Trickle-down housing economics

Charles G. Nathanson*

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Abstract

This paper provides a quantitative framework for estimating the effects on house prices and household welfare of building different types of housing within a city or metropolitan area. According to our estimates, low-income households without a college degree benefit more from the construction of low-quality rather than high-quality housing, but low-quality construction makes many other households worse off. These conclusions depend on household mobility across cities, the strength of urban spillovers, the indivisibility of housing, and the differential preferences of households with and without a college degree.

Since 1980, the inflation-adjusted price of housing has significantly risen in many large cities around the world. In the United States, many households with low incomes or lacking a college degree have migrated away from such cities in response to rising house prices (Gyourko et al., 2013; Diamond, 2016; Ganong and Shoag, 2017). Policymakers have called this situation an "affordability crisis" (e.g., White House, 2016). Several economists recommend that cities ease permitting rules so that the construction of new housing units can bring down house prices. Using fast tracking, inclusionary zoning, or tax credits, many governments relax permitting rules specifically for the construction of smaller units or units in neighborhoods with low-income households. While there exist empirical studies of these policies, little theoretical work has estimated how the type of construction affects house prices and the composition of households within a metropolitan area.

To answer this question, we model a city with different qualities of housing that is home to households of varying education and income. We study the effects of raising the quantity of each type of housing, which we interpret as a targeted relaxation in permitting rules. When we estimate our model using data from Boston in 2016, we find that low-quality construction increases the welfare of low-education, low-income households twice as much as high-quality construction. However, low-quality construction makes many other households worse off because in-migration of low-education, low-income households lowers the city's wages and amenities.

The model features a continuum of households choosing one of multiple cities in which to live and work. Cities differ in the types of available housing, in their amenities, and their labor productivities. Firms in each city demand both low-education and high-education labor. Households differ in their education and their labor endowment, so that households with a greater labor endowment earn more income within each education group. Households split their income between housing and non-housing consumption. They take house prices, amenities, and labor prices as given and choose a city and type of housing to maximize utility. Household preferences may depend on their education. For instance, high-education households may value amenities over non-housing consumption relatively more than low-education households, which seems to hold empirically (Bayer et al., 2007; Diamond, 2016).

In the model, the population of households in the city may affect city-wide amenities and labor prices. Such "spillovers" have been the focus of much of the literature in urban economics, and we adopt the literature's estimates of these spillovers when we quantify the effects of construction. The combination of spillovers and preference heterogeneity drives our result that construction may make some households worse off. As an example, low-quality construction brings more low-education households into the city. According to estimates from Diamond (2016), this in-migration of low-education households lowers city amenities. House prices fall to compensate the average household for the loss of amenities. If high-education households value amenities more than low-education households, then the decline in house prices insufficiently compensates high-education households for the loss of amenities, and they become worse off.

We estimate the model by fitting the joint distribution of education, house prices, and income in the American Community Survey for the Boston-Cambridge-Newton, MA-NH metropolitan area in 2016. Using estimates of household preferences from Diamond (2016) as well as the aforementioned spillover estimates, we compute the effects of construction and labor productivity shocks on house prices and household welfare. As an application, we feed in the skill-biased productivity shock that Diamond (2016) estimates occurred in Boston between 1980 and 2000. This annualized shock raises average house prices 3.2% in the model and causes substantial out-migration of low-income and low-education households. We then calculate the minimum quantity of construction such that the combination of the productivity shock and the construction makes

no one worse off. The resulting construction equals 1.4% of the housing stock, more than triple the annual construction that occurs in our data. Furthermore, the model-based construction involves building much more low-quality housing than what we observe in the data.

Our results hold because households are mobile across cities and because housing is indivisible. When households are immobile, construction makes all of the city's households better off, and building a unit of the highest quality improves welfare more than building a unit of any other quality. The same holds when housing is divisible—households can costlessly divide any housing unit into two new units of lesser quality—for a subset of the spillovers we consider. Papers have studied indivisibility with immobility (Braid, 1981; Määttänen and Terviö, 2014; Landvoigt et al., 2015) and divisibility with mobility (Henderson, 1974; Glaeser and Gottlieb, 2009). The *combination* of mobility and indivisibility breaks the trickle-down mechanism in these frameworks.

The key contribution of our paper is incorporating urban spillovers into a quantitative model of heterogeneous housing quality. Spillovers distinguish our paper from an older theoretical literature building on Sweeney (1974b), who theoretically analyzes the effects of constructing different qualities of indivisible housing on house prices (see Arnott, 1987 for a literature review). Braid (1981), for instance, corresponds to the special case of our indivisible model without spillovers and mobility. In these older models, neither incomes nor amenities depend on the composition of households in the city. These spillovers drive our key theoretical and quantitative results. Like us, Davis and Dingel (2018) incorporate spillovers into a spatial equilibrium model with heterogeneous housing quality. All households in their model have identical preferences, whereas preference heterogeneity is responsible for the most important results in our framework.

Another contribution is to explain why certain households oppose construction in their city of residence. According to our estimates, high-income and high-education households oppose low-quality construction because it attracts low-education households, who lower amenities and labor prices. This mechanism differs from that in Hilber and Robert-Nicoud (2013) and Ortalo-Magné and Prat (2014). These papers formalize the "homevoter hypothesis" of Fischel (2001) with models in which homeowners oppose construction in order to increase their home equity wealth. This incentive is absent from our static model. Our results might explain why the restrictiveness of zoning correlates more closely with demographics than with homeownership in multiple empirical studies (Gyourko and Molloy, 2015).

Several empirical papers estimate the effects of subsidies for low-income housing on housing supply and house prices. Schwartz et al. (2006), Baum-Snow and Marion (2009), and Diamond and McQuade (2017) find that these subsidies increase the supply of low-income housing in the neighborhoods they target and increase the value of surrounding homes in low-income and declining neighborhoods. Unlike these studies, our paper estimates the effect of low-quality housing on an entire city or metropolitan area. This distinction matters, as many of these subsidies fail to increase the supply of low-income housing at the metropolitan area level (Eriksen and Rosenthal, 2010; Schuetz et al., 2011). As a result, existing subsidy programs may not identify the effects of constructing low-quality housing on a metropolitan area. Our structural approach does.

Another paper estimating the effect of construction on house prices with a structural approach is Anenberg and Kung (2018). Our papers differ in how we treat the cross-city migration that occurs in response to construction. Anenberg and Kung (2018) assume that households from outside the city occupy all newly built housing, irrespective of any adjustment to house prices. In contrast, migration demand in our model is a downward-sloping function of house prices that we endogenously derive. Anenberg and Kung (2018) estimate that building high-quality housing has no effect on the prices of other types of housing, while we find large effects of high-quality

construction on low-quality house prices.

Our framework does not directly model filtering, the process through which high-quality construction eventually houses low-income households after years of depreciation (Rosenthal, 2014). Despite this omission, there is a simple way to map our framework into a filtering model. Filtering models admit a unique steady-state distribution for housing quality that depends on the intensity of construction of each type of housing (Sweeney, 1974a). The stock of housing in our static model corresponds to the steady-state distribution in a filtering model. Increasing the housing stock in our framework maps to the change in construction intensity in a filtering model that would cause the corresponding shift to the steady-state distribution of housing quality.

The paper proceeds as follows. Section 1 lays out the economic environment, defines equilibrium, and characterizes equilibrium house prices with and without divisibility. Section 2 theoretically analyzes the equilibrium effects of construction and productivity shocks. Section 3 discusses our strategy for estimating the model, and Section 4 describes the data we use. Section 5 presents the quantitative results, and Section 6 concludes. Supplements to Sections 1–3 appear, respectively, in Appendices A–C.

1 Environment and equilibrium

1.1 Housing supply

The economy consists of T cities indexed by t. In city t, available housing qualities are $q_{j,t} > 0$, where $j \in \mathcal{J}_t = \{0, ..., J_t\}$ and $q_{j,t}$ strictly increases in j. The measure of housing of quality $q_{j,t}$ is $h_{j,t} > 0$, and this housing trades in competitive markets at a price $p_{j,t}$.

There are two types of agents: households and rentiers. Rentiers are endowed with the entire housing stock and have utility that is a linear function of a composite non-housing consumption good c, whose price we normalize to 1. They take house prices as given and choose how much housing to sell and how much c to consume subject to a budget constraint.

1.2 The distribution of households

Households differ in their education, $e \in \{L, H\}$, labor endowment, $z > 0$, and taste for each city t, ϵ_t. Across households and cities, the ϵ_t are distributed independently as identical Gumbel distributions (McFadden, 1973). The distribution of z among households of education e equals $\widetilde{n}_e(z)$, about which we assume the following:

Assumption 1. *For each $e \in \{L, H\}$,*

 (a) *the support of $\widetilde{n}_e$ is convex;*
 (b) *the greatest lower bound of the support of $\widetilde{n}_e$ equals zero;*
 (c) *$\widetilde{n}_e$ is continuous; and*
 (d) *$\int_0^\infty \widetilde{n}_e(z) dz > 0$.*

Assumptions 1(a) and 1(b) mean that there are no gaps in the distribution of labor endowments and that there are households with arbitrarily small labor endowments. These conditions ensure that equilibrium house prices are locally unique. Assumption 1(c) rules out mass points, in which

a positive measure of households have the same education and labor endowment, as well as jumps in the endowment distributions. This assumption allows us to define comparative statics with respect to an equilibrium.

Each household lives in one city. We denote the measure of households of education e with labor endowment z living in city t by $n_{e,t}(z)$. The population of households of education e in city t equals $N_{e,t} = \int_0^\infty n_{e,t}(z)dz$, and the total population of households in city t is $N_t = N_{L,t} + N_{H,t}$. The total labor endowment of education group e in city t is $Z_{e,t} = \int_0^\infty z n_{e,t}(z)dz$. We restrict attention to allocations of households across cities in which $N_{e,t} > 0$ for each $e \in \{L, H\}$ and $t \in \{1,...T\}$; that is, some households of each education group must live in each city. This restriction allows us to divide by these populations when we specify amenity and productivity spillovers. Such allocations are possible due to Assumption 1(d), which guarantees a nonzero population of households with each education.

1.3   Household preferences and constraints

Households have preferences over four goods—composite non-housing consumption c, housing quality q, city amenities a, and an idiosyncratic taste for each city ϵ—represented by the utility function

$$u_e(c, q, a, \epsilon) = c^{\beta_{c,e}} q^{\beta_{q,e}} a^{\beta_{a,e}} \exp(\beta_{\epsilon,e}\epsilon), \tag{1}$$

where $\beta_{c,e}, \beta_{q,e}, \beta_{a,e}, \beta_{\epsilon,e} > 0$ for each $e \in \{L, H\}$.

Cobb-Douglas preferences over housing and non-housing consumption, such as in (1), appear in many equilibrium models of city choices (e.g., Glaeser and Gottlieb, 2009; Gennaioli et al., 2013; Diamond, 2016), and are consistent with the stability of housing expenditure as a share of income across places and time (Davis and Ortalo-Magné, 2011). The term involving ϵ is present in some recent work (Kline and Moretti, 2014; Hsieh and Moretti, 2018) and limits household mobility across cities in response to changes in utility coming from c, q, and a. As in Diamond (2016), preferences may differ across education groups. For instance, low-education households may care relatively more about non-housing consumption than amenities compared to high-education households. Differences in preferences across the groups are quite important for our results.

Amenities in each city are non-rival and non-excludable to city households. As in Diamond (2016), amenities depend on exogenous characteristics of the city as well as the relative population of households with education H:

$$a_t = \widetilde{a}_t \left(\frac{N_{H,t}}{N_{L,t}} \right)^{\gamma_a}, \tag{2}$$

where $\gamma_a \geq 0$ and $\widetilde{a}_t > 0$ for each t. When $\gamma_a > 0$, city amenities increase when more high education households arrive. Households may simply enjoy meeting high-education households. Alternatively, consumption by high-education households may produce non-excludable benefits to other households, as is the case with philanthropy. Diamond (2016) discusses (2) further.

Labor in each city trades in competitive markets, and we denote the price of labor of education e in city t by $w_{e,t}$. A household's income then equals $w_{e,t}z$, which we denote $y_{e,t}(z)$. Each household takes house prices, labor prices, and amenities as given and chooses a city t, non-housing

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consumption c, and a quantity x_j of each housing quality $q_{j,t}$ subject to the following constraints:

$$c + \sum_{j \in \mathcal{J}_t} p_{j,t} x_j \leq w_{e,t} z \tag{3}$$

$$q = \sum_{j \in \mathcal{J}_t} x_j q_{j,t} \tag{4}$$

$$t \in \{1, ..., T\} \tag{5}$$

$$0 \leq c \tag{6}$$

$$0 \leq x_j, \qquad \forall j \in \mathcal{J}_t. \tag{7}$$

Within these constraints, households can combine fractional amounts of housing of different qualities into a single effective housing unit. These activities might take the form of splitting time between different locations, renting a single room in a larger house, or knocking down walls between neighboring apartment units.

Many of these activities, however, involve costs not present in our model, such as the privacy lost from occupying a single room in a larger house. Another cost, which is particularly relevant in the cities that motivate this paper, are regulations prohibiting divisibility like minimum lot sizes and maximum occupancy constraints (Gyourko et al., 2008; Glaeser and Ward, 2009). To capture these costs, we introduce two additional constraints:

$$(x_0, ..., x_{J_t}) \in \{0, 1\}^{J_t + 1} \tag{8}$$

$$\sum_{j=0}^{J_t} x_j = 1. \tag{9}$$

We call (8) and (9) the *indivisibility constraints*. Within them, each household must choose exactly one unit of one type of housing. Recent papers featuring these constraints include Määttänen and Terviö (2014) and Landvoigt et al. (2015).

1.4 Firms

Firms in each city t combine low-education and high-education labor to produce the non-housing consumption good c according to the production function

$$F_t(Z_L, Z_H) = ((A_{L,t} Z_L)^\rho + (A_{H,T} Z_H)^\rho)^{\frac{1}{\rho}}, \tag{10}$$

where Z_e is the quantity of labor of education e a firm uses, and $0 < \rho \leq 1$. A large literature in labor economics adopts (10) to explain the evolution of wages for workers with and without a college degree (Goldin and Katz, 2008; Card, 2009). Firms in t take $A_{L,t}$ and $A_{H,t}$ as given and choose labor inputs Z_L and Z_H. The resulting profits, $F_t(Z_L, Z_H) - w_{L,t} Z_L - w_{H,t} Z_H$, accrue to the rentiers living in city t.

The only differences in production technology across cities come from variation in $A_{L,t}$ and $A_{H,t}$, which govern the productivity of each type of labor. As with amenities, productivity depends on exogenous characteristics of the city as well as the city's population:

$$A_{e,t} = \widetilde{A}_{e,t} N_t^{\gamma_N} \left(\frac{N_{H,t}}{N_t} \right)^{\gamma_H}, \tag{11}$$

where $\gamma_N \geq 0$, $\gamma_H \geq 0$, and $\widetilde{A}_{e,t} > 0$ for each t. When $\gamma_N > 0$, productivity increases when the population of the city goes up and the relative share of each education group remains constant. Labor productivity is indeed higher in more populous cities, and an extensive literature in urban economics finds that part of this phenomenon is a causal effect of population on productivity (Combes and Gobillon, 2015).

When $\gamma_H > 0$, productivity increases when the share of high-education households in the city rises. The functional form of this effect matches that in Lucas (1988), who posits a constant elasticity of productivity spillover with respect to the average human capital in the population. While productivity is higher in cities and states with more human capital (Moretti, 2004a,b; Gennaioli et al., 2013), some of this effect may arise from the decisions of relatively productive firms to locate in regions with more human capital. We explore the implications of both positive and zero values.

1.5   Equilibrium definitions

Local equilibrium consists of house prices $p_{j,t}$, labor prices $w_{e,t}$, amenity levels a_t, productivity levels $A_{e,t}$, populations $n_{e,t}(z)$, and housing demands $x_{j,t}$ for each $e \in \{L, H\}$, $j \in \mathcal{J}_t$, and $t \in \{1, ..., T\}$ satisfying six conditions:

- the measure of households of education e and labor endowment z choosing city t equals $n_{e,t}(z)$, while the sum of x_j across households choosing city t equals $x_{j,t}$;

- households choose c and x_j to maximize utility subject to the constraints (3)–(7) and also (8)–(9) in the case of indivisible housing;

- rentiers in each city t choose a quantity of housing to sell to maximize utility subject to their budget constraints, and the quantity they choose of housing of quality q_j equals $x_{j,t}$;

- each firm in each city chooses Z_L and Z_H to maximize profits, and the total labor of education e firms in city t choose equals $Z_{e,t}$;

- the population of households of education e, $N_{e,t}$, is positive for each city t; and

- the amenity and productivity equations, (2) and (11), hold for each city t.

In local equilibrium, markets clear, and households maximize utility conditional on their city choices. An *equilibrium* is a local equilibrium in which household city choices maximize utility.

1.6   Equilibrium characterization

We begin by characterizing equilibrium prices of labor. Firm profit maximization implies that these labor prices coincide with marginal products:

$$w_{e,t} = ((A_{L,t}Z_{L,t})^\rho + (A_{H,t}Z_{H,t})^\rho)^{\frac{1}{\rho}-1} A_{e,t}^\rho Z_{e,t}^{\rho-1} \tag{12}$$

for each $e \in \{L, H\}$ and $t \in \{1, ..., T\}$. Because the population of each education group in each city, $N_{e,t}$, is positive, the productivities and labor endowments, $A_{e,t}$ and $Z_{e,t}$, for the corresponding households are positive as well. As a result, $w_{e,t} > 0$ for each education group in each city, meaning that all households earn positive income in equilibrium.

