A Unifying Framework for Farrell Profit Efficiency Measurement

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Abstract

Measuring profit efficiency is a challenging task and many different approaches were suggested in the literature. This paper synthesizes existing approaches to form a general Farrell-type approach of profit efficiency measurement. Our derivations help us unveil new and useful relationships between existing profit efficiency measures and the proposed new Farrell-type profit efficiency measures. In turn, this helps us establishing a generalized and unifying framework for studying efficiency behavior of firms, where the profit efficiency measure satisfies some desirable properties and contains the conventional Farrell measures of technical efficiency and allocative efficiency as multiplicative components, along with a new measure we introduce and refer to as 'revenue efficient allocative efficiency measure'. The proposed new approach also encompasses and is coherent with the profit-maximizing behavior as a benchmark case.

Keywords: profit efficiency, Farrell-type measures, revenue efficient allocative efficiency, invariance property of allocative efficiency measure

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1. Introduction

In measuring technical efficiency of production two types of scaling inputs and outputs are frequently used: (i) scaling each element of a vector by a potentially distinct scalar; and (ii) scaling all elements of a vector by the same scalar. In this paper, efficiency measures based on these two approaches are referred to as the Russell-type measures and the Farrell-type measures, respectively. We apply the above scaling methods to the profit function and introduce several new Farrell-type efficiency measures for the profit efficiency measurement context.

Farrell (1957) adopted the scaling method (ii) to characterize the concepts of overall (cost or revenue) efficiency and its decomposition into multiplicative components of technical and allocative efficiencies. His framework was then generalized and popularized by Charnes, Cooper and Rhodes (1978), Färe and Lovell (1978), and Banker, Charnes and Cooper (1984), just to mention a few. Although other types of technical efficiency measures have been introduced, the Farrell technical efficiency measurement appears to be the most widely adopted one. The dominating popularity of Farrell-type efficiency measures in practice is reflected by thousands of empirical works in the past several decades. Somewhat paradoxically, such important class of efficiency measures has not been extended to input- and output-oriented measures of profit efficiency. Existing Farrell efficiency measures have so far not been embraced explicitly in the decomposition of profit efficiency according to our best knowledge of the literature. In particular, the various existing measures of profit efficiency were not defined explicitly as radial scaling of inputs and/or outputs. Thus, there appears to be a clear gap in the literature. The main goal of this paper therefore is to fill this gap by developing a coherent framework of profit efficiency measurement where the popular Farrell-type measures are expressed as components of profit efficiency.

It is worth noting from the start that measuring profit efficiency is a particularly challenging task. For example, the maximal profit level can be zero, positive, or even undefined. Meanwhile, the actual profit level, besides possibility to be positive, can also be zero or even negative. This complicates the problem of defining a suitable measure of profit efficiency so that it is well defined mathematically as well as has a convenient (e.g., percentage) interpretation.

Various measures of profit efficiency have been suggested in the literature. Incorporating Farrell's scaling idea, here we will synthesize some of existing key ideas in the literature (including Banker and Maindiratta (1988), Chavas and Cox (1994), Färe and Primont (1995), Chambers et al. (1998), and Ray (2004)) into a general and unified approach of Farrell-type efficiency measures. Importantly, this paper will explicitly relate to profit maximization principle, embracing it as a benchmark case. In particular, we introduce three Farrell-type measures of profit efficiency that have some theoretical advantages over existing measures as well as possess useful practical interpretations.

The general model introduced in this paper helps us unveil new and interesting relationships between existing profit efficiency measures and the new Farrell-type profit efficiency measures. A new component in the decomposition of the output-oriented profit efficiency, which we dub as the 'revenue-efficient allocative efficiency measure', is introduced to help the firm determine its production scale and its input mix. Altogether this paper establishes a coherent and unified framework that extends Farrell's (1957) measures to profit efficiency and embraces or unifies many other useful measures as special cases.

The rest of this paper is structured as follows: In the next section, we introduce a general framework of profit efficiency measure. Then section 3 focuses on the Farrell output-oriented profit efficiency measure specifically and in section 4, some interpretations of this measure are provided. In section 5, we show that the relationship among different measures in Farrell-type framework, technical efficiency, revenue efficiency and cost efficiency are just some special cases of the profit efficiency measure under various constraints. Section 6 expresses multiplicative decomposition of the profit efficiency measure and Section 7 links this Farrell-type profit efficiency measure to some existing profit efficiency measures. Section 8 illustrates the Farrell-type profit efficiency measure and its decomposition while section 9 concludes.

2. A General Framework for Farrell-type Measures of Profit Efficiency

Let $x \in \mathbb{R}_+^N$ and $q \in \mathbb{R}_+^M$ be input and output vectors, respectively. The corresponding prices are denoted with vectors $w \in \mathbb{R}_+^N$ and $p \in \mathbb{R}_+^M$, respectively. The production technology is given by $\mathfrak{I} = \{(x, q): x\ can\ produce\ q\}$ in which the following axioms are satisfied: (i) $(x, 0) \in \mathfrak{I}\ \forall x \in \mathbb{R}_+^N$

and $(0,q) \notin \Im \; \forall q \neq 0$. (ii) For all $(x,q) \in \Im$, if $(-x',q') \leqq (-x,q)$, then $(x',q') \in \Im$. (iii) For all $(x^0,q^0) \in \Im$, the output set $P(x^0) \coloneqq \{q: (x^0,q) \in \Im\}$ and input set $L(q^0) \coloneqq \{x: (x,q^0) \in \Im\}$ are convex sets. (iv) For all $x \in \mathbb{R}_+^N$, the output set $P(x)$ is closed and bounded.[1] Let (x^0,q^0) be an observed input-output vector and let the observed total revenue (or sales) at output prices p^0 and observed total costs at input prices w^0 be denoted by $R^0 = p^0 \cdot q^0$ and $C^0 = w^0 \cdot x^0$ respectively. Further, let π^0 be the observed profit, i.e., $\pi^0 = p^0 \cdot q^0 - w^0 \cdot x^0$. Facing prices (p^0,w^0) and using technology $\Im$, the maximal profit a firm can attain is given by

$$\pi(p^0,w^0) := \sup_{x,q} \{p^0 \cdot q - w^0 \cdot x: (x,q) \in \Im\} \tag{1}$$

In the following discussions, a firm will be referred to as *profit efficient* if and only if it achieves maximal profit, i.e., when $\pi(p^0,w^0) = \pi^0$.

Before going further with a definition of profit efficiency, it is important to note that there are quite many cases where using $\pi(p,w)$ is problematic. This includes the case of global increasing returns to scale (IRS), where $\pi(p,w) = +\infty$. Similar situation is with the constant returns to scale (CRS), where $\pi(p,w) = +\infty$ or $\pi(p,w) = 0$, depending on the prices (p,w) relative to the shape of frontier of technology set $\Im$. One potential remedy to such cases is to allow for additional constraints because in practice, firms indeed often face various constraints: e.g., fixed inputs or endowments at a given period, minimal employment requirement, maximal amount of inputs available, budget limits, etc. We will denote such constraints with a constraint set Z, where $Z \coloneqq \{(x,q): R_{lb} \leqq p \cdot q \leqq R_{ub}, q_{lb} \leqq q \leqq q_{ub}, C_{lb} \leqq w \cdot x \leqq C_{ub}, \; and \; x_{lb} \leqq x \leqq x_{ub}\}$. The maximum profit under those constraints is given by

$$\pi(p,w|Z) \coloneqq \sup_{x,q} \{p \cdot q - w \cdot x : (x,q) \in \Im \cap Z\} \tag{2}$$

One could also add many other types of constraints.[2] In the special case when each lower bound is zero and each upper bound is positive infinity, the constraint set becomes $Z = \mathbb{R}_+^{M+N}$ and $\pi(w,p|Z)$ will be the usual profit function in (1). We will refer to $\pi(w,p)$ and $\pi(p,w|Z)$ as *unrestricted and restricted maximal profit function*, respectively. Furthermore, let $\mathfrak{D} = \{(x,q;w,p) \in \mathbb{R}_+^N \times \mathbb{R}_+^M \times \mathbb{R}_{++}^N \times \mathbb{R}_{++}^M: (x,q) \in \Im, x \neq 0_N \; and \; q \neq 0_M\}$ be a relevant domain for measuring efficiency. Thus we rule out the extreme cases that $p^0 \cdot q^0 = 0$ and $w^0 \cdot x^0 = 0$. A general measure of profit efficiency for the observed quantity vector (x^0, q^0) and price vector (w^0, p^0) is a function $\phi: \mathfrak{D} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, and subject to Z, defined as

$$\phi(x^0, q^0; w^0, p^0 | Z) := \sup_{\theta,\lambda,x,q} \{f(\lambda_1, \dots, \lambda_N; \theta_1, \dots, \theta_M) :$$

$$\sum_{m=1}^{M} p_m^0 \cdot (\theta_m q_m^0) - \sum_{n=1}^{N} w_n^0 \cdot (\lambda_n x_n^0) \leqq p^0 \cdot q - w^0 \cdot x,$$

$$(x, q) \in \mathfrak{I} \cap Z\} \tag{3}$$

where $f(\lambda_1, \dots, \lambda_N; \theta_1, \dots, \theta_M)$ is a generic objective function chosen by the researcher. The objective function in (3) is bounded from above when $\pi(p, w|Z)$ is a finite number.

