Endogenous Business Cycles in the Overlapping Generations Market Game Model

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Abstract

This paper examines an overlapping generations version of the Shapley-Shubik market game model with production. We show the existence of equilibria for the model and study the price dynamics generated in the model through strategic interactions. Conditions for the existence of complex and chaotic equilibrium dynamics are characterized, which are more complicated than in the overlapping generations market game model without production. Our analysis shows that production combined with price-taking behavior by households locks down the ratio of output and input prices, which then reduces the nonlinearity that arises in the pure exchange model.

1 Introduction

In this paper, we study whether strategic interactions contribute to instabilities of the economic dynamics in the overlapping generations (OLG) version of the Shapley-Shubik market game model with production (see, for example, Dubey and Shubik [4], or Chen et al.[3]).

The study of complex dynamics in economic models has focused historically on the question of whether modern capitalist economies are inherently stable or unstable. The argument for instability is based on the fact of business cycles, which first appear historically with the onset of industrialization in the early 1800's in the West. The argument for stability first appears in early real business cycle models where exogenous aggregate shocks generate deviations from an otherwise stable steady-state of the model.

Grandmont [6] was one of the first papers to raise the possibility that endogenous complex dynamics might provide an alternative explanation for business cycle fluctuations, by showing that such dynamics could arise in conventional OLG models, although only for the case of sufficiently large risk aversion on the part of old agents in the model. Shortly after Grandmont's work appeared, Boldrin and Montrucchio [2] showed that complex dynamics could also arise in the neoclassical capital model. In both of these modeling approaches, however, the parameter values required for chaotic trajectories to arise were unrealistic. For OLG models, Grandmont required relative risk aversion coefficients in excess of 8, while in the capital model, discount factors were required to be less than 0.35. Neither of these assumptions seemed at all realistic.

Goenka et al. [5] showed in the context of a pure exchange OLG market game that the nonlinearities introduced by imperfect competition were such that one could obtain chaotic dynamics even for log utility, as long as markets were thin in terms of the amount of endowment agents offered.

Goenka et al. note that extensions of their work with this kind of model suggests that production smooths the model in the sense that complex dynamics are not as easily generated as in the pure exchange model. In this paper, we show that Goenka et. al.'s observation is true.

Specifically, we will show in the paper that when incorporating production in the market game OLG model, the price dynamics depend on market thickness, general equilibrium price ratios, individual offers and particular choices of utility functions. Unlike standard CRRA utility functions assumed in Grandmont [6] and Goenka et.al.[5], for complex dynamics to occur, the preferences in our model must be a mix of preferences, for example, a combination of preferences with constant relative risk aversions and increasing relative risk aversions. Endowment assumptions and market thinness alone cannot ensure the existence of complicated dynamics. In our paper, we also show impossibility of such price dynamics to occur for log-linear preferences. In other words, the case for complex dynamics to occur with particular production functions and utility functions is much more limited. As a result, complex dynamics are not as easily observable as in models without production.

Finally, we are able to confirm the results from Goenka, et. al. on the Pareto rankability of Nash equilibria in terms of market thickness, which has important welfare implications for business cycle-like activity based on the coordination equilibria that can arise in market game models (again, see Chen et.al. [3] for details on these equilibria).

The remainder of the paper is organized as follows. Section 2 specifies the model and market equilibrium. Section 3 displays the dynamics analysis. Section 4 studies the the special case when the preferences are log-linear. Section 5 concludes. Some proofs are contained in the appendices.

2 The model

2.1 Agents

We consider a market game model of m firms with a single type of input goods and a single type of output goods. In an OLG setting, in each period there are n young agents and n old agents. The old own the firms that produce the same type of consumption goods. The young agents are endowed with labor and offer an exogenously fixed amount of labor. In each period, the young agents offer labor, make bids on consumption goods and purchase share ownerships of the two firms. The old would purchase consumption goods funded by selling their shares of two firms and profits of two firms.

Assume consumer i born time t, would offer labor $l_{i,t}$ when young. At time $s = t, t+1$, consumer i's bid for consumption goods is denoted $b^t_{i,s}$. The share of firm k purchased by consumer i is $a^k_{i,t}$ and

$$\sum_{i=1}^{n} a^k_{i,t} = 1. \tag{1}$$

Let the asset price for firm k be $q_{k,t}$, $k = 1, 2, ..., m$.

The aggregate bid on output good is the sum of bids from young agents and bids from old agents, which is

$$B_t = \sum_{i=1}^{n} (b^t_{i,t} + b^{t-1}_{i,t}). \tag{2}$$

Let the aggregate output at time t be Q_t. The price of output good at time t is

$$p_t = \frac{B_t}{Q_t}. \tag{3}$$

The consumption of consumer i at time $s = t, t+1$ is

$$x^t_{i,s} = \frac{b^t_{i,s}}{p_s} = \frac{b^t_{i,s} \cdot Q_s}{B_s}. \tag{4}$$

The aggregate input good is

$$L_t = \sum_{i=1}^{n} l_{i,t}.$$

Let firm k's bid for input good be $w_{k,t}$. The aggregate bid on input good is

$$W_t = \sum_{j=1}^{m} w_{j,t}. \tag{5}$$

The input good price is thus given by

$$r_t = \frac{W_t}{L_t}. \tag{6}$$

Let the production function for firm k be $f_k(x)$ with input $L_{k,t}$, $k = 1, 2$. The profit function for firm k at time t is given by

$$\pi_{k,t} = p_t f_k(L_{k,t}) - r_t L_{k,t}, \qquad k = 1, 2.$$

2.2 Strategic interactions

2.2.1 Consumers side

At time t, consumer i uses labor income to purchase consumption goods and share ownerships of two firms. At time $t+1$, consumer i sells his share ownerships of two firms, and his consumption at time $t+1$ is funded by the time $t+1$ profits of the shares of the firms he owns and selling his shares of two firms.

We also make the following assumptions on consumers' utility function

Assumption 2.1. We assume

1. Utility is additively separable: $u(x_t^t, x_t^{t+1}) = U(x_t^t) + V(x_{t+1}^t)$.
2. U and V are smooth strictly increasing, strictly concave and satisfy Inada conditions.

Consumer i's utility maximization problem is thus given as follows:

$$\max_{\{x_{i,t}^t,\ x_{i,t+1}^t,\ a_{i,t}^1,\dots,\ a_{i,t}^m\}} U(x_{i,t}^t) + V(x_{i,t+1}^t)$$

subject to

$$p_t x_{i,t}^t + \sum_{j=1}^{m} a_{i,t}^j q_{j,t} = r_t l_{i,t} \tag{7}$$

and

$$p_{t+1} x_{i,t+1}^t = \sum_{j=1}^{m} (q_{j,t+1} + \pi_{j,t+1}) a_{i,t}^j. \tag{8}$$

In our model, consumers are assumed to be perfectly competitive, i.e. they take consumption good price p_t and asset prices $q_{k,t},\ k = 1, 2, ..., m$ as given. The input good price r_t is not affected by their optimization behavior, therefore it is taken as given, too. Consumption good and input good prices taken as given, the consumers take firms' profits $\pi_{k,t+1},\ k = 1, 2$ as given too.

2.2.2 Firms side

At time t, firms will take the aggregate offer of labor L_t as given but will take account of the other firm's bid for labor. Firm i's best response to the other firm's action is determined by the solution to the optimization problem as follows:

$$\max_{w_{i,t}} \frac{B_t}{Q_t} f_i(L_{i,t}) - w_{i,t}$$

subject to

$$w_{i,t} \leq \frac{B_t}{Q_t} f_i(L_{i,t})$$

$$L_{i,t} = w_{i,t} \frac{L_t}{W_t}, \quad i = 1, 2, ..., m$$

The constraint just implies that the net profit should be nonnegative. The profits of two firms at time $t+1$ are given respectively as follows

$$\pi_{1,t+1} = p_{t+1} f_1(\frac{w_{1,t+1}}{W_{t+1}} L_{t+1}) - w_{1,t+1}$$
$$\pi_{2,t+1} = p_{t+1} f_2(\frac{w_{2,t+1}}{W_{t+1}} L_{t+1}) - w_{2,t+1}.$$

2.3 Best responses

2.3.1 Consumers

Replacing $x_{i,t}^t$ and $x_{i,t+1}^t$ in terms of $a_{i,t}^k,\ k = 1, 2$, the two-period utility function can be written as

$$U(\frac{r_t l_{i,t} - \sum_{j=1}^m a_{i,t}^j q_{j,t}}{p_t}) + V(\frac{\sum_{j=1}^m (q_{j,t+1} + \pi_{j,t+1}) a_{i,t}^j}{p_{t+1}}).$$

The best response functions for consumer i are solved from the following F.O.Cs:
F.O.Cs for $a_{i,t}^j$, $j = 1, 2, ..., m$:

$$\frac{U'(x_{i,t}^t)}{V'(x_{i,t+1}^t)} = \frac{q_{j,t+1} + \pi_{j,t+1}}{q_{j,t}} \cdot \frac{p_t}{p_{t+1}}. \tag{9}$$

According to (9), for $\forall j, l = 1, 2, ..., m$, we also have

$$\frac{q_{j,t+1} + \pi_{j,t+1}}{q_{j,t}} = \frac{q_{l,t+1} + \pi_{l,t+1}}{q_{l,t}}.$$

2.3.2 Firms

Firm i's best response to the other firm's action is determined by the solution to the optimization problem at time t as follows

$$\max_{w_{i,t}} \frac{B_t}{Q_t} f_i(w_{i,t} \frac{L_t}{W_t}) - w_{i,t}$$

subject to

$$w_{i,t} \leq \frac{B_t}{Q_t} f_i(w_{i,t}\frac{L_t}{W_t}).$$

The constraint just implies that the net profit should be nonnegative. The F.O.C of the optimization problem of firm i with respect to $w_{i,t}$ shows:

$$\frac{B_t}{Q_t} f_i^{'}(L_{i,t})(\frac{L_t}{W_t} - w_{i,t}\frac{L_t}{W_t^2}) + B_t f_i(L_{i,t})(-\frac{1}{Q_t^2}) \cdot \big(f_i^{'}(L_{i,t})(\frac{L_t}{W_t} - w_{i,t}\frac{L_t}{W_t^2})\big) - 1 = 0,$$

or

$$\frac{p_t}{r_t} \cdot f_i'(L_{i,t}) \cdot \frac{Q_{-i,t}}{Q_t}\frac{W_{-i,t}}{W_t} = 1, \tag{10}$$

where $Q_{-i,t} = Q_t - f_i(L_{i,t})$ and $W_{-i,t} = W_t - w_{i,t}$.

Since there are m firms, there are m F.O.Cs like (10). Since production functions are assumed as known, the only variables in those m equations are input shares $\frac{w_{i,t}}{W_t}$, $i = 1, 2, ..., m$. Derived from (10), we also have the following $m-1$ equations, for $j = 2, 3, .., m$:

$$f_1'(L_{1,t}) \cdot \frac{Q_{-1,t}}{Q_t}\frac{W_{-1,t}}{W_t} = f_j'(L_{j,t}) \cdot \frac{Q_{-j,t}}{Q_t}\frac{W_{-j,t}}{W_t}. \tag{11}$$

Since $\sum_{i=1}^m \frac{w_{i,t}}{W_t} = 1$, there are only $m-1$ variables in the above $m-1$ equations, thus we are likely to solve for input shares $\frac{w_{i,t}}{W_t}$, $i = 1, 2, ..., m$, from these $m-1$ equations. Substituting the input shares back to (10), we see that (10) determines the price ratio p_t/r_t, given the production function forms.