We next characterize the equilibrium city choice of each household. To do so, we define indirect utility, $v_{e,t}(z)$ to be the maximized utility of a household in city t whose idiosyncratic taste for

that city is zero. Specifically,

$$v_{e,t}(z) = \max_{c,x_j} u_e(c,q,a_t,0) \tag{13}$$

subject to the relevant constraints between (3) and (9). In the indivisible case, some households may be unable to choose any combinations of c and x_j within these constraints, in which case the right side of (13) does not exist. This situation arises precisely for households too poor to afford even the cheapest housing unit, that is, when $w_{e,t}z < \min(p_{0,t},...,p_{J_t,t})$, and for these households, we define $v_{e,t}(z) = 0$. Indirect utility also equals zero for households in the indivisible case who must spend all of their income to live in city t, in which case $w_{e,t}z = \min(p_{0,t},...,p_{J_t,t})$.

To study the average welfare of households in the economy, we introduce another measure of indirect utility, $\overline{v}_{e,t}(z)$, that delivers the average utility of all households in city t with a given e and z. Specifically, we define $\overline{v}_{e,t}(z) = \exp(E(\log u_e(c,q,a_t,\epsilon_t) \mid e,z,t))$ whenever $n_{e,t}(z) > 0$. Putting the log inside the expectation is necessary because the expected level of utility fails to exist due to the thickness of the Gumbel distribution's tail.

Because households choose cities to maximize utility, a household chooses city t when, for each $t' \neq t$, $v_{e,t}(z)\exp(\beta_{\epsilon,e}\epsilon_t) > v_{e,t'}(z)\exp(\beta_{\epsilon,e}\epsilon_{t'})$. Standard results on Gumbel distributions (McFadden, 1978) allow us to solve for the population distributions and average utilities, $n_{e,t}$ and $\overline{v}_{e,t}$, in terms of the indirect utilities, $v_{e,t}$. Lemma 1 provides these solutions and uses Assumptions 1(a) and (c) to prove facts about each $n_{e,t}$ that are useful for characterizing equilibrium.

Lemma 1. *In equilibrium, the following hold for each $z > 0$, $e \in \{L,H\}$, and $t \in \{1,...,T\}$:*

(a) *there exists $t' \in \{1,...,T\}$ such that $v_{e,t'}(z) > 0$;*

(b) *$n_{e,t}$ is a continuous distribution with convex support satisfying*

$$n_{e,t}(z) = \frac{\widetilde{n}_e(z)v_{e,t}(z)^{\beta_{\epsilon,e}^{-1}}}{\sum_{t'=1}^{T} v_{e,t'}(z)^{\beta_{\epsilon,e}^{-1}}}; \tag{14}$$

(c) *$\inf\{v_{e,t}(z') \mid n_{e,t}(z') > 0\} = 0$; and*

(d) *if $n_{e,t}(z) > 0$, then $\overline{v}_{e,t}(z) = \exp(\beta_{\epsilon,e}\gamma)\left(\sum_{t'=1}^{T} v_{e,t'}(z)^{\beta_{\epsilon,e}^{-1}}\right)^{\beta_{\epsilon,e}}$, where γ is Euler's constant.*

Proof. Appendix A.1. □

By Lemma 1(a), every household can achieve positive utility in one of the economy's cities. As shown in the proof, this result depends on Assumption 1(b). Lemma 1(a) guarantees that the denominator in (14) is positive so that the solution for $n_{e,t}(z)$ is well-defined. By Lemma 1(b), there are no mass points, discontinuities, or gaps in the distributions of labor endowments among households of a given education within each city. These properties hold for the distributions within the entire economy, $\widetilde{n}_e$, so Lemma 1(b) proves that the within-city distributions inherit these properties. Lemma 1(c) states that indirect utility comes arbitrarily close to zero within each education group in each city. This result relies on Assumptions 1(a) and 1(b).

The formula that Lemma 1(d) gives for $\overline{v}_{e,t}(z)$ does not depend on t. Therefore, the average utility of households of a given e and z is the same in all cities where they live. In urban models with perfect mobility, utility is equal in all cities for households of a given skill level (Roback, 1982; Glaeser and Gottlieb, 2009). Imperfect mobility generalizes this equivalence by requiring only that the average level of utility is equal across cities (see Hsieh and Moretti, 2018 for further discussion of this point). We label this average $\overline{v}_e(z)$.

To finish characterizing equilibrium, we solve for equilibrium house prices and indirect utilities within each city, $p_{j,t}$ and $v_{e,t}(z)$, as functions of the distributions of households in the city, $n_{e,t}$. We consider the divisible and indivisible cases separately.

1.6.1 Divisible housing

When housing is divisible, the price per unit of quality, $p_{j,t}/q_{j,t}$, must be equal in equilibrium across the different types of housing in the city. To demonstrate this result, we write each household's first-order condition with respect to x_j (the amount of housing of quality $q_{j,t}$) as

$$\frac{\partial u_e/\partial q}{\partial u_e/\partial c} \leq \frac{p_{j,t}}{q_{j,t}}, \tag{15}$$

with consumption of x_j only in the case of equality. If $p_{j,t}/q_{j,t}$ exceeds $p_{j',t}/q_{j',t}$ for any $j' \neq j$, then no household will choose quality $q_{j,t}$. This situation cannot hold in equilibrium because the market for that type of housing must clear.

Proposition 1 solves for the equilibrium price-to-quality ratio.

Proposition 1. *In equilibrium, $p_{j,t} = \mu_t q_{j,t}$ for each $j \in \mathcal{J}_t$, where*

$$\mu_t = \frac{\sum_{e \in \{L,H\}} \left(\beta_{c,e} + \beta_{q,e}\right)^{-1} \beta_{q,e} w_{e,t} Z_{e,t}}{\sum_{j' \in \mathcal{J}_t} h_{j',t} q_{j',t}}. \tag{16}$$

Proof. Appendix A.2. □

Each household in city t can purchase a unit of housing quality at an effective price μ_t. Because preferences over c and q are Cobb-Douglas, each household chooses to spend a constant share of income on housing quality, and this share equals $\beta_{q,e}/(\beta_{c,e} + \beta_{q,e})$. As a result, the numerator of (16) gives the total expenditure on housing in city t. Plugging the c and q choices of each household into (1) yields each household's indirect utility:

$$v_{e,t}(z) = \beta_c^{\beta_c} \beta_q^{\beta_q} \left(\beta_c + \beta_q\right)^{-(\beta_c + \beta_q)} (w_{e,t} z)^{\beta_{c,e} + \beta_{q,e}} \mu_t^{-\beta_{q,e}} a_t^{\beta_{a,e}}. \tag{17}$$

1.6.2 Indivisible housing

In the indivisible case, each household occupies exactly one housing unit in equilibrium. Because households are optimizing, house prices must strictly increase in quality among occupied units. Furthermore, the set of occupied units must equal the N_t highest quality units in city t. Otherwise, a rentier endowed with a high quality vacant unit would not be optimizing. Finally, the housing quality a household chooses must weakly increase in that household's labor endowment, z, within each education group, e. In other words, households sort on labor endowment, and hence income, within each education group. As shown by Määttänen and Terviö (2014) for more general utility functions, this sorting condition holds whenever the marginal rate of substitution from housing to non-housing consumption increases in non-housing consumption.

Proposition 2 formalizes these statements.

Proposition 2. *In equilibrium,*

$$j_{0,t} = \sup\left\{ j \in \mathcal{J}_t \ \middle| \ \sum_{j'=j}^{J_t} h_{j',t} \geq N_t \right\} \tag{18}$$

exists, $p_{j,t}$ strictly increases over $j \geq j_{0,t}$ and equals zero for j such that $\sum_{j'=j}^{J_t} h_{j',t} > N_t$, and $x_{j,t}$ equals zero for $j < j_{0,t}$ and $h_{j,t}$ for $j > j_{0,t}$. The quality chosen by a household of education e and labor endowment z weakly increases in z for each e.

Proof. Appendix A.3. □

Proposition 2 pins down the price of the lowest quality occupied unit, $p_{j_{0,t},t}$, when the housing stock exceeds the city's population:

$$p_{j_{0,t},t} = 0 \tag{19}$$

if $\sum_{j \in \mathcal{J}_t} h_{j,t} > N_t$. The prices of the city's higher quality units solve the system that equates household demand for these units to the rentiers' endowments. Under the conditions in the following lemma, this system of equations admits a unique solution.

Lemma 2. *Suppose $n_{L,t}$ and $n_{H,t}$ are continuous distributions whose supports are convex sets with greatest lower bound zero. If $\sum_{j \in \mathcal{J}_t} h_{j,t} > N_t$, then a unique local equilibrium exists.*

Proof. Appendix A.4. □

In equilibrium, $n_{L,t}$ and $n_{H,t}$ satisfy the continuity and convexity properties by Lemma 1(b). Furthermore, when (19) holds, the greatest lower bounds of the distributions' supports are zero by Lemma 1(c). Therefore, in any equilibrium in which $\sum_{j \in \mathcal{J}_t} h_{j,t} > N_t$, the population distributions $n_{L,t}$ and $n_{H,t}$ uniquely determine the local equilibrium in city t.

We characterize this local equilibrium when a positive measure of households from each education group choose each type of occupied housing. Such equilibria are the focus of our comparative statics analysis in Section 2 and our estimation in Section 3. The other case, in which a positive measure of only one education group chooses some types of housing, complicates the comparative statics analysis and does not hold for the data we analyze in Section 5.

For each chosen quality level—that is, for each $j \geq j_{0,t}$—we define $z_{e,j,t}$ to be the greatest lower bound of labor endowments z among households of education e choosing $q_{j,t}$. When $j > j_{0,t}$, $z_{e,j,t}$ also equals the least upper bound of labor endowments among households of education e choosing the quality one step below, $q_{j-1,t}$, because of sorting and because the support of $n_{e,t}$ is convex. A household with this endowment and education level is indifferent between $q_{j,t}$ and $q_{j-1,t}$:

$$\left(w_{e,t} z_{e,j,t} - p_{j,t} \right)^{\beta_{c,e}} q_{j,t}^{\beta_{q,e}} = \left(w_{e,t} z_{e,j,t} - p_{j-1,t} \right)^{\beta_{c,e}} q_{j-1,t}^{\beta_{q,e}} \tag{20}$$

for each $j \in \{j_{0,t} + 1, ..., J_t\}$ and $e \in \{L, H\}$. By Proposition 2, the measure of households choosing each such quality level coincides with the total housing stock available:

$$h_{j,t} = \sum_{e \in \{L,H\}} \int_{z_{e,j,t}}^{z_{e,j+1,t}} n_{e,t}(z) dz \tag{21}$$

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for each $j \in \{j_{0,t} + 1, ..., J_t\}$, where $z_{e,J_t+1,t}$ equals the least upper bound (or ∞, if no upper bound exists) of the labor endowments of households of education e in city t.

By (20), the endowment cutoffs are linear functions of house prices. Substituting these functions into (21) delivers, together with (19), $J_t - j_{0,t} + 1$ equations in $J_t - j_{0,t} + 1$ unknown house prices. As we show in Appendix A.5, these equations always admit a unique solution for house prices, and prices in this solution strictly increase in quality. When the resulting endowment cutoffs also strictly increase in quality, the unique local equilibrium is one in which households from each education group choose each type of housing. The indirect utility of each household in city t is

$$v_{e,t}(z) = u_e(w_{e,t}z - p_{j,t}, q_{j,t}, a_t, 0), \quad z \in (z_{e,j,t}, z_{e,j+1,t}]. \tag{22}$$

2 Equilibrium effects of construction

In this section, we study the effects of constructing housing in a single city, t^*, on equilibrium house prices and welfare there. Constructing housing of quality q_{j,t^*} in city t^* corresponds to increasing h_{j,t^*}. We interpret such construction as the outcome of a relaxation of permitting rules for housing of quality q_{j,t^*}. House prices exceed the marginal costs of land and structure in many metropolitan areas (Glaeser and Gyourko, 2003), suggesting that easier permitting would lead developers to build more housing.

We later explore the equilibrium effects of exogenous changes to amenities and productivity. To encompass these changes as well, we study the equilibrium effect of marginally increasing each h_{j,t^*} by $\delta_{h,j}$, each $\log \widetilde{A}_{e,t^*}$ by $\delta_{A,e}$, and $\log a_{t^*}$ by δ_a. We denote the combined equilibrium effect of these changes by ∂, which we call a comparative static.

The effect of these changes on house prices consists of $\partial p_{j,t^*}$ for $j \in \mathcal{J}_{t^*}$. For households who strictly prefer t^* to all other cities, the effect on the log of their indirect utility equals $\partial \log v_{e,t^*}(z)$. This effect does not depend on the household's idiosyncratic taste for the city, ϵ_{t^*}, which appears as an additive constant in the log of the household's indirect utility. We adopt $\partial \log v_{e,t^*}(z)$ as our measure of the effect of the changes in t^* on welfare.

2.1 Equilibrium assumptions

To avoid edge cases, we proceed under two assumptions about the equilibrium around which we compute comparative statics. First, a positive measure of households of each education choose each occupied quality of housing in city t^*:

Assumption 2. *For each $e \in \{L, H\}$ and $j \in \{j_{0,t^*}, ..., J_{t^*}\}$, $x_{e,j,t^*} > 0$.*

Due to Assumption 2, we may differentiate (20) and (21) to calculate comparative statics. The data we analyze in Section 5 satisfy this assumption. Second, some of the lowest quality occupied housing remains vacant:

Assumption 3. $x_{j_{0,t^*},t^*} < h_{j_{0,t^*},t^*}$.

If this lowest quality, q_{j_{0,t^*},t^*}, represents outdoor locations, then Assumption 3 holds as long as there remain some unoccupied outdoor locations where households could feasibly reside. In the data we present in Section 4, we identify q_{j_{0,t^*},t^*} as locations where the homeless live. Assumption

3 implies that $\sum_{j \in \mathcal{J}_t} h_{j,t} > N_t$. As a result, (19) holds, and n_{L,t^*} and n_{H,t^*} uniquely determine local equilibrium by Lemma 2.

2.2 Local approximation

An exact solution for comparative statics in city t^* necessitates computing comparative statics in all other cities as well. This interconnectedness is apparent from (14), which shows that a change to $v_{e,t^*}(z)$, the indirect utility in t^*, alters the population levels in all other cities, $n_{e,t}(z)$. These population changes move indirect utilities, $v_{e,t}(z)$, and these changes feed back to the population levels in t^*, $n_{e,t^*}(z)$.

Computing comparative statics in every city substantially complicates both the theoretical and quantitative analysis. To avoid these complications, we propose an approximation that allows us to solve for comparative statics in t^* exclusively using the equilibrium allocations in t^*:

$$\partial \log \overline{v}_e(z) \approx 0 \tag{23}$$

for each $e \in \{L, H\}$ and z such that $n_{e,t^*}(z) > 0$. When (23) holds, changes in t^* do not affect the average utility in the economy of households of a given education and labor endowment, or they affect this average only a small amount. Models with perfect mobility (e.g., Roback, 1982), sometimes make the analogous approximation that changes in one city do not affect household utility. This approximation may hold because the city is a relatively small share of the economy.

2.3 Derivatives of equilibrium conditions

To solve for the effects on house prices and welfare, we differentiate the model's equilibrium conditions to obtain a system of linear equations that we can directly solve.

Differentiating (14) while applying (23) and Lemma 1(d) yields

$$\partial \log n_{e,t^*}(z) = \beta_{\epsilon,e}^{-1} \partial \log v_{e,t^*}(z) \tag{24}$$

for each $e \in \{L, H\}$ and z such that $n_{e,t^*}(z) > 0$. Due to the approximation in (23), changes to the population in city t^* depend only on changes to indirect utility there and not in any other city. The population of households of a given education and labor endowment moves in the same direction as their indirect utility, rising if it rises and falling if it falls. This relation is stronger when $\beta_{\epsilon,e}$ is smaller because the idiosyncratic city taste, ϵ, has less of an effect on utility.

Differentiating (2) produces

$$\partial \log a_{t^*} = \delta_a + \gamma_a \partial \log N_{H,t^*} - \gamma_a \partial \log N_{L,t^*}. \tag{25}$$

The city's amenities increase with an exogenous shock, δ_a. In the presence of amenity spillovers—that is, when $\gamma_a > 0$—amenities also increase when the population of high-education households endogenously rises or when the population of low-education households endogenously falls.

Differentiating (11) delivers

$$\partial \log A_{e,t^*} = \delta_{A,e} + (\gamma_N - \gamma_H) \frac{N_{L,t^*}}{N_t^*} \partial \log N_{L,t^*} + \left(\gamma_N \frac{N_{H,t^*}}{N_{t^*}} + \gamma_H \frac{N_{L,t^*}}{N_{t^*}} \right) \partial \log N_{H,t^*} \tag{26}$$

for each $e \in \{L, H\}$. Productivity increases with an exogenous shock, $\delta_{A,e}$. If population spillovers are stronger than human capital spillovers, then $\gamma_N > \gamma_H$ and productivity also increases when the population of low-education households endogenously rises. Under the reverse, $\gamma_N < \gamma_H$ and productivity falls when the population of low-education households rises. Productivity rises along with the population of high-education households as long as $\gamma_N > 0$ or $\gamma_H > 0$.

Differentiating (12) gives

$$\partial \log w_{e,t^*} = \partial \log A_{e,t^*} + (1-\rho)\frac{Y_{\sim e,t^*}}{Y_{t^*}}\partial \log\left(\frac{Z_{\sim e,t^*}}{Z_{e,t^*}}\right) + (1-\rho)\frac{Y_{\sim e,t^*}}{Y_{t^*}}\partial \log\left(\frac{A_{\sim e,t^*}}{A_{e,t^*}}\right) \quad (27)$$

for each $e \in \{L, H\}$, where $\sim e$ denotes the other education group, $Y_{e,t^*} = w_{e,t^*}Z_{e,t^*}$ is the total income of households in that group, and $Y_{t^*} = Y_{L,t^*} + Y_{H,t^*}$ is the total income of the city's households. When $\rho = 1$, high- and low-education labor are perfect substitutes, so each labor price depends only on that labor's productivity. When $\rho < 1$, each labor price increases with the relative scarcity of that type of labor in the city, given by the endogenous ratio $Z_{\sim e,t^*}/Z_{e,t^*}$. It also decreases with any relative increase in that labor's productivity, $A_{\sim e,t^*}/A_{e,t}$, meaning that an increase in $\log A_{e,t^*}$ raises $\log w_{e,t^*}$ more when a similar shock occurs to $\log A_{\sim e,t^*}$.