Different forms of $f(\lambda_1, \dots, \lambda_N; \theta_1, \dots, \theta_M)$ in (3) can be chosen to derive particular efficiency measures. In this paper we will focus on the case where $\theta_m = \theta$ for $m = 1, \dots, M$, and $\lambda_n = \lambda$ for $n = 1, \dots, N$. Thus, (3) becomes a general Farrell-type measure of profit efficiency:

$$\phi(x^0, q^0; w^0, p^0 | Z)$$

$$= \sup_{\theta,\lambda,x,q} \{f(\lambda, \theta): p^0 \cdot (\theta q^0) - w^0 \cdot (\lambda x^0) \leqq p^0 \cdot q - w^0 \cdot x, (x, q) \in \mathfrak{I} \cap Z\}. \tag{4}$$

If in addition we have $f(\lambda, \theta) = 0.5(\theta + 1/\lambda)$ and $\lambda = 1/\theta$, then (4) becomes

$$\phi(x^0, q^0; w^0, p^0 | Z)$$

$$= \sup_{\theta,x,q} \{\theta: p^0 \cdot (\theta q^0) - w^0 \cdot (x^0/\theta) \leqq p^0 \cdot q - w^0 \cdot x, (x, q) \in \mathfrak{I} \cap Z\}.$$

This special case is the reciprocal of the overall graph efficiency measure introduced by Färe, Grosskopf and Lovell (1985, p. 119).

Furthermore, if we let $f(\lambda, \theta) = (\lambda + \theta)/2$ and $\theta = \lambda = \beta$ instead, then (4) becomes

$$\phi(x^0, q^0; w^0, p^0 | Z)$$

$$= \sup_{\beta,x,q} \{\beta: \beta p^0 \cdot q^0 - \beta w^0 \cdot x^0 \leqq p^0 \cdot q - w^0 \cdot x, (x, q) \in \mathfrak{I} \cap Z\}$$

$$= \sup_{\beta,x,q} \left\{\beta: \beta \leq \frac{p^0 \cdot q - w^0 \cdot x}{p^0 \cdot q^0 - w^0 \cdot x^0}, (x, q) \in \mathfrak{I} \cap Z\right\}$$

$$= \frac{\pi(p^0, w^0 | Z)}{\pi^0}.$$

This special case is the reciprocal of the profit efficiency measure defined in Banker and Maindiratta (1988).[3]

A special attention of this paper is on $f(\theta,\lambda) = \theta - \lambda + 1$ where $\theta \geqq 1$ and $\lambda \leqq 1$. In this case, the measure of profit efficiency (4) becomes

$$\phi(x^0, q^0; w^0, p^0 | Z)$$
$$= \sup_{\theta,\lambda,x,q} \{\theta - \lambda + 1 : p^0 \cdot (\theta q^0) - w^0 \cdot (\lambda x^0) \leqq p^0 \cdot q - w^0 \cdot x, (x,q) \in \Im \cap Z, \theta \geqq 1, \lambda \leqq 1\}. \quad (5)$$

Although it is a special case of (4), formulation (5) also contains many interesting special cases.

One important case is when setting $\lambda = 1$ in (5) to have a radial scaling in outputs only. We will use subscript "o" to identify the output-oriented nature and "PE_o" to denote the measure in this special case, given by

$$PE_o = \phi_o(x^0, q^0; w^0, p^0 | Z)$$
$$:= \sup_{\theta,x,q} \{\theta : p^0 \cdot (\theta q^0) - w^0 \cdot x^0 \leqq p^0 \cdot q - w^0 \cdot x, (x,q) \in \Im \cap Z\}. \quad (6)$$

This measure resembles the Farrell efficiency measure, except that it benchmarks not with respect to the technology, but with respect to the maximal profit and thus embraces the profit maximizing behavior as the benchmark. We will refer to PE_o as the *Farrell output-oriented measure of profit efficiency*. Noteworthy, simple derivations give another useful interpretation

$$PE_o = \sup_{\theta,x,q} \{\theta : \theta(p^0 \cdot q^0) - w^0 \cdot x^0 \leqq p^0 \cdot q - w^0 \cdot x, (x,q) \in \Im \cap Z\}$$
$$= \sup_{\theta,x,q} \{\theta : \theta R^0 - C^0 \leqq \pi(p^0, w^0 | Z)\}$$
$$= \frac{\pi(p^0, w^0 | Z) + C^0}{R^0} = 1 + \frac{\pi(p^0, w^0 | Z) - \pi^0}{R^0}. \quad (7)$$

Since $\pi(p^0, w^0 | Z) - \pi^0 \geqq 0$, we have $PE_o \geqq 1$. This measure tells us that, with prices (w^0, p^0) and input-output vector (x^0, q^0), the firm can raise profit level by $(PE_o - 1) \times 100\%$ of its observed total revenue, R^0, by eliminating profit inefficiency.

Similarly, if $\theta = 1$ in (5), we will use subscript "i" to identify the input-oriented nature of $\phi(.)$ in (5) and "PE_i" to denote the measure, which we call *Farrell input-oriented measure of profit efficiency.*

Similar to the derivation of (7), this measure also has interesting interpretation that can be seen from the following derivations:

$$\begin{aligned} PE_i &= \phi_i(x^0, q^0; w^0, p^0 | \, Z) \\ &:= \sup_{\lambda,x,q}\{2-\lambda: p^0 \cdot q^0 - w^0 \cdot (\lambda x^0) \leqq p^0 \cdot q - w^0 \cdot x, (x,q) \in \Im \cap Z\} \\ &= 2 - \inf_{\lambda,x,q}\{\lambda: R^0 - \lambda C^0 \leqq \pi(p^0, w^0|Z)\} \\ &= 1 + \frac{\pi(p^0, w^0|Z) - \pi^0}{C^0}. \end{aligned} \tag{8}$$

This measure indicates that, if profit inefficiency is eliminated, the firm can raise its profit level by $(PE_i - 1) \times 100\%$ of its observed total costs, C^0 (also note that $PE_i \geqq 1$).

Incorporating both output and input sides simultaneously is also possible. One example is to let $\lambda = 2 - \theta$ in (5). We will refer to the resulting function as the *Farrell jointly oriented measure of profit efficiency*, denoting it with "PE_{io}" and deriving it as

$$\begin{aligned} PE_{io} &= \phi_{io}(x^0, q^0; w^0, p^0 | \, Z) \\ &:= \sup_{\theta,x,q}\left\{\theta - (2-\theta) + 1: p^0 \cdot (\theta q^0) - w^0 \cdot \left((2-\theta)x^0\right) \leqq p^0 \cdot q - w^0 \cdot x, (x,q) \in \Im \cap Z\right\} \\ &= \sup_{\theta}\{2\theta - 1: \theta(R^0 + C^o) - 2C^0 \leqq \pi(p^0, w^0|Z)\} \\ &= \sup_{\theta}\left\{2\theta - 1: \theta \leqq \frac{\pi(p^0, w^0|Z) + 2C^o}{R^0 + C^o} = \frac{\pi(p^0, w^0|Z) - \pi^0}{R^0 + C^o} + 1\right\} \\ &= 1 + \frac{\pi(p^0, w^0|Z) - \pi^0}{(R^0 + C^o)/2}. \end{aligned} \tag{9}$$

If $(R^0 + C^o)/2$ is treated as a measure of the size of the firm, then the profit efficiency measure ϕ_{io} indicates that the firm can increase profit level by $(PE_{io} - 1) \times 100\%$ of the size of this firm after profit inefficiency is eliminated.

It is worth noting that the special cases described in (7), (8), and (9) can also be related to the directional technology measure of profit efficiency, introduced by Chambers, Chung and Färe (1998), which we restate here as

$$\phi_T^d(x^0, q^0; w^0, p^0; g_x, g_q) := \sup_{\beta,x,q}\{\beta: p^0 \cdot (q^0 + \beta g_q) - w^0 \cdot (x^0 - \beta g_x) \leqq \pi(p^0, w^0|Z)\}$$

$$= \frac{\pi(p^0, w^0) - \pi^0}{p^0 \cdot g_q + w^0 \cdot g_x}.$$

By setting $g_x = x^0 \; or \; 0$ and $g_q = q^0 \; or \; 0$, we get three equivalent cases:

$$PE_o = 1 + \phi_T^d(x^0, q^0; w^0, p^0; 0, q^0).$$

$$PE_i = 1 + \phi_T^d(x^0, q^0; w^0, p^0; x^0, 0).$$

$$PE_{io} = 1 + 2\phi_T^d(x^0, q^0; w^0, p^0; x^0, q^0).$$

We will come back to these relationships in Section 7. Meanwhile, before moving on, it is worth summarizing the key points made so far. First, we have introduced a new general *Farrell-type measure of profit efficiency*, stated in (5), which includes radial changes in outputs (see (6)) or inputs (see (8)) as special cases. Further, it also allows for simultaneous radial changes of inputs and outputs (see (9)). Each of these three measures ((6), (8), and (9)) are greater than or equal to one (and so their reciprocal is in (0,1]). If total revenue, total costs and the average of revenue and costs are regarded as three different indicators of firm size, then $PE_i, PE_o, and \; PE_{io}$ all have the interpretation that the potential increase in profit level as a percentage of the firm size.