2.4 Market clearing condition

2.4.1 Goods market

The market clear condition is

$$\sum_{i=1}^{n}(x_{i,t}^t + x_{i,t}^{t-1}) = \sum_{j=1}^{m} f_j(L_{j,t}) = Q_t. \tag{12}$$

2.4.2 Money market

Assume the money supply in the market is $\bar{M}$, then

$$B_t + \sum_{j=1}^{m} q_{j,t} = \bar{M}. \tag{13}$$

The reason we only consider the consumer side is because money circulates between firms and agents, while the old agents own the firms. Firms pay wages to the young agents since they provide labor. The profit that amounts to $B_t - W_t$, along with the amount of money equal to $\sum_{j=1}^{m} q_{j,t}$ from sales of assets, is shared by old agents since they own the firms. So the old agents earn in total $B_t - W_t + \sum_{j=1}^{m} q_{j,t}$. Young agents give back all their wages W_t to the firms for consumption and share ownerships. Old agents also give back their earnings to the firms to purchase consumption. So in total the amount of money in circulation is equal to $W_t + (B_t - W_t + \sum_{j=1}^{m} q_{j,t}) = B_t + \sum_{j=1}^{m} q_{j,t}$. Hence we have (13).

2.4.3 Asset market

The asset market clearing condition is equation (1):

$$\sum_{i=1}^{n} a_{i,t}^{k} = 1, \quad k = 1, 2, ..., m.$$

2.5 Market equilibrium

The market equilibrium is defined as follows

Definition 2.1. The Nash equilibrium for the market game OLG model is a sequence of bids $\{b_{i,t}^{t}, b_{i,t}^{t-1}\}_{t=1,2,...}^{i=1,2,...,n}$, $\{w_{j,t}\}_{t=1,2,...}^{j=1,2,...,m}$ and $\{q_{k,t}\}_{t=1,2,...}^{k=1,2,...,m}$ such that

1. labor inputs are exogenously given by the sequence $\{l_{i,t}\}_{t=1,2,...}^{i=1,2,...,n}$;
2. agents and firms' best responses conditions (i.e. (7)-(10)) are satisfied subject to their budget constraints considered;
3. both money market and good markets are clear (i.e. (12)-(13));
4. the aggregate money supply $\bar{M}$ is also exogenously given.

3 Equilibrium dynamics

For this part of analysis, we make the following assumption

Assumption 3.1. All the exogenous labor offers are identical and independent of time, i.e. $l_{i,t} = l_i = l$ for all i and t.

Assumption 3.2. Agents born at time t are identical.

These assumptions together with the stationarity of population imply that the aggregate labor input is given by $nl = L$. Given the specific production function forms, we also have

1. according to (11), the input labor shares are independent of time, so we denote $\frac{w_{j,t}}{W_t} = s_j$;
2. according to (10), $\frac{p_t}{r_t}$ is independent of time, so we denote $\frac{p_t}{r_t} = \frac{p}{r}$; also we notice $\frac{r}{p}$ is a function of L, we denote this function $g(L)$.
3. for each firm, the output $f_j(s_j L)$ is independent of time, hence the aggregate output $Q = \sum_{j=1}^{m} f_j(s_j L)$ is also independent of time.

Therefore equation (7) and (8) are reduced to

$$x_{i,t}^t = \frac{r}{p} \cdot l - \frac{1}{n} \sum_{j=1}^{m} \frac{q_{j,t}}{p_t}, \tag{14}$$

and

$$\begin{aligned} x_{i,t+1}^t &= \frac{1}{n} \sum_{j=1}^{m} \frac{q_{j,t+1} + \pi_{j,t+1}}{p_{t+1}} \\ &= \frac{1}{n} \sum_{j=1}^{m} (\frac{q_{j,t+1}}{p_{t+1}} + f_j(s_j L) - \frac{r_{t+1}}{p_{t+1}} \cdot s_j L) \\ &= \frac{1}{n} \sum_{j=1}^{m} \frac{q_{j,t+1}}{p_{t+1}} + \frac{Q}{n} - \frac{r}{p} \cdot l. \end{aligned} \tag{15}$$

Notice here $Q = \sum_j^m f_j(s_j \cdot nl)$ is a function of population n. According to (9), the F.O.C for a typical consumer's optimization problem is reduced to

$$\frac{U'(x_{i,t}^t)}{V'(x_{i,t+1}^t)} = \frac{U'(\frac{r}{p}\cdot l - \frac{1}{n}\sum_{j=1}^m \frac{q_{j,t}}{p_t})}{V'(\frac{1}{n}\sum_{j=1}^m \frac{q_{j,t+1}}{p_{t+1}} + \frac{Q}{n} - \frac{r}{p}\cdot l)}$$
$$= \frac{\frac{q_{j,t+1}}{p_{t+1}} + \frac{\pi_{j,t+1}}{p_{t+1}}}{\frac{q_{j,t}}{p_t}}$$
$$= \frac{\frac{q_{j,t+1}}{p_{t+1}} + f_j(s_j L) - \frac{r}{p}\cdot s_j \cdot L}{\frac{q_{j,t}}{p_t}}$$

Notice in the above equation, j is arbitrary, so we have

$$\frac{\frac{q_{i,t+1}}{p_{t+1}} + f_i(s_i L) - \frac{r}{p}\cdot s_i \cdot L}{\frac{q_{i,t}}{p_t}} = \frac{\frac{q_{j,t+1}}{p_{t+1}} + f_j(s_j L) - \frac{r}{p}\cdot s_j \cdot L}{\frac{q_{j,t}}{p_t}}$$

for $\forall i, j = 1, 2, ..., m$. Thus we have the following result by summing over j in both numerator and denominator

$$\frac{\frac{q_{j,t+1}}{p_{t+1}} + \frac{\pi_{j,t+1}}{p_{t+1}}}{\frac{q_{j,t}}{p_t}} = \frac{\sum_{j=1}^m \frac{q_{j,t+1}}{p_{t+1}} + Q - \frac{r}{p}\cdot L}{\sum_{j=1}^m \frac{q_{j,t}}{p_t}}.$$

Hence

$$\frac{U'(x_{i,t}^t)}{V'(x_{i,t+1}^t)} = \frac{\frac{1}{n}\sum_{j=1}^m \frac{q_{j,t+1}}{p_{t+1}} + \frac{Q}{n} - \frac{r}{p}\cdot l}{\frac{1}{n}\sum_{j=1}^m \frac{q_{j,t}}{p_t}}. \tag{16}$$

We notice from (16) that the equation governing the law of motion for the perfectly competitive consumer side has incorporated production, so $\frac{U'(x_{i,t}^t)}{V'(x_{i,t+1}^t)}$ is not simply $\frac{p_t}{p_{t+1}}$. But since consumer side is perfectly competitive, we view our model as a generalization of Grandmont's model [6].

It's useful to make the following change of variables. We let

$$\theta_t = \frac{1}{n}\sum_{j=1}^m \frac{q_{j,t}}{p_t},$$

$$\tilde{Q} = \frac{Q}{n},$$

$$\tilde{l} = \frac{r}{p}\cdot l = g(L)\cdot l.$$

Notice $\theta_t \in (0, \tilde{l})$ since consumption at time t should be positive and $\tilde{Q} \in [\tilde{l}, +\infty)$ since per capita profit should be nonnegative. Also according to (13)

$$p_t = \frac{\bar{M}}{Q + n\theta_t} \tag{17}$$

and

$$r_t = \frac{r}{p} \cdot p_t = g(L) \cdot \frac{\bar{M}}{Q + n\theta_t}. \tag{18}$$

Note $\tilde{Q}$ and $\tilde{l}$ are functions of L and independent of any prices, hence a full range of θ_t can be obtained independently of $\tilde{Q}$ or $\tilde{l}$ through input prices p_t.

The F.O.C for a typical consumer's optimization problem becomes

$$-\theta_t U'(\tilde{l} - \theta_t) + (\theta_{t+1} + \tilde{Q} - \tilde{l})V'(\theta_{t+1} + \tilde{Q} - \tilde{l}) = 0. \tag{19}$$

Equation (19) shows an implicit characterization of the dynamics of our model in terms of state variable θ_t. Before analyzing the dynamics of the model, first we show the existence of steady state equilibria in the model.

Lemma 3.1. *There exists a steady-state equilibrium.*

Proof. We define a function

$$h(\theta) = -\theta U'(\tilde{l} - \theta) + (\theta + \tilde{Q} - \tilde{l})V'(\theta + \tilde{Q} - \tilde{l}).$$

Then

$$\lim_{\theta \to 0^+} h(\theta) = (\tilde{Q} - \tilde{l})V'(\tilde{Q} - \tilde{l}) \geq 0.$$

The above inequality is strict if and only if $\tilde{Q} > \tilde{l}$.
First we consider the case when $\tilde{Q} > \tilde{l}$, we have

$$\lim_{\theta \to 0^+} h(\theta) = (\tilde{Q} - \tilde{l})V'(\tilde{Q} - \tilde{l}) > 0.$$

On the other hand, we have

$$\lim_{\theta \to \tilde{l}^-} h(\theta) = -\tilde{l} \lim_{\theta \to \tilde{l}^-} U'(\tilde{l} - \theta) + \tilde{Q}V'(\tilde{Q}) < 0$$

according to Inada condition. Since $h(\theta)$ is continuous, it must have a zero in $(0, \tilde{l})$.
Second we consider the case when $\tilde{Q} = \tilde{l}$, we have

$$h(\theta) = \theta(V'(\theta) - U'(\tilde{l} - \theta)).$$

Since $\theta \neq 0$, it suffices to find zeros of function $V'(\theta) - U'(\tilde{l} - \theta)$. Notice

$$\lim_{\theta \to 0^+} V'(\theta) - U'(\tilde{l} - \theta) = \lim_{\theta \to 0^+} V'(\theta) - U'(\tilde{l}) > 0,$$

and

$$\lim_{\theta \to \tilde{l}^-} V'(\theta) - U'(\tilde{l} - \theta) = V'(\tilde{l}) - \lim_{\theta \to \tilde{l}^-} U'(\tilde{l} - \theta) < 0$$

according to Inada condition. Since $V'(\theta) - U'(\tilde{l} - \theta)$ is continuous, there must have a zero in $(0, \tilde{l})$ and the same is for the function $h(\theta)$. In both cases, $h(\theta)$ has a zero in $(0, \tilde{l})$, which implies that there exists a steady-state equilibrium. □

Lemma 3.2. *The steady state equilibrium $\hat{\theta}$ is a nontrivial function of the average output per worker $\tilde{Q}$ if $\frac{dh}{d\tilde{Q}}|_{\theta=\hat{\theta}} \neq 0$.*

Proof. Let $\hat{\theta}$ satisfies

$$h(\hat{\theta}, \tilde{Q}) = -\hat{\theta} U'(\tilde{l} - \hat{\theta}) + (\hat{\theta} + \tilde{Q} - \tilde{l}) V'(\hat{\theta} + \tilde{Q} - \tilde{l}) \equiv 0. \tag{20}$$

Notice $\tilde{l} = \frac{r}{p} \cdot l$ is also a function of $\tilde{Q} = \frac{Q}{n}$ because $\frac{r}{p}$ is a function of Q according to (10). Differentiating h with respect to $\hat{\theta}$, we get

$$\frac{dh}{d\hat{\theta}} = -U'(\tilde{l} - \hat{\theta}) + V'(\hat{\theta} + \tilde{Q} - \tilde{l}) + \hat{\theta} U''(\tilde{l} - \hat{\theta}) + (\hat{\theta} + \tilde{Q} - \tilde{l}) V''(\hat{\theta} + \tilde{Q} - \tilde{l}).$$

Since $h(\hat{\theta}, \tilde{Q}) \equiv 0$, $V' > 0$ and $\tilde{Q} \geq \tilde{l}$, we have

$$-U'(\tilde{l} - \hat{\theta}) + V'(\hat{\theta} + \tilde{Q} - \tilde{l}) = \frac{-(\tilde{Q} - \tilde{l}) V'(\hat{\theta} + \tilde{Q} - \tilde{l})}{\hat{\theta}} \leq 0. \tag{21}$$