Equations (25)–(27) express changes to amenities and labor prices in terms of changes to the total populations and labor endowments of each education group in city t^*. The changes in these totals come from aggregating (24), which gives the change in the population of every household in the city by education e and labor endowment z. Integrating (24) over all households in each education group yields

$$\partial \log N_{e,t^*} = N_{e,t^*}^{-1} \int_{z|n_{e,t^*}(z)>0} \partial \log n_{e,t^*}(z)n_{e,t^*}(z)dz, \quad (28)$$

which states that the total log change equals the average of the log changes for households with each labor endowment. The change in the total labor endowment follows a similar formula, but now the average involves weighting by the income of each household:

$$\partial \log Z_{e,t^*} = Y_{e,t^*}^{-1} \int_{z|n_{e,t^*}(z)>0} \partial \log n_{e,t^*}(z)y_{e,t^*}(z)n_{e,t^*}(z)dz. \quad (29)$$

The remaining equilibrium conditions are those determining house prices and indirect utilities. We differentiate these separately in the divisible and invisible cases.

2.3.1 Divisible housing

By Proposition 1, the change in equilibrium house prices is given by

$$\partial \log p_{j,t^*} = \partial \log \mu_{t^*} \quad (30)$$

for each $j \in \mathcal{J}_{t^*}$. Across all quality levels present in the city, log house prices move lock-step according to fluctuations in the city's price-to-quality ratio, μ_{t^*}. Proposition 1 allows us to derive the change in this ratio as

$$\partial \log \mu_{t^*} = -\frac{\sum_{j\in\mathcal{J}_{t^*}} \delta_{h,j}q_{j,t^*}}{\sum_{j\in\mathcal{J}_{t^*}} h_{j,t^*}q_{j,t^*}} + \sum_{e\in\{L,H\}} \frac{(\beta_{c,e}+\beta_{q,e})^{-1}\beta_{q,e}Y_{e,t}(\partial \log w_{e,t}+\partial \log Z_{e,t})}{(\beta_{c,L}+\beta_{q,L})^{-1}\beta_{q,L}Y_{L,t}+(\beta_{c,H}+\beta_{q,H})^{-1}\beta_{q,H}Y_{H,t}}. \quad (31)$$

The first term gives the log change to the total housing quality present in city t^*. An log increase to this quantity enters the equation negatively. The effect of building a unit of housing is more negative when that unit is of larger quality, as the new unit then has a larger effect on the total housing quality in the city. The second term gives the log change in the average income in the city. Households spend more on housing when they have more income, so this term is positive.

To calculate the change in indirect utility, we differentiate (17) to obtain

$$\partial \log v_{e,t^*}(z) = (\beta_{c,e} + \beta_{q,e})\partial \log w_{e,t^*} - \beta_{q,e}\partial \log \mu_{t^*} + \beta_{a,e}\partial \log a_{t^*} \tag{32}$$

for all $z > 0$ and for each $e \in \{L, H\}$. Indirect utility rises with labor prices, falls with the house price-to-quality ratio, and rises with amenities. These changes do not depend on the labor endowment z, so all households of the same education experience the same log change in welfare.

2.3.2 Indivisible housing

In equilibrium, the measure of households choosing each quality level above the lowest occupied one coincides with the measure of housing at that quality level. Differentiating this condition, which appears in (21), gives

$$\delta_{h,j} = \sum_{e \in \{L,H\}} \left(\underbrace{z_{e,j+1,t^*} n_{e,t^*}(z_{e,j+1,t^*})\partial \log z_{e,j+1,t^*}}_{\text{trickle-up}} - \underbrace{z_{e,j,t} n_{e,t^*}(z_{e,j,t})\partial \log z_{e,j,t^*}}_{\text{trickle-down}} \right. \tag{33}$$
$$\left. + \underbrace{\int_{z_{e,j,t^*}}^{z_{e,j+1,t^*}} \partial \log n_{e,t^*}(z) n_{e,t^*}(z)dz}_{\text{migration}} \right)$$

when $j_{0,t^*} < j < J_{t^*}$. When $j = J_{t^*}$, (33) holds without the trickle-up term. Three forces combine to absorb the $\delta_{h,j}$ units of new housing. The first, which we call trickle-up, involves marginal households from the next highest quality level switching down to q_{j,t^*}. The second, which we call trickle-down, involves marginal households from the next lowest quality level switching up to q_{j,t^*}. The final term, migration, represents households who move to the city and choose q_{j,t^*}.

We relate the trickle-up and trickle-down terms to house prices by differentiating (20):

$$\partial \log z_{e,j,t^*} = \frac{(y_{e,j,t^*} - p_{j-1,t^*})\partial p_{j,t^*}}{y_{e,j,t^*}(p_{j,t^*} - p_{j-1,t^*})} - \frac{(y_{e,j,t^*} - p_{j,t^*})\partial p_{j-1,t^*}}{y_{e,j,t^*}(p_{j,t^*} - p_{j-1,t^*})} - \partial \log w_{e,t^*}, \tag{34}$$

where $y_{e,j,t^*} = y_{e,t^*}(z_{e,j,t^*})$, for each $e \in \{L, H\}$ and $j \in \{j_{0,t^*} + 1, ..., J_{t^*}\}$ The endowment cutoff—that is, the left side of (34)—rises when the price of the higher quality level rises or the price of the lower quality level falls. Both price changes induce marginal households to switch from the higher quality to the lower one. The endowment cutoff also rises when the price of the education group's labor, w_{e,t^*}, falls, as a decline in income leads marginal households to switch to the lower quality.

Due to Assumption 3, (19) holds. Furthermore, $x_{j,t^*} < h_{j,t^*}$ continues to hold for $j = j_{0,t^*}$ under perturbations to the local equilibrium given the strict inequality. As a result, the identity of the lowest occupied quality does not change, so

$$\partial p_{j,t^*} = 0 \tag{35}$$

for $j = j_{0,t^*}$.

For households not on the margin between two qualities, differentiating (22) yields

$$\partial \log v_{e,t^*}(z) = \frac{\beta_{c,e} y_{e,t^*}(z) \partial \log w_{e,t^*}}{y_{e,t^*}(z) - p_{j,t^*}} - \frac{\beta_{c,e} \partial p_{j,t^*}}{y_{e,t^*}(z) - p_{j,t^*}} + \beta_{a,e} \partial \log a_{t^*}, \qquad z \in (z_{e,j,t^*}, z_{e,j+1,t^*}) \tag{36}$$

for each $e \in \{L, H\}$. A household's welfare rises with the price of its labor and with the amenities in the city. Welfare falls with the price of the housing the household currently consumes. Any changes in the prices of other qualities of housing have no direct effect on the household's welfare. The change for marginal households depends on whether the corresponding endowment cutoff increases or decreases. If $\partial \log z_{e,j,t^*} > 0$, then the marginal households all choose the lower quality level as a result of the changes to primitives, so the relevant house price for them in (36) is p_{j-1,t^*}. If $\partial \log z_{e,j,t^*} < 0$, then the marginal households all choose the higher quality level as a result of the changes to primitives, so the relevant house price in (36) is p_{j,t^*}.

2.4 When trickle-down economics works

We present two special cases in which trickle-down housing economics works. That is, construction improves the welfare of all households in city t^*, and a single new housing unit improves welfare most when the quality of the unit is $q_{J_{t^*},t^*}$, the highest quality available. Both properties fail in the model we estimate in Section 5. For the rest of this section, $\delta_{A,L} = \delta_{A,H} = \delta_a = 0$.

2.4.1 Divisible housing

In the divisible case, the total quality-adjusted housing stock, $\sum_{j \in \mathcal{J}_t} q_{j,t} h_{j,t}$, determines local equilibrium. As a result, the effects of construction depend on the change to this total, $\sum_{j \in \mathcal{J}_t} q_{j,t} \delta_{h,j}$. An increase raises household welfare under two conditions. First, the equilibrium is *locally stable*, meaning that perturbations to the city's population raise the welfare of departing households or decrease the welfare of arriving households. A formal definition appears in Appendix B.1. Second, $\gamma_N = 0$, which limits spillovers to those depending on the relative population of households with different education.

Proposition 3. *In the divisible model, for each $e \in \{L, H\}$ and $z > 0$,*

$$\partial \log v_{e,t^*}(z) \propto \sum_{j \in \mathcal{J}_{t^*}} q_{j,t^*} \delta_{h,j}, \tag{37}$$

where $\propto$ denotes proportionality that is positive if the equilibrium is locally stable and $\gamma_N = 0$.

Proof. Appendix B.2. □

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According to Proposition 3, the benefits constructing high quality housing trickle down to lower income households. Furthermore, each household benefits most when the quality of a single new unit is the highest quality in the city. A rich household may move into this new unit, and then poor households may move into partitions of the rich household's vacated unit whose quality exceeds that of their previous housing. This reallocation is one way that the benefits of high quality construction can trickle down to lower income households.

2.4.2 Indivisible housing with nearly no mobility

In the limit without mobility, the outcomes that depend on the city population remain fixed. These fixed outcomes include labor prices and amenities. The effect on welfare, $\partial \log v_{e,t^*}(z)$, depends only on house price changes, as is apparent from (36). Welfare increases more when the price of the housing that a household is choosing decreases more. The following proposition characterizes the effect of construction on house prices and welfare:

Proposition 4. *Suppose $\delta_{h,j} \geq 0$ for $j > j_{0,t^*}$, with at least one strict inequality, in the immobile limit of the indivisible model. For each e and $z \geq z_{e,j_{0,t^*}+1,t^*}$, $\partial \log v_{e,t^*}(z)/\partial \delta_{h,j'}$ is positive, increases over $j' \in \{j_{0,t^*}+1, ..., j\}$, and stays constant over $j' \in \{j, ..., J_{t^*}\}$. If $z < z_{e,j_{0,t^*}+1,t^*}$, then $\partial \log v_{e,t^*}(z) = 0$.*

Proof. Appendix B.3. □

An analogous result appears in Section 3C of Braid (1981).

The benefits of constructing high quality housing again trickle down to lower income households, with the exception of non-marginal households choosing the lowest unoccupied quality, who are indifferent. For other households, the decline in one's house price is largest when construction occurs at the city's highest quality, $q_{J_{t^*},t^*}$. As in the divisible case, the strict Pareto optimum for constructing a set amount of housing is to build all of it at this highest quality. Another similarity is that constructing any unit whose price is positive lowers all positive house prices and increases the welfare of all households choosing housing with a positive price.

The mechanism behinds these results is similar to the one in the divisible case. When construction occurs at a high quality, the price of this housing falls so that some households on the margin with the next lowest quality choose the new housing. This choice creates vacancy at the next lowest quality, so the price of this housing falls to induce poorer marginal households to switch. This process continues down to the lowest occupied quality, lowering the price of all occupied housing in the city above the lowest level. Even the prices of housing of quality higher than the construction must fall; otherwise, richer households would choose to switch down to the new housing, which Proposition 2 rules out. However, a given inframarginal household benefits most when construction occurs at or above the quality it currently chooses.

3 Estimation strategy

To solve the system of linear equations determining comparative statics in the indivisible case, we require values of the following parameters: ρ, γ_a, γ_N, γ_H, $\beta_{c,e}/\beta_{\epsilon,e}$, and $\beta_{a,e}/\beta_{\epsilon,e}$ for $e \in \{L, H\}$. We also require values of the following equilibrium outcomes: $N_{e,t^*}/N_{t^*}$, $Y_{e,t^*}/Y_{t^*}$, y_{e,j,t^*}, and p_{j,t^*} for $e \in \{L, H\}$ and $j \in \{j_{0,t^*}, ..., J_{t^*}\}$. Two other equilibrium outcomes appear in the equations: z_{e,j,t^*}

and n_{e,t^*} for $e \in \{L, H\}$ and $j \in \{j_{0,t^*}, ..., J_{t^*}\}$. By changing variables from the labor endowment, z, to income, y, we replace these outcomes with the probability density functions of income within each education group, f_{L,t^*} and f_{H,t^*}. Details of this change in variables appear in Appendix C.1.

The remainder of Section 3 describes how we estimate the parameters and equilibrium outcome we need using household-level data from city t^* and prior estimates from the literature.

3.1 Observations

We observe a representative sample of households indexed by i in a single metropolitan area with the following data. First is a sample weight, g_i. Second is a dummy variable, e_i, equal to one if the household has education H and zero if the household has education L. Third is the income that the household reports for the prior year, y_i. Fourth is a categorical variable, $o_i \in \{-1, 0, 1\}$, giving the status of the household's ownership of its place of residence. This variable equals minus one for households who reside in a house of the lowest occupied quality, q_{j_{0,t^*}, t^*}. For households residing in houses of higher quality, o_i equals zero for renters and one for owner-occupants. When $o_i = 0$, we observe the monthly rent that the household pays, r_i, and when $o_i = 1$, we observe the value of the house, v_i.

3.2 House prices and housing demands

Our first step is imputing an annual price of housing for each household, which we call p_i. Due to the assumption above, we know from the model that $p_{j_{0,t^*}, t^*} = 0$, so we set $p_i = 0$ when $o_i = -1$. We set $p_i = 12r_i$ when $o_i = 0$, meaning we multiply the monthly rent by twelve. For owner-occupants, the annual price of housing equals the cost of capital times v_i, plus maintenance costs and taxes, less expected capital gains. If the latter are proportional to v_i, then there exists some constant, ϕ, such that the annual price for each owner-occupant is $p_i = \phi v_i$. We assume that such proportionality holds, and we take ϕ from data we present in Section 4.

The second step is assigning households, i, to housing quality indices, j. In the absence of measurement error, each distinct value of the annual house price, p_i, corresponds to a distinct quality by Proposition 2. Measurement error may arise for a variety of reasons, including misreporting and unmodeled search frictions leading to price dispersion for similar housing units. To smooth this error, we assign quality indices by binning households along annual house prices, p_i. Specifically, we set $j(i) = 0$ when $o_i = -1$. When $o_i \in \{0, 1\}$, we set $j(i)$ equal to the quantile of p_i among households in this group, out of a total of 50 quantiles. We assume that $\mathcal{J}_t = \{0, ..., 50\}$, meaning that we observe households in every occupied quality level. We estimate the price for each quality as the average of p_i among households choosing that quality: $\widehat{p}_{j,t^*} = \sum_i g_i \delta_{i,j} p_i / \sum_i g_i \delta_{i,j}$, where $\delta_{i,j}$ is an indicator equal to one if and only if $j(i) = j$. For each $j \in \mathcal{J}_{t^*}$, our estimate of the share of households choosing housing of quality q_{j,t^*} is $\sum_i g_i \delta_{i,j} / \sum_i g_i$.

3.3 The distribution of households

To fit income distributions to the data, we specify f_{L,t^*} and f_{H,t^*} as double Pareto-lognormal distributions, a four-parameter family that Reed (2003) and Reed and Jorgensen (2004) propose to characterize income distributions. Because it allows for Pareto behavior in both the upper and lower tails, the double Pareto-lognormal distribution describes income distributions better than

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the lognormal distribution.

Proposition 2 sharply restricts the joint distribution of income and housing quality by requiring that $y_i \geq y_{i'}$ whenever $j(i) \geq j(i')$ and $e_i = e_{i'}$. The data violate this restriction if there are any pairs of households of the same education in which the poorer household lives in a higher quality house than the richer household. To fit the model to data that may violate this restriction, we assume that the income we observe, y_i, does not necessarily equal the income from the model, $y_{e_i,t^*}(z_i)$, where z_i is the unobserved labor endowment of household i. Instead, y_i equals $y_{e_i,t^*}(z_i)$ plus noise. This noise may come from temporary fluctuations in income that, due to adjustment costs, do not cause households to change the quality of the housing they choose (Chetty and Szeidl, 2007). Our empirical strategy identifies several parameters of the model under the following assumption about this noise:

Assumption 4. $0 = E\delta_{i,j}e_i(y_i - y_{e_i,t^*}(z_i)) = E\delta_{i,j}(1 - e_i)(y_i - y_{e_i,t^*}(z_i))$ *for each* $j \in \mathcal{J}_{t^*}$.

By Assumption 4, the noise not only has a mean of zero but also is uncorrelated with a household's education, housing choice, and the interaction of the two.

We exploit Assumption 4 to estimate several of the remaining parameters using the generalized method of moments (Hansen, 1982). These parameters consist of the eight double Pareto-lognormal parameters as well as the rent-to-price ratio, ϕ, the population shares of each education group, $N_{L,t^*}/N_{t^*}$ and $N_{H,t^*}/N_{t^*}$, and the ratio $(\beta_{q,L}/\beta_{c,L})/(\beta_{q,H}/\beta_{c,H})$, which we call ζ. This ratio describes the relative preference of low-education households for housing versus non-housing consumption relative to high-education households. We denote the vector of these twelve parameters by θ, and we denote the set over which we search for this vector by Θ.