Clearly, it is beyond the space permitted by any journal to consider all these (and other) cases in reasonable details in one paper and so from now on we will focus only on one of them—the output-oriented measure of Farrell profit efficiency. Our goal is to develop a general framework of profit efficiency measurement that encompasses existing Farrell output-oriented efficiency measures as special cases and as multiplicative components in the decomposition of the profit efficiency measure. Similar results for the other two measures (the *Farrell input-oriented measure of profit efficiency* and the *Farrell jointly oriented measure of profit efficiency*) are briefly summarized in Appendices 1 and 2.

3. The Farrell Output-oriented Measure of Profit Efficiency and its Properties

The Farrell output measure of profit efficiency can be restated in terms of the max-max (or sup-sup) principle—as a two-stage optimization strategy, first optimizing for profit and then optimizing for measuring the distance between the profit efficient allocation and the actual allocation. From (6),

$$PE_o = \phi_o(x^0, q^0; w^0, p^0 | \, Z)$$

$$= \sup_{\theta} \left\{ \sup_{x,q} \{ \theta : p^0 \cdot (\theta q^0) - w^0 \cdot (x^0) \leqq p^0 \cdot q - w^0 \cdot x, (x,q) \in \Im \cap Z \} \right\}$$

$$= \sup_{\theta} \left\{ \sup_{x,q} \left\{ \frac{p^0 \cdot q - w^0 \cdot x + w^0 \cdot x^0}{p^0 \cdot q^0} : (x,q) \in \Im \cap Z \right\} \geqq \theta \right\}. \qquad (10)$$

Intuitively, this measure expands observed outputs equi-proportionally or radially, in the spirit of the Farrell measurement, to raise the profit level to the maximal possible profit level under the reference technology. The constraint set Z (e.g., on revenue, cost, outputs and inputs) is included for the sake of generality, to encompass various practical cases where a firm is required to satisfy some conditions. In the following discussions, let θ^* be a solution in (10) (note that $(x^0, \theta^* q^0)$ may be an infeasible input-output vector).

A geometric illustration of the Farrell profit efficiency measure is in order. Consider Figure 1, where "*abcde*" denotes the production frontier and f_1 is the observed input-output vector (x^0, q^0). Profit maximization can be realized at point $c = (x^*, q^*)$. The output-oriented technical efficiency is measured by q^{te}/q. While pp' is the iso-profit line, one can hypothetically expand the output to raise profit until it coincides pp'. Then $q^{\pi}/q^0 = \theta^*$ measures the Farrell output-oriented profit efficiency, while the gap between technical efficiency and the profit efficiency can be regarded as the (profit) allocative efficiency, which is measured via q^{π}/q^{te} in Figure 1.[4]

[Insert Figure 1 here]

One should note the difference between identification of an efficient point and the computation of the efficiency measure. The explanation of Figure 1 may suggest that (x^0, q^{π}) is the profit efficient point. Indeed output level q^{π} is just for exposition and computation. The output-oriented profit efficiency is actually comparing the profit level at point *c* and the observed profit level at (x^0, q^0). Given the observed prices, the profit levels at point *c* and (x^0, q^{π}) are equal. The value of increasing the profit level as a proportion of the observed revenue is equal to the proportional expansion of output from q^0 to q^{π}. Both points c and (x^0, q^0) are technically feasible but the production activity (x^0, q^{π}) is not feasible.

How good is this measure? To answer such a question, various researchers have stated, explicitly or implicitly, various desirable properties of the profit efficiency measure, e.g., see Nahm and Vu (2013, p. 46), Asmild et al. (2007, p. 318), among others. To adapt them to our context, let Φ be a profit efficiency measure. Here we summarize several such desirable properties:

P.1 Φ is a well-defined function for all finite maximal and observed profit levels, and for all $(x^0, q^0) \in \Im$. When the maximum profit is a finite value, the value of Φ is also finite.

P.2 When the maximum profit is a finite value, there exists some real number ψ such that a firm is identified as profit efficient if and only if $\Phi = \psi$. Further, for any $(w^0, p^0) > 0$, there do not exist input-output bundles $(x', q') \in \Im$ and $(x", q") \in \Im$ such that

$$\Phi(x', q'; w^0, p^0) < \psi < \Phi(x", q"; w^0, p^0)$$

P.3 Φ is homogeneous of degree zero in input and output prices.

P.4 Φ is independent of units of measurement, i.e., commensurable.

The intuition behind these properties and their importance are self-explanatory. It also appears natural to accept them as axioms that one would expect any profit efficiency measure to satisfy. Our measure does satisfy all the above four properties. In particular, suppose the observed total revenue is positive and the maximum profit level is a finite number. I.e., $R^0 > 0$ and $\pi(p^0, w^0|Z) < +\infty$. From (7), we have

$$PE_o = 1 + \frac{\pi(p^0, w^0|Z) - \pi^0}{R^0}. \tag{11}$$

Since the denominator is positive and the difference between the maximum and observed profits is finite, the Farrell output-oriented profit efficiency measure is well-defined for all observed input and output quantities and prices. Further, in view of (11), it is clear that $\phi_o(x^0, q^0; w^0, p^0| Z) = 1$ if and only if $\pi(p^0, w^0|Z) - \pi^0 = 0$ which means profit efficient. In view of $PE_o \geqq 1$, P.2 is clearly satisfied for $\psi = 1$. Also note that from (7) we have $\phi_o(x^0, q^0; w^0, p^0| Z) = [\pi(p^0, w^0|Z) + w^0 x^0]/p^0 q^0$. Since the profit function is homogeneous of degree one in input and output prices, we have, for $t > 0$,

$$\phi_o(x^0, q^0; tw^0, tp^0| Z) = [\pi(tp^0, tw^0|Z) + tw^0 \cdot x^0]/tp^0 \cdot q^0$$
$$= [\pi(p^0, w^0|Z) + w^0 \cdot x^0]/p^0 \cdot q^0 = \phi_o(x^0, q^0; w^0, p^0| Z).$$

I.e., the Farrell output-oriented measure of profit efficiency is homogeneous of degree zero in input and output prices. Finally, let (x^*, q^*) be a profit-maximizing input-output vector. Then $\phi_o(x^0, q^0; w^0, p^0 | Z) = [(p^0 \cdot q^* - w^0 \cdot x^*) + w^0 \cdot x^0]/p^0 \cdot q^0$. Since after changes of units of measurement, prices and quantities will adjust so that $p_m q_m$ and $w_n x_n$ remain unchanged for all m and n, the commensurability of $\phi_o(x^0, q^0; w^0, p^0 | Z)$ follows. Hence the Farrell output-oriented profit efficiency measure satisfies all the four desirable properties P.1 to P.4.

In addition to the above properties for any profit efficiency measure, the output-oriented profit efficiency measure is homogeneous of degree *-1* in the observed output vector q^0. It is also continuous in input and output prices as well as in inputs and outputs.

4. Intuition of the Farrell Output-oriented Profit Efficiency Measure

To simplify further discussions and notations, here we will focus on the case where the maximum profit level exists at the input and output prices under consideration. Several interesting interpretations of the Farrell output-oriented profit efficiency measure can be found by rearranging its formula.

Firstly, from (7), we immediately get

$$PE_o = \frac{C^0}{R^0} + \frac{\pi(p^0, w^0 | Z)}{R^0}. \tag{12}$$

The interpretation of (12) is very useful because PE_o is expressed in terms of two factors: (i) the realized cost-revenue ratio, and (ii) the best possible profit margin for the firm. Both of these factors are well understood in business analysis by practitioners in business and investment community. Indeed, the cost-revenue ratio (also called efficiency ratio in finance and is also a version of cost-benefit ratio in business analysis) is among the most important key performance indicators (KPI) in business practice, as is the profit margin indicator.

Secondly, subtracting and adding R^0 to the numerator of (7), we get

$$PE_o = \frac{\pi(p^0, w^0 | Z) - \pi^0}{R^0} + 1. \tag{13 }$$

Thus, the Farrell output-oriented profit efficiency measure PE_o in the form of (13) tells about the unrealized (or lost) profit measured in percentage to actual total revenue or sales and so it can be understood by practitioners as a measure of 'lost profit margin' relative to sales.

Thirdly, let R^* and C^* be the profit-maximizing total revenue and total costs respectively (under prices (p^0, w^0), technology $\mathfrak{I}$ and constraints Z) and noting that $\pi(p^0, w^0|Z) = R^* - C^*$, one can rewrite (7) as follows:

$$PE_o = \frac{R^*}{R^0} + \frac{C^0 - C^*}{R^0} \tag{14}$$

This particular interpretation is also very useful because it shows decomposition into two important performance indicator: (i) ratio of the best possible revenue to the actual total revenue (sales) and (ii) excessive cost (beyond the possible minimum) measured as percentage relative to the actual sales. Note that R^*/R^0 might be larger, equal or small than unity, which can be interpreted respectively as 'undersell', 'efficient sale' and 'oversell', all measured relative to the profit maximizing efficiency criterion. Similarly, $(C^0 - C^*)/R^0$ might be larger, equal to or smaller than zero, which can be interpreted respectively as 'overspend', 'efficient cost' and 'underspend', all benchmarked with respect to the profit maximizing efficiency criterion.

