r_{\ell}} (\frac{1}{x_\ell}), \text{with } y > 0, z > 0\), the Hessian principal minor
\[|H_r| = \left| \frac{\partial^2 y(z)}{\partial z \partial z^r} \right| = (1 - \delta_r(z)) \prod_{\ell=1}^{r} \frac{1}{\sigma_\ell z_\ell} (-\frac{\partial y}{\partial z_\ell}), \ r = 1, \ldots, L. \tag{13}\]

**Proof.** Writing
\[|H_L| = \left| \begin{array}{cc} H_{L-1} & h_L \\ h_L & H_{LL} \end{array} \right| = |H_{L-1}|(H_{LL} - h_L H_{L-1}^{-1} h^L) = |H_{L-1}|(H_{LL} + \frac{1}{\sigma_0 y} \frac{\partial y}{\partial z_L} \frac{\partial y}{\partial z_L} \delta_{L-1})\]
implies
\[\frac{|H_L|}{|H_{L-1}|} = -\frac{1}{\sigma_L z_L} \frac{\partial y}{\partial z_L} (1 - \frac{\sigma_L z_L}{\sigma_0 y} \frac{\partial y}{\partial z_L}) + \frac{1}{\sigma_0 y} \frac{\partial y}{\partial z_L} \frac{\partial y}{\partial z_L} \delta_{L-1}\]
\[= -\frac{1}{\sigma_L z_L} \frac{\partial y}{\partial z_L} \frac{1 - \delta_{L-1} - \frac{\sigma_L z_L}{\sigma_0 y} \frac{\partial y}{\partial z_L}}{1 - \delta_{L-1}} = \frac{1}{\sigma_L z_L} \frac{-\partial y}{\partial z_L} \frac{1 - \delta_L}{1 - \delta_{L-1}}\]
and (70), given \(|H_1| = (1 - \delta_1(z)) \frac{1}{\sigma_1 z_1} (-\frac{\partial y}{\partial z_1})\).

The principal minor \(|H_r|\) of order \(r\) of a concave-convex VS function must be not zero, having \(0 < \delta_r(z) < 1\) and the same sign as the principal minor of order \(r - 1\) if \(z_r\) is a joint output and the opposite alternating sign if \(z_r\) is an input. Under quasi-concavity-convexity when \(1 < \delta_1(z)\) and returns to scale are increasing, principal minors are the same but non diagonal elements in the inverse change signs.
3.3  Bordered Hessian and bordered Hessian Inverse

The bordered Hessian inverse is also of such recursive inductive simplicity that the reader can readily verify it. Given the bordered Hessian

\[ B_L = \begin{pmatrix}
0 & \frac{\partial y}{\partial z_1} & \frac{\partial y}{\partial z_2} & \cdots & \frac{\partial y}{\partial z_L} \\
\frac{\partial y}{\partial z_1} & \sigma_1 z_1 & \sigma_1 z_2 & \cdots & \sigma_1 z_L \\
\frac{\partial y}{\partial z_2} & \sigma_2 z_1 & \sigma_2 z_2 & \cdots & \sigma_2 z_L \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial y}{\partial z_L} & \sigma_L z_1 & \sigma_L z_2 & \cdots & \sigma_L z_L 
\end{pmatrix}
\]

the bordered Hessian inverse

\[ B_L^{-1} = \frac{1}{\delta_L(z)} \begin{pmatrix}
\sigma_1 z_1 
\sigma_2 z_1 & \sigma_2 z_2 & \cdots & \sigma_2 z_L \\
\sigma_1 z_2 & \sigma_1 z_1 & \sigma_1 z_3 & \cdots & \sigma_1 z_L \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_L z_2 & \sigma_L z_1 & \sigma_L z_3 & \cdots & \sigma_L z_L 
\end{pmatrix}
\]

(14)

provided \(0 < y, \ 0 < \delta_L(z) = \sum_{\ell=1}^L \sigma_\ell z_\ell \frac{\partial y}{\partial z_\ell}, \) where \(\delta_L, = \sum_{\ell=1, \neq \lambda}^L \sigma_\ell z_\ell \frac{\partial y}{\partial z_\ell}.