Given our estimates of house prices and housing demands, the components of θ uniquely determine the incomes of the households on the margin between the occupied quality levels, y_{e,j,t^*} for $e \in \{L, H\}$ and $j \in \{1, ..., 50\}$. To illustrate this correspondence, we divide (21) by the city population, N_{t^*}, and change variables from z to y to obtain

$$\frac{\sum_i g_i \delta_{i,j}}{\sum_i g_i} = \sum_{e \in \{L,H\}} \frac{N_{e,t^*}}{N_{t^*}} \int_{y_{e,j,t^*}}^{y_{e,j+1,t^*}} f_{e,t^*}(y)dy \tag{38}$$

for each $j \in \{1, ..., 50\}$. We also equate the two solutions to (20) for $q_{j,t^*}/q_{j-1,t^*}$:

$$\left(\frac{y_{L,j,t^*} - \widehat{p}_{j,t^*}}{y_{L,j,t^*} - \widehat{p}_{j-1,t^*}} \right)^{\frac{\beta_{c,L}}{\beta_{q,L}}} = \left(\frac{y_{H,j,t^*} - \widehat{p}_{j,t^*}}{y_{H,j,t^*} - \widehat{p}_{j-1,t^*}} \right)^{\frac{\beta_{c,H}}{\beta_{q,H}}} \tag{39}$$

for each $j \in \{1, ..., 50\}$. For certain θ, these equations give unique solutions for the income cutoffs:

Lemma 3. *If*

$$\frac{\sum_{j' \in \{j,...,50\}} \sum_i g_i \delta_{i,j'}}{\sum_i g_i} \leq \sum_{e \in \{L,H\}} \frac{N_{e,t^*}}{N_{t^*}} \int_{\widehat{p}_{j,t^*}}^{\infty} f_{e,t^*}(y)dy \tag{40}$$

for each $j \in \{1, ..., 50\}$, *then unique values of* y_{e,j,t^*} *for* $e \in \{L, H\}$ *and* $j \in \{1, ..., 50\}$ *solve* (38)–(39). *When* $\zeta = 1$, *these values strictly increase in* j *for each* e.

Proof. Appendix C.2. □

Lemma 3 guarantees unique solutions for the income cutoffs when θ satisfies (40), which states, for each j, that the share of housing of quality at least q_{j,t^*} in the data cannot exceed the share of households in the model with income at least $\widehat{p}_{j,t^*}$. Assumption 2 requires that the resulting income cutoffs strictly increase in j for each e. To abide by this restriction, we limit Θ to θ that satisfy (40) and for which the resulting income cutoffs strictly increase in j. Because monotonicity holds when $\zeta = 1$, Θ is nonempty. For computational reasons, we further restrict Θ to lie within a neighborhood of our initial guess, θ_0.

To estimate θ, we compare several conditional expectations of income and education in the model to the data. In the model, the average income of households of education e choosing housing quality q_{j,t^*}, which we denote $\overline{y}_{e,j,t^*}(\theta)$, equals the conditional mean under the distribution f_{e,t^*} over the interval $[y_{e,j,t^*}(\theta), y_{e,j+1,t^*}(\theta))$. Similarly, the average education of households choosing housing quality q_{j,t^*}, which we denote $\overline{e}_{j,t^*}$, equals the ratio of the measure under f_{e,t^*} of the interval $[y_{e,j,t^*}(\theta), y_{e,j+1,t^*}(\theta))$ to the sum of these measures across both values of e. The following moment conditions equate empirical realizations of these conditional expectations to their model-based counterparts:

$$0 = E\delta_{i,j}(1 - e_i)(y_i - \overline{y}_{L,j,t^*}(\theta)) \tag{41}$$

$$0 = E\delta_{i,j}e_i(y_i - \overline{y}_{H,j,t^*}(\theta)) \tag{42}$$

$$0 = E\delta_{i,j}(e_i - \overline{e}_{j,t^*}(\theta)) \tag{43}$$

for each $j \in \mathcal{J}_{t^*}$. These moment conditions hold due to Assumption 4.

As in Hansen (1982), we estimate θ by minimizing a quadratic form of the realizations of these moments in the data. The covariance of each pair of distinct moments equals zero, so a valid weighting matrix for the estimation is the diagonal matrix consisting of the inverses of the sample variances of each moment under our initial guess, θ_0. The resulting estimator is then

$$\theta(\theta_0) = \arg\min_{\theta \in \Theta} \sum_{j \in \mathcal{J}_t} \frac{(\sum_i g_i \delta_{i,j}(1 - e_i)(y_i - \overline{y}_{L,j,t^*}(\theta)))^2}{\sum_i g_i \delta_{i,j}(1 - e_i)(y_i - \overline{y}_{L,j,t^*}(\theta_0))^2}$$
$$+ \frac{(\sum_i g_i \delta_{i,j}e_i(y_i - \overline{y}_{H,j,t^*}(\theta)))^2}{\sum_i g_i \delta_{i,j}e_i(y_i - \overline{y}_{H,j,t^*}(\theta_0))^2} + \frac{(\sum_i g_i \delta_{i,j}(e_i - \overline{e}_{j,t^*}(\theta)))^2}{\sum_i g_i \delta_{i,j}(e_i - \overline{e}_{j,t^*}(\theta_0))^2}. \tag{44}$$

We iterate the generalized method of moments estimation by using $\theta(\theta_0)$ as the initial guess to arrive at a final estimate of $\widehat{\theta} = \theta(\theta(\theta_0))$. This estimator chooses the value of θ that best fits the joint distribution of income, education, and housing quality we observe in the data.

3.4 Remaining parameters

We take the production function parameters—ρ, γ_N, and γ_H—from prior estimates in the literature. The remaining parameters we need are $\beta_{c,L}/\beta_{\epsilon,L}$, $\beta_{c,H}/\beta_{\epsilon,H}$, $\beta_{a,L}/\beta_{\epsilon,L}$, $\beta_{a,H}/\beta_{\epsilon,H}$, and γ_a.

We rely on three sets of estimates for these remaining parameters. To describe these existing estimates, we let $\beta_{w,e} = \beta_{c,e} + \beta_{q,e}$ denote the sum of the utility weights on non-housing and housing consumption for each education $e \in \{L, H\}$. The first estimate is the average of the ratio $\beta_{q,e}/\beta_{w,e}$ in a subset of the population. This estimate, which we denote α, equals the average share of income spent on housing in the divisible model. In the indivisible model, this ratio satisfies the equation

$$1 - \alpha = (1 - \overline{e}_\alpha)\frac{\beta_{c,L}}{\beta_{w,L}} + \overline{e}_\alpha \frac{\beta_{c,H}}{\beta_{w,H}}, \tag{45}$$

where $\bar{e}_\alpha$ equals the share of households of education H among the subset in which α is the average value of $\beta_{q,e}/\beta_{w,e}$. This education share constitutes the second of the three estimates. Together with the equation defining ζ, (45) determines the ratios $\beta_{c,L}/\beta_{w,L}$ and $\beta_{c,H}/\beta_{w,H}$:

Lemma 4. *Given ζ, unique values of $\beta_{c,L}/\beta_{w,L}$ and $\beta_{c,H}/\beta_{w,H}$ jointly in $(0,1)$ solve (45).*

Proof. Appendix C.3. □

Using our estimate of ζ from $\widehat{\theta}$, we solve for these unique values of $\beta_{c,L}/\beta_{w,L}$ and $\beta_{c,H}/\beta_{w,H}$.

The final source is a paper that jointly estimates $\beta_{w,L}/\beta_{\epsilon,L}$, $\beta_{w,H}/\beta_{\epsilon,H}$, $\beta_{a,L}/\beta_{\epsilon,L}$, $\beta_{a,H}/\beta_{\epsilon,H}$, and γ_a, the vector of which we denote ψ. These joint estimates identify the remaining parameters when we multiply the first two estimates by the ratios $\beta_{c,L}/\beta_{w,L}$ and $\beta_{c,H}/\beta_{w,H}$. The equilibrium under all of the parameters we have estimated may not be stable. The paper whose estimates we are using provides a sampling distribution for ψ in addition to point estimates. To ensure stability, we sample 10,000 times from the sampling distribution for ψ and use the mean of the estimates for which the marginal effects of δ_a on a_{t^*} and of each $\delta_{A,e}$ on A_{e,t^*} are positive.

4 Data

4.1 American Community Survey

4.1.1 Data description

Household-level data come from the American Community Survey (ACS), which the U.S. Census Bureau has conducted annually since 2005 to provide current economic information about the United States (see U.S. Census Bureau, 2014 for the most recent available documentation). We use the public use microdata sample that is part of the Integrated Public Use Microdata Series (Ruggles et al., 2018). The data are a weighted random sample of the U.S. population.

The Census Bureau categorizes observations depending on the type of residence where the surveyed person lives. The names of the two types of residences are "housing units" and "group quarters." When a sampled person resides at a housing unit, we observe all persons living at that place of residence. The respondent designates one of these persons—someone owning or renting the unit, if possible—as the "householder." We observe no other linked persons in the case of a person residing in group quarters.

Group quarters consist of institutional (e.g., prisons and hospitals) and non-institutional facilities. Non-institutional group quarters fall into one of three types: college dormitories, military facilities, and "other." The ACS excludes seven sub-categories of other non-institutional group quarters. The non-excluded sub-categories are homeless shelters, religious group quarters, adult group homes, adult residential treatment centers, and workers' living quarters (U.S. Census Bureau, 2012). For sampled persons in other non-institutional group quarters, the ACS sample weights reflect the full population in other non-institutional group quarters. To construct these weights, the Census Bureau uses population estimates from the decennial census, which counts persons in both the excluded and included sub-categories.

We aggregate all persons in a housing unit into a single household observation, i, while a person in group quarters constitutes a single observation. We use the "household weight" variable,

which assigns a weight to each group quarters person and housing unit, as g_i. We use total personal income, which is available for all persons at least 15 years old, as y_i for persons in group quarters. For each housing unit, we use the sum of this variable across persons as y_i. We set $e_i = 1$ if a person in group quarters or the householder of a housing unit has a bachelor's degree. For housing units, we set $o_i = 0$ if the residents rent the housing unit and $o_i = 1$ if the residents own or are purchasing the housing unit. We set $o_i = -1$ for persons in group quarters. For renters paying cash rent, we use monthly contract rent as r_i. We use the survey respondent's estimate of the house value as v_i when $o_i = 1$. The Census Bureau top-codes rent and home values above separate thresholds in each state with the average of the variable above that threshold in that state. Finally, we label a housing unit as new construction if the household reports the construction year as the one directly before the survey year.

4.1.2 Sample selection

For renter households not paying cash rent, we do not observe the value of any non-cash goods and services they pay to reside in their housing unit. Because our estimation strategy depends on observing this information, we drop such households.

The group quarters observations we keep should correspond to persons living in the lowest quality housing in each city. Ideally, we could limit group quarters persons to the homeless, but we do not observe homelessness in the ACS. To attempt this refinement, we drop persons in institutional group quarters, and then among the remaining group quarters population, we drop minors, students, and the employed (which includes the armed forces). The remaining persons likely reflect adults in other non-institutional group quarters who do not have a job.

We limit our estimation sample to the Boston-Cambridge-Newton, MA-NH metropolitan area in 2016 (the most recent data year when we starting writing this paper). Among the largest 20 metropolitan areas in the United States in 2016, Boston has some of the lowest construction activity and highest home values and rents. Some of the large metropolitan areas more extreme than Boston in these dimensions are New York-Newark-Jersey City, NY-NJ-PA and San Francisco-Oakland-Hayward, CA. We choose Boston over these metropolitan areas because New York and California allow rent control while Massachusetts and New Hampshire prohibit it. Our model assumes that a competitive market determines rents.

4.1.3 Summary statistics

Table 1 lists summary statistics for the data from the Boston metropolitan area in 2016. As shown in Panel A, we drop about 91% of the group quarters persons when selecting our estimation sample. This share is large because most group quarters persons reside in an institution or are minors, students, or employed. In contrast, we drop only about 3% of renter households, corresponding to those not paying cash rent. We keep all owner-occupant households.

Panel B shows weighted means of the variables we use. Income of owner-occupant households is about twice that of renter households, on average. The mean income of group quarters persons is much lower, at $7,531. Our flag for a college degree is also highest for owner-occupant households and lowest for group quarters persons. Among renters, the average rent is $1,284, and the Census Bureau censors less than 2% of rents. Among owner-occupants, the average home value is $491,724, with the Census Bureau censoring less than 1% of home values. Almost all

of our weighted sample consists of renter and owner-occupant households, with about twice as many owner-occupant households as renters. The total number of unweighted observations in the estimation sample is 18,269.

4.2 Real Capital Analytics

Real Capital Analytics (RCA) provides quarterly estimates of the capitalization rate (i.e., the annual income return) for multifamily rental properties with at least ten units in the Boston metropolitan area that sold in 2016. RCA also provides quarterly estimates of the rental revenue and net operating income per square foot for multifamily properties held by institutional investors. For each quarter in 2016, we calculate the annual rental return by multiplying the capitalization rate by the institutional rental revenue per square foot and then dividing by the institutional operating income per square foot. The average of the rental yield estimate for each quarter, 0.09, serves as our estimate for ϕ.

4.3 Estimates from other work

Several papers in labor economics estimate the inverse elasticity of substitution between college and non-college labor to be about 0.7 (see the discussion in Card, 2009). This inverse elasticity corresponds to $1 - \rho$, so we set $\rho = 0.3$. Combes and Gobillon (2015) review the empirical literature on productivity spillovers and find that the typical estimate of the elasticity of productivity with respect to population density lies between 0.04 and 0.07. These estimates correspond to γ_N, so we set $\gamma_N = 0.055$, which is the midpoint of this range.

Moretti (2004b) estimates that log output in an industry within a city rises about 0.0055 ("a 0.5–0.6-percentage-point increase") when the college share in other industries in the same city rises by one percentage point. Interpreting this estimate as 100 times the derivative of log productivity with respect to $N_{H,t}/N_t$, we obtain $0.55 = \gamma_H N_t/N_{H,t}$. The college shares in the two years in the sample in Moretti (2004b) are 0.161 and 0.191. Setting $N_{H,t}/N_H$ equal to the average of these two numbers, 0.176, gives us $\gamma_H = 0.097$.[1]

Diamond (2016) estimates $\psi = (\beta_{w,L}/\beta_{\epsilon,L}, \beta_{w,H}/\beta_{\epsilon,H}, \beta_{a,L}/\beta_{\epsilon,L}, \beta_{a,H}/\beta_{\epsilon,H}, \gamma_a)$ using the generalized method of moments. Her data include labor incomes, rental payments, and city choices of workers with and without a college degree in the United States between 1980 and 2000, and her model is close to the divisible housing framework in our paper. We use the estimate from her "full model," corresponding to column 3 of her Table 5 (ignoring differential effects for Blacks and immigrants), which is $\psi = (4.026, 2.116, 0.274, 1.1012, 2.60)$. We obtain the sampling distribution of ψ from her replication files. Under her point estimate, low-education households care more than high-education households about income when choosing a city. The reverse is true for amenities.

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[1]A similar approach to calibrating γ_H uses numbers from Gennaioli et al. (2013), who estimate that an additional year of average schooling in a sub-national region raises log productivity in that region by 0.074. In their data, the average college share is 0.11, the years of schooling for individuals with a college degree is 16, and the average years of schooling for all individuals is 6.52. The average years of schooling among individuals without a college degree is thus $(6.52 - 0.11 * 16)/0.89 = 5.35$. A one percentage point increase in the college share increases log productivity by $0.01 * (16 - 5.35) * 0.074 = 0.0079$ if we assume that the commensurate one percentage point decrease in the non-college share is orthogonal to their years of schooling. Multiplying this estimate by 100 and by the college share (as we did in the Moretti, 2004b calculation) yields $\gamma_H = 0.087$. Reassuringly, this number is close to the one from the Moretti (2004b) calculation. We use the larger estimate, 0.097, to explore the full effects of human capital spillovers in the model.

In the divisible model that Diamond (2016) estimates, $\beta_{q,e}/\beta_{w,e}$ coincides with the share of income that a household of education e spends on housing. Davis and Ortalo-Magné (2011) estimate that renters in the United States between 1980 and 2000 (the setting in which Diamond, 2016 estimates her model) spend about 24% of their income on housing, so we set $\alpha = 0.24$. The final estimate is $\bar{e}_\alpha$, the share of renter households in the United States between 1980 and 2000 in which the householder has a college degree. In the U.S. Census from 1980, 1990, and 2000 (the same data that Davis and Ortalo-Magné (2011) and Diamond (2016) use), $\bar{e}_\alpha = 0.18$.

5 Quantitative results

5.1 Estimation

Table 2 reports our estimates of the model parameters as well as some of the equilibrium outcomes. Households split roughly evenly between low and high education, with about 51% in the former category and 49% in the latter. However, the low education households earn only about one third of the city's income. On average, then, high education households earn about double what low education households earn.

The components of ψ—the vector of parameters whose sampling distribution we take from Diamond (2016)—substantially differ from the point estimates in Diamond (2016) only for γ_a, the amenity spillover. Our equilibrium is unstable under her point estimates.[2] When we take the mean of the resamples that lead to stability, we obtain an estimate for γ_a of 1.103, which is less than half the point estimate of 2.6 from Diamond (2016). The other estimates change much less, indicating that the strength of the amenity spillover in the point estimate prevents stability. These estimates imply that high-education households value amenities versus non-housing consumption 8.20 times more than low-education households do.

Figure 1 displays the average incomes and education shares of households choosing each quality of housing against the price of the corresponding quality. We plot these averages in both the model and the data. Our estimation minimizes the differences between the model and data averages, so the closeness of these outcomes indicates the goodness of fit.

As Panel A shows, the model matches the empirical income averages quite well with the notable exception of over-predicting the incomes of the few low-education households choosing very high quality housing and under-predicting the incomes of the few high-education households choosing very low quality housing. Conditional on housing quality, the incomes of high-education households exceed the incomes of low-education households in both the model and the data. This outcome is consistent with our estimated value of $\zeta = 1.603$, which indicates that low-education households value housing versus non-housing consumption relatively more than high-education households.

Panel B shows that the model likewise closely fits the education shares for each type of housing, with the exception of under-predicting the high-education share choosing very low quality housing and over-predicting the high-education share choosing very high quality housing. Only

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[2]Because our models differ, this instability does not imply that the equilibrium in Diamond (2016) is unstable. In particular, by imposing the local approximation in (23), we assume away economy-wide changes in utility levels from migrations into and out of Boston. Diamond (2016) estimates her model for the entire United States, meaning that such migrations change economy-wide utility levels. As a result, re-allocations of households that cause instability in our approximate model may not do so in her more complete framework.

about 10% of households choosing the lowest quality levels have high education, while more than 80% of the households choosing the highest quality levels have high education.