5. Some Important Special Cases

Many important special cases can be derived from the proposed Farrell output-oriented profit efficiency measure in (6) by setting various restrictions to cater to particular situations that a firm may be facing in practice. We will present just a few here, which are perhaps the most interesting to general audience.

First of all, when the maximum profit level is zero (e.g., which can happen in the long-run competitive equilibrium), the profit efficiency measure in (6) will coincide with the very intuitive and frequently used in practice cost-revenue ratio (or the reciprocal to the 'return to dollar' efficiency measure), i.e., $PE_o = C^0/R^0$.

When we impose restrictions on quantities and prices, we find that the Farrell profit efficiency measure is closely related to other Farrell measures. Specifically, let the maximum revenue at (x^0, p^0) be $R(x^0, p^0) \coloneqq \max_q \{p^0 q: (x^0, q) \in \mathfrak{I}\}$ and recall the following output-oriented efficiency measures:

Farrell output-oriented measure of technical efficiency:

$$TE_o(x^0, q^0) \coloneqq \max_\theta \{\theta: (x^0, \theta q^0) \in \mathfrak{I}\}. \tag{15a}$$

Farrell revenue efficiency measure:

$$RE(x^0, q^0; p^0) := R(x^0, p^0)/p^0 \cdot q^0. \quad (15b)$$

Farrell output-oriented measure of allocative efficiency:

$$AE_o(x^0, q^0; p^0) := RE(x^0, q^0; p^0)/TE_o(x^0, q^0). \quad (15c)$$

Now, consider the constraint $x_{lb} = x_{ub} = x^0$ in Z (and for simplicity suppose there are no other constraints in Z). That is, the firm is required (e.g., in the short-run) to utilize the already available resources, no more and no less. Let $Z(x^0) = \{(x, q): (x, q) \in Z, x = x^0\}$. Then the output-oriented measure of profit efficiency coincides with the standard revenue efficiency measure, *i.e.*, from (10), we get

$$\begin{aligned}
&\phi_o\big(x^0, q^0; w^0, p^0 | Z(x^0)\big) \\
&= \sup_{\theta}\left\{\sup_{x,q}\left\{\frac{p^0 \cdot q - w^0 \cdot x^0 + w^0 \cdot x^0}{p^0 \cdot q^0} : (x, q) \in \Im \cap Z(x^0)\right\} \geqq \theta\right\} \\
&= \sup_{\theta}\left\{\sup_{x,q}\left\{\frac{p^0 \cdot q}{p^0 \cdot q^0} : (x^0, q) \in \Im\right\} \geqq \theta\right\} \\
&= \frac{R(x^0, p^0)}{p^0 \cdot q^0} := RE(x^0, q^0; p^0). \qquad (16)
\end{aligned}$$

for all $(x^0, q^0) \in \Im$ and $p^0 \cdot q^0 > 0$, where $R(x^0, p^0)$ is the maximum revenue at (x^0, p^0).

Second, in addition to $x_{lb} = x_{ub} = x_0$, consider the supremum of the constrained problem in which all restrictions must be satisfied for all positive output prices. Then, the right hand side of (6) becomes

$$\begin{aligned}
&\sup_{\theta,q}\{\theta: p \cdot (\theta q^0) - w^0 \cdot x^0 \leqq p \cdot q - w^0 \cdot x^0, (x, q) \in \Im \cap Z(x^0), p > 0\} \\
&= \sup_{\theta,q}\{\theta: p \cdot (\theta q^0) \leqq p \cdot q, (x^0, q) \in \Im \; for\ all\ p > 0\} \\
&= \sup_{\theta}\{\theta: p \cdot (\theta q^0) \leqq R(x^0, p), for\ all\ p > 0\}
\end{aligned}$$

Furthermore, for the case where the output set $P(x^0) = \{q: (x^0, q) \in \Im\}$ is convex, it is well-known from duality theory (see Färe and Primont (1995, p. 49) that

$$P(x^0) = \{q: p \cdot q \leqq R(x^0, p) \; for\ all\ p > 0\}.$$

Therefore, "$p \cdot (\theta q^0) \leqq R(x^0, p), for\ all\ p > 0$" can be replaced with $\theta q^0 \in P(x^0)$, i.e.,

$$\sup_{\theta,q}\{\theta: p\cdot(\theta q^0) - w^0\cdot x^0 \leqq p\cdot q - w^0\cdot x^0, (x,q)\in \Im \cap Z(x^0), p>0\}$$

$$= \sup_{\theta}\{\theta: \theta q^0 \in P(x^0)\}$$

$$= TE_o(x^0, q^0)$$

where $TE_o(x^0, q^0)$ is the *Farrell output-oriented measure of technical efficiency*. We also note that

$$TE_o(x^0, q^0) = \sup_{\theta}\{\theta: p\cdot(\theta q^0) \leqq R(x^0,p), for\ all\ p>0\}$$

$$= \sup_{\theta}\left\{\theta: \theta \leqq \frac{R(x^0,p)}{p\cdot q^0}\ \ for\ all\ p>0\right\}$$

$$= \inf_{p>0}\frac{R(x^0,p)}{p\cdot q^0}.$$

Suppose that this infimum is achieved at some p^s. Then $TE_o(x^0, q^0) = R(x^0, p^s)/p^s\cdot q^0$, implying

$$\phi_o\big(x^o, q^o; w^0, p^s | Z(x^0)\big) = TE_o(x^0, q^0).^5 \tag{17}$$

Therefore the Farrell output-oriented profit efficiency measure encompasses the Farrell output-oriented measures of revenue efficiency and of technical efficiency as special cases.[6] In particular, the Farrell output-oriented profit efficiency, revenue efficiency and Farrell output-oriented technical efficiency can be expressed in the form of equation (13). Hence all of them can be interpreted as the percentage of inefficiency losses in profit relative to the total revenue. Such consistent interpretation is very convenient for practitioners.

6. Decomposition of the Output –oriented Farrell Profit Efficiency Measure

The relationships in (16) and (17) provide a unified framework that unites the output-oriented profit efficiency with revenue efficiency and Farrell technical efficiency, as well as helps establishing various decompositions of the former. In view of the increasing restrictions imposed on the Farrell output-oriented profit efficiency measure in (16) and (17), we have

$$\phi_o(x^0, q^0; w^0, p^0 | Z) \geqq \phi_o\big(x^0, q^0; w^0, p^0 | Z(x^0)\big)$$
$$\geqq \phi_o\big(x^0, q^0; w^0, p^s | Z(x^0)\big). \tag{18}$$

Applying (6), (16) and (17) to (18), it follows that for all feasible input-output combinations (x^0, q^0) and for all input-output prices (w^0, p^0), we have

$$PE_o(x^0, q^0; w^0, p^0) \geqq RE(x^0, q^0; p^0) \geqq TE_o(x^0, q^0).^7 \tag{19}$$

Output-oriented profit allocative efficiency measure, $AE_o^\pi(x^0, q^0; w^0, p^0|Z)$, is defined in (20) to capture the loss of profits due to suboptimal input-output mix,

$$AE_o^\pi = AE_o^\pi(x^0, q^0; w^0, p^0|Z) := \frac{\phi_o(x^0, q^0; w^0, p^0|Z)}{\phi_o\big(x^0, q^0; w^0, p^s|Z(x^0)\big)} = \frac{PE_o}{TE_o}. \tag{20}$$

It is clear from (19) and (20) that $AE_o^\pi \geqq 1$. Further, $AE_o^\pi = 1$ if and only if $TE_o = PE_o$, i.e., no increase in the profit level is possible by changing input-output mix.

Thus, rewriting (20), we derive our first decomposition:

$$PE_o = AE_o^\pi \times TE_o. \tag{21}$$

Furthermore, define the *revenue efficient allocative efficiency*, $AE_o^{re}(x^o, q^o; w^0, p^0|Z)$, as

$$AE_o^{re} = AE_o^{re}(x^0, q^0; w^0, p^0|Z) := \frac{AE_o^\pi(x^0, q^0; w^0, p^0|Z)}{AE_o(x^0, q^0, p^0)}, \tag{22}$$

then the second decomposition follows from the first decomposition

$$PE_o = \frac{AE_o^\pi}{AE_o} \times AE_o \times TE_o = AE_o^{re} \times AE_o \times TE_o. \tag{23}$$

Meanwhile, combining (15c), (20) and (22), we get

$$AE_o^{re} = \frac{AE_o^\pi}{AE_o} = \frac{PE_o/TE_o}{RE/TE_o} = \frac{PE_o}{RE}. \tag{24}$$

Since $PE_o \geqq RE$ in (19), we have $AE_o^{re} \geqq 1$. Intuitively, AE_o^{re} shows the improvement of profits from the maximum revenue for a given input vector to maximum profits when all inputs and outputs are variable.