According to Inada condition $U'' < 0, V'' < 0$, together with $\hat{\theta} + \tilde{Q} - \tilde{l} > 0$ we have

$$\frac{dh}{d\hat{\theta}} < 0.$$

On the other hand, if

$$\begin{aligned}
\frac{dh}{d\tilde{Q}} =& -\hat{\theta} U''(\tilde{l} - \hat{\theta}) \cdot \frac{d\tilde{l}}{d\tilde{Q}} + (1 - \frac{d\tilde{l}}{d\tilde{Q}}) V'(\hat{\theta} + \tilde{Q} - \tilde{l}) + (\hat{\theta} + \tilde{Q} - \tilde{l})(1 - \frac{d\tilde{l}}{d\tilde{Q}}) V''(\hat{\theta} + \tilde{Q} - \tilde{l}) \\
=& V'(\hat{\theta} + \tilde{Q} - \tilde{l}) + (\hat{\theta} + \tilde{Q} - \tilde{l}) V''(\hat{\theta} + \tilde{Q} - \tilde{l}) \\
&- \frac{d\tilde{l}}{d\tilde{Q}} \left(\hat{\theta} U''(\tilde{l} - \hat{\theta}) + V'(\hat{\theta} + \tilde{Q} - \tilde{l}) + (\hat{\theta} + \tilde{Q} - \tilde{l}) V''(\hat{\theta} + \tilde{Q} - \tilde{l}) \right) \\
\neq& 0,
\end{aligned}$$

and according to $h(\hat{\theta}, \tilde{Q}) \equiv 0$ along with

$$\frac{dh}{d\tilde{Q}} + \frac{dh}{d\hat{\theta}} \cdot \frac{d\hat{\theta}}{d\tilde{Q}}|_{\hat{\theta}=\hat{\theta}(\tilde{Q})} \equiv 0,$$

we have

$$\frac{d\hat{\theta}}{d\tilde{Q}}|_{\hat{\theta}=\hat{\theta}(\tilde{Q})} = -\frac{\frac{dh}{d\tilde{Q}}}{\frac{dh}{d\hat{\theta}}} \neq 0,$$

hence $\hat{\theta}$ is a nontrivial function of $\tilde{Q}$. □

According to the definition of market thickness in [8], when Q is small/large relative to L, we are tempted to say that the market is thin/thick. Therefore $\frac{\tilde{Q}}{l} = \frac{\frac{Q}{n}}{\frac{L}{n}} = \frac{Q}{L}$ is a good measure of market thickness. Assuming individual labor offer l is fixed, we use $\tilde{Q}$ to measure the "thickness" of the market.

We have the following result on market thickness.

Proposition 3.1. If $\tilde{Q} = \tilde{l}$ or

$$\tilde{Q} \neq \tilde{l}, \quad \frac{d\hat{\theta}}{d\tilde{Q}} - \frac{d\tilde{l}}{d\tilde{Q}} < \frac{\hat{\theta}}{\tilde{Q} - \tilde{l}},$$

then thick markets are Pareto superior to thin markets.

Proof. Let the lifetime utility function associated with steady-state be

$$W(\hat{\theta}, \tilde{Q}) = U(\tilde{l} - \hat{\theta}) + V(\hat{\theta} + \tilde{Q} - \tilde{l}).$$

Then

$$\begin{aligned}
\frac{dW}{d\tilde{Q}} &= U'(\tilde{l} - \hat{\theta})(\frac{d\tilde{l}}{d\tilde{Q}} - \frac{d\hat{\theta}}{d\tilde{Q}}) + V'(\hat{\theta} + \tilde{Q} - \tilde{l})(\frac{d\hat{\theta}}{d\tilde{Q}} + 1 - \frac{d\tilde{l}}{d\tilde{Q}}) \\
&= \big(- U'(\tilde{l} - \hat{\theta}) + V'(\hat{\theta} + \tilde{Q} - \tilde{l})\big)(\frac{d\hat{\theta}}{d\tilde{Q}} - \frac{d\tilde{l}}{d\tilde{Q}}) + V'(\hat{\theta} + \tilde{Q} - \tilde{l}) \\
&\overset{(21)}{=} \frac{-(\tilde{Q} - \tilde{l})V'(\hat{\theta} + \tilde{Q} - \tilde{l})}{\hat{\theta}} \cdot (\frac{d\hat{\theta}}{d\tilde{Q}} - \frac{d\tilde{l}}{d\tilde{Q}}) + V'(\hat{\theta} + \tilde{Q} - \tilde{l}) \\
&= \Big(1 - \frac{(\tilde{Q} - \tilde{l})(\frac{d\hat{\theta}}{d\tilde{Q}} - \frac{d\tilde{l}}{d\tilde{Q}})}{\hat{\theta}}\Big)V'(\hat{\theta} + \tilde{Q} - \tilde{l})
\end{aligned}$$

By Inada condition, $V'(\hat{\theta}+\tilde{Q}-\tilde{l}) > 0$. If $\tilde{Q} = \tilde{l}$, then $\frac{dW}{d\tilde{Q}} > 0$. If $\tilde{Q} \neq \tilde{l}$, since $\frac{d\hat{\theta}}{d\tilde{Q}} - \frac{d\tilde{l}}{d\tilde{Q}} < \frac{\hat{\theta}}{\tilde{Q}-\tilde{l}}$ is equivalent to $1 - \frac{(\tilde{Q}-\tilde{l})(\frac{d\hat{\theta}}{d\tilde{Q}} - \frac{d\tilde{l}}{d\tilde{Q}})}{\hat{\theta}} > 0$, we still have $\frac{dW}{d\tilde{Q}} > 0$. Since the steady state marginal utility with respect to average output per worker is strictly positive under the assumption, it follows that utility increases as the market gets thick. □

3.1 Backward dynamics

Now we analyze the backward dynamics of the market game OLG model. We write the backward dynamics as $\theta_t = \varphi(\theta_{t+1})$. We further define two functions

$$v_1(\theta) = \theta U'(\tilde{l} - \theta)$$

and

$$v_2(\theta) = (\theta + \tilde{Q} - \tilde{l})V'(\theta + \tilde{Q} - \tilde{l}).$$

And we have

$$v_1(\varphi(\theta)) = v_2(\theta).$$

The first derivatives of $v_1(\theta)$ and $v_2(\theta)$ yield

$$v_1'(\theta) = U'(\tilde{l} - \theta) - \theta U''(\tilde{l} - \theta) > 0$$

since $U' > 0$ and $U'' < 0$ and

$$v_2'(\theta) = V'(\theta + \tilde{Q} - \tilde{l}) + (\theta + \tilde{Q} - \tilde{l})V''(\theta + \tilde{Q} - \tilde{l}).$$

We see that $v_1(\theta)$ is positive and strictly increasing on the interval $(0, \tilde{l})$.Let

$$R_2(x) = -\frac{V''(x)x}{V'(x)}$$

be the relative risk aversion of the old agent. We make the following assumption on $R_2(x)$.

Assumption 3.3. We assume

1. $R_2(x)$ is continuous, positive and strictly increasing on the interval $[\tilde{Q} - \tilde{l}, \tilde{Q}]$;

2. $R_2(\tilde{Q} - \tilde{l}) < 1 < R_2(\tilde{Q})$;

Remark. Assumption 3.3 implies that $\tilde{Q}$ has a positive lower bound under assumption. This is because $\tilde{Q}$ must be greater than $\hat{x}$, which is a value satisfying $R(\hat{x}) = 1$, by monotonicity. The assumption that $\tilde{Q}$ has a positive lower bound is equivalent to the assumption that the number of agents n must satisfy $\sum_{j=1}^{m} f_j(s_j \cdot nl)/n$ has a positive lower bound, i.e. the average output per labor is bounded away from zero.

Then we have the following results

Proposition 3.2. Under Assumption 2.1, 3.1, 3.2, 3.3, there exists a unique critical point θ^* of $\varphi(\theta)$ on interval $(0, \tilde{l})$.

Proof. Since

$$
\begin{aligned}
v_2'(\theta) &= V'(\theta + \tilde{Q} - \tilde{l})\big(1 + \frac{V''(\theta + \tilde{Q} - \tilde{l})}{V'(\theta + \tilde{Q} - \tilde{l})} \cdot (\theta + \tilde{Q} - \tilde{l})\big) \\
&= V'(\theta + \tilde{Q} - \tilde{l})\Big(1 - R_2(\theta + \tilde{Q} - \tilde{l})\Big),
\end{aligned}
$$

then under Assumption 3.3, $\exists\ \theta^* \in (0, \tilde{l})$ such that $R_2(\theta^* + \tilde{Q} - \tilde{l}) = 1$ and $v_2'(\theta^*) = 0$. Since $R_2(x)$ is strictly increasing, such θ^* is unique, and $R_2(\theta + \tilde{Q} - \tilde{l}) < 1$ for $\theta \in (0, \theta^*)$ and $R_2(\theta + \tilde{Q} - \tilde{l}) > 1$ for $\theta \in (\theta^*, \tilde{l})$. Since $V'(\theta + \tilde{Q} - \tilde{l}) > 0$ on the interval $(0, \tilde{l})$, we have $v_2'(\theta) > 0$ for $\theta \in (0, \theta^*)$ and $v_2'(\theta) < 0$ for $\theta \in (\theta^*, \tilde{l})$.
Notice $v_1(\varphi(\theta)) = v_2(\theta)$, hence $\varphi(\theta) = v_1^{-1}(v_2(\theta))$ and $\varphi'(\theta) = \frac{v_2'(\theta)}{v_1'(\varphi(\theta))}$. Since $v_1(\theta)$ is strictly increasing on $(0, \tilde{l})$, it follows that $\varphi(\theta)$ has a unique critical point at θ^* and $\varphi'(\theta) > 0$ for $\theta \in (0, \theta^*)$ and $\varphi'(\theta) < 0$ for $\theta \in (\theta^*, \tilde{l})$.

□

Remark. Assumption 3.3 ensures the uniqueness of critical point θ^*, but this assumption also sets strict restrictions on the choice of utility function $V(x)$. We show in Appendix I how to construct a utility function that satisfies Assumption 3.3 and Inada conditions, and we will see that the construction is nontrivial but such utility function exists, which is a combination of preferences with constant relative risk aversions and increasing relative risk aversions. As such utility functions are far more complicated than standard CRRA utility functions showed in Goenka, *et al.*, we have the hypothesis that for complex dynamics to occur, the choice of utility functions must be very special and the phenomenon of wide dynamics will not be as commonly seen as in similar models without production.

Corollary 3.1. There exists a unique interior steady-state equilibrium point $\hat{\theta}$ on interval $(0, \tilde{l})$.

Proof. According to (19), we have the following equality

$$
-\varphi(\theta_{t+1})U'(\tilde{l} - \varphi(\theta_{t+1}) + (\theta_{t+1} + \tilde{Q} - \tilde{l})V'(\theta_{t+1} + \tilde{Q} - \tilde{l}) = 0.
$$

If $\tilde{Q} > \tilde{l}$, then

$$
\varphi(0) \cdot U'\big(\tilde{l} - \varphi(0)\big) = (\tilde{Q} - \tilde{l})V'(\tilde{Q} - \tilde{l}) > 0,
$$

which implies

$$
\varphi(0) > 0.
$$

If $\tilde{Q} = \tilde{l}$, then

$$
\varphi(0) \cdot U'\big(\tilde{l} - \varphi(0)\big) = 0,
$$

which implies

$$
\varphi(0) = 0.
$$

We then analyze $\varphi'(0)$ in this case, which is

$$\varphi'(0) = \frac{v_2'(0)}{v_1'(\varphi(0))} = \frac{v_2'(0)}{v_1'(0)} = \frac{V'(0)}{U'(\tilde{l})} > 1.$$

From the analysis of $\varphi(\theta)$, which is unimodal on the interval $(0, \tilde{l})$ in Proposition 3.2., we see that in both cases there exists a unique interior steady-state equilibrium $\hat{\theta}$ such that $\varphi(\hat{\theta}) = \hat{\theta}$.

□

3.2 Cycles of period 2

Now we want to analyze the conditions for the existence of cycles of period 2. We define $\bar{\theta} = \frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})}$. Cyclic equilibria of order 2 are important because their existence implies the existence of sunspot equilibria. We make the following assumptions

Assumption 3.4. We assume

1. $\bar{\theta} < 1$.
2. $\tilde{Q} > \tilde{l}$.

According to [5] and [6], under Assumption 2.1, 3.4, the sufficient condition for existence of a cycle of period 2 is $\varphi'(\hat{\theta}) < -1$.