5.2 Effects of new construction

We estimate the effects of construction in two ways. Figure 2 describes the effects of the prior year's construction in the data, while Figure 3 summarizes the effects of building at different quality levels.

To produce Figure 2, we set $\delta_{h,j}$ equal to the weighted number of new housing units we observe in the data at the quality corresponding to each $j \in \{1, ..., 50\}$. We then calculate comparative statics by solving the linear system of equations from Section 2. The effects we estimate represent the difference between the equilibrium we observe and a counter-factual in which no construction occurred.

Panel A displays the resulting percentage change in each positive house price, which we derive from $\partial \log p_{j,t^*}$ for $j \in \{1, ..., 50\}$. According to these results, construction lowered all positive house prices. Low quality prices display particular sensitivity, with the lowest price falling 8%. However, the average price of housing fell by only 0.82%, and the median house price fell only 1.05%. These numbers are small relative to the 4.0% increase in real house prices from 2015 to 2016 in Boston. According to our estimates, real house prices would have risen about 5.0% without the construction.

In Panel B, we plot the percentage changes corresponding to the average $\beta_{\epsilon,e}^{-1} \partial \log v_{e,t^*}(z)$ among the households of each education group choosing each housing quality, including the lowest occupied one. By (24), these outcomes correspond to the percentage change in the measure of households of education e with $z \in (z_{e,j,t^*}, z_{e,j+1,t^*})$, so we refer to them as population changes. According to our estimates, construction made many rich households worse off. It made poor households better off, with the exception of the poorest households, who choose the lowest occupied quality. Construction decreased the population of high-education households by 0.02% while increasing the population of low-education households by 0.63%. The resulting changes to city-wide outcomes, such as amenities and labor prices, made rich households worse off.

To explore the effects of building at q_{j,t^*}, we set $\delta_{h,j}$ equal to the total quantity of construction in 2015 while keeping the other $\delta_{h,j'}$ equal to zero. This exercise explores the counter-factual in which all construction occurred at q_{j,t^*} instead of at the qualities where it actually occurred. By comparing the outcomes for different values of j, we discover how the quality of construction affects house prices and welfare.

In Panel A of Figure 3, we plot the percentage change in average city-wide house prices for each construction quality, which we represent on the x-axis with the price of the housing where the construction occurs. Construction lowers average house prices between 0.9% and 1% for construction at many low qualities, while lowering house prices between 0.7% and 0.8% for construction at many high qualities. The quality where construction occurs has little effect on average house prices, given the quantity of construction that occurs.

Although the price responses do not depend much on construction quality, welfare responses do. We compute these welfare responses as the changes in the populations of four groups that we form by categorizing households according to whether they are in the top or bottom half of the income distribution and whether they are high or low education. If the rich high-education population falls, for instance, that means that on average, construction makes high-education

households in the top half of the city income distribution worse off.

Panel B displays the changes in each of the four population groups for each construction quality, which we again represent on the x-axis with the price of the housing where the construction occurs. Most low-quality construction makes both rich groups worse off, and very low-quality construction also makes poor high-education households worse off. All three of these groups tend to prefer higher quality construction, which makes them better off. Conversely, poor low-education households prefer lower quality construction. Although even high quality construction makes them better off, low quality construction makes them twice as well off as high quality construction. The choice of construction quality has both winners and losers.

In Figure 4, we investigate the roll of each spillover in generating the results in Figure 3. We replicate Figure 3 while setting each of γ_a, γ_H, and γ_N equal to zero. We include a final panel in which all three spillovers equal zero.

As Figure 4 shows, low-quality construction makes high-education households worse off mostly due to the amenity spillover. When we turn this spillover off, even the lowest quality construction now makes high-education households better off. In contrast, low-quality construction continues to make them worse off when we set either γ_H or γ_N equal to zero. With $\gamma_a = 0$, low-quality construction still makes rich low-education households worse off because they compete in the labor market with households who migrate to the city in response to low-quality construction, who mostly are low-education households. However, this effect is about six times smaller without the amenity spillover than with it.

5.3   Responses to skill-biased productivity shocks

We consider how construction might mitigate the adverse welfare effects of house price growth. One reason house prices have grown so much in certain cities is because productivity has risen while construction has been low (Hsieh and Moretti, 2018). To produce a house price increase in our model, we use non-zero values of $\delta_{A,L}$ and $\delta_{A,H}$, the productivity shifters for low- and high-education labor. Diamond (2016) estimates that, for the period between 1980 and 2000, the values of these productivity shifters are −0.314 and 0.075 for the Boston metropolitan area (see Table A.6 of her online appendix). According to these estimates, the Boston production function changed to make high-education labor more productive and low-education labor less. We annualize these numbers by setting $\delta_{A,L} = -0.0157$ and $\delta_{A,H} = 0.0038$. During this period, real house prices in Boston doubled by growing at an annual rate of 3.78%.

Panels A–C of Figure 5 plot the endogenous house price and population changes that occur in response to this productivity shock. These graphs correspond to the outcomes in Figure 2, although the change to primitives is now a productivity shock rather than construction. We hold the housing stock constant by setting $\delta_{h,j} = 0$ for $j \in \{1, ..., 50\}$, which Panel A displays.

The productivity shock raises house prices, particular for low quality housing. It raises average house prices by 3.20% and median house prices by 4.20%. These changes come close to the empirical annualized house price growth of 3.78% that occurred in the period that Diamond (2016) estimates the productivity shocks. Panel C shows that, in response to this shock, the high-education population increases across the income distribution as households migrate to the city. In contrast, out-migration occurs among low-education households for nearly all incomes. The only exceptions are very rich and extremely poor low-education households, who benefit from the shock.

Panels D–F show the marginal effect on these outcomes from the construction that occurred in 2015. We plot this construction in Panel D by showing the amount of construction at each quality as a share of the total housing stock. In Panel E, we see that the house price response is lower than what occurs without construction, but by only a small amount. The average house price grows by 2.33%, and the median grows by 3.11%. Nearly all low-education households continue to suffer from the combined effect of the productivity shock and the construction, as Panel F shows.

In Panels G–I, we plot these outcomes for a different set of construction amounts, $\delta_{h,j}$ for $j \in \{1, ..., 50\}$. Specifically, we minimize $\delta_{h,1} + ... \delta_{h,50}$ relative to the constraints that $\partial \log v_{e,t^*}(z) \geq 0$ for all $e \in \{L, H\}$ and $z > 0$ and that $\delta_{h,j} \geq 0$ for all $j \in \{1, ..., 50\}$. In so doing, we find the minimal amount of construction such that no household suffers in response to the combined effect of the construction and the productivity shock.

Panel G displays the resulting construction amounts for each quality, $\delta_{h,j}$ for $j \in \{1, ..., 50\}$. The total construction as a share of the housing stock is 1.42%. This amount is about three times greater than the quantity of construction that occurred in 2015, appearing in Panel D and equal to 0.45% of the housing stock. However, a 1.42% annual expansion in the housing stock is typical of many growing metropolitan areas in the United States, and is below the growth rates of many of them.

The construction in Panel G further differs from that in Panel C along the distribution of quality. The distribution is more extreme in Panel G, with a greater share of both the highest quality and the lowest quality levels. Specifically, Panel G has 8.84% of construction occurring at the highest quality, while Panel C has 6.14%. Of the remaining construction, the average value in Panel G equals $246,975, which is much smaller than the corresponding value of $428,835 in Panel C.

Panels H and I report the effects of the productivity shock and the construction in Panel G on house prices and populations. House price growth is now much smaller than in Panels B and E throughout the quality distribution. House prices never rise by more than 2%, with an average increase of 0.31% and a median increase of 0.49%. Panel I shows that the low-education populations remain about the same, while the high-education populations continue to grow. Low-education households now neither suffer or gain, while high-education households continue to gain.

As a robustness check, we perform a similar exercise in which we minimize $p_{1,t^*}\delta_{h,1} + ... + p_{50,t^*}\delta_{h,50}$. It may cost more to construct higher quality housing, and our model does not consider these costs. This minimand takes the equilibrium prices as the costs and seeks to minimize the aggregate construction cost of the new housing. When we repeat the exercise with this new minimand, we again find that optimizing construction involves expanding the housing stock by 1.42%. Construction spreads out a bit more over different high qualities, with only 3.63% of the construction at the highest quality. However, the average price of the new construction is even lower than in Panels D or G, meaning that construction involves building even more low-quality housing.

While the combination of the productivity shock and the construction in Panel G makes no one worse off, the construction itself does. This result is apparent from comparing Panels C and I. Middle- to high-income high-education households are better off in Panel B than in Panel I, as their populations grow more. When we search for a construction vector such that both construction in isolation and construction together with the productivity shock make no one worse off, we fail to find one. This null result suggests that cities face unavoidable trade-offs when deciding the

quality of new construction.

6 Conclusion

To conclude, we return to the topic with which we began: what can stem the rapid growth of house prices in many cities? Like many economists, our analysis recommends expanding the stock of available housing. The novelty of our approach is to consider the quality of construction. We calculate that expanding the housing stock in Boston by 1.4% annually would eliminate welfare losses if the new units primarily are at the lower end of the housing market. This finding provides one rationale for cities to relax permitting rules specifically for low-quality construction. However, building such units makes high-education and high-income households worse off, which might explain why expensive cities have not already relaxed permitting rules for low-income housing enough to eliminate welfare lost by low-income, low-education households.
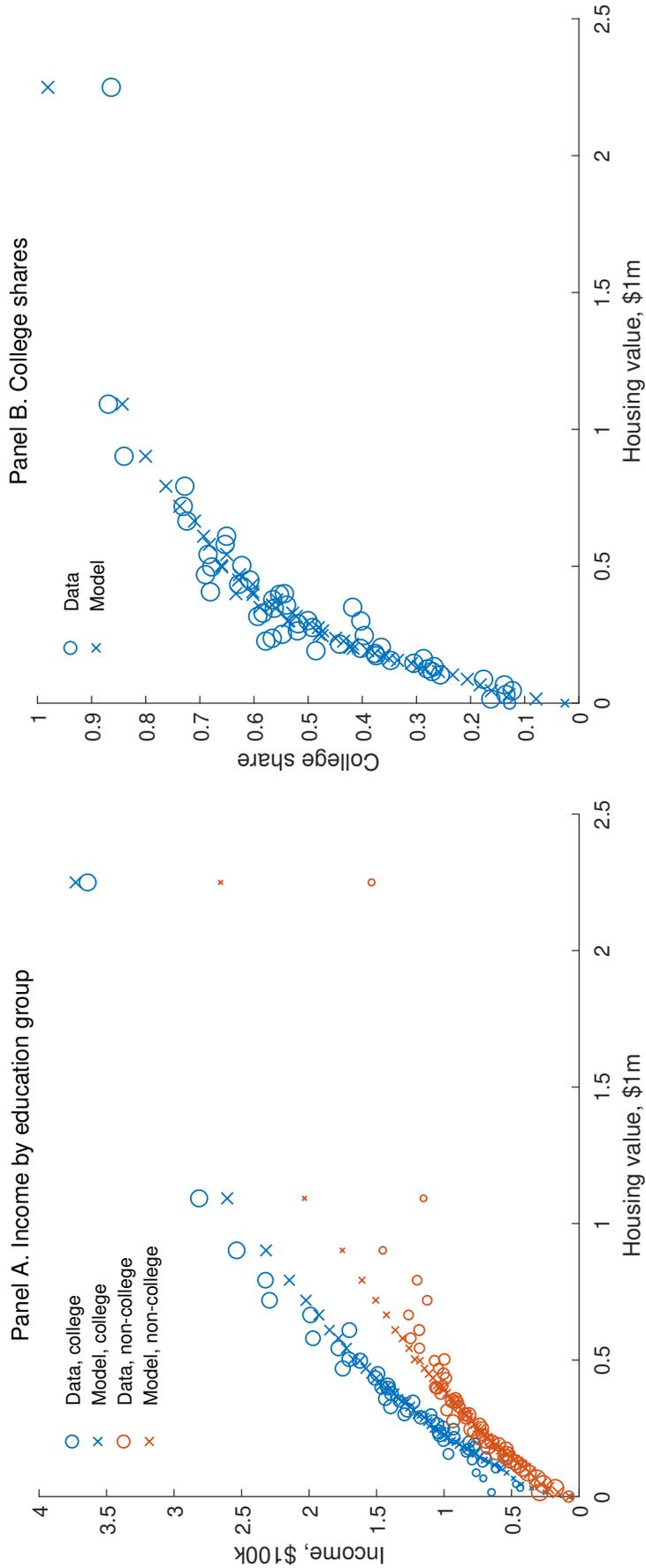
Panel A. Income by education group

Panel B. College shares

FIGURE 1. GOODNESS OF FIT

Notes: The two-dimensional size of each marker is proportional to the number of housing units in each housing value and education bin.
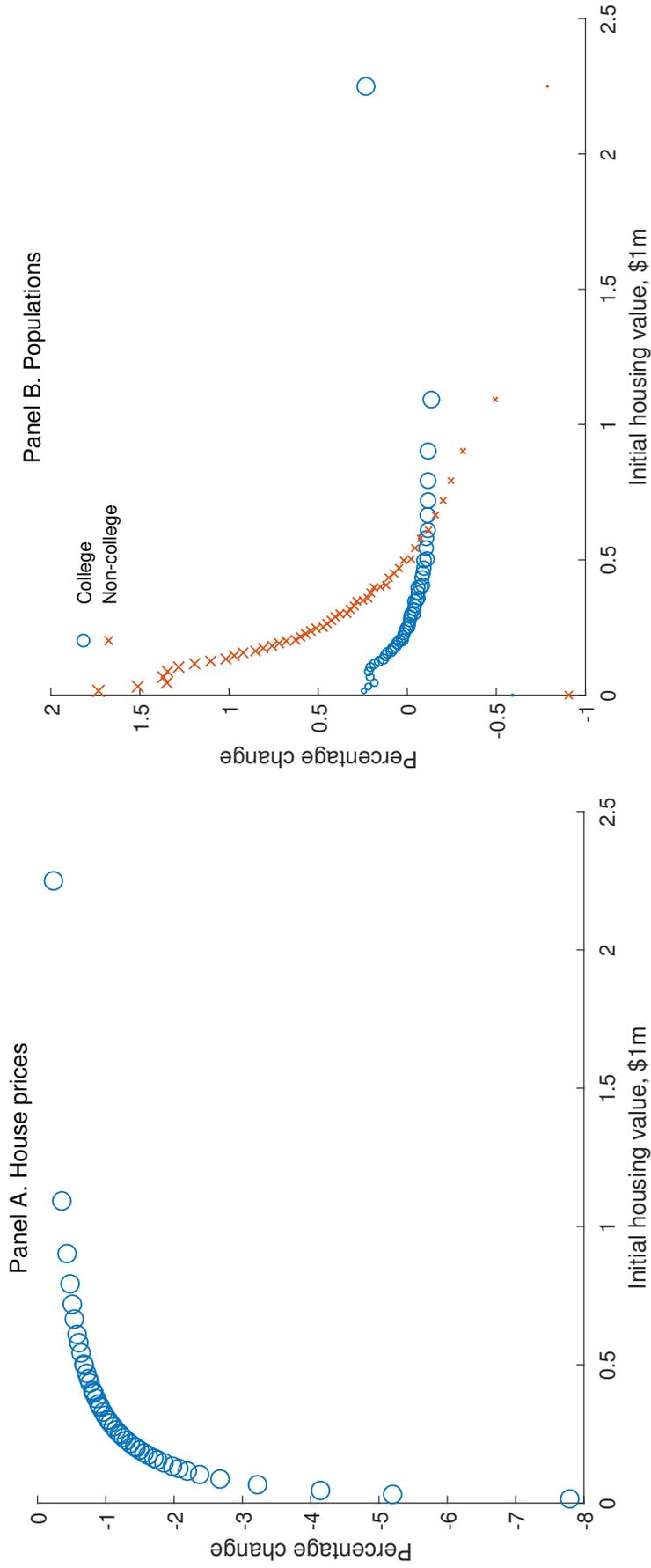
Panel A. House prices

Panel B. Populations

FIGURE 2. EFFECT OF 2015 CONSTRUCTION ON 2016 HOUSING MARKET, BOSTON METROPOLITAN AREA

Notes: The two-dimensional size of each marker is proportional to the number of housing units in each housing value and education bin.
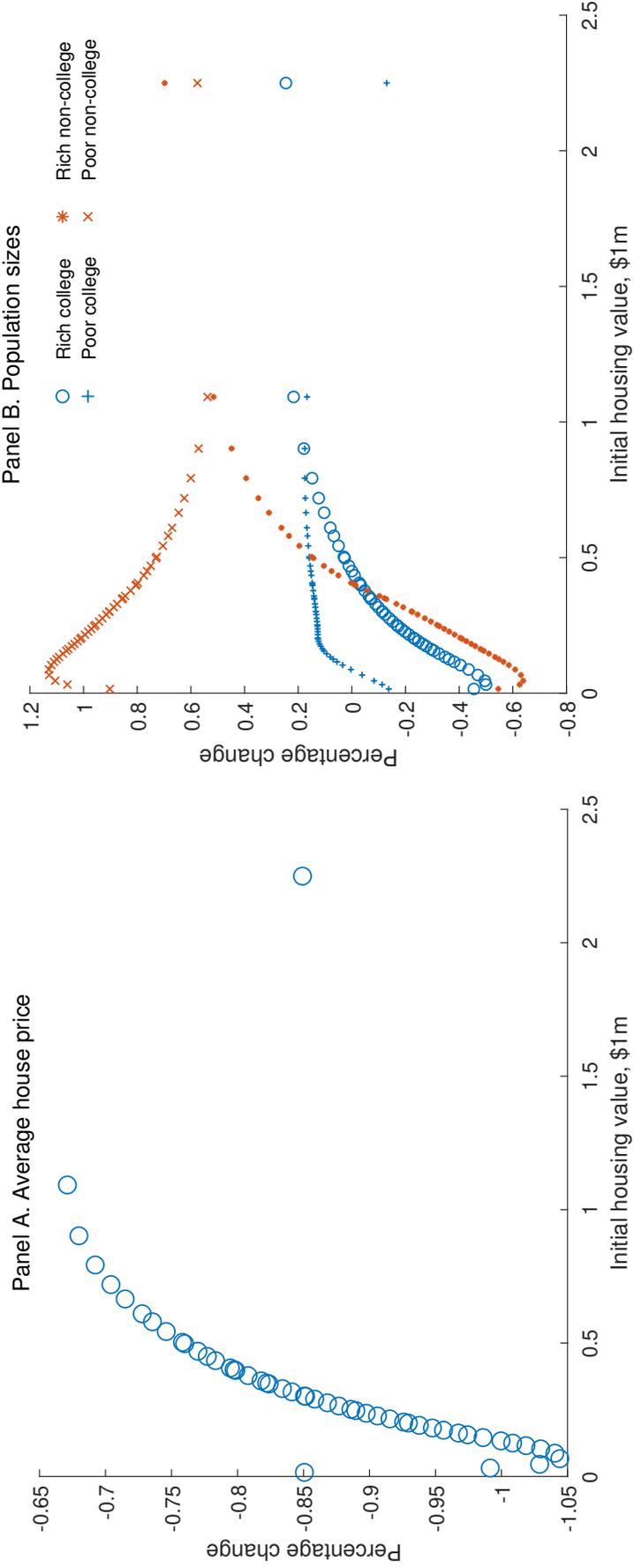
FIGURE 3. EFFECT OF CONSTRUCTION QUALITY ON 2016 HOUSING MARKET, BOSTON METROPOLITAN AREA