Decomposing a profit efficiency measure provides useful information to decision makers. Yet, the desirable properties of the decomposition of an efficiency measure have not been paid sufficient attention in the literature. A notable recent exception is the work of Aparicio, Pastor and Zofio (2017), who suggested a new property of decomposition, which is satisfied only by some existing measures of overall efficiency. Here, we generalize this useful property to profit efficiency and state it as the fifth desirable property:

P.5 Let $TE(x,q)$ be a measure of technical efficiency and $\Phi(x,q;w,p)$ be a measure of profit efficiency. Consider a technically inefficient input-output vector $a=(x^a,q^a)\in\Im$. Let $b=\left(x^b,q^b\right)\in\Im$ be a corresponding technically efficient input-output vector in the computation of $TE(x^a,q^a)$. If $\Phi(x,q;w,p)$ contains allocative efficiency as a component and AE^a and AE^b are the values of allocative efficiency of observations a and b respectively, then $AE^a=AE^b$.

We will refer to this property as the *invariance property of allocative efficiency measure*.

In words, property P. 5 postulates that, given input and output prices, the values of allocative efficiency for an observed technically inefficient point and the corresponding technically efficient reference point are equal. It follows that if a measure of technical efficiency projects two non-identical firms to the same technically efficient point, then these two firms should have the same allocative efficiency.

Importantly, we note that the Farrell output-oriented measure of profit efficiency satisfies this property. To see this, consider $(x^0,q^a)\in\Im$ and $TE_o(x^0,q^a)=\theta^{te}>1$. Let $q^b=\theta^{te}q^a$. Then $\left(x^0,q^b\right)\in\Im$ and $TE_o\left(x^0,q^b\right)=1$. From (20), we have

$$\begin{aligned}AE_o^{\pi}\left(x^0,q^b;w^0,p^0|Z\right)&=\frac{PE_o\left(x^0,q^b;w^0,p^0|Z\right)}{TE_o(x^0,q^b)}\\&=\frac{PE_o(x^0,\theta^{te}q^a;w^0,p^0|Z)}{TE_o(x^0,\theta^{te}q^a)}\\&=\frac{PE_o(x^0,q^a;w^0,p^0|Z)/\theta^*}{TE_o(x^0,q^a)/\theta^*}\\&=AE_o^{\pi}(x^0,q^a;w^0,p^0|Z),\end{aligned}$$

where the third inequality comes from the fact that both TE_o and PE_o are homogenous of degree –1 in outputs. Since Aparicio, Pastor and Zofio (2017) have proved that AE_o satisfies this property, it then follows that all components related to allocative efficiency, $AE_o^{\pi}, AE^{re}, and\ AE_o$, satisfy property P.5.

It is also worth noting here that decompositions of profit efficiency that appeared in the literature of efficiency analysis are typically additive. The only decomposition that included some multiplicative components that we are aware of was suggested by Aparicio, Pastor and Ray (2013) and even there, the decomposition of profit efficiency consists of both additive and multiplicative elements. Moreover, it is also important to emphasize that none of those decompositions explicitly included the Farrell technical and allocative efficiency measures as components.

Some authors did use Farrell measures in their decomposition. For examples, both Banker and Maindiratta (1988) and Ray (2004) identified the technically efficient input vector by the Farrell measure but they then used profit levels to define the technical efficiency. Importantly, the Farrell measure was not a component in their decomposition of profit efficiency. Both of them have not defined explicitly their profit efficiency measure as radial scaling of inputs and/or outputs. Our decomposition in (23) fills in this apparent gap in the literature: It provides decomposition of the profit efficiency that explicitly includes the Farrell technical and allocative efficiency measures as multiplicative components with meaningful interpretations for business and economic analysis, which we briefly discussed above.

7. Relationship to other measures

As we mentioned in the introduction, some roots to the framework of profit efficiency that we developed here can be found in various earlier works in different contexts. The most prominent and perhaps most closely related formula is found in Chavas and Cox (1994). Specifically, in their insightful paper, Chavas and Cox (1994) used the following formula

$$D_o(x^0, q^0) = \min_{\theta}\left\{\theta : \; p^0 \cdot \left(\frac{q^0}{\theta}\right) - w_0 \cdot x_0 \;\leqq\; \pi(p^0, w^0)\right\}. \tag{25}$$

After fixing some details (e.g., changing *min* to *inf*, re-defining the domain and the range of the function to circumvent peculiar cases, etc.), one can see that the $D_o(x^0, q^0)$ in (25) is the reciprocal of the Farrell output-oriented measure of profit efficiency in the unrestricted case. However, it is also important to note that the attention of Chavas and Cox (1994) was not on the profit efficiency measurements but on the outer bounds of the input and output distance functions. They have not explored the possibility of using this formula as a profit efficiency measure.[8]

Furthermore, Ray (2004, pp. 233 – 234) proposed the following profit efficiency measure

$$\phi_{Ray}(x^0, q^0; w^0, p^0) = (\pi(p^0, w^0) - \pi^0) / C^0.$$

Clearly, this measure is equivalent to the Farrell input-oriented measure of profit efficiency in the unrestricted case (i.e., we have $\phi_o = \phi_{Ray} + 1$). It is also worth noting that both Chambers, Chung and Färe (1998) and Ray (2004) treated the denominator in their formulation as an arbitrary normalization. In this paper, we explicitly show that such normalization can be justified by a certain orientation choice, as we did in (6). A suitable normalization can be used to relate the profit efficiency measure to the output-oriented Farrell measure of technical efficiency for decomposing the profit efficiency measure into various sources, as we did in the previous section.

It is also important to note that Ray (2004) mentioned normalizing the difference by total costs, which has been applied by Färe, Grosskopf and Weber (2004) and Das and Ghosh (2009).[9] It can be shown by similar reasoning and derivations as we have done above that normalization by total cost can be justified if one instead use the input oriented Farrell profit efficiency measure.

Another important model of profit efficiency measure is the directional technology measure of profit efficiency mentioned in Section 2. It is worth comparing it to the Farrell measures of profit efficiency we introduced in this paper. Let us first list some advantages of expressing efficiency in the forms like (7) to (9).

a. As shown in the previous section, the Farrell measures of profit efficiency contain multiplicative elements of other existing Farrell measures.
b. All profit efficiency measures are expressed in the same style. In particular, each measure is expressed as:

   efficiency measure = 1 + percentage increase in profits.

c. The measures PE_o, PE_i, and PE_{io} are derived from the Farrell-type general model (4). This general model contains other measures as special cases which are not implied by the directional technology measure of profit efficiency.

The directional technology measure of profit efficiency has its own merits, too. When the observed output vector and input vector are chosen as directions, it can be shown that properties P.1 to P.4 are satisfied. If the objective of the decision maker is known, then the directions can be chosen to reflect the values of the decision maker. However, the directional distance function approach has two challenges:

1. The direction must be chosen by the researcher. Currently, the most frequently adopted directions are the observed inputs and outputs. Although there are attempts to search for a direction (e.g., see Färe, . Pasurka and Vardanyan (2017), Färe, Grosskopf, and Whittaker (2013), among others), there is no ultimate or unambiguous rule for choosing the directions.
2. The magnitude of the directional vector is another issue. Once a direction is chosen, its scaling affects the numerical value of the measure. For example, from the point of view of efficiency ranking, the two directions (1, 2) and (1000, 2000) are equivalent, but the implied efficiency scores may be very different, e.g., 0.5 and 0.0005. Although this is not an issue in theoretical study, it sometimes causes difficulties of interpretation in empirical studies.
3. When the directions are not the output and input vectors, some desirable properties may not hold. For example, if $(g_q, g_x) = (1,1)$, it can be shown that the property of commensurability (P.4) is not satisfied.

Further, it can be shown that the decomposition of the directional technology measure of profit efficiency does not satisfy the invariance property of allocative efficiency measure P.5. To see this, let $\vec{D}_T(x^0, q^0; g_x, g_q) = \sup_\beta\{\beta: (x^0 - \beta g_x, q^0 + \beta g_q) \in \Im\}$ be the directional technology distance function which is used as an indicator of technical efficiency. Following Chambers, Chung and Färe (1998, p. 361), the corresponding allocative efficiency is given by

$$\overrightarrow{AE}_T(x^0, q^0; w^0, p^0; g_x, g_q) \coloneqq \phi_T^d(x^0, q^0; w^0, p^0; 0, q^0) - \vec{D}_T(x^0, q^0; g_x, g_q).$$

Choosing $(g_x, g_q) = (x^0, q^0)$ and consider $(x^a, q^a) \in \Im$ such that $\vec{D}_T(x^a, q^a; x^a, q^a) = \beta^a > 0$. Let $(x^b, q^b) = (x^a - \beta^a x^a, q^a + \beta^a q^a)$. Then $(x^b, q^b) \in \Im$ and $\vec{D}_T(x^b, q^b; x^b, q^b) = 0$. Both a and b face the same price vector (w^0, p^0). Let $\pi^j = p^0 q^j - w^0 x^j$ be the profit level of firm $j, j = a, b$. Since (x^b, q^b) is technically efficient,

$$
\begin{aligned}
&\overrightarrow{AE}_T\big(x^b, q^b; w^0, p^0; x^b, q^b\big) \\
&= \phi_T^d\big(x^b, q^b; w^0, p^0; x^b, q^b\big) - \vec{D}_T\big(x^b, q^b; x^b, q^b\big) \\
&= \phi_T^d\big(x^b, q^b; w^0, p^0; x^b, q^b\big).
\end{aligned}
$$