Proposition 3.3. Under Assumption 2.1, 3.4, if

$$R_2(\hat{\theta} + \tilde{Q} - \tilde{l}) > (2 + \frac{\tilde{Q} - \tilde{l}}{\hat{\theta}}) + (\frac{\tilde{Q}}{\tilde{l} - \hat{\theta}} - 1) \cdot R_1(\tilde{l} - \hat{\theta}),$$

then there exists a cycle of period 2.

Proof. Note at steady state, $\varphi(\hat{\theta}) = \hat{\theta}$. We have

$$\begin{aligned}
\varphi'(\hat{\theta}) &= \frac{v_2'(\hat{\theta})}{v_1'(\varphi(\hat{\theta}))} \\
&= \frac{V'(\hat{\theta} + \tilde{Q} - \tilde{l})}{U'(\tilde{l} - \hat{\theta})} \cdot \frac{1 - R_2(\hat{\theta} + \tilde{Q} - \tilde{l})}{1 + \frac{\hat{\theta}}{\tilde{l} - \hat{\theta}} \cdot R_1(\tilde{l} - \hat{\theta})} \\
&\overset{(20)}{=} \frac{\hat{\theta}}{\hat{\theta} + \tilde{Q} - \tilde{l}} \cdot \frac{1 - R_2(\hat{\theta} + \tilde{Q} - \tilde{l})}{1 + \frac{\hat{\theta}}{\tilde{l} - \hat{\theta}} \cdot R_1(\tilde{l} - \hat{\theta})}.
\end{aligned}$$

Hence $\varphi'(\hat{\theta}) < -1$ is equivalent to

$$R_2(\hat{\theta} + \tilde{Q} - \tilde{l}) > (2 + \frac{\tilde{Q} - \tilde{l}}{\hat{\theta}}) + (\frac{\tilde{Q}}{\tilde{l} - \hat{\theta}} - 1) \cdot R_1(\tilde{l} - \hat{\theta}), \tag{22}$$

i.e. the sufficient condition for the existence of cycles of period 2. □

Notice from Proposition 3.3. the condition for the existence of cycles of period 2 is that old agents are sufficiently risk averse. Apparently $R_2(\hat{\theta} + \tilde{Q} - \tilde{l})$ has to be greater than 2.

We can change variables and write (22) in a more convenient form. Denote

$$k = \frac{\tilde{Q} - \tilde{l}}{\hat{\theta}} > 0$$

and

$$t = \frac{\tilde{Q}}{\tilde{l} - \hat{\theta}} > 1.$$

Then

$$\hat{\theta} = \frac{t-1}{kt+t}\tilde{Q} \tag{23}$$

and

$$\tilde{l} = \frac{k+t}{kt+t}\tilde{Q}. \tag{24}$$

By substituting $\hat{\theta}$ and $\tilde{l}$ using (23) and (24) into

$$\frac{V'(\hat{\theta} + \tilde{Q} - \tilde{l})}{U'(\tilde{l} - \hat{\theta})} = \frac{\hat{\theta}}{\hat{\theta} + \tilde{Q} - \tilde{l}} = \frac{1}{1 + \frac{\tilde{Q} - \tilde{l}}{\hat{\theta}}},$$

we have

$$k = \frac{U'(\frac{\tilde{Q}}{t})}{V'(\frac{t-1}{t}\tilde{Q})} - 1. \tag{25}$$

By substituting (23), (24) and (25) into (22), we have an alternative form for (22):

$$R_2(\frac{t-1}{t}\tilde{Q}) > \Big(\frac{U'(\frac{\tilde{Q}}{t})}{V'(\frac{t-1}{t}\tilde{Q})} + 1\Big) + (t-1)R_1(\frac{\tilde{Q}}{t}), \qquad t > 1. \tag{26}$$

We also have some further restrictions on k and t. First, since $k > 0$, according to 25,

$$\frac{U'(\frac{\tilde{Q}}{t})}{V'(\frac{t-1}{t}\tilde{Q})} > 1. \tag{27}$$

Second, the first part of Assumption 3.4. yields

$$\frac{U'(\frac{k+t}{kt+t}\tilde{Q})}{V'(\frac{kt-k}{kt+t}\tilde{Q})} < 1. \tag{28}$$

Assumption 3.5. $R_1(\theta)$ is positive and nondecreasing.

Under Assumption 3.5., restraints (25), (27), (28) and $k > 0$, $t > 1$, we can analyze under what conditions (26) holds such that a cycle of period 2 occurs. We study a series of $\{\tilde{Q}^j, t^j\}^{j=1,2,\cdots}$ on the following cases.

Case 1: $\exists\, c_1, c_2$ such that $c_1 < \frac{\tilde{Q}}{t}(= \tilde{l} - \hat{\theta}) < c_2,\ c_1 > 0$.

The assumption means that $\frac{\tilde{Q}}{t}(= \tilde{l} - \hat{\theta})$ is bounded away from 0 and $\tilde{Q}$ is greater than c_1. We assume in the series, $\exists \tilde{Q}$ sufficiently big such that $\tilde{Q} > c_2$. Then we have the following inequalities on the two sides of (26):

$$\text{LHS} = R_2(\tilde{Q} - \frac{\tilde{Q}}{t}) > R_2(\tilde{Q} - c_2),$$

and

$$\text{RHS} < \big(\frac{U'(c_1)}{V'(\tilde{Q} - c_1)} + 1\big) + (t-1)R_1(c_2) < \big(\frac{U'(c_1)}{V'(\tilde{Q} - c_1)} + 1) + (\frac{\tilde{Q}}{c_1} - 1)R_1(c_2).$$

Hence if U and V satisfy

$$R_2(\tilde{Q} - c_2) > \big(\frac{U'(c_1)}{V'(\tilde{Q} - c_1)} + 1) + (\frac{\tilde{Q}}{c_1} - 1)R_1(c_2) \tag{29}$$

for some $\tilde{Q}^j$ and t^j in the series, then a cycle of period 2 occurs.

Case 2: $\exists\, t_0, t_1$ such that $1 < t_0 < t < t_1$.

The assumption means that $t(= \frac{\tilde{Q}}{\tilde{l} - \hat{\theta}})$ is bounded away from 1. Then we have the following inequalities on the two sides (26):

$$\text{LHS} > R_2\big((1 - \frac{1}{t_0})\tilde{Q}\big)$$

and

$$\text{RHS} < \big(\frac{U'(\frac{\tilde{Q}}{t_1})}{V'((1-\frac{1}{t_1})\tilde{Q})}+1\big)+t_1R_1(\frac{\tilde{Q}}{t_0}).$$

Hence if U and V satisfy

$$R_2\big((1-\frac{1}{t_0})\tilde{Q}\big) > \big(\frac{U'(\frac{\tilde{Q}}{t_1})}{V'((1-\frac{1}{t_1})\tilde{Q})}+1\big)+t_1R_1(\frac{\tilde{Q}}{t_0}) \tag{30}$$

for some $\tilde{Q}^j$ and t^j in the series, then a cycle of period 2 occurs.

Case 3: $t \to 1$.

If there's a subsequence $\{t\}$ of $\{t^j\}^{j=1,2,\cdots}$ such that $t \to 1$, first we can show that corresponding sequence $\{\tilde{Q}\}$ satisfies $\frac{t-1}{t}\tilde{Q} \to +\infty$.

By remark below Assumption 3.3, we assume that $\tilde{Q}$ cannot approach 0. If $\exists\ c_1, c_2$ such that $0 < c_1 < \tilde{Q} < c_2$, then

$$1 \overset{(27)}{<} \frac{U'(\frac{\tilde{Q}}{t})}{V'(\frac{t-1}{t}\tilde{Q})} < \frac{U'(\frac{c_1}{t})}{V'(\frac{t-1}{t}c_2)}.$$

Then by letting $t \to 1$, $V'(\frac{t-1}{t}c_2) \to +\infty$, $U'(\frac{c_1}{t}) \to U'(c_1)$, and therefore $\frac{U'(\frac{c_1}{t})}{V'(\frac{t-1}{t}c_2)} \to 0$, a contradiction. Hence $\tilde{Q} \to +\infty$. We can prove that $\frac{t-1}{t}\tilde{Q} \to +\infty$.
Suppose not, i.e. $\exists\ M$, such that $\frac{t-1}{t}\tilde{Q} < M$, then

$$1 \overset{(27)}{<} \frac{U'(\frac{\tilde{Q}}{t})}{V'(\frac{t-1}{t}\tilde{Q})} < \frac{U'(\frac{\tilde{Q}}{t})}{V'(M)}.$$

Then by letting $t \to 1$, $U'(\frac{\tilde{Q}}{t}) \to 0$, and therefore $\frac{U'(\frac{\tilde{Q}}{t})}{V'(M)} \to 0$, a contradiction. Hence if $t \to 1$, it must be that $\frac{t-1}{t}\tilde{Q} \to +\infty$. Assume $\lim_{t\to 1}\frac{U'(\frac{\tilde{Q}}{t})}{V'(\frac{t-1}{t}\tilde{Q})}$ exists and is bounded above by some constant $\Delta > 1$. Then if U and V satisfy

$$R_2(\frac{t-1}{t}\tilde{Q}) > (\Delta+1)+(t-1)R_1(\tilde{Q}) \tag{31}$$

for some $\tilde{Q}$ and t in the series, then a cycle of period 2 exists.

Case 4: $t \to +\infty$.

If $t \to +\infty$, first we can show that $\tilde{Q} \to +\infty$. Suppose not, then $\exists c_1, c_2$ such that $0 < c_1 < \tilde{Q} < c_2$ (we assumed $\tilde{Q}$ cannot approach 0). Then the two sides of (26) become

$$\text{LHS} < R_2(c_2) < +\infty$$

and

$$\text{RHS} > \big(\frac{U'(\frac{c_2}{t})}{V'(\frac{t-1}{t}c_1)} + 1\big) + (t-1)R_1(\frac{c_1}{t}).$$

When $t \to +\infty$,

$$\big(\frac{U'(\frac{c_2}{t})}{V'(\frac{t-1}{t}c_1)} + 1\big) + (t-1)R_1(\frac{c_1}{t}) \to \big(\frac{U'(0)}{V'(c_1)} + 1\big) + (t-1)R_1(0),$$

hence

$$\big(\frac{U'(\frac{c_2}{t})}{V'(\frac{t-1}{t}c_1)} + 1\big) + (t-1)R_1(\frac{c_1}{t}) \to +\infty.$$

Hence there's a contradiction since LHS has to be greater than RHS. Therefore $\tilde{Q}$ must satisfy that $\tilde{Q} \to +\infty$.

We consider different cases of $\frac{\tilde{Q}}{t}$. If $\frac{\tilde{Q}}{t}$ is bounded above and below, then it reduces to the Case 1 discussed already. We only consider the cases when $\frac{\tilde{Q}}{t} \to 0$ and $\frac{\tilde{Q}}{t} \to +\infty$.

If $\frac{\tilde{Q}}{t} \to 0$, then LHS $\to R_2(\tilde{Q})$, RHS $\to \big(\frac{U'(\frac{\tilde{Q}}{t})}{V'(\tilde{Q})} + 1\big) + (t-1)R_1(0)$ and $\frac{U'(\frac{\tilde{Q}}{t})}{V'(\tilde{Q})} \to +\infty$. Then if U and V satisfy that

$$R_2(\tilde{Q}) > \Big(\frac{U'(\frac{\tilde{Q}}{t})}{V'(\tilde{Q})} + 1\Big) + (t-1)R_1(0) \tag{32}$$

for some $\tilde{Q}$ and t in the series such that $\frac{\tilde{Q}}{t}$ is sufficiently close to 0, a cycle of period 2 exists.

If $\frac{\tilde{Q}}{t} \to +\infty$, and if $\frac{U'(\frac{\tilde{Q}}{t})}{V'(\frac{t-1}{t}\tilde{Q})}$ is bounded above by some constant Ω, then if U and V satisfy that

$$R_2(\frac{t-1}{t}\tilde{Q}) > \Omega + 1 + (t-1)R_1(\frac{\tilde{Q}}{t}) \tag{33}$$

for some $\tilde{Q}$ and t in the series, a cycle of period 2 exists.