Notes: The two-dimensional size of each marker is proportional to the number of housing units in each housing value and education bin.
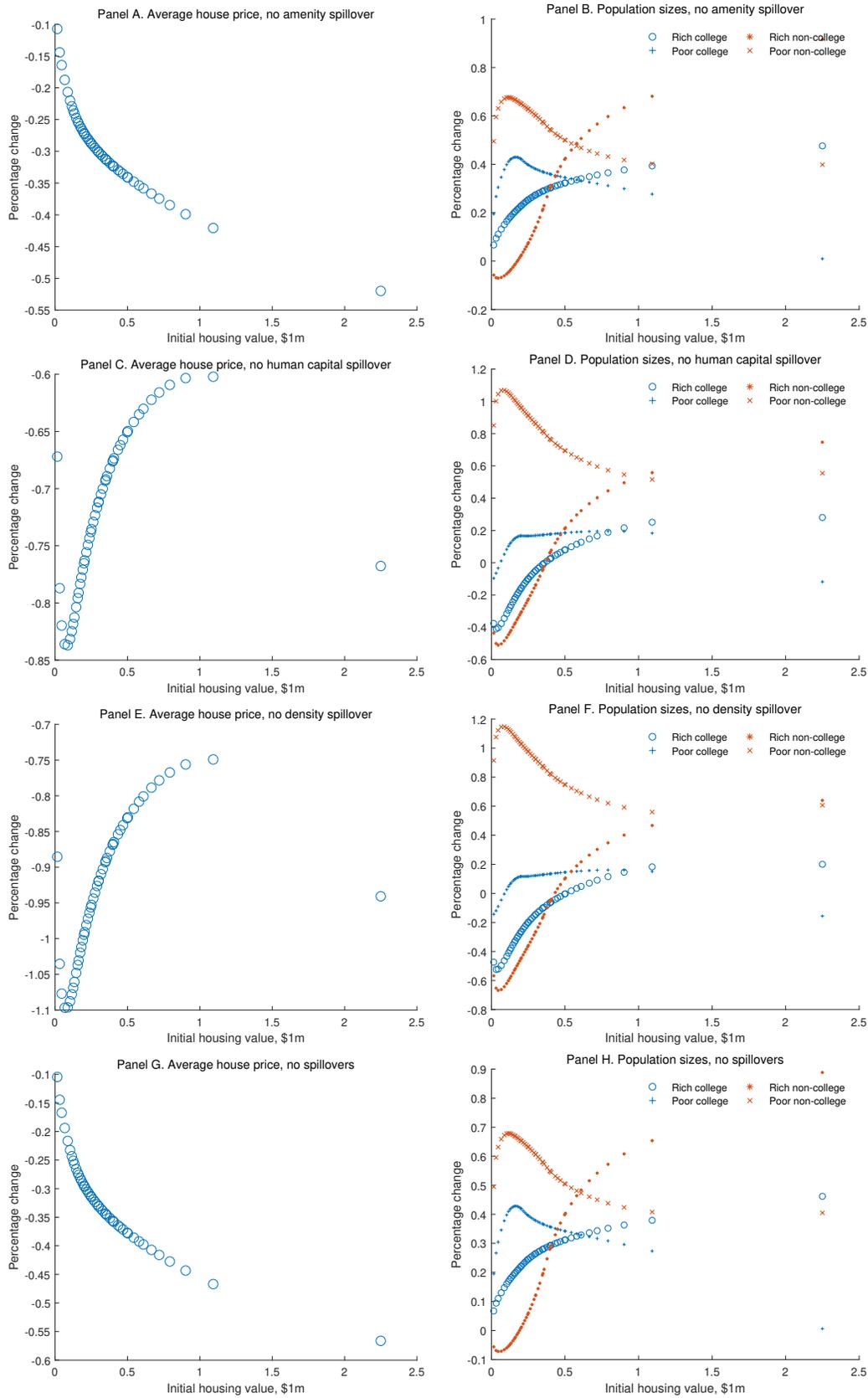
FIGURE 4. EFFECT OF CONSTRUCTION QUALITY ON 2016 HOUSING MARKET, BOSTON METROPOLITAN AREA, FOR DIFFERENT SPILLOVERS
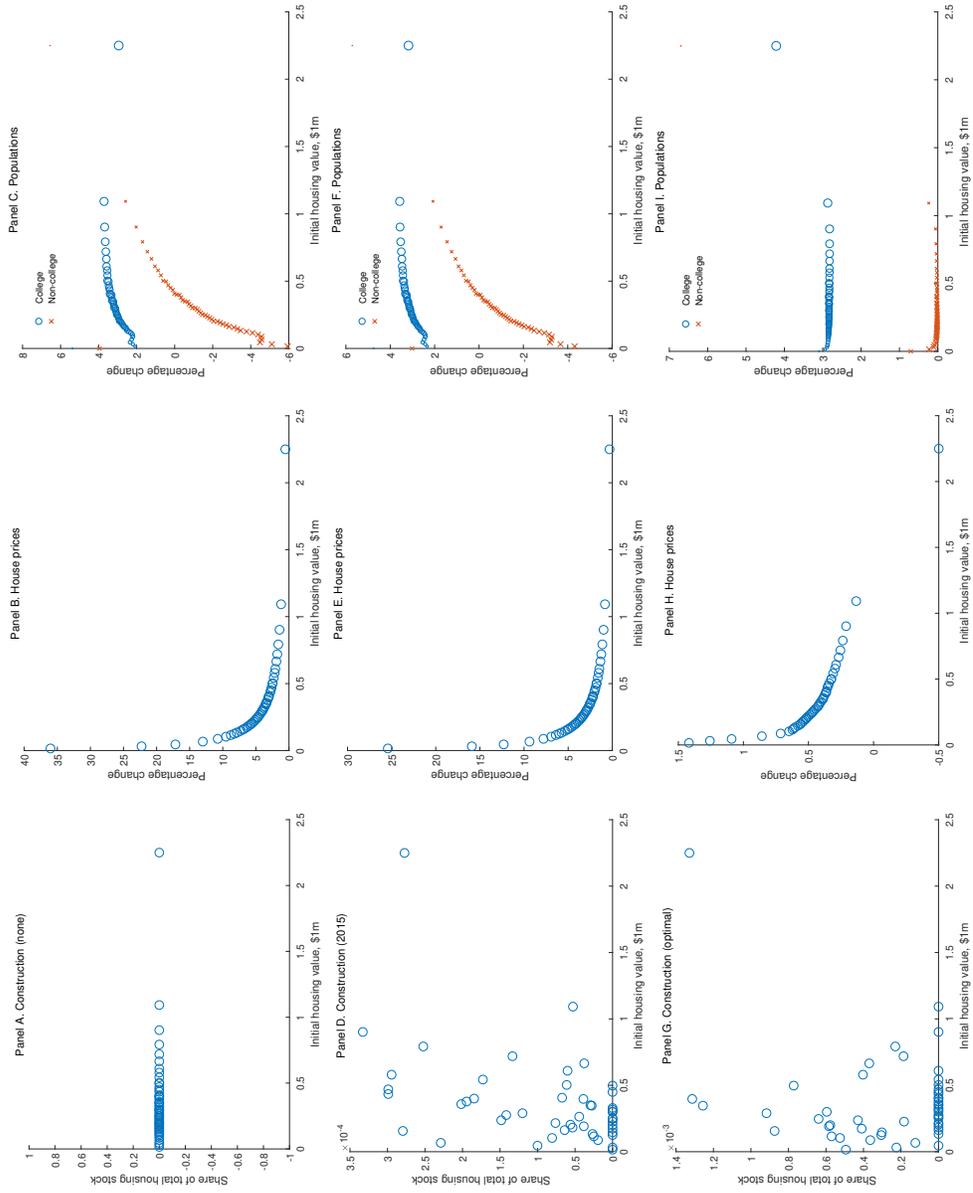
30

Notes: The two-dimensional size of each marker is proportional to the number of housing units in each housing value and education bin.

FIGURE 5. EFFECT OF PRODUCTIVITY SHOCK GIVEN DIFFERENT CONSTRUCTION ON 2016 HOUSING MARKET, BOSTON METROPOLITAN AREA

Notes: The two-dimensional size of each marker is proportional to the number of housing units in each housing value and education bin.

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TABLE 1

SUMMARY STATISTICS FOR BOSTON-CAMBRIDGE-NEWTON, MA-NH METROPOLITAN AREA IN 2016

	Group quarters	Renters	Owners
Panel A: All observations			
Present in estimation sample	0.091	0.968	1
Weighted observations	170,602	701,301	1,120,455
Unweighted observations	3,223	6,156	11,893
Panel B: Estimation sample			
Ownership status	−1	0	1
Income	$7,531	$64,961	$138,223
Education	0.128	0.397	0.554
Rent	-	$1,284	-
Rent censoring	-	0.019	-
Home value	-	-	$491,724
Home value censoring	-	-	0.009
New construction	-	0.005	0.004
Weighted observations	15,486	678,917	1,120,455
Unweighted observations	430	5,946	11,893

TABLE 2

Estimates of model parameters

	Low education	High education
Panel A: Parameters differing by education group		
$N_{e,t^*}/N_{t^*}$	0.509	0.491
$Y_{e,t^*}/Y_{t^*}$	0.334	0.666
$\beta_{w,e}/\beta_{\epsilon,e}$	4.398	2.078
$\beta_{c,e}/\beta_{\epsilon,e}$	3.280	1.713
$\beta_{a,e}/\beta_{\epsilon,e}$	0.245	1.047
Panel B: Parameters not differing by education group		
ζ	1.603	
γ_a	1.103	
γ_N	0.055	
γ_H	0.098	

A Supplements to Section 1

A.1 Proof of Lemma 1

For a contradiction, suppose (a) fails, so that $v_{e,t}(z) = 0$ for all $t \in \{1, ... T\}$ for some $e \in \{L, H\}$ and $z > 0$. This situation may hold only in the indivisible and requires that $w_{e,t} z \leq \min(p_{0,t}, ..., p_{J_t,t})$ for all $t \in \{1, ..., T\}$. By Assumption 1(b), there exists households of education e with labor endowment $z' < z$. For such households, $w_{e,t} z' < w_{e,t} z \leq \min(p_{0,t}, ..., p_{J_t,t})$ for all $t \in \{1, ..., T\}$, contradicting the budget constraint for such households and proving (a).

We next prove (14). Consider households of education e with endowment z. No such household chooses t if $v_{e,t}(z) \exp(\beta_{\epsilon,e} \epsilon_t) < v_{e,t'}(z) \exp(\beta_{\epsilon,e} \epsilon_{t'})$ for any $t' \neq t$. If $v_{e,t}(z) = 0$, then this inequality holds by part (a), so $n_{e,t}(z) = 0$. Such households choose city t if $v_{e,t}(z) > 0$ and $\epsilon_t - \epsilon_{t'} > \beta_{\epsilon,e}^{-1} \log v_{e,t}(z') - \beta_{\epsilon,e}^{-1} \log v_{e,t}(z)$ for all $t' \neq t$ such that $v_{e,t}(z) > 0$. Train (2009), on pages 36 and 74–75, shows that the probability that independent draws from a Gumbel distribution satisfy this inequality is $v_{e,t}(z)^{\beta_{\epsilon,e}^{-1}} / \sum_{t' | v_{e,t'}(z) > 0} v_{e,t'}(z)^{\beta_{\epsilon,e}^{-1}}$. Because the measure of such households in the economy is $\widetilde{n}_e(z)$, (14) follows.

To finish proving (b), we note that the denominator of (14) is always positive and that $v_{e,t}(z)^{\beta_{\epsilon,e}^{-1}}$ is either zero or positive. Therefore, $n_{e,t}$ can have a mass point only if $\widetilde{n}_e$ does, which Assumption 1(c) rules out, so $n_{e,t}$ is atomless. Consider $z' < z''$ such that $n_{e,t}(z'), n_{e,t}(z'') > 0$. It follows thats $\widetilde{n}_e(z'), \widetilde{n}_e(z''), v_{e,t}(z') > 0$. If $z \in (z', z'')$, then $\widetilde{n}_e(z) > 0$ by Assumption 1(a). Furthermore, $v_{e,t}(z) > v_{e,t}(z')$ because $w_{e,t} z > w_{e,t} z'$, meaning that a household with endowment z can choose the same housing bundle and greater non-housing consumption as a household with endowment z'. It follows that $n_{e,t}(z) > 0$, implying that the support of $n_{e,t}$ is convex.

We now prove (c). In equilibrium, $N_{e,t} > 0$ (by definition), so $z_{e,t}^{lb} = \inf\{z \mid n_{e,t}(z) > 0\}$ exists. By Assumption 1(b), $\inf\{z \mid \widetilde{n}_e(z) > 0\} = 0$. By (14), $n_{e,t}(z) > 0$ if $v_{e,t}(z) > 0$ and $\widetilde{n}_e(z) > 0$. It follows that $z_{e,t}^{lb} = \inf\{z \mid v_{e,t}(z) > 0\}$, which equals $z_{e,t}^{lb} = 0$ in the divisible case and $z_{e,t}^{lb} = w_{e,t}^{-1} \min(p_{0,t}, ..., p_{J_t,t})$ in the indivisible case. Consider $\delta_z > 0$. When $z = z_{e,t}^{lb} + \delta_z$, the maximal value of c that satisfies the household constraints is $w_{e,t} \delta_z$, so as $\delta_z \to 0$, $v_{e,t}(z_{e,t}^{lb} + \delta_z) \to 0$. It follows that $\inf\{v_{e,t}(z) \mid n_{e,t}(z) > 0\} = \inf\{v_{e,t}(z) \mid v_{e,t}(z) > 0\} = 0$.

To prove (d), we denote $V_{e,t}(z) = \beta_{\epsilon,e}^{-1} \log v_{e,t}(z)$. Using the simplifications on page 74 of Train (2009), we have

$$E(\epsilon_t \mid e, z, t) = \frac{\int_{-\infty}^{\infty} \epsilon_t \exp\left(-\exp(-\epsilon_t) \sum_{t' | v_{e,t'}(z) > 0} \exp(V_{e,t'}(z) - V_{e,t}(z))\right) \exp(-\epsilon_t) d\epsilon_t}{\int_{-\infty}^{\infty} \exp\left(-\exp(-\epsilon_t) \sum_{t' | v_{e,t'}(z) > 0} \exp(V_{e,t'}(z) - V_{e,t}(z))\right) \exp(-\epsilon_t) d\epsilon_t}. \tag{A1}$$

Changing variables to $\tau = \exp(-\epsilon_t)$ and simplifying, as on page 75 of Train (2009), gives

$$E(\epsilon_t \mid e, z, t) = -\frac{\int_0^{\infty} \log(\tau) \exp\left(-\tau \sum_{t' | v_{e,t'}(z) > 0} \exp(V_{e,t'}(z) - V_{e,t}(z))\right) d\tau}{\int_0^{\infty} \exp\left(-\tau \sum_{t' | v_{e,t'}(z) > 0} \exp(V_{e,t'}(z) - V_{e,t}(z))\right) d\tau}. \tag{A2}$$

Evaluating these integrals yields $E(\epsilon_t \mid e, z, t) = \gamma - V_{e,t} + \log \sum_{t' | v_{e,t'}(z) > 0} \exp(V_{e,t'}(z))$. It follows that $\log \overline{v}_{e,t}(z) = \beta_{\epsilon,e} V_{e,t}(z) + \beta_{\epsilon,e} E(\epsilon_t \mid e, z, t) = \beta_{\epsilon,e} \gamma + \beta_{\epsilon,e} \log \sum_{t'} v_{e,t'}(z)^{\beta_{\epsilon,e}^{-1}}$. Exponentiating proves (d).

A.2 Proof of Proposition 1

The text above Proposition 1 proves that $p_{j,t} = \mu_t q_{j,t}$ for each $j \in \mathcal{J}_t$. Using this result, we re-write the budget constraint, (3), as $c + \mu_t q \leq w_{e,t} z$. Given t, maximizing (1) is equivalent to maximizing the Cobb-Douglas utility function $c^{1-\alpha} q^\alpha$ given income $w_{e,t} z$ and prices one and μ_t for c and q, where $\alpha = \beta_{q,e}(\beta_{c,e} + \beta_{q,e})^{-1}$. If $\mu_t \leq 0$, then households cannot optimize, so $\mu_t > 0$ in equilibrium. As shown on pages 55–56 of Mas-Collel et al. (1995), the optimal consumption bundle in this environment consists of expending a share α of income on q, so that $q = \beta_{q,e}(\beta_{c,e} + \beta_{q,e})^{-1} \mu_t^{-1} w_{e,t} z$. Given that $p_{j,t} > 0$ for all $j \in \mathcal{J}_t$, rentiers maximize utility by selling all of their housing. Equating the total housing quality rentiers sell to the total quality households demand produces (16).