The directional allocative efficiency for a is

$$
\begin{aligned}
&\overrightarrow{AE}_T(x^a, q^a; w^0, p^0; x^a, q^a) \\
&= \phi_T^d(x^a, q^a; w^0, p^0; x^a, q^a) - \vec{D}_T(x^a, q^a; x^a, q^a) \\
&= \frac{\pi(p^0, w^0) - \pi^a}{p^0 q^a + w^0 x^a} - \beta^a \\
&= \frac{\pi(p^0, w^0) - \Big(p^0 \cdot \big(q^b - \beta^a q^a\big) - w^0 \cdot \big(x^b + \beta^a x^a\big)\Big) - \beta^a(p^0 \cdot q^a + w^0 \cdot x^a)}{p^0 \cdot q^a + w^0 \cdot x^a} \\
&= \frac{\pi(p^0, w^0) - \big(p^0 \cdot q^b - w^0 \cdot x^b\big)}{p^0 \cdot q^a + w^0 \cdot x^a} \\
&= \frac{\pi(p^0, w^0) - \pi^b}{p^0 \cdot q^b + w^0 \cdot x^b} \cdot \frac{p^0 \cdot q^b + w^0 \cdot x^b}{p^0 \cdot (q^a + \beta^a q^a) + w^0 \cdot (x^a - \beta^a x^a) - \beta^a(p^0 \cdot q^a - w^0 \cdot x^a)} \\
&= \phi_T^d\big(x^b, q^b; w^0, p^0; x^b, q^b\big) \frac{p^0 \cdot q^b + w^0 \cdot x^b}{p^0 \cdot q^b + w^0 \cdot x^b - \beta^a \pi^a} \\
&\neq \overrightarrow{AE}_T\big(x^b, q^b; w^0, p^0; x^b, q^b\big) \quad for \;\; \beta^a \pi^a \neq 0.
\end{aligned}
$$

Hence $\overrightarrow{AE}_T(x^a, q^a; w^0, p^0; x^a, q^a)$ depends on the technical efficiency and initial profit level, and so the invariance property of allocative efficiency measure P.5 is not satisfied for the allocative efficiency measure based on the directional distance function.

The similarities between the Farrell-type measures of profit efficiency and the measures from the directional distance function highlight the difference between identification of an efficient point and the formula for computing the value of the efficiency measure. The directional distance function approach needs to select a certain directional vector. Our Farrell-type measures keep input mix and output mix fixed and focus on radial scaling. In terms of final results, the main difference is on the formulae of the efficiency measure adopted. In particular, the three measures discussed in this paper can be expressed in similar formulae and the potential profit gains can be interpreted as a percentage of the observed "firm size." We also note that the normalization "$p^0 \cdot q^0 + w^0 \cdot x^0$" was criticized for involving double counting by Cherchye, Kuosmanen and Leleu (2010) although such criticism is less relevant if our purpose is related to ranking only. All in all, if one seeks for meaningful interpretation, then the measure like PE_{io} is a good candidate as it has an appealing economic intuition.

Finally, since Farrell measures of technical efficiency are widely adopted in empirical studies, then it is desirable to also have a measure of profit efficiency that contains such technical efficiency as a component. As mentioned above, an important value of the new profit efficiency measures is that they can contain the very popular (both in theory and practice) Farrell technical and allocative efficiency measures as multiplicative components.

8. A Numerical Illustration

The theoretical developments we presented above are general rather than specifically related to a particular estimation approach. In this section we present a numerical illustration using a hypothetical two-input-two-output production technology. When data are available, computing profit efficiency by various approaches, such as the data envelopment analysis (DEA) approach, stochastic frontier analysis approach, etc., can also be used.

Specifically, consider the following production technology:

$$\mathfrak{I} \coloneqq \{(x_1, x_2; q_1, q_2) : (0.5q_1^2 + 0.5q_2^2)^{0.5} \leqq 15x_1x_2 - (x_1x_2)^{1.5} \;\; for \;\; x_1x_2 \leqq 100;$$
$$q_1 = q_2 = 0 \quad for \;\; x_1x_2 > 100\}$$

This technology satisfies strong disposability in outputs and inputs, and has convex output sets and input sets. For comparison, we include the computations of several selected measures of profit efficiency. They are listed below:

1. Following Chambers, Chung and Färe (1998), let $(g_x, g_q) = (x^0, q^0)$, then

$$\phi_{ccf} = \frac{\pi(x^0, q^0; w^0, p^0) - \pi^0}{p^0 g_q + p^0 g_x}.$$

2. Following Coopers, Pastor, Aparicio, and Borras (2011), let $v_n^x = \frac{1}{x_n^0}, n = 1, \cdots, N$ and $v_m^q = \frac{1}{q_m^0}, m = 1, \cdots, M,$ then

$$\phi_{cpab} = \frac{\pi(x^0, q^0; w^0, p^0) - \pi^0}{\min\left\{\frac{w_1^0}{v_1^x}, \cdots, \frac{w_N^0}{v_N^x}; \frac{p_1^0}{v_1^q}, \cdots, \frac{p_N^0}{v_M^q}\right\}}$$

where v_n^x and v_m^q are the weights for input *n* and output *m* respectively.

3. Following Portela and Thanassoulis (2005), let (x^*, q^*) be the profit-maximizing input-output bundle at prices (p^0, w^0), then

$$\phi_{pt} = \left(\prod_{n=1}^{N} \frac{x_n^*}{x_n^0}\right)^{\frac{1}{N}} \Bigg/ \left(\prod_{m=1}^{M} \frac{q_m^*}{q_m^0}\right)^{\frac{1}{M}}$$

4. Following Färe, Grosskopf and Lovell (1985), we have

$$\phi_{fgl} = \inf_{\lambda}\left\{\lambda: p\frac{y^0}{\lambda} - w(\lambda x^0) \leqq \pi^*\right\}$$

5. Following Banker and Maindiratta (1988), we have

$$\phi_{bm} = \inf_{x,q}\left\{\frac{\pi^0}{p^0 q - w^0 x} : (x, q) \in \Im, p^0 q - w^0 x > 0\right\}.$$

A hypothetical data set is listed in Table 1. The numerical values of various measures of profit efficiency are presented in Table 2. The results of PE_o and PE_{io} are similar. In particular, Firms 1 and 7 are profit efficient. Firm 6 is nearly profit efficient. The other firms are very inefficient in the sense that the observed profit level can be raised by more than 20% of the observed revenue.

[Insert Table 1 here]

[Insert Table 2 here]

The ranking of PE_o, PE_{io}, ϕ_{ccf} and ϕ_{fgl} are consistent with each other. The following particular cases draw our attentions:

a. The measures ϕ_{cpab} and ϕ_{pt} include the observed quantities as denominator. The observed quantity of output 1 is zero for Firm 9. These two measures are undefined for this firm.

b. The measure ϕ_{bm} is defined for positive profit levels only. Zero or negative profits appear in Firms 5, 7, 8 and 9. So this measure is not defined for these firms.

c. The range of ϕ_{pt} should be between 0 and 1 but its value is larger than 1 for Firm 6.

Table 2 shows that some properties of P.1 to P.5 may not be satisfied by some measures of profit efficiency. Appendix 3 includes a comparison of the measures listed in this section.

Next we compute the decomposition of the output-oriented profit efficiency only. From Table 3, the advantages of our new decomposition are clear. First, conventional Farrell technical efficiency and allocative efficiency are included as components. As they are used most frequently in empirical studies, Equation (23) explicitly link the profit efficiency to existing Farrell measures. Second, changing input scale or input mix are important management decisions. The measure of revenue efficient allocative efficiency indicates the benefits of such changes. Consider Firm 8, which is almost revenue efficient. If we use the output prices only, we will identify this firm as revenue efficient and nothing need to be done. However, with $AE_o^{re} = 1.393$, we see that this firm can increase its profit level by 39.3% of the observed revenue through adjusting the scale or/and mix of inputs. On the other hand, although Firm 4 is also very profit inefficient with $PE_o = 1.296$, input scale and mix can kept unchanged because $AE_o^{re} = 1$.

[Insert Table 3 here]

Now we note that the output-oriented measures AE_o and TE_o are related to changing outputs only whereas the potential improvement indicated by AE_o^{re} can be realized by changing the input scale and the input mix only. Since $RE = TE_o \times AE_o$ we can drive meaningful implications from comparing AE_o^{re} and RE. In particular, it can be seen from Table 3 that firms 2 – 5, 8 and 9 are very profit inefficient. Upon observing AE_o^{re} and RE, we can conclude on three possible patterns for improvement:

a. Output-oriented changes are sufficient (Firms 2 – 4)
b. Improvement can be achieved through changing input mix and input scale only (Firm 8)
c. Both output-oriented and input-oriented changes must be done to eliminate all profit inefficiency (Firm 9)

9. Conclusion

By synthesizing various approaches in the literature, this paper presents a cohesive framework of profit efficiency measurement. We start with a general model of Farrell-type measure of profit efficiency, and then consider several special cases of particular interest: the output-oriented, input-oriented, and jointly oriented measure of profit efficiency. They all are larger than or equal to unity and indicate full (100%)

profit efficiency when equal to unity. These measures have intuitive interpretation that the observed profit level can be raised by a certain percentage.