We've analyzed all cases such that under certain conditions, e.g.(29),(30),(31), (32) and (33), the existence of a cycle of period 2 is justified. In the next section, we want to analyze the conditions under which the existence of cycles of period 3 is justified.

3.3 Cycles of period 3

According to [7], if a map φ has a cycle of period 3, then it would have a cycle of any period. We then apply results of [5] and [6] to find conditions under which a cycle of period 3 occurs. We want to show that for some $\tilde{l}$ and the unique critical point θ^* of φ the following conditions are satisfied

1. $\varphi(\theta^*) \leq \tilde{l}$ (or equivalently, $v_2(\theta^*) \leq v_1(\tilde{l})$)
2. $\exists\ \theta_0 \geq \theta^*$, s.t. $v_2(\theta^*) > v_1(\theta_0)$
3. $v_2(\theta_0) \leq v_1\big(\frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*\big)$

If the above conditions are satisfied, a cycle of period 3 exists. Therefore we have the following proposition

Proposition 3.4. If there exist $\tilde{Q}$ and $\tilde{l}$ such that the following conditions are satisfied:

1. $\theta^* < \varphi(\theta^*)$,
2. $\varphi(\varphi(\theta^*)) < \frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*$,

where θ^* is the critical point of $\varphi(\theta)$, then there exists a cycle of period 3.

Proof. According to Proposition 3.2, $\varphi'(\theta) < 0$ for $\theta \in (\theta^*, \tilde{l})$. Since $\theta^* < \varphi(\theta^*)$, then $\varphi'(\theta) < 0$ on interval $(\theta^*, \varphi(\theta^*)]$. Under the first part of Assumption 3.4., $\theta^* > \frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*$, then $\varphi(\theta^*) > \frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^* > \varphi(\varphi(\theta^*))$, and further we have $\theta^* < \varphi^{-1}(\frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*) < \varphi(\theta^*)$ as $\varphi'(\theta) < 0$ on interval $(\theta^*, \varphi(\theta^*)]$. We can choose arbitrary θ_0 such that $\varphi^{-1}(\frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*) < \theta_0 < \varphi(\theta^*)$ and show that θ_0 satisfies:

1. $\theta_0 \geq \theta^*$, s.t. $v_2(\theta^*) > v_1(\theta_0)$;
2. $v_2(\theta_0) \leq v_1\big(\frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*\big)$.

First since $\theta^* < \varphi^{-1}(\frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*)$ and $\varphi^{-1}(\frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*) < \theta_0$, we have $\theta_0 > \theta^*$. Since $\theta_0 < \varphi(\theta^*)$ and $v_1(\theta)$ is strictly increasing, we have $v_1(\theta_0) < v_1(\varphi(\theta^*)) = v_2(\theta^*)$. Second since $\theta^* < \varphi^{-1}(\frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*) < \theta_0 < \varphi(\theta^*)$ and $\varphi'(\theta) < 0$ on interval $(\theta^*, \varphi(\theta^*)]$, we have $\varphi(\theta_0) < \varphi\big(\varphi^{-1}(\frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*)\big) = \frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*$. Again since $v_1(\theta)$ is strictly increasing, $v_2(\theta_0) = v_1(\varphi(\theta_0)) < v_1\big(\frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*\big)$. Hence both 1&2 are satisfied.

We can also show that $0 < \varphi(\theta^*) < \tilde{l}$. Denote $k(\theta) = v_1(\theta) - v_2(\theta^*)$. When $\theta \to 0^+$, $k(\theta) \to -v_2(\theta^*) < 0$; when $\theta \to \tilde{l}^-$, $k(\theta) \to +\infty$. Therefore by continuity of $k(\theta)$ there exists $\breve{\theta}$ such that $k(\breve{\theta}) = 0$. Then $v_1(\breve{\theta}) = v_2(\theta^*)$ yields $0 < \breve{\theta} = \varphi(\theta^*) < \tilde{l}$.

The above analysis shows that there exists a cycle of period 3. □

Remark. We have the following observations

1. Since θ^* is a critical point of $\varphi(\theta)$, θ^* is also a critical point of $v_2'(\theta)$, hence θ^* can be solved from $v_2'(\theta^*) = 0$.

2. $\theta^* < \varphi(\theta^*)$ is equivalent to $v_1(\theta^*) < v_1(\varphi(\theta^*)) = v_2(\theta^*)$, hence equivalent to $\frac{U'(\tilde{l}-\theta^*)}{V'(\theta^*+\tilde{Q}-\tilde{l})} < \frac{\theta^*+\tilde{Q}-\tilde{l}}{\theta^*}$.

3. $\varphi(\theta)$ is strictly increasing when $\theta < \theta^*$, hence $\varphi(\varphi(\theta^*)) < \frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*$ is equivalent to $\varphi(\theta^*) < \varphi^{-1}(\frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*)$.

Hence in order to find out whether the economy will exhibit chaos, we can check the conditions in the following proposition which is equivalent to Proposition 3.4:

Proposition 3.5. If there exist $\tilde{Q}$ and $\tilde{l}$ such that the following conditions are satisfied:

1. $\frac{U'(\tilde{l}-\theta^*)}{V'(\theta^*+\tilde{Q}-\tilde{l})} < \frac{\theta^*+\tilde{Q}-\tilde{l}}{\theta^*}$,

2. $v_2(\theta^*) < v_1 \circ v_2^{-1} \circ v_1(\frac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})} \cdot \theta^*)$,

where θ^* is solved from $v_2'(\theta^*) = 0$, then there exists a cycle of period 3.

Remark. From the above analysis in section 3.2 and section 3.3, we can see for complex dynamics to occur, the utility functions must be very special with particular values of $\tilde{Q}$ and $\tilde{l}$.

4 Special cases: log-linear preferences

We study the price dynamics when utility functions are log-linear. The importance and convenience of log-linearity assumption of preferences on studying overlapping generations models are addressed in [1]. Notice log-linear utility functions do not satisfy Assumption 3.3. Hence the analysis of price dynamics in section 3 does not apply to log-linear utility functions.

We examine the following cases of firms: (1). when $m = 2$, (2). when $m = 3$, (3). when the production functions have the form $f(L) = L$, which is to approximate a pure exchange economy. Without loss of generality, we assume in the two firms case, the production functions are given by:

$$\begin{aligned} \text{firm1}: \quad & q_{1,t} = f_1(L_{1,t}) = A \cdot L_{1,t}^2, \\ \text{firm2}: \quad & q_{2,t} = f_2(L_{2,t}) = L_{2,t}^\alpha, \qquad 0 < \alpha < 1. \end{aligned}$$

We also assume in the three firms case, the production functions are given by:

$$\begin{aligned} \text{firm1}: \quad & q_{1,t} = f_1(L_{1,t}) = A \cdot L_{1,t}^2, \\ \text{firm2}: \quad & q_{2,t} = f_2(L_{2,t}) = B \cdot L_{2,t}^\alpha, \qquad 0 < \alpha < 1 \\ \text{firm3}: \quad & q_{3,t} = f_3(L_{3,t}) = C \cdot L_{3,t}. \end{aligned}$$

When firms have constant returns to scale (CRTS) production functions $f(L) = L$, at time t, firm i's best response is determined by the solution to the optimization problem as follows:

$$\max_{w_{i,t}} \frac{B_t}{Q_t} L_{i,t} - w_{i,t}$$

subject to

$$w_{i,t} \leq \frac{B_t}{Q_t} L_{i,t},$$

and

$$L_{i,t} = \frac{w_{i,t}}{W_t} L_t.$$

The constraint just implies that the net profit should be nonnegative. However, since we all know that each firm has the same production function $f(L) = L$, the aggregate output Q_t is exactly L_t, then the optimization problem setting and solving is a bit different with the arbitrary m firms case. In other words, we can write the optimization problem of firm i as follows:

$$\max_{w_{i,t}} \frac{B_t}{L_t} \frac{w_{i,t}}{W_t} L_t - w_{i,t},$$

equivalently

$$\max_{w_{i,t}} B_t \frac{w_{i,t}}{W_t} - W_{i,t}.$$

The F.O.C with respect to $w_{i,t}$ is

$$\frac{B_t W_{-i,t}}{W_t^2} = 1, \tag{34}$$

where $W_{-i,t} = W_t - w_{i,t}$. Since $p = \frac{B_t}{Q_t} = \frac{B_t}{L_t}$ and $r = \frac{W_t}{L_t}$, we can also write equation (30) as

$$\frac{p_t}{r_t} \cdot \frac{W_{-i,t}}{W_t} = 1 \tag{35}$$

Since i is arbitrary, we can conclude that $\frac{w_{i,t}}{W_t} = \frac{1}{m}$ and

$$\frac{p_t}{r_t} = \frac{m}{m-1}.$$

Hence

$$\begin{aligned}\pi_{i,t} &= p_t l_{i,t} - r_t l_{i,t} \\ &= (p_t - r_t)\frac{w_{i,t}}{W_t} L_t \\ &= (p_t - \frac{m-1}{m} p_t) \cdot \frac{1}{m} L_t \\ &= \frac{p_t L_t}{m^2}.\end{aligned}$$

In the following analysis, we assume $U(x) = \text{log}x$ and $V(x) = \beta\text{log}x$.

4.1 Price dynamics

We can show in the appendices that for m firms and the CRTS firms, $\varphi(\theta)$ is a linear function. It is well-known in Li and Yorke's Chaos Theorem ([7]) that with the difference equation $x_{t+1} = f(x_t)$, for chaos to occur, there must exist x in the domain such that $f^3(x) \leq x < f(x) < f^2(x)$. Since the price dynamics equations for both m firms and the CRTS firms case are linear, we conclude that neither cycles of period 2 or cycles of period 3 will occur in these cases.

4.2 Analysis

We consider two cases when $m = 2$ and when $m = 3$.

4.2.1 Two firms case

We set the parameters of the two production functions f_1 and f_2 of firm 1 (increasing return to scale) and firm 2(decreasing return to scale) respectively to be $A = B = 1, \alpha = 0.5, \beta = 0.8$. We do calculations to find average consumption of the young/old, demand for real balances, price ratios q_i/r, q_i/p, $i = 1, 2$ and r/p.

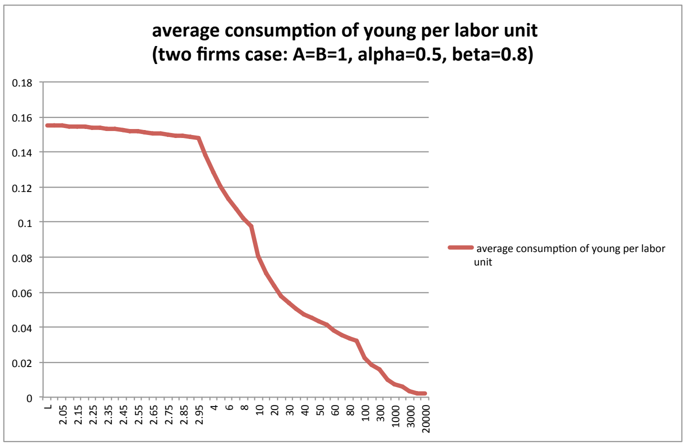
In the two firms case, as labor increases, the average consumption of the young decreases to almost zero, while the average consumption of the old almost doubles as input good labor doubles according to Figure 1. The demand for real balances almost quadruples as labor doubles according to Figure 3. The price ratios between two asset prices and input price as labor changes are shown in Figure 4. For IRTS firm, q_1/r almost doubles as labor doubles. For DRTS firm, q_2/r converges to zero. So if input price r is fixed, the asset price for IRTS firm rises higher and higher, while the asset price for DRTS firm decreases to almost zero as labor increases. Price ratios r/p and q_i/p, $i = 1, 2$, are shown in Figure 5. We see that r/p converges to zero, q_1/p increases and q_2/p converges to zero as labor increases. This shows that if r is fixed, the output price p increases as labor increases and the asset price q_1 for IRTS firm increases faster than output price p. As labor increases, the mark-up of IRTS firm increases drastically while the mark-up for DRTS firm converges to 1.