A.3 Proof of Proposition 2

In equilibrium, the housing market clears, so the measure of households must be no greater than the total measure of housing rentiers may sell: $\sum_{j \in \mathcal{J}_t} h_{j,t} \geq N_t$. It follows that $j_{0,t}$ exists.

We next prove the statements about housing demand. Because rentiers optimize, they sell $h_{j,t}$ units of housing of quality $q_{j,t}$ when $p_{j,t} > 0$, they sell any amount of such housing when $p_{j,t} = 0$, and $p_{j,t} < 0$ is impossible. Consider $j < j_{0,t}$ such that $x_{j,t} > 0$. Because $\sum_{j'=j_{0,t}}^{J_t} h_{j',t} \geq N$, there exists $j' \geq j_{0,t}$ such that $x_{j',t} < h_{j',t}$. It follows that $p_{j',t} = 0$, which implies that households choosing $q_{j,t}$ are failing to optimize because $q_{j',t} > q_{j,t}$ and $0 = p_{j',t} \leq p_{j,t}$. This contradiction implies that no such j exist, so $x_{j,t} = 0$ when $j < j_{0,t}$. Similarly, consider j such that $j > j_{0,t}$ and $x_{j,t} < h_{j,t}$. Then $p_{j,t} = 0$. Furthermore, because $\sum_{j'=j_{0,t}+1}^{J_t} h_{j',t} < N_t$, there exists $j' \leq j_{0,t}$ such that $x_{j',t} > 0$. Households choosing $q_{j',t}$ cannot be optimizing, as $p_{j,t} \leq p_{j',t}$ and $q_{j,t} > q_{j',t}$. This contradiction implies that there are no such j, proving that $x_{j,t} = h_{j,t}$ when $j > j_{0,t}$.

To prove the statements about house prices, we first consider j such that $\sum_{j'=j}^{J_t} h_{j',t} > N_t$ and $h_{j,t} > 0$. By the definition of $j_{0,t}$, $j \leq j_{0,t}$. If $j < j_{0,t}$, then $0 = x_{j,t} < h_{j,t}$, so $p_{j,t} = 0$. If $j = j_{0,t}$, then $x_{j,t} = N_t - \sum_{j'=j_{0,t}+1}^{J_t} h_{j',t} < h_{j_{0,t},t} = h_{j,t}$, again implying $p_{j,t} = 0$. Next, consider j, j' such that $h_{j,t}, h_{j',t} > 0$ and $j_{0,t} \leq j < j'$. Because $x_{j,t} = h_{j,t} > 0$, there exist households choosing $q_{j,t}$. If $p_{j,t} \geq p_{j',t}$, these households are failing to optimize because $q_{j',t} > q_{j,t}$. It follows that $p_{j,t} < p_{j',t}$.

We prove the final statement by contradiction. Consider two households with education e and respective labor endowments $z < z'$ and quality choices $q_{j,t} > q_{j',t}$. Optimality for each household implies the inequalities $(w_{e,t} z - p_{j,t})^{\beta_{c,e}} q_{j,t}^{\beta_{q,e}} \geq (w_{e,t} z - p_{j',t})^{\beta_{c,e}} q_{j',t}^{\beta_{q,e}}$ and $(w_{e,t} z' - p_{j',t})^{\beta_{c,e}} q_{j',t}^{\beta_{q,e}} \geq (w_{e,t} z' - p_{j,t})^{\beta_{c,e}} q_{j,t}^{\beta_{q,e}}$. If $w_{e,t} z = p_{j,t}$, then the first optimality inequality is violated because $p_{j',t} < p_{j,t}$, which implies that $w_{e,t} z - p_{j',t} > w_{e,t} z - p_{j,t} = 0$. By the budget constraint, $w_{e,t} z \geq p_{j,t}$, so we must have $w_{e,t} z > p_{j,t}$. Because $z < z'$, $w_{e,t} z < w_{e,t} z'$, so $w_{e,t} z' - p_{j,t} > w_{e,t} z - p_{j,t} > 0$. We may therefore rearrange the optimality inequalities to produce

$$\frac{w_{e,t} z - p_{j',t}}{w_{e,t} z - p_{j,t}} \leq \left(\frac{q_{j,t}}{q_{j',t}} \right)^{\frac{\beta_{q,e}}{\beta_{c,e}}} \leq \frac{w_{e,t} z' - p_{j',t}}{w_{e,t} z' - p_{j,t}}. \tag{A3}$$

Cross-multiplying the outer terms yields the contradiction $(p_{j,t} - p_{j',t})(z' - z) \leq 0$.

A.4 Proof of Lemma 2

Denote by $p_t = (p_{0,t}, \dots, p_{J_t,t})$ a vector of potential house prices in city t. Define $\mathcal{P}_t = \{p_t \mid 0 \le p_{0,t} \le \dots \le p_{J_t,t}\}$. We derive housing demand as a function of $p_t \in \mathcal{P}_t$.

For $j, j' \in \mathcal{J}_t$ distinct, a household of education e and endowment z prefers and can afford $q_{j,t}$ over $q_{j',t}$ only if two conditions hold. First, $w_{e,t}z \ge p_{j,t}$. Second, either $w_{e,t}z < p_{j',t}$ or both $w_{e,t}z \ge p_{j',t}$ and

$$\left(w_{e,t}z - p_{j,t}\right)^{\beta_{c,e}} q_{j,t}^{\beta_{q,e}} \ge \left(w_{e,t}z - p_{j',t}\right)^{\beta_{c,e}} q_{j',t}^{\beta_{q,e}}, \tag{A4}$$

with strict preference in the case of strict inequality. The point where (A4) becomes an equality is

$$z_{e,j,j',t}(p_t) = \frac{q_{j,t}^{\frac{\beta_{q,e}}{\beta_{c,e}}} p_{j,t} - q_{j',t}^{\frac{\beta_{q,e}}{\beta_{c,e}}} p_{j',t}}{\left(q_{j,t}^{\frac{\beta_{q,e}}{\beta_{c,e}}} - q_{j',t}^{\frac{\beta_{q,e}}{\beta_{c,e}}}\right) w_{e,t}}. \tag{A5}$$

If $j > j'$, then (A4) holds if and only if $z \ge z_{e,j,j',t}(p_t)$. Furthermore, because $p_{j',t} \le p_{j,t}$, $z_{e,j,j',t}(p_t) \ge w_{e,t}^{-1} p_{j,t} \ge w_{e,t}^{-1} p_{j',t}$. Therefore, if $j > j'$, the household prefers and can afford $q_{j,t}$ over $q_{j',t}$ only if $z_{e,j,j',t}(p_t)$, with strict preference when $z > z_{e,j,j',t}(p_t)$. By symmetry, when $j < j'$, these preference relations hold when $z \le z_{e,j',j,t}(p_t) = z_{e,j,j',t}(p_t)$, strictly with strict inequality. Let $z_{e,j,t}^{min}(p_t) = \max\{z_{e,j,j',t}(p_t) \mid j > j'\}$ for $j > 0$ and zero for $j = 0$. Let $z_{e,j,t}^{max}(p_t) = \min\{z_{e,j,j',t}(p_t) \mid j < j'\}$ for $j < J_t$ and infinity for $j = J_t$. Demand for $q_{j,t}$ equals

$$x_{j,t}(p_t) = \sum_{e \in \{L,H\}} x_{e,j,t}(p_t), \tag{A6}$$

where

$$x_{e,j,t}(p_t) = \int_{z_{e,j,t}^{min}(p_t)}^{z_{e,j,t}^{max}(p_t)} n_{e,t}(z) dz. \tag{A7}$$

The following lemma collects useful facts about demand.

Lemma A1. *For each $j \in \mathcal{J}_t$, $x_{j,t}$ is continuous. Let $p_t, p'_t \in \mathcal{P}_t$. If $p'_{j,t} \ge p_{j,t}$ and $p'_{j',t} \le p_{j',t}$ for $j' \ne j$, then $x_{j,t}(p'_t) \le x_{j,t}(p_t)$, with strict inequality if $p'_{j,t} < p_{j,t}$ and $x_{j,t}(p_t) > 0$. Finally,*

$$\bar{p}_t = \inf\left\{\widetilde{p} \ge 0 \; \middle| \; \sum_{e \in \{L,H\}} \int_{w_{e,t}^{-1}\widetilde{p}}^{\infty} n_{e,t}(z) dz < h_{J_t,t}\right\} \tag{A8}$$

exists, and if $p_{J_t+1,t} = \bar{p}_t$, then $x_{j,t}(p_t) \le h_{j,t}$ if $p_{j,t} = p_{j+1,t}$.

Proof. We prove the first two sentences for each $x_{e,j,t}$; they then hold for $x_{j,t}$ immediately. Both $z_{e,j,t}^{min}$ and $z_{e,j,t}^{max}$ (the latter only for $j < J_t$), are continuous functions of p_t. Because $n_{e,t}$ is continuous by assumption, and because the composition of continuous functions is continuous, the fundamental theorem of calculus implies that $x_{e,j,t}$ is continuous.

From (A5), $z_{e,j,j',t}(p_t)$ strictly increases in $p_{j,t}$ and decreases in $p_{j',t}$ if $j > j'$ and strictly decreases in $p_{j,t}$ and increases in $p_{j',t}$ if $j < j'$. Therefore, $z_{e,j,t}^{min}(p'_t) \ge z_{e,j,t}^{min}(p_t)$ and $z_{e,j,t}^{max}(p'_t) \le z_{e,j,t}^{max}(p_t)$. It

follows that $x_{e,j,t}(p'_t) \leq x_{e,j,t}(p_t)$. Now suppose that $x_{e,j,t}(p_t) > 0$ and that $p'_{j,t} < p_{j,t}$. We claim that $0 \leq z^{min}_{e,j,t}(p_t) < \sup\{z \mid n_{e,t}(z) > 0\}$. The first inequality follows because for $j > j'$, $z_{e,j,j',t}(p_t) \geq w^{-1}_{e,t}p_{j,t} \geq 0$. If the second inequality fails, then $[z^{min}_{e,j,t}(p_t), z^{max}_{e,j,t}(p_t)] \cap \{z \mid n_{e,t}(z) > 0\}$ is either $\emptyset$ or $\{\sup\{z \mid n_{e,t}(z) > 0\}\}$, both of which have measure zero under $n_{e,t}$, contradicting $x_{e,j,t}(p_t) > 0$. Then $\inf\{z \mid n_{e,t}(z) > 0\} = 0 \leq z^{min}_{e,j,t}(p_t) < \sup\{z \mid n_{e,t}(z) > 0\}$. Therefore, because $z^{min}_{e,j,t}(p_t) < z^{min}_{e,j,t}(p'_t)$, the convexity of the support of $n_{e,t}$ implies that $x_{e,j,t}(p'_t) < x_{e,j,t}(p_t)$, as desired.

Let $j < J_t$. If $p_{j,t} = p_{j+1,t}$, then $z_{e,j,j+1,t}(p_t) = w^{-1}_{e,t}p_{j,t}$. A household of education can afford $q_{j,t}$ only if $z \geq w^{-1}_{e,t}p_{j,t}$ and prefers $q_{j+1,t}$ over $q_{j,t}$ only if $z \leq z_{e,j,j+1,t}(p_t) = w^{-1}_{e,t}p_{j,t}$. Therefore, only those with $z = w^{-1}_{e,t}p_{j,t}$ may choose $q_{j,t}$, and the measure of such households equals zero, so $x_{j,t}(p_t) = 0 \leq h_{j,t}$. We turn to the case $j = J_t$. The limit of the integral in (A8) as $\widetilde{p} \to \infty$ equals zero, so $\overline{p}_t$ exists. A household chooses $q_{J_t,t}$ only if $z \geq w_{e,t}p_{J_t,t}$, so the summation in (A8) provides an upper bound on $x_{J_t,t}(p_t)$ when $\widetilde{p} = p_{j,t}$. Therefore, $x_{J_t,t}(p_t) \leq h_{J_t,t}$ when $p_{J_t,t} \leq \overline{p}_t = p_{J_t+1,t}$. □

We next prove local equilibrium existence by constructing a sequence $p^i_t \in \mathcal{P}_t$ whose limit provides a local equilibrium. For all $i \geq 0$, set $p_{J_t+1,t} = \overline{p}_t$. We set $p^0_t = 0$. If $i + j \equiv 0 \pmod{J_t + 1}$ and $x_{j,t}(p^i_t) > h_{j,t}$, then $p^i_{j,t}$ is the unique solution to $h_{j,t} = x_{j,t}(p^i_{j,t}; p^{i-1}_{-j,t})$. Uniqueness and existence follow from the intermediate value theorem and Lemma A1 because $x_{j,t}(p^{i-1}_{j,t}; p^{i-1}_{-j,t}) > h_{j,t} \geq x_{j,t}(p^{i-1}_{j+1,t}; p^{i-1}_{-j,t})$ and $h_{j,t} > 0$. Otherwise, we set $p^i_{j,t} = p^{i-1}_{j,t}$. At each step, $p^i_{j,t} \in [p^{i-1}_{j,t}, p^{i-1}_{j+1,t}]$, so $p^i \in \mathcal{P}_t$ for all $i \geq 0$. Furthermore, $p^i_{j,t}$ weakly increases in i for each $j \in \{0,...,J_t\}$, and $p^i_{j,t} \leq p^i_{J_t+1,t} = \overline{p}$ for all such j and all $i \geq 0$. By the monotone converge theorem, p^i_t converges. We denote the limit p^*_t.

Consider j such that $p^*_{j,t} > 0$. Let i_j be the first i such that $p^i_{j,t} > 0$. We claim that $x_{j,t}(p^i_t) \geq h_{j,t}$ for $i \geq i_j$. When $i = i_j$, $p^i_{j,t} > p^{i-1}_{j,t} = 0$, so $x_{j,t}(p^i_t) = h_{j,t}$. We proceed by induction. For each $i > i_j$, $p^i_{j',t} \geq p^{i-1}_{j',t}$ for $j' \neq j$. If $p^i_{j,t} = p^{i-1}_{j,t}$, then by Lemma A1, $x_{j,t}(p^i_t) \geq x_{j,t}(p^{i-1}_t) \geq h_{j,t}$. If $p^i_{j,t} > p^{i-1}_{j,t}$, then $x_{j,t}(p^i_t) = h_{j,t}$. Therefore, $x_{j,t}(p^i_t) \geq h_{j,t}$ for all $i \geq i_j$, as claimed. It follows that $x_{j,t}(p^i_t) = h_{j,t}$ for all $i \equiv -j \pmod{J_t + 1}$. Because $x_{j,t}$ is continuous by Lemma A1, $x_{j,t}(p^i_t)$ converges because p^i_t does. This limit must equal $h_{j,t}$ because it appears infinitely often in the sequence. Therefore, $x_{j,t}(p^*_t) = \lim_{i \to \infty} x_{j,t}(p^i_t) = h_{j,t}$.

Consider j such that $p^*_{j,t} = 0$. For all $i > 0$ such that $i \equiv -j \pmod{J_t+1}$, $x_{j,t}(p^i_t) \leq h_{j,t}$. Because $x_{j,t}$ is continuous by Lemma A1, $x_{j,t}(p^i_t)$ converges. This limit cannot exceed $h_{j,t}$ because $x_{j,t}(p^i_t) \leq h_{j,t}$ for infinitely many i. Therefore, $x_{j,t}(p^*_t) = \lim_{i \to \infty} x_{j,t}(p^i_t) \leq h_{j,t}$.

Given that $w_{L,t}$ and $w_{H,t}$ clear the labor market when (12) holds, local equilibrium holds if and only if prices clear the housing market. Rentiers sell all housing at positive prices and any amount of housing at zero prices, so the market clears at p^*_t if and only if $x_{j,t}(p^*_t) = h_{j,t}$ when $p^*_{j,t} > 0$ and $x_{j,t}(p^*_t) \leq h_{j,t}$ when $p^*_{j,t} = 0$. Therefore, p^*_t provides local equilibrium house prices.

We finally prove that this local equilibrium is unique. Consider two local equilibrium price vectors p_t and p'_t. For a contradiction, suppose that $p_t \neq p'_t$. By Proposition 2 and (19), $p_{j,t} = p'_{j,t} = 0$ for $j \leq j_{0,t}$, $p_{j,t}, p'_{j,t} > 0$ for $j > j_{0,t}$, and $p_t, p'_t \in \mathcal{P}_t$. Furthermore, for $j > j_{0,t}$, $x_{j,t}(p_t) = x_{j,t}(p'_t) = h_{j,t}$. Without loss of generality, suppose that $p_{j,t} < p'_{j,t}$ for some $j > j_{0,t}$. Let $\mathcal{J}'_t = \{j \in \{0,...,J_t\} \mid p_{j,t} < p'_{j,t}\}$. Given p_t, a household of education e and endowment z prefers and can afford $q_{j,t}$ over all $q_{j',t}$, where $j \in \mathcal{J}'_t$ and $j' \notin \mathcal{J}'_t$, only if $\max\{z_{e,j,j',t}(p_t) \mid j > j' \notin \mathcal{J}'_t\} \leq z \leq \min\{z_{e,j,j',t}(p_t) \mid j < j' \notin \mathcal{J}'_t\}$.