We discussed the Farrell output-oriented measure of profit efficiency in more details and show that this measure satisfies some desirable properties discussed by other researchers in the literature. Importantly, this measure includes Farrell output-oriented measure of technical efficiency, and revenue efficiency as special cases. To the best of our knowledge, this is the first time in the literature that a profit efficiency measure contains Farrell technical efficiency and allocative efficiency explicitly as multiplicative components.

We also pointed out that the three special cases of the new profit efficiency measures are related to the directional distance function approach in the sense that they identify the same efficient point for the same direction. The formula of the new measures of profit efficiency has the advantages of convenient interpretations and, for the output-oriented measure, satisfying the invariance property for allocative efficiency measures, inspired by Aparicio, Pastor and Zofo (2017).

We also illustrate our approach with a hypothetical example. In this example, when a firm is not profit efficient, the measure of revenue efficient allocative efficiency introduced in this paper clearly indicated whether the input scale or its mix must be changed to increase the firm's profit level. Noteworthy, such important insight on performance cannot be obtained from other existing decompositions of profit efficiency.

The theoretical results of this paper are general and various estimation methods can be used to implement it in practice, mote notable the DEA and the SFA. Our approach can also be adapted to other variants, such as the IDEA (Cooper, Park and Yu (2001) and Zhu (2004)), CAR-DEA (Cook and Zhu (2008)), DEA with non-homogeneous firms (Cook et al. (2013)), and based on DEA nonparametric models of optimizing behavior (Cherchye et al. (2008)), to mention just a few.

Notes

1. This ensures that input-oriented and output-oriented efficiency measures are well-defined. No other assumptions are needed for general discussions of profit efficiency. One can assume additional regularity conditions (for example, see Färe and Primont (1995), Table 2.1, p. 27).
2. Similar constraint sets have been adopted in the measurement of technical efficiency. E.g., when there are some nondiscretionary input variables (Kopp 1981, and Banker and Morey 1986), the lower and upper bounds for some inputs are equal. If each input n and output m is associated with a 'weight' w_n or p_m, the assurance region method introduced by Thompson et al. (1986) imposes constraints like $\alpha \leqq w_i/w_j \leqq \beta$ where α and β are two positive numbers.
3. It is worth noting that this special case of the general profit efficiency measure can give negative values (since the observed profit can be negative). Moreover, when the maximal profit is zero (which is an important case that embraces perfectly competitive equilibrium), then this measure will give zero regardless of the observed profit, which can be viewed as a limitation.
4. It is also worth reminding again that without some upper bounds on x, q, $w \cdot x$ or $p \cdot q$, there will be no maximum profit level if technology exhibits IRS and so, as can be shown, $PE_o = +\infty$. Similar situation may also occur under CRS. In a sense, the upper constraints we have in the definition help regularizing the profit function and, in turn, the related Farrell profit efficiency measure for such cases, besides helping to tailor these concepts closer to the reality.
5. Färe and Primont (1995, p. 129) has a similar expression, stating that $1/D_o(x^0, q^0) = \sup_{\theta}\{\theta : p \cdot (\theta q^0) - w \cdot x^0 \leqq \pi(p, w), (p, w) \geq 0\}$. Their derivation requires convex technology set. In our formulation, convex technology set is not needed but inputs are fixed in finding the maximum profits and the shadow price vector p^s. When the technology set is not convex, it is possible for the left-hand side in the above equation smaller than the right-

hand side but the equality of (17) still holds.

6. Note that the Farrell output-oriented measure of technical efficiency is not a special case of the directional measure of technical efficiency. Instead, they are equivalent measures in the sense that the rankings implied by both measure are identical.

7. The left- and right-hand side of (19) can be rewritten as $(\pi(p^0, w^0) + w^0 \cdot x^0)/(p^0 \cdot q^0) \geqq 1/D_o(x^0, q^0)$ where $D_o(x^0, q^0)$ is the output distance function. Färe and Primont (1995, p. 131) have mentioned this inequality from duality relation. Färe and Grosskopf (2004, p. 39) have stated the same expression through the relation between the profit function and the directional output distance function.

8. In a related work, Asmild et al. (2007, Theorem 4, p. 316) showed that the same formula is the limit of the so-called 'linked cone model' that they discussed.

9. Aparicio, Pastor, and Ray (2013) also proposed another normalization - using the average total costs of all firms.

10. The profit-maximizing quantities are $x_1^* = x_2^* = 9$ and $q_1^* = q_2^* = 486$ for $w_1 = w_2 = 27$ and $p_1 = p_2 = 1$; and $x_1^* = x_2^* = 7.5$ and $q_1^* = q_2^* = 421.875$ for $w_1 = w_2 = 56.25$ and $p_1 = p_2 = 1$.

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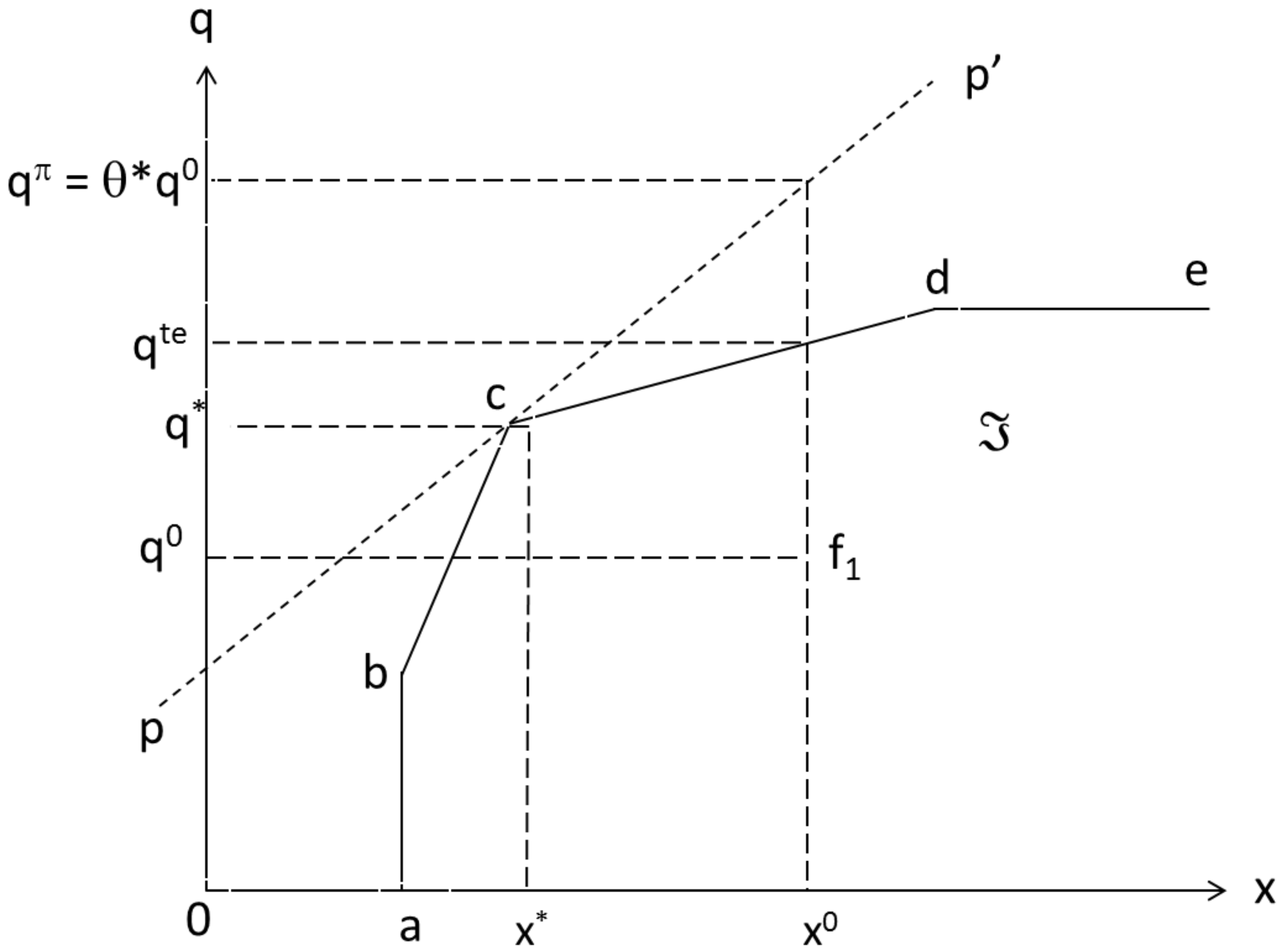