These are conditions under which the bordered Hessian is nonsingular, all off-diagonal elements in its inverse are positive regardless of its diagonal elements. These need to be restricted to negative in input and to positive in joint output rows. Given the recursive structure of \(B_L\) in \(L\), if (67) is correct when \(L = 2\) it is correct for all \(L\). Verify that

\[
\begin{align*}
(B_L B_L^{-1})_{1,1} &= \frac{1}{\delta_2(z)} \sum_{\ell=1}^2 \frac{\partial y}{\partial z_\ell} \frac{\partial y}{\partial z_\ell}, \quad \frac{1}{\delta_2(z)} (\sigma_1 z_1 - \delta_{2,1}(z)), \quad \frac{1}{\delta_2(z)} (\sigma_2 z_2 - \delta_{2,2}(z)) = (1, 0, 0) \\
(B_L B_L^{-1})_{2,2} &= \frac{1}{\delta_2(z)} \left( \frac{\partial y}{\partial z_1} \sigma_1 z_1 + (1 - \sigma_1 z_1) \frac{\partial y}{\partial z_2} \right), \quad \frac{1}{\delta_2(z)} \left( \sigma_2 z_2 - \delta_{2,2}(z) \right) = 1 \\
(B_L B_L^{-1})_{2,3} &= \frac{1}{\delta_2(z)} \left( \frac{\partial y}{\partial z_2} + \sigma_1 z_2 \frac{\partial y}{\partial z_2} \right), \quad \frac{1}{\delta_2(z)} \left( \sigma_2 z_2 - \delta_{2,2}(z) \right) = 0 \\
(B_L B_L^{-1})_{3,3} &= \frac{1}{\delta_2(z)} \left( \sigma_2 z_2 \frac{\partial y}{\partial z_2} + \sigma_1 z_2 \frac{\partial y}{\partial z_2} \right), \quad \frac{1}{\delta_2(z)} \left( \sigma_2 z_2 - \delta_{2,2}(z) \right) = 1
\end{align*}
\]

The \(VS(z)\) is quasi-concave in inputs and quasi-convex in joint outputs, with all off-diagonal elements in the bordered Hessian positive, only if

\[ 0 < y(z), \ 0 < \delta_L, \ 0 < \delta_{L,\lambda} = \sum_{\ell=1, \neq \lambda}^L \sigma_\ell z_\ell \frac{\partial y}{\partial z_\ell}, \ \lambda = 1, \ldots, L. \]
Theorem 3  The bordered Hessian of \( y = \sum_{\ell=1}^{L} q_{\ell} z^{r_{\ell}} \), \( y > 0, z > 0 \), has determinant

\[
|\mathcal{B}_L| = \left| \begin{array}{cc}
0 & \frac{\partial y(z)}{\partial z} \\
\frac{\partial y(z)}{\partial z} & p \frac{\partial^2 y(z)}{\partial z^2}
\end{array} \right| = \sigma y p^{L-1} \delta_L(z) \prod_{\ell=1}^{L} \left( \frac{\partial y}{\partial z_{\ell}} \right), \quad 1 \leq L
\]

(16)

Proof. Writing

\[
|\mathcal{B}_L| = \left| \begin{array}{cc}
\mathcal{B}_{L-1} & b^L \\
b_L & \mathcal{B}_{LL}
\end{array} \right| = |\mathcal{B}_{L-1}|(\mathcal{B}_{LL} - b_L \mathcal{B}_{L-1} b^L) = |\mathcal{B}_{L-1}|(\mathcal{B}_{LL} - \frac{p}{\sigma y} \frac{\partial y}{\partial z_{\ell}} \frac{\partial y}{\partial z_{\ell}}) \frac{1 + \delta_{L-1}}{\delta_{L-1}}
\]

implies

\[
\frac{|\mathcal{B}_L|}{|\mathcal{B}_{L-1}|} = -\frac{p}{\sigma y} \frac{\partial y}{\partial z_{\ell}} \left( 1 - \frac{\sigma_L z_L}{\sigma y} \frac{\partial y}{\partial z_L} \right) - \frac{p}{\sigma y} \frac{\partial y}{\partial z_{\ell}} \frac{1 + \delta_{L-1}}{\delta_{L-1}} = \frac{p}{\sigma y} \frac{\partial y}{\partial z_{\ell}} \frac{\delta_L}{\delta_{L-1}}
\]

and (70) given \( \mathcal{B}_1 = -(\frac{\partial y}{\partial z_{\ell}})^2 = \sigma y \delta_1(z) \frac{1}{\delta_{L-1}}(-\frac{\partial y}{\partial z_{\ell}}) \).

When \( L = 2 \) the bordered Hessian inverse obtained by the cofactor method

\[
\mathcal{B}_2^{-1} = \frac{1}{|\mathcal{B}_2|} \left( \begin{array}{ccc}
p^2 \left( \frac{\partial^2 y}{\partial z_{\ell}^2} \frac{\partial y}{\partial z_{\ell}} \frac{\partial y}{\partial z_{\ell}} \right) & \frac{\partial y}{\partial z_{\ell}} \frac{\partial y}{\partial z_{\ell}} & \frac{\partial y}{\partial z_{\ell}} \\
\frac{\partial y}{\partial z_{\ell}} & \frac{\partial y}{\partial z_{\ell}} & \frac{\partial y}{\partial z_{\ell}} \\
\frac{\partial y}{\partial z_{\ell}} & \frac{\partial y}{\partial z_{\ell}} & \frac{\partial y}{\partial z_{\ell}}
\end{array} \right)
\]

is \( \mathcal{B}_2^{-1} \) at (69) after dividing \( |\mathcal{B}_2| \) and all entries by \( \sigma y p \prod_{\ell=1}^{2} \frac{1}{\sigma_{\ell} z_{\ell}}(-\frac{\partial y}{\partial z_{\ell}}) \).