4.2.2 Three firms case

We set the parameters of the three productions function f_1,f_2 and f_3 of firm 1(increasing returns to scale), firm 2 (decreasing returns to scale) and firm 3 (constant returns to scale) respectively to be $A = B = C = 1, \alpha = 0.5, \beta = 0.8$. We do calculations to find average consumption of the young/old, demand for real balances, price ratios q_i/r, q_i/p, $i = 1, 2, 3$ and r/p.

In the three firms case, as labor increases, the average consumption of the young increases and converges to some constant as labor increases, while the average consumption of the old almost doubles as labor doubles. The demand for real balances almost quadruples as labor doubles. The price ratios between three asset prices and input price as labor changes are shown in Figure 4. For IRTS firm, q_1/r almost doubles as labor doubles. For DRTS firm, q_2/r converges to zero. For CRTS firm, q_3/r slowly increases and converges to a constant. So if input price r is fixed, the asset price for IRTS firm rises high, the asset price for DRTS firm decreases to almost zero as labor increases and the asset price for CRTS firm converges to almost a constant. Price ratios r/p and q_i/p, $i = 1, 2, 3$, are shown in Figure 5. We see that r/p converges to almost a constant. For IRTS firm, q_1/p almost doubles as labor input doubles. For DRTS firm, q_2/p converges to zero. For CRTS firm, q_3/p is almost a constant as labor increases. This shows if r is fixed, p will converges to a constant since r/p will converge to a constant. Then q_1 almost doubles as labor input doubles, q_2 converges to zero and q_3 converges to a constant assuming r is fixed, as labor increases. As labor increases, the mark-up of IRTS firm increases drastically while the mark-ups for DRTS and CRTS firms converge to 1.

In both cases, if the money supply M is fixed, then if L increases, the output price p has to be deflated since M/p is increasing.



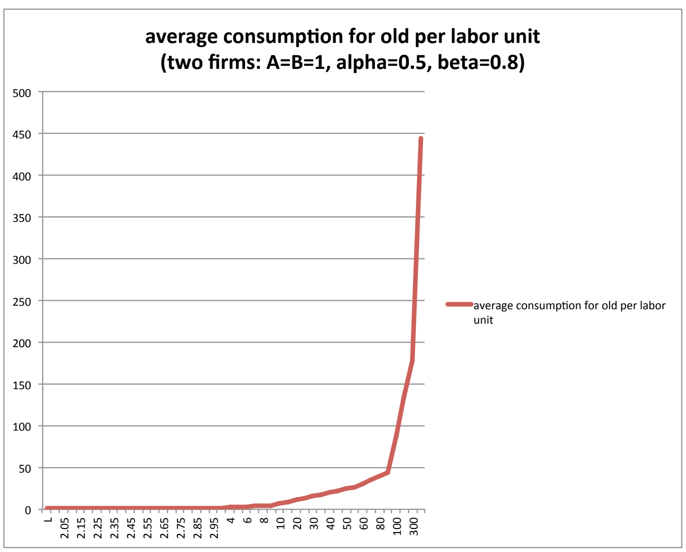
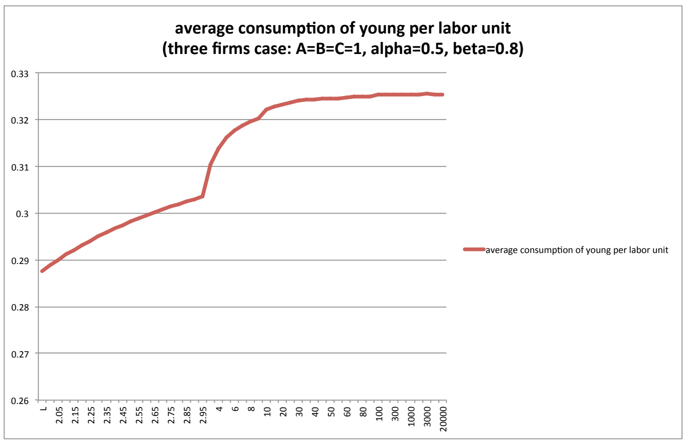


Figure 1: Average consumption of young and average consumption of old in two firms case



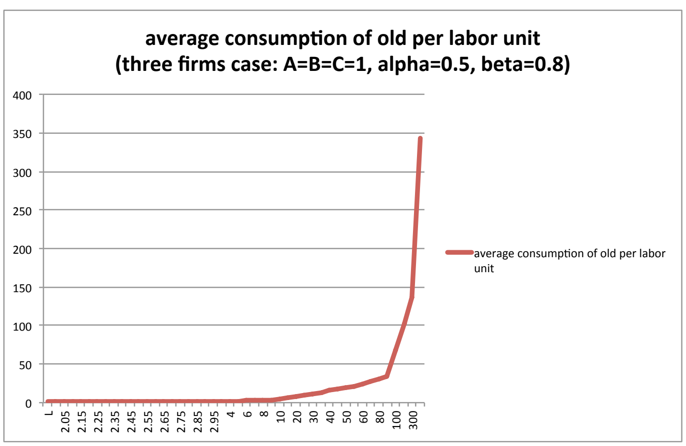


Figure 2: Average consumption of young and average consumption of old in three firms case

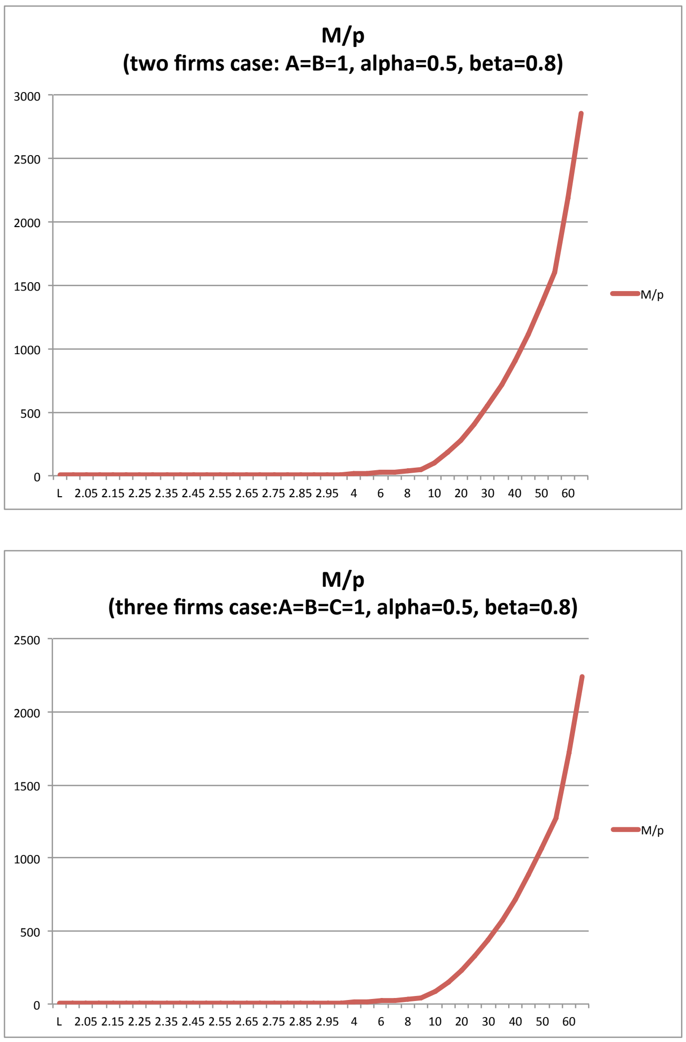
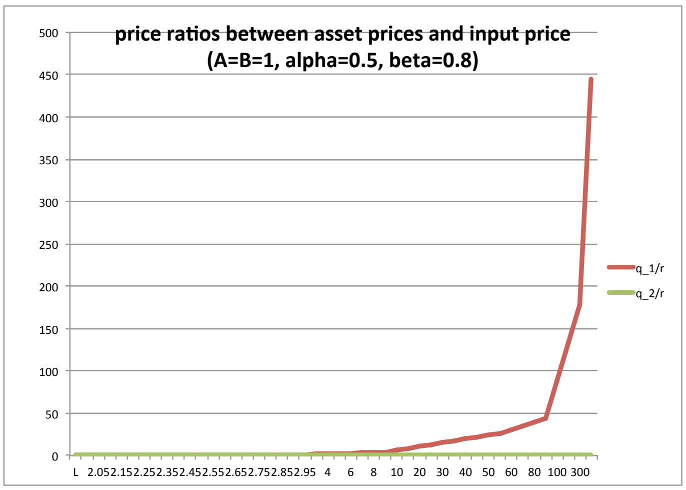


Figure 3: Demand for real balances in two firms case (above) and three firms case (below)



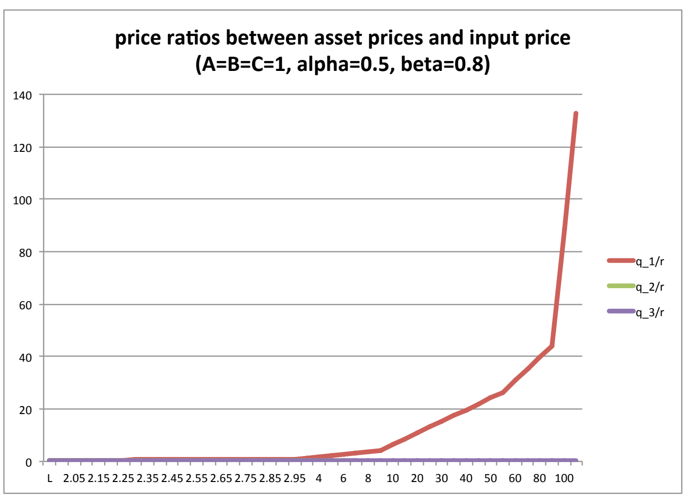


Figure 4: Price ratios q_i/r, $i = 1, 2$, in two firms case (above) and three firms case (below)

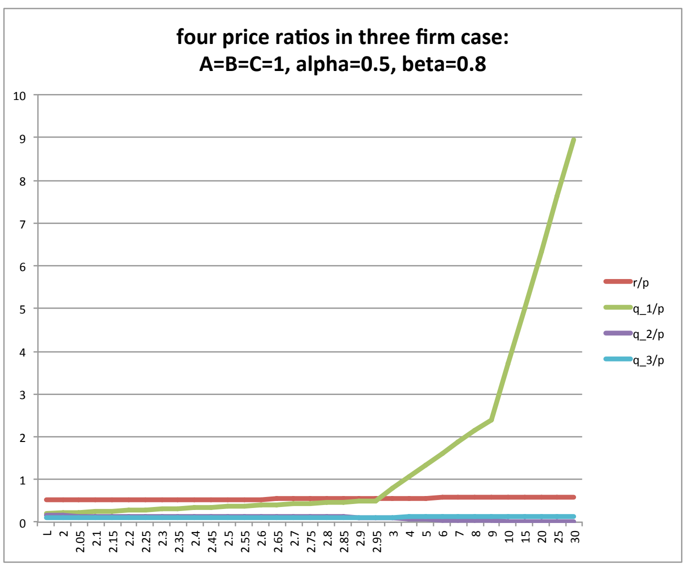
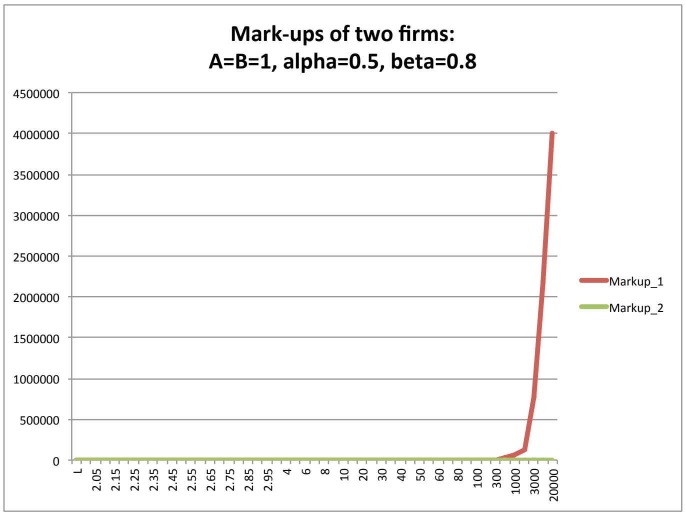