Figure 1. Illustration of the output-oriented Farrell profit efficiency measure

Table 1: A hypothetical data set

	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5	Firm 6	Firm 7	Firm 8	Firm 9
x_1^0	9	8	9	9	10	7	7.5	4	3
x_2^0	9	10	10	9	15	10	7.5	20	5
q_1^0	486	200	400	300	200	440	421.88	450	0
q_2^0	486	600	400	450	400	480	421.88	500	235
w_1^0	27	27	27	27	27	27	56.25	56.25	56.25
w_2^0	27	27	27	27	27	27	56.25	56.25	56.25
p_1^0	1	1	1	1	1	1	1	1	1
p_2^0	1	1	1	1	1	1	1	1	1
π^0	486	314	287	264	-75	461	0	-400	-215
π^*	486	486	486	486	486	486	0	0	0

Table 2: Various Measures of Profit Efficiency[10]

	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5	Firm 6	Firm 7	Firm 8	Firm 9
C^0	486	486	513	486	675	459	843.75	1350	450
R^0	972	800	800	750	600	920	843.75	950	235
π^0	486	314	287	264	-75	461	0	-400	-215
π^*	486	486	486	486	486	486	0	0	0
$\Phi \geqq 1$ (larger values mean more inefficient)									
PE_o	1	1.215	1.249	1.296	1.935	1.027	1	1.421	1.915
PE_{io}	1	1.268	1.303	1.359	1.880	1.036	1	1.348	1.628
$\Phi \geqq 0$ (larger values mean more inefficient)									
ϕ_{ccf}	0	0.134	0.152	0.180	0.440	0.018	0	0.174	0.314
ϕ_{cpab}	0	0.860	0.819	0.914	2.805	0.132	0	1.778	undef
$0 \leqq \Phi \leqq 1$ (smaller values mean more inefficient)									
ϕ_{pt}	1	0.717	0.781	0.756	0.428	1.017	1	0.943	undef
ϕ_{fgl}	1	0.877	0.862	0.839	0.649	0.982	1	0.839	0.723
ϕ_{bm}	1	0.646	0.591	0.543	undef	0.949	undef	undef	undef

Table 3: Decomposition of the Farrell Output-oriented Measure of Profit Efficiency

	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5	Firm 6	Firm 7	Firm 8	Firm 9
PE_o	1	1.215	1.249	1.296	1.935	1.027	1	1.421	1.915
AE_o^{re}	1	1.003	1.007	1	1.406	1.018	1	1.393	1.348
AE_o	1	1.118	1	1.020	1.054	1.001	1	1.001	1.414
TE_o	1	1.083	1.241	1.271	1.306	1.009	1	1.019	1.004
RE	1	1.211	1.241	1.296	1.376	1.009	1	1.020	1.421

Appendix 1: Farrell input-oriented measure of profit efficiency

Input-oriented measures are listed below. Note that the measures of technical efficiency and allocative efficiency are different from the original Farrell measures.

A.1.1 Farrell input-oriented measure of profit efficiency

$$PE_i = \phi_i(x^0, q^0; w^0, p^0 | Z)$$

$$= \sup_{\lambda,x,q}\{2-\lambda: p^0 \cdot (q^0) - w^0 \cdot (\lambda x^0) \leqq p^0 \cdot q - w^0 \cdot x, (x,q) \in \Im \cap Z\}$$

$$= 1 + \frac{\pi(p^0, w^0 | Z) - \pi^0}{C^0}.$$

A1.2 Farrell input-oriented measure of technical efficiency

Let costs be minimized at input price vector w^s for $x = x^{te}$ when outputs are fixed at q^0, where x^{te} is the technically efficient input vector. The Farrell-type input-oriented measure of technical efficiency in the profit efficiency framework is

$$TE_i = \sup_{\lambda}\{2-\lambda: (\lambda x^0, q^0) \in \Im \cap Z\}$$

$$= 2 - \inf_{w>0} \frac{C(q^0, w)}{w \cdot x^0}$$

$$= \phi_i\big(x^0, q^0; w^s, p^0 \big| Z(q = q^0)\big)$$

$$= 1 + \frac{\pi(p^0, w^s | Z(q = q^0)) - \pi^0}{C^0}.$$

Farrell's original formulation of input-oriented measure of technical efficiency is $2 - TE_i \leqq 1$.

A1.3 Cost efficiency:

$$CE = \sup_{\lambda,q}\{2-\lambda: w^0 \cdot (\lambda x^0) \geqq w^0 \cdot x, (x, q^0) \in \Im \cap Z\}$$

$$= \phi_i\big(x^0, q^0; w^0, p^0 \big| Z(q = q^0)\big)$$

$$= 1 + \frac{\pi(p^0, w^0 | Z(q = q^0)) - \pi^0}{C^0}.$$

Farrell's original formulation of cost efficiency is $2 - CE \leqq 1$.

A1.4 Farrell input-oriented measure of allocative efficiency:

$$AE_i = \frac{CE}{TE_i}.$$

Farrell's original formulation of allocative efficiency is $(2 - CE)/(2 - TE_i) \leqq 1$.

A1.5 Output-oriented profit measure of allocative efficiency:

$$AE_i^{\pi} = \frac{PE_i}{TE_i}.$$

A1.6 Cost-efficient measure of allocative efficiency

$$AE^{ce} = \frac{PE_i}{CE}.$$

A1.7 Decomposition

$$PE_i = AE^{ce} \times AE_i \times TE_i$$

Appendix 2: Farrell jointly-oriented measure of profit efficiency

A2.1 Farrell jointly-oriented measure of profit efficiency

$$PE_{io} = \phi_{io}(x^0, q^0; w^0, p^0 | Z)$$

$$= \sup_{\beta,x,q}\{2\beta - 1 : p^0 \cdot (\beta q^0) - w^0 \cdot \left((2-\beta)x^0\right) \leqq p^0 \cdot q - w^0 \cdot x, (x,q) \in \Im \cap Z\}$$

$$= 1 + \frac{\pi(p^0, w^0|Z) - \pi^0}{(R^0 + C^o)/2}.$$

A2.2 Farrell jointly-oriented measure of technical efficiency:

Let profits be maximized at price vector (p^s, w^s) for $(x,q) = (x^{te}, q^{te})$ where (x^{te}, q^{te}) is the technically efficient input-output vector. The Farrell-type jointly oriented measure of technical efficiency in the profit efficiency framework is

$$TE_{io} = \sup_{\beta}\{\beta : \left((2-\beta)x^0, \beta q^0\right) \in \Im\}$$

$$= \phi_i(x^0, q^0; w^s, p^s | Z)$$

$$= 1 + \frac{\pi(p^s, w^s|Z) - (p^s \cdot q^0 - w^s \cdot x^0)}{(p^s \cdot q^0 + w^s \cdot x^0)/2}.$$

A2.3 Jointly-oriented measure of allocative efficiency:

$$AE_{io} = \frac{PE_{io}}{TE_{io}}.$$

A2.4 Decomposition:

$$PE_{io} = AE_{io} \cdot TE_{io}.$$

Appendix 3: Comparing different measures of profit efficiency

The following table indicates the properties listed in the paper satisfied by the various measures of profit efficiency.

Table A.1: Properties of various measures of profit efficiency

	PE_o	ϕ_{fgl}	ϕ_{bm}	ϕ_{ccf}	ϕ_{pt}	ϕ_{cpab}
P.1	✓	✓	✗	✓	✗	✗
P.2	✓	✓	✓	✓	✗	✓
P.3	✓	✓	✓	✓	✓	✓
P.4	✓	✓	✓	✓*	✓	✓
P.5	✓	✗	✓	✗	✗	✗

* This property may not hold when the directions are not output and input vectors.

Farrell output-oriented profit efficiency measure:

$$PE_o := \underset{\theta,x,q}{sup}\{\theta: p^0 \cdot (\theta q^0) - w^0 \cdot (x^0) \leqq p^0 \cdot q - w^0 \cdot x, (x,q) \in \mathfrak{I} \cap Z\}.$$

Färe, Grosskopf and Lovell (1985)

$$\phi_{fgl} = \inf_{\lambda}\left\{\lambda: p \cdot \frac{y^0}{\lambda} - w \cdot (\lambda x^0) \leqq \pi^*\right\}$$

Banker and Maindiratta. (1988)

$$\phi_{bm} = \inf_{x,q}\left\{\frac{\pi^0}{p^0 \cdot q - w^0 \cdot x} : (x,q) \in \mathfrak{I}, p^0 \cdot q - w^0 \cdot x > 0\right\}.$$

Chambers, Chung and Färe (1998)

$$\phi_{ccf} = \frac{\pi(x^0, q^0; w^0, p^0) - \pi^0}{p^0 \cdot g_q + p^0 \cdot g_x}.$$

Portela and Thanassoulis (2005)

$$\phi_{pt} = \left(\prod_{n=1}^{N} \frac{x_n^*}{x_n^0}\right)^{\frac{1}{N}} \Bigg/ \left(\prod_{m=1}^{M} \frac{q_m^*}{q_m^0}\right)^{\frac{1}{M}}$$

Coopers, Pastor, Aparicio, and Borras (2011)

$$\phi_{cpab} = \frac{\pi(x^0, q^0; w^0, p^0) - \pi^0}{\min\left\{\frac{w_1^0}{v_1^x}, \cdots, \frac{w_N^0}{v_N^x}; \frac{p_1^0}{v_1^q}, \cdots, \frac{p_N^0}{v_M^q}\right\}}$$