If \( 0 < \delta_r(z) \) the principal minor of order \( r \) is not zero, having the same sign as the principal minor of order \( r - 1 \) if \( z_r \) is a joint output and the opposite sign if \( z_r \) is an input.

If \( 0 < \delta_r(z) < 1 \) the principal minor \( |\mathcal{H}_r| \) in (67) and \( |\mathcal{B}_r| \) in (70) have the same sign. If \( 1 < \delta_r(z) \) the principal minors in (67) and (70) have opposite signs.

### 3.4 Necessary and Sufficient quasi-concave-convex Conditions

To complete the analysis the necessary conditions stated at (40) and (42) can be shown to be sufficient also by deriving that they imply that principal submatrices in the Hessian and bordered Hessian are nonsingular. Arrow-Endhoven [?] in Theorem 5 proved that quasi-concave production functions require that a bordered principal minor of order \( r \) has the sign of \((-1)^r \) over all \( z \). In an obvious extension to the quasi-concave-convex functions with joint outputs bordered principal minors must have the same sign over all \( z \) not crossing zero. From (70) the bordered principal minors are

\[
(\mathcal{B}_1, |\mathcal{B}_2|, \ldots, |\mathcal{B}_L|) = \sigma y \left( \delta_1 \frac{1}{\sigma_1 z_1}(-\frac{\partial y}{\partial z_1}), \delta_2 \prod_{\ell=1}^{2} \frac{1}{\sigma_\ell z_\ell}(-\frac{\partial y}{\partial z_\ell}), \ldots, \delta_L \prod_{\ell=1}^{L} \frac{1}{\sigma_\ell z_\ell}(-\frac{\partial y}{\partial z_\ell}) \right).
\]

(17)
The partial derivative $\frac{\partial y}{\partial z}$ is never zero, positive for an input and negative for a joint output for all $z$. Under quasi-concavity-convexity $\delta_1, \delta_2, \ldots, \delta_L$ must all be different from zero and $|B_r|$ has the sign opposite to the sign of $|B_{r-1}|$ if $z_r$ is an input and the same sign if $z_r$ is a joint output.

**Theorem 4** The VS function $y = \left(\sum_{\ell=1}^{L} q_\ell z_\ell^r\right)^{\frac{1}{r_0}}, y > 0, z > 0)$, is quasi-concave-convex if and only if $(\delta_1(z), \delta_2(z), \ldots, \delta_L(z))$ is a vector of positive numbers for all $z$.

**Proof.** From (68) and (70) ($\delta_r > 0, r > 1$) is the condition for the bordered principal submatrix of order $r$ to be nonsingular. Substituting $(\delta_1(z), \delta_2(z), \ldots, \delta_L(z))$ into (71)

$$
\left(\begin{array}{c}
|B_1|, |B_2|, \ldots, |B_L|
\end{array}\right) =

\left(\begin{array}{c}
\sigma_0 y \frac{\sigma_1 z_1}{\sigma_0 y} \frac{\partial y}{\partial z_1} \\
\frac{\partial y}{\partial z_1} \frac{1}{\sigma_1 z_1} (-\frac{\partial y}{\partial z_1}), \sum_{\ell=1}^{2} \frac{\sigma_\ell z_\ell}{\sigma_0 y} \frac{\partial y}{\partial z_\ell} \prod_{\ell=1}^{2} \frac{1}{\sigma_\ell z_\ell} (-\frac{\partial y}{\partial z_\ell}), \ldots, \sum_{\ell=1}^{L} \frac{\sigma_\ell z_\ell}{\sigma_0 y} \frac{\partial y}{\partial z_\ell} \prod_{\ell=1}^{L} \frac{1}{\sigma_\ell z_\ell} (-\frac{\partial y}{\partial z_\ell})
\end{array}\right)
$$

is an array of numbers none zero over all $z > 0$ with $|B_r|$ having the opposite sign of $|B_{r-1}|$ if $z_r$ is an input and the same sign if $z_r$ is a joint output. The Arrow-Enthoven Theorem 5 applied to the VS with joint outputs proves Theorem 8. The interpretation of their theorem here is that the VS function is quasi-concave-convex if and only if the function is quasi-concave-convex in any subset of its variables. □
References