Figure 5: Price ratios r/p, q_i/r, $i = 1, 2, 3$, in two firms case (above)and three firms case (below)



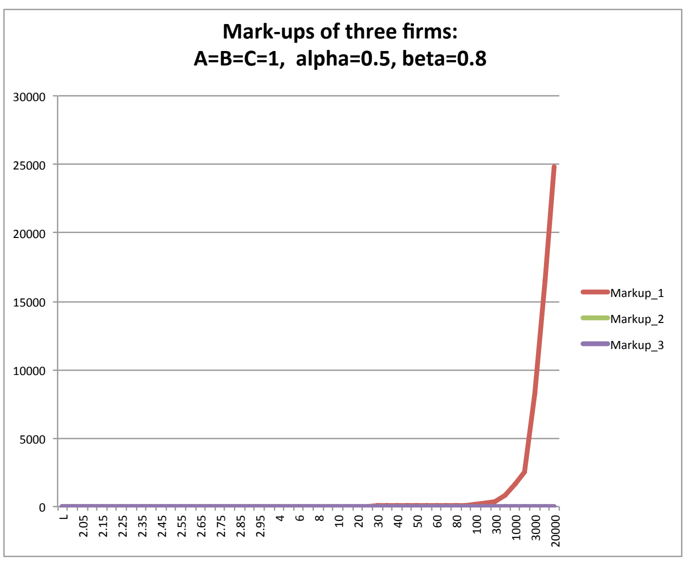


Figure 6: mark-ups in two firms case (above)and three firms case (below)

5 Discussion and conclusions

In this paper, we characterize conditions under which complex and chaotic equilibrium dynamics are possible in the overlapping generations market game model with production, which are more complicated than in the OLG market game model without production. For such complex dynamics to occur, the consumers preferences in the second period of the overlapping generations model have to be preferences of a mix of relative risk aversions, e.g. a mix of preferences with constant relative risk aversions and increasing relative risk aversions, which are more complicated than in Goenka et al. [5] or Grandmont [6]. For cycles of period 2 to occur, the old agents must be sufficiently risk-averse, which is similar to the conclusions in Goenka et al.[5] and the Grandmont [6]. The number of agents must ensure that the average output per worker is bounded away from zero. In general, it must be very special for complex dynamics to occur in our model with particular choices of production functions and utility functions.

We show the impossibility of such complex dynamics to occur for log-linear preferences, as price dynamics under such preferences are linear. Our analysis confirmed Goenka, et al.'s claim that some preliminary work with similar model suggests productions add "smoothing" to the model as wide dynamics coming out of pure exchange model are not easily observed.

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6 Appendix I: Construction of utility function $u(x)$

We aim to construct a utility function $u(x)$ such that:

- $u(x)$ satisfies Inada conditions;
- $R(x), x > 0$ is strictly increasing in an interval $\mathcal{H}$;
- $\exists\ x^*$ such that $x^* \in \mathcal{H}$ and $R(x^*) = 1$.

One guess of the function form for $u(x)$ is

$$u(x) = \frac{x^{1-\theta}}{1-\theta} + \epsilon e^{-kx},$$

where θ, ϵ, k are parameters to be fixed ($0 < \theta < 1, k > 0$ and ϵ need not to be small).

6.1 choice of ϵ

First, we analyze the conditions under which $u(x)$ satisfies Inada conditions. We want

1. $u(x)$ is twice differentiable on $(0, \infty)$,
2. $u'(x) > 0$ and $u''(x) < 0$ for $0 < x < \infty$,
3. $\lim_{x\to 0} u'(x) = +\infty$,
4. $\lim_{x\to +\infty} u'(x) = 0$.

We can always replace $u(x)$ with $u(x) - u(0)$ to satisfy $u(0) = 0$. Note

$$u'(x) = x^{-\theta} - k\epsilon e^{-kx}$$
$$u''(x) = -\theta x^{-\theta-1} + k^2\epsilon e^{-kx}.$$

Then

$$\lim_{x\to 0} u'(x) = \lim_{x\to 0}(x^{-\theta} - k\epsilon e^{-kx}) = +\infty$$

and

$$\lim_{x\to +\infty} u'(x) = \lim_{x\to +\infty}(x^{-\theta} - k\epsilon e^{-kx}) = 0,$$

hence 1,3,4 of Inada conditions are satisfied. The second condition among Inada conditions is equivalent to

$$k\epsilon e^{-kx} < x^{-\theta},\ \forall x > 0, \tag{36}$$

$$k^2\epsilon e^{-kx} < \theta x^{-\theta-1},\ \forall x > 0, \tag{37}$$

and is thus equivalent to

$$\epsilon < \min\{\frac{x^{-\theta}}{ke^{-kx}},\ \frac{\theta x^{-\theta-1}}{k^2e^{-kx}}\},\ \forall x > 0$$

and is equivalent to

$$\epsilon < \min\Big\{\min\{\frac{x^{-\theta}}{ke^{-kx}}\},\ \min\{\frac{\theta x^{-\theta-1}}{k^2e^{-kx}}\}\Big\}.$$

Next we try to find values for $\min\{\frac{x^{-\theta}}{ke^{-kx}}\}$ and $\min\{\frac{\theta x^{-\theta-1}}{k^2e^{-kx}}\}$.

To find $\min\{\frac{x^{-\theta}}{ke^{-kx}}\}$, we do some change of variable and let $x = \alpha \cdot \frac{\theta}{k}$, $\alpha > 0$. Therefore

$$\min\{\frac{x^{-\theta}}{ke^{-kx}}\} = \frac{1}{k(\frac{\theta}{k})^{\theta}} \cdot \min\{(\frac{e^{\alpha}}{\alpha})^{\theta}\},$$

then it suffices to find α such that $\frac{e^{\alpha}}{\alpha}$ is minimized. Let $g(\alpha) = \frac{e^{\alpha}}{\alpha}$, we see that

$$g'(\alpha) = \frac{e^{\alpha} \cdot \alpha - e^{\alpha}}{\alpha^2} = \frac{e^{\alpha}}{\alpha^2} \cdot (\alpha - 1).$$

Thus

$$\begin{aligned} g'(\alpha) < 0 &\quad \text{if } \alpha < 1, \\ g'(\alpha) = 0 &\quad \text{if } \alpha = 1, \\ g'(\alpha) > 0 &\quad \text{if } \alpha > 1, \end{aligned}$$

which implies that $g(\alpha)$ reaches minimum at $\alpha = 1$. Hence

$$\min\{\frac{x^{-\theta}}{ke^{-kx}}\} = \frac{e^{\theta}}{k(\frac{\theta}{k})^{\theta}}.$$

Similarly, to find $\min\{\frac{\theta x^{-\theta-1}}{k^2e^{-kx}}\}$, we still do the change of variable and let $x = \alpha \cdot \frac{\theta}{k}$, $\alpha > 0$. Therefore

$$\min\{\frac{\theta x^{-\theta-1}}{k^2e^{-kx}}\} = \frac{\theta}{k^2(\frac{\theta}{k})^{\theta+1}} \cdot \min\{\big(\frac{e^{\alpha}}{\alpha^{\frac{\theta+1}{\theta}}}\big)^{\theta}\}.$$

It suffices to find α such that $\frac{e^\alpha}{\alpha^{\frac{\theta+1}{\theta}}}$ is minimum. Let $h(\alpha) = \frac{e^\alpha}{\alpha^{\frac{\theta+1}{\theta}}}$, then

$$h'(\alpha) = \frac{e^\alpha \alpha^{\frac{1}{\theta}}(\alpha - \frac{\theta+1}{\theta})}{(\alpha^{\frac{\theta+1}{\theta}})^2}.$$

Then

$$\begin{aligned} h'(\alpha) < 0 \quad \text{if } \alpha < \frac{\theta+1}{\theta}, \\ h'(\alpha) = 0 \quad \text{if } \alpha = \frac{\theta+1}{\theta}, \\ h'(\alpha) > 0 \quad \text{if } \alpha > \frac{\theta+1}{\theta}, \end{aligned}$$

which implies that $h(\alpha)$ is minimized at $\alpha = \frac{\theta+1}{\theta}$. Hence

$$\min\{\frac{\theta x^{-\theta-1}}{k^2 e^{-kx}}\} = \frac{\theta e^{\theta+1}}{k^2(\frac{\theta+1}{k})^{\theta+1}}.$$

In all, if ϵ satisfies

(38)
$$\epsilon < \min\{\frac{e^\theta}{k(\frac{\theta}{k})^\theta}, \frac{\theta e^{\theta+1}}{k^2(\frac{\theta+1}{k})^{\theta+1}}\},$$

then the second condition of Inada conditions is satisfied.

6.2 $x^* : R(x^*) = 1$

Given $u(x)$, the relative risk aversion is

$$R(x) = -\frac{u''(x)x}{u'(x)} = \frac{\theta x^{-\theta} - k^2\epsilon \cdot xe^{-kx}}{x^{-\theta} - k\epsilon \cdot e^{-kx}}$$

Let x^* be the point that satisfies $R(x^*) = 1$. Then x^* satisfies

(39)
$$k\epsilon \cdot e^{-kx}(1 - kx) = (1-\theta)x^{-\theta}.$$

According to equation (36), we have

$$(1-\theta)x^{*-\theta} < x^{*-\theta}(1 - kx^{*-\theta}),$$

hence x^* satisfies

(40)
$$x^* < \frac{\theta}{k}.$$

6.3 $\mathcal{H}$: the interval in which $R'(x^*) > 0$

That $R'(x) > 0$ is equivalent to

$$(\theta x^{-\theta} - k^2\epsilon \cdot xe^{-kx})' \cdot (x^{-\theta} - k\epsilon e^{-kx}) - (x^{-\theta} - k\epsilon \cdot e^{-kx})' \cdot (\theta x^{-\theta} - k^2\epsilon \cdot xe^{-kx}) > 0,$$

which is equivalent to

$$kx + \frac{\theta^2}{kx} + k\epsilon \cdot x^\theta e^{-kx} - 2\theta - 1 > 0. \tag{41}$$

Let $f(x) = kx + \frac{\theta^2}{kx} + k\epsilon \cdot x^\theta e^{-kx} - 2\theta - 1$. Notice when $x = \frac{\theta}{k}$, $f(\frac{\theta}{k}) = k\epsilon \cdot (\frac{\theta}{k})^\theta e^{-\theta} - 1$. According to equation (38), $\epsilon < \frac{e^\theta}{k(\frac{\theta}{k})^\theta}$, hence $f(\frac{\theta}{k}) < 0$. Also note $\lim_{x\to 0} f(x) = +\infty$.

We aim to find an interval in which $R'(x^*) > 0$, then by continuity of $R'(x)$, there is an interval containing x^* such that $R(x)$ is strictly increasing on this interval. To find such conditions, first x^* must satisfy equation (41), i.e. $f(x^*) > 0$. Second, we want to locate the lower and upper bound of the targeted interval, which are two points $\underline{x}$ and $\bar{x}$ that are nearest to x^* and satisfy $f(x) = 0$ or $x = 0$. Both lower bound $\underline{x}$ and upper bound $\bar{x}$ exist: the upper bound is greater than x^* and less than $\frac{\theta}{k}$ because $f(x)$ changes signs between x^* and $\frac{\theta}{k}$; since $\lim_{x\to 0} f(x) = +\infty$ and $f(x^*) > 0$, the lower bound is either 0 or is the largest zero of $f(x)$ less than x^*.

We can still do some change of variables to make things simpler. Let $\bar{x} = \beta_1 \cdot \frac{\theta}{k}$ and $x^* = \beta_2 \cdot \frac{\theta}{k}$. Thus equation (40) is equivalent to $\beta_2 < 1$. That $\bar{x}$ satisfies $R'(\bar{x}) = 0$, i.e. $f(\bar{x}) = 0$ is equivalent to

$$(\beta_1 + \frac{1}{\beta_1} - 2)\theta + k\epsilon \cdot (\beta_1 \frac{\theta}{k})^\theta e^{-\beta_1\theta} - 1 = 0. \tag{42}$$

And $R'(x^*) > 0$ is equivalent to

$$(\beta_2 + \frac{1}{\beta_2} - 2)\theta + k\epsilon \cdot (\beta_2 \frac{\theta}{k})^\theta e^{-\beta_2\theta} > 1. \tag{43}$$

Equation (39) is equivalent to

$$k\epsilon \cdot e^{-\beta_2\theta}(\beta_2 \frac{\theta}{k})^\theta = \frac{1-\theta}{1-\beta_2\theta} \tag{44}$$

Substituting (44) into (43) we have that $R'(x^*) > 0$ is equivalent to

$$(\beta_2 + \frac{1}{\beta_2} - 2)\theta + \frac{1-\theta}{1-\beta_2\theta} > 1 \tag{45}$$

Let $q(x) = (x + \frac{1}{x} - 2)\theta + \frac{1-\theta}{1-\theta x}$. $\forall\theta$, $\lim_{x\to 0} q(x) = +\infty$, hence given θ, $\{x : q(x) > 1\}$ is a nonempty open set.

Since $x^* < \bar{x}$ and $f(1 \cdot \frac{\theta}{k}) < 0$, we have that

$$\beta_2 < \beta_1 < 1 \tag{46}$$

6.4 cycles of period 2

In the main article, from equation (22) we know for cycles of period 2 to occur, it must be that $R(\bar{x}) > 2$. By substituting $\bar{x}$ with $\beta_1 \cdot \frac{\theta}{k}$, we have that $R(\bar{x}) > 2$ is equivalent to

$$k\epsilon \cdot e^{-\beta_1\theta}(\frac{\beta_1\theta}{k})^\theta > \frac{2-\theta}{2-\beta_1\theta}. \tag{47}$$

By substituting according to equation (42), we have further that $R(\bar{x}) > 2$ is equivalent to

$$(\beta_1 + \frac{1}{\beta_1} - 2)\theta + \frac{2-\theta}{2-\beta_1\theta} < 1. \tag{48}$$

Let $p(x) = (x + \frac{1}{x} - 2)\theta + \frac{2-\theta}{2-\theta x}$. We can prove that $\forall\theta < 1$, $\{x : p(x) < 1\}$ is nonempty. Consider $p'(x) = (1 - \frac{1}{x^2})\theta + \frac{\theta(2-\theta)}{(2-\theta x)^2}$. We see that $\lim_{x\to 0} p'(x) = -\infty$ and $\lim_{x\to 1} p'(x) = \frac{\theta}{2-\theta} > 0$. Then the critical point of $p(x)$ is within interval (0,1) and hence $\min_{x\in(0,1)} p(x) < p(1) = 1$, which implies that $\exists\ x'$ such that $p(x') < 1$. By continuity of $p(x)$, given θ, $\{x : p(x) < 1\}$ is a nonempty open set.

6.5 summary

Our method is to try many values of k, θ, ϵ and keep the right ones by checking whether the resulting utility function satisfies the desired conditions according to the following procedure:

Step 1: Select k, θ and ϵ such that (38) holds.

Step 2: Solve β_2 from (44) (note there may be multiple roots). Keep those roots if $\beta_2 < 1$ and continue. If not, return to Step 1.

Step 3: Check if (45) holds. If yes, we find a critical point $x^* = \beta_2 \cdot \frac{\theta}{k}$ such that $R'(x^*) > 0$. If not, return to Step 1.

Step 4: Solve β_1 from (42) (note there may be multiple roots), choose the ones which are nearest to (greater than or less than) β_2 and less than 1. If $\beta_1 > \beta_2$, then the upper bound for $\mathcal{H}$ is $\bar{x} = \beta_1 \cdot \frac{\theta}{k}$. If $\beta_1 < \beta_2$, then the lower bound for $\mathcal{H}$ is $\underline{x} = \beta_1 \cdot \frac{\theta}{k}$. If we only have one value for β_1 which is greater than β_2, then the lower bound for $\mathcal{H}$ is 0.

Step 5: We can further examine the possibility of the occurrence of cycles of period 2 by checking if (48) holds . If not, then there's no possibility of cycles of period 2.

Numerical results show that there exist k, θ, ϵ such that $R(x)$ is strictly increasing in $\mathcal{H}$. Thus we find utility functions that satisfy the three conditions at the beginning of the appendix. The problem of finding right values for parameters of the proposed utility function $u(x)$ for the possibility of cycles of period 2 to occur is reduced to solving five variables $k, \theta, \epsilon, \beta_1, \beta_2$ from two equations (42) and (44), two inequalities (45) and (48), and constraints (38), (46) and $k > 0, 0 < \theta < 1$.

7 Appendix II: log-linear preferences

7.1 m firms

According to (9), we have

$$\frac{x^t_{i,t+1}}{x^t_{i,t}} = \beta \frac{q_{1,t+1} + \pi_{1,t+1}}{q_{1,t}} \cdot \frac{p_t}{p_{t+1}} \tag{49}$$

$$= \beta \frac{q_{j,t+1} + \pi_{j,t+1}}{q_{j,t}} \cdot \frac{p_t}{p_{t+1}}, \quad j = 1, 2, ..., m. \tag{50}$$

Hence

$$\begin{aligned} p_{t+1} x^t_{i,t+1} &= \frac{x_{i,t+1}}{x^t_{i,t}} \cdot \frac{p_{t+1}}{p_t} \cdot p_t x^t_{i,t} \\ &\overset{(50)}{=} \beta \frac{q_{1,t+1} + \pi_{1,t+1}}{q_{1,t}} \cdot \frac{p_t}{p_{t+1}} \cdot \frac{p_{t+1}}{p_t} \cdot p_t x^t_{i,t} \\ &= \beta \frac{q_{1,t+1} + \pi_{1,t+1}}{q_{1,t}} \cdot p_t x^t_{i,t} \\ &\overset{(7)}{=} \beta \frac{q_{1,t+1} + \pi_{1,t+1}}{q_{1,t}} \cdot (r_t l_{i,t} - \sum_{j=1}^m a^j_{i,t} q_{j,t}), \end{aligned}$$

we have

$$\begin{aligned} p_{t+1} x^t_{i,t+1} &= \beta \frac{q_{1,t+1} + \pi_{1,t+1}}{q_{1,t}} \cdot (r_t l_{i,t} - \sum_{j=1}^m a^j_{i,t} q_{j,t}) \\ &\overset{(8)}{=} \sum_{j=1}^m (q_{j,t+1} + \pi_{j,t+1}) a^j_{i,t} \\ &\overset{(50)}{=} \frac{q_{1,t+1} + \pi_{1,t+1}}{q_{1,t}} \cdot \sum_{j=1}^m a^j_{i,t} q_{j,t}. \end{aligned}$$

Hence

$$\beta \cdot (r_t l_{i,t} - \sum_{j=1}^{m} a_{i,t}^{j} q_{j,t}) = \sum_{j=1}^{m} a_{i,t}^{j} q_{j,t}.$$

We have

$$x_{i,t}^{t} = \frac{r_t}{p_t} \cdot \frac{l_{i,t}}{1+\beta}$$
$$x_{i,t+1}^{t} = \frac{\beta}{1+\beta} \cdot \frac{q_{1,t+1} + \pi_{1,t+1}}{q_{1,t}} \cdot \frac{r_t}{p_{t+1}} \cdot l_{i,t}$$

Hence

$$x_{i,t}^{t-1} = \frac{\beta}{1+\beta} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} \cdot \frac{r_{t-1}}{p_t} \cdot l_{i,t-1}.$$

We assume at each period labor L_t is constant, i.e. $L_t = L$ and also $\frac{r_t}{p_t} = \frac{r_{t-1}}{p_{t-1}} = g(L)$. The money market clearing condition becomes

$$\begin{aligned}
\sum_{i=1}^{n} (x_{i,t}^{t} + x_{i,t}^{t-1}) &= \frac{r_t}{p_t} \cdot \frac{1}{1+\beta} \cdot \sum_{i=1}^{n} l_{i,t} + \frac{\beta}{1+\beta} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} \cdot \frac{r_{t-1}}{p_t} \cdot \sum_{i=1}^{n} l_{i,t-1} \\
&= (\frac{r_t}{p_t} \cdot \frac{1}{1+\beta} + \frac{\beta}{1+\beta} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} \cdot \frac{r_{t-1}}{p_t}) \cdot L \\
&= (\frac{r_t}{p_t} \cdot \frac{1}{1+\beta} + \frac{\beta}{1+\beta} \cdot \frac{r_{t-1}}{p_{t-1}} \cdot \frac{p_{t-1}}{p_t} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}}) \cdot L \\
&= \frac{r}{p} \cdot \frac{1}{\beta+1} \cdot \Big(1 + \beta(\frac{p_{t-1}}{p_t} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}})\Big) \cdot L \\
&= g(L) \cdot L \cdot \frac{1}{\beta+1} \cdot \Big(1 + \beta(\frac{p_{t-1}}{p_t} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}})\Big) \\
&= Q.
\end{aligned}$$

Hence

$$\frac{p_{t-1}}{p_t} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} = \frac{\beta+1}{\beta} \cdot \frac{Q}{g(L) \cdot L} - \frac{1}{\beta}. \tag{51}$$

Denote $\frac{q_{i,t}}{p_t} = \tilde{q}_{i,t}$, $i = 1, 2, 3$. Notice

$$\begin{aligned}
\frac{p_{t-1}}{p_t} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} &= \frac{\frac{q_{1,t}}{p_t} + \frac{\pi_{1,t}}{p_t}}{\frac{q_{1,t-1}}{p_{t-1}}} \\
&= \frac{\tilde{q}_{1,t} + \frac{p_t \cdot f_1(s_1 L) - r_t s_1 L}{p_t}}{\tilde{q}_{1,t-1}} \\
&= \frac{\tilde{q}_{1,t} + f_1(s_1 L) - \frac{r_t}{p_t} s_1 L}{\tilde{q}_{1,t-1}} \\
&= \frac{n\theta_t + Q - g(L) \cdot L}{n\theta_{t-1}} \\
&= \frac{\theta_t + \tilde{Q} - \tilde{l}}{\theta_{t-1}}.
\end{aligned}$$

According to (51), notice $\tilde{Q}$ and $\tilde{l}$ are functions of L,

$$\theta_t = \frac{\theta_{t+1} + \tilde{Q} - \tilde{l}}{\frac{\beta+1}{\beta} \cdot \frac{Q}{g(L) \cdot L} - \frac{1}{\beta}}, \tag{52}$$

which shows price dynamics is linear.

7.2 Special case: CRTS firms

We study price dynamics like in the previous section. Without loss of generality, we consider the case when utility function is log linear. The money market clearing condition becomes

$$\begin{aligned}
\sum_{i=1}^{n} (x_{i,t}^{t} + x_{i,t}^{t-1}) &= \frac{r}{p} \cdot \frac{1}{1+\beta} \cdot \left(1 + \beta \cdot \frac{p_{t-1}}{p_t} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}}\right) \cdot L \\
&= \frac{m-1}{m} \cdot \frac{1}{1+\beta} \left(1 + \beta \cdot \frac{p_{t-1}}{p_t} \cdot \frac{q_{1,t} + \frac{p_t L_t}{m^2}}{q_{1,t-1}}\right) \cdot L \\
&= L.
\end{aligned}$$

Denote $\frac{q_{i,t}}{p_t} = \tilde{q}_t$, $i = 1, 2, ..., m$, since all asset prices are identical. Therefore

$$\frac{\tilde{q}_t + \frac{L_t}{m^2}}{\tilde{q}_{t-1}} = \frac{p_{t-1}}{p_t} \cdot \frac{q_{1,t} + \frac{p_t L_t}{m^2}}{q_{1,t-1}} = \frac{m}{m-1} \cdot \frac{1+\beta}{\beta} - \frac{1}{\beta} = \frac{n\theta_t + \frac{L_t}{m}}{n\theta_{t-1}},$$

and hence

$$\theta_t = \frac{\theta_{t+1} + \frac{l}{m}}{\frac{m}{m-1} \cdot \frac{1+\beta}{\beta} - \frac{1}{\beta}}, \tag{53}$$

which shows price dynamics is linear and independent of L.