Therefore, total demand for qualities $q_{j,t}$, where $j \in \mathcal{J}'_t$, equals

$$\sum_{j \in \mathcal{J}'_t} h_{j,t} = \sum_{j \in \mathcal{J}'_t} x_{j,t}(p_t) = \sum_{e \in \{L,H\}} \int_{\mathcal{Z}_{e,t}(p_t)} n_{e,t}(z) dz, \tag{A9}$$

where

$$\mathcal{Z}_{e,t}(p_t) = \bigcup_{j \in \mathcal{J}'_t} \left[\max \left\{ z_{e,j,j',t}(p_t) \mid j > j' \notin \mathcal{J}'_t \right\}, \min \left\{ z_{e,j,j',t}(p_t) \mid j < j' \notin \mathcal{J}'_t \right\} \right]. \tag{A10}$$

We define $\mathcal{Z}_{e,t}(p'_t)$ similarly. Because $p_{j,t} < p'_{j,t}$ for $j \in \mathcal{J}'_t$ and $p_{j,t} \geq p'_{j,t}$ for $j \notin \mathcal{J}'_t$, $z_{e,j,j',t}(p_t) < z_{e,j,j',t}(p'_t)$ if $\mathcal{J}'_t \ni j > j' \notin \mathcal{J}'_t$ and $z_{e,j,j',t}(p_t) > z_{e,j,j',t}(p'_t)$ if $\mathcal{J}'_t \ni j < j' \notin \mathcal{J}'_t$. Therefore, $\mathcal{Z}_{e,t}(p'_t) \subset \mathcal{Z}_{e,t}(p_t)$. Furthermore, $z_{e,j,j',t}(p'_t) > 0$ when $\mathcal{J}'_t \ni j > j'$ because $p'_{j,t} > 0$ and $p'_t \in \mathcal{P}_t$. It follows that $\min \mathcal{Z}_{e,t}(p_t) > \min \mathcal{Z}_{e,t}(p'_t) > 0$. Because the greatest lower bounds of the supports of $n_{L,t}$ and $n_{H,t}$ equal zero,

$$\int_{\mathcal{Z}_{e,t}(p'_t)} n_{e,t}(z) dz \leq \int_{\mathcal{Z}_{e,t}(p_t)} n_{e,t}(z) dz, \tag{A11}$$

with strict inequality if $\mathcal{Z}_{e,t}(z) \neq \emptyset$, which holds for some e due to (A9). Therefore, by (A9), $\sum_{j \in \mathcal{J}'_t} h_{j,t} = \sum_{j \in \mathcal{J}'_t} x_{j,t}(p'_t) < \sum_{j \in \mathcal{J}'_t} x_{j,t}(p_t) = \sum_{j \in \mathcal{J}'_t} h_{j,t}$, a contradiction.

A.5 Solution to (19)–(21)

For each $j \in \{j_{0,t} + 1, ..., J_t\}$, we rewrite (20) and (21) as

$$N_t - \sum_{j'=j}^{J_t} h_{j',t} = \sum_{e \in \{L,H\}} \int_0^{z_{e,j,t}(p_{j,t}, p_{j-1,t})} n_{e,t}(z) dz, \tag{A12}$$

where $z_{e,j,t}(p_{j,t}, p_{j-1,t}) = z_{e,j,j-1,t}(p_t)$. By the definition of $j_{0,t}$, the minimum value of the left side of (A12) is positive. The maximal value is less than N_t because $h_{J_t,t} > 0$. Therefore, the left side of (A12) lies in $(0, N_t)$ for each $j \in \{j_{0,t} + 1, ..., J_t\}$. Because $z_{e,j,t}(\cdot, \cdot)$ is an increasing linear function of its first argument, and because the support of each $n_{e,t}$ is convex, the right side of (A12) strictly and continuously increases in $p_{j,t}$ over the range $[0, N_t)$. Therefore, given $p_{j-1,t}$, a unique value of $p_{j,t}$ solves (A12) for each $j \in \{j_{0,t} + 1, ..., J_t\}$. Because (19) determines $p_{j_{0,t},t}$, induction shows that unique values of $p_{j,t}$ solve (A12).

We now prove that this unique solution for $p_{j,t}$ strictly increases over $j \in \{j_{0,t}, ..., J_t\}$. Because $z_{e,j_{0,t}+1,t}(p_{j_{0,t},t}, p_{j_{0,t},t}) = 0$ and the left of (A12) exceeds zero, $p_{j_{0,t}+1,t} > 0 = p_{j_{0,t},t}$. Proceeding inductively, we note that

$$z_{e,j,t}(p_{j-1,t}, p_{j-1,t}) = w_{e,t}^{-1} p_{j-1,t} = z_{e,j-1,t}(p_{j-1,t}, p_{j-1,t}) < z_{e,j-1,t}(p_{j-1,t}, p_{j-2,t}) \tag{A13}$$

as $z_{e,j-1,t}(\cdot, \cdot)$ strictly falls in its second argument. The left of (A12) strictly rises in j, so $p_{j,t} > p_{j-1,t}$.

Finally, we characterize the condition under which $x_{e,j,t} > 0$ for each $e \in \{L, H\}$ and $j \in \{j_{0,t}, ..., J_t\}$. Denote $z_{e,j,t} = z_{e,j,t}(p_{j,t}, p_{j-1,t})$, where the prices are the unique solutions to (19)–(21). We claim that, in local equilibrium, a positive measure of households of each education choose each type of occupied housing if and only if $z_{e,j,t}$ strictly increases over $j \in \{j_{0,t}, ..., J_t + 1\}$ for each $e \in \{L, H\}$.

Suppose such monotonicity holds. Appendix A.4 shows that for $j \in \mathcal{J}_t$, a household of education e and endowment z prefers and can afford to choose $q_{j,t}$ over $q_{j-1,t}$ if and only if $z \geq z_{e,j,t}$

when $j > 0$; that same household prefers and and can afford to choose $q_{j,t}$ over $q_{j+1,t}$ if and only if $z \leq z_{e,j+1,t}$ when $j < J_t$. By transitivity and the assumed monotonicity, a household of education e with endowment $z \in [z_{e,j,t}, z_{e,j+1,t}]$ prefers and can afford to choose $q_{j,t}$ over all $q_{j',t}$ for $j' \in \mathcal{J}_t$ because either $z_{e,j',t} \leq ... \leq z_{e,j,t} \leq z$ or $z \leq z_{e,j+1,t} \leq ... \leq z_{e,j',t}$. Therefore, all households optimize their housing choices when prices solve (19)–(21) and a household of education e and endowment z chooses $q_{j,t}$ only if $z \in [z_{e,j,t}, z_{e,j+1,t}]$. Because housing markets clear by (21), these prices and housing demands constitute a local equilibrium, and due to Lemma 2, they give the unique local equilibrium. In this local equilibrium, $x_{e,j,t} > 0$ for each $e \in \{L, H\}$ and $j \in \{j_{0,t}, ..., J_t\}$ because $\inf\{z \mid n_{e,t}(z) > 0\} = z_{e,j_0,t} \leq z_{e,j,t} < z_{e,j+1,t} \leq z_{e,J_t+1,t} = \sup\{z \mid n_{e,t}(z) > 0\}$ and $n_{e,t}$ has convex support.

Conversely, suppose such monotonicity fails to hold. As argued in the text, in any local equilibrium in which $x_{e,j,t} > 0$ for $e \in \{L, H\}$ and $j \in \{j_{0,t}, ..., J_t\}$, a household of education e chooses $q_{j,t}$ for such j only if $z \in [z_{e,j,t}, z_{e,j+1,t}]$. If monotonicity fails, then there exists some such e and j such that $x_{e,j,t} = 0$, a contradiction.

B   Supplements to Section 2

B.1   Local stability in the divisible model

In any equilibrium of the divisible model, $n_{e,t}(z)/N_{e,t} = \widetilde{n}_e(z)/\widetilde{N}_e$, where $\widetilde{N}_e$ equals the mass of households of education e in the economy. This relation follows from substituting (17) into (14). As a result, $Z_{e,t} = \int_0^\infty z n_{e,t}(z) dz = N_{e,t} \int_0^\infty z \widetilde{N}_e^{-1} \widetilde{n}_e(z) dz = N_{e,t} \overline{z}_e$, where $\overline{z}_e$ equals the average z among households of education e. Therefore, in equilibrium, $N_{L,t}$ and $N_{H,t}$ determine $Z_{L,t}$ and $Z_{H,t}$. Because amenities, productivities, and prices in t are all functions of $N_{L,t}$, $N_{H,t}$, $Z_{L,t}$, and $Z_{H,t}$, the populations $N_{L,t}$ and $N_{H,t}$ pin down the local equilibrium in any equilibrium. By (17), there exist functions $v_{e,t}^N(N_{L,t}, N_{H,t})$ for each $e \in \{L, H\}$ and $t \in \{1, ..., T\}$ such that $v_{e,t}(z) = z^{\beta_{c,e}+\beta_{q,e}} v_{e,t}^N(N_{L,t}, N_{H,t})$ for all $z > 0$ in any equilibrium.

An equilibrium is *locally stable* in city t if the Jacobian of $v_t^N = (v_{L,t}^N, v_{H,t}^N)$ at $(N_{L,t}, N_{H,t})$ is Volterra-Lyapunov stable (Cross, 1978). Equivalently for a two-by-two Jacobian, the diagonal is negative and its determinant is positive (Cross, 1978).

Because the diagonal is negative, perturbing $N_{e,t}$ moves equilibrium welfare of households of education e in the opposite direction when local stability holds. Furthermore, perturbing both $N_{L,t}$ and $N_{H,t}$ moves equilibrium welfare of at least one of the education groups in the opposite direction. If not, then there exist $dN_{L,t}, dN_{H,t} \neq 0$ and a non-negative diagonal matrix D_t^N such that $(Dv_t^N)(dN_{L,t}, dN_{H,t}) = D_t^N(dN_{L,t}, dN_{H,t})$. As a result, $Dv_t^N - D_t^N$ has an eigenvalue of zero, meaning that dv_t^N is not strongly stable and hence is not Volterra-Lyapunov stable, a contradiction (Cross, 1978). We prove Proposition 3 using these results.

B.2   Proof of Proposition 3

In the divisible case, each $\delta_{h,j}$ enters the derivatives of the equilibrium conditions only in (31), where it appears with a coefficient proportional to q_{j,t^*}. Due to the linearity of the system of equations, $\partial \log v_{e,t^*}(z)$ is proportional to $\sum_{j \in \mathcal{J}_{t^*}} q_{j,t^*} \delta_{h,j}$, as claimed.

Suppose that $\gamma_N = 0$ and that the equilibrium is locally stable in t^*. Define $Q_t = \sum_{j \in \mathcal{J}_t} q_{j,t} h_{j,t}$. To prove positive proportionality in (37), we assume without loss of generality that $\partial Q_{t^*} > 0$. For a

contradiction, suppose that $\partial \log v_{e',t^*}(z) \leq 0$ for some $e' \in \{L, H\}$ and $z > 0$. By (32), $\partial \log v_{e',t^*}(z') = \partial \log v_{e',t^*}(z)$ for all $z' > 0$, which, by (24), implies that $\partial N_{e',t^*} \leq 0$. Consider the following perturbation to the initial equilibrium:

$$d \log N_{e,t^*} = \partial \log N_{e,t^*} - \begin{cases} 0, & \partial \log N_{\sim e',t^*} \in (-\infty, 0] \\ \partial \log N_{\sim e',t^*}, & \partial \log N_{\sim e',t^*} \in (0, \partial \log Q_{t^*}] \\ \partial \log Q_{t^*}, & \partial \log N_{\sim e',t^*} \in (\partial \log Q_{t^*}, \infty) \end{cases} \tag{B1}$$

for each $e \in \{L, H\}$, where $\sim e' = H$ when $e' = L$ and $\sim e' = L$ when $e' = H$. In equilibrium, $Z_{e,t^*} = \bar{z}_e N_{e,t^*}$, so $Z_{L,t^*}/Z_{H,t^*} = N_{L,t^*}/N_{H,t^*}$. As a result, $N_{L,t^*}/N_{H,t^*}$ determines labor prices and amenities in t^* in equilibrium when $\gamma_N = 0$. Because $d \log(N_{L,t^*}/N_{H,t^*}) = \partial \log(N_{L,t^*}/N_{H,t^*})$, $da_{t^*} = \partial a_{t^*}$, $dw_{L,t^*} = \partial w_{L,t^*}$, and $dw_{H,t^*} = \partial w_{H,t^*}$. Combining (31) with (B1) yields

$$d \log \mu_{t^*} = \partial \log \mu_{t^*} + \begin{cases} \partial \log Q_{t^*}, & \partial \log N_{\sim e',t^*} \in (-\infty, 0] \\ \partial \log Q_{t^*} - \partial \log N_{\sim e',t^*}, & \partial \log N_{\sim e',t^*} \in (0, \partial \log Q_{t^*}] \\ 0, & \partial \log N_{\sim e',t^*} \in (\partial \log Q_{t^*}, \infty). \end{cases} \tag{B2}$$

When $\partial \log N_{\sim e',t^*} \leq 0$, $d \log v_{e,t^*}^N < \partial \log v_{e,t^*}^N \leq 0$ for each $e \in \{L, H\}$. These inequality contradicts local stability because $d \log N_{e,t^*} \leq 0$ for each $e \in \{L, H\}$. When $0 < \partial \log N_{\sim e',t^*} \leq \partial \log Q_{t^*}$, $d \log v_{e',t^*}^N < \partial \log v_{e',t^*}^N \leq 0$. This inequality contradicts local stability because $d \log N_{e',t^*} \leq 0$ and $d \log N_{\sim e',t^*} = 0$. When $\partial \log N_{\sim e',t^*} > \partial \log Q_{t^*}$, $d \log N_{e',t^*} < \partial \log N_{e',t^*} \leq 0$ and $d \log N_{\sim e',t^*} > 0$. These inequalities contradict local stability because $d \log v_{e',t^*}^N = \partial \log v_{e',t^*}^N \leq 0$ and $d \log v_{\sim e',t^*}^N = \partial \log v_{\sim e',t^*}^N > 0$.

B.3   Proof of Proposition 4

In the limit, (24) becomes $\partial \log n_{e,t^*}(z) = 0$ for $e \in \{L, H\}$ and $z > 0$. Using this result, we differentiate (A12) for $j \in \{j_{0,t^*} + 1, ..., J_{t^*}\}$. Because z_{e,j,t^*} linearly increases in p_{j,t^*} and decreases in p_{j-1,t^*}, solving for $\partial p_{j,t^*}$ yields

$$\partial p_{j,t^*} \propto \chi_{j,t^*} \partial p_{j-1,t^*} - \sum_{j'=j}^{J_t^*} \delta_{h,j'}, \tag{B3}$$

for each such j, where $\chi_{j,t^*} > 0$. Using this equation, we induct on j' to prove that $\partial p_{j,t^*}/\partial \delta_{h,j'}$ is negative for $j, j' \in \{j_{0,t^*} + 1, ..., J_{t^*}\}$, strictly decreases over $j' \in \{j_{0,t^*} + 1, ..., j\}$, and remains constant over $j' \in \{j, ..., J_{t^*}\}$. The claim is immediate for $j' = j_{0,t^*} + 1$ because of (35). The inductive step follows immediately as well.

Because $\partial \log n_{e,t^*}(z) = 0$ for $e \in \{L, H\}$ and $z > 0$, $\partial N_{e,t^*} = \partial Z_{e,t^*} = 0$ for each $e \in \{L, H\}$, giving $\partial \log a_{t^*} = 0$ from (2), $\partial \log A_{e,t^*} = 0$ from (26), and $\partial \log w_{e,t^*} =$ from (27). Therefore, (36) indicates that $\partial \log v_{e,t^*}(z) = -\beta_{c,e} \partial p_{j,t^*}/(y_{e,t^*}(z) - p_{j,t^*})$ when $z \in (z_{e,j,t^*}, z_{e,j+1,t^*})$. This equation holds for $z = z_{e,j,t^*}$ because $\partial \log z_{e,j,t^*} \leq 0$, which comes from differentiating (A12) and applying the assumption that $\delta_{h,j'} \geq 0$ for $j' > j_{0,t^*}$. Proposition 4 follows from the earlier statements proved about $\partial p_{j,t^*}$.

C Supplements to Section 3

C.1 Changing variables from z to y

For each $e \in \{L,H\}$, the cumulative distribution function of income equals $N_{e,t^*}^{-1} \int_0^{w_{e,t^*}^{-1} y} n_{e,t^*}(z) dz$. Differentiating gives the probability density function, $f(y) = N_{e,t^*}^{-1} w_{e,t^*}^{-1} n_{e,t^*}(w_{e,t^*}^{-1} y)$. For y such that $n_{e,t^*}(w_{e,t^*}^{-1} y) > 0$, define $\sigma_{e,t^*}(y) = (\partial \log n_{e,t^*})(w_{e,t^*}^{-1} y)$. Applying these equations and the change of variables $y = w_{e,t^*} z$, we transform (28) to

$$\partial \log N_{e,t^*} = \int_{y | \sigma_{e,t^*}(y) > 0} \sigma_{e,t^*}(y) f_{e,t^*}(y) dy \tag{C1}$$

(29) to

$$\partial \log Z_{e,t^*} = N_{e,t^*} Y_{e,t^*}^{-1} \int_{y | \sigma_{e,t^*}(y) > 0} \sigma_{e,t^*}(y) y f_{e,t^*}(y) dy, \tag{C2}$$

and (33) to

$$\frac{\delta_{h,j}}{N_{t^*}} = \sum_{e \in \{L,H\}} \frac{N_{e,t^*}}{N_{t^*}} \left(\underbrace{y_{e,j+1,t^*} f_{e,t^*}(y_{e,j+1,t^*}) \partial \log z_{e,j+1,t^*}}_{\text{trickle-up}} - \underbrace{y_{e,j,t^*} f_{e,t^*}(y_{e,j,t^*}) \partial \log z_{e,j,t^*}}_{\text{trickle-down}} \right.$$

$$\left. + \underbrace{\int_{y_{e,j,t^*}}^{y_{e,j+1,t^*}} \sigma_{e,t^*}(y) f_{e,t^*}(y) dy}_{\text{migration}} \right). \tag{C3}$$

To calculate $\sigma_{e,t^*}(y)$, we combine (24) and (36) to obtain

$$\sigma_{e,t^*}(y) = \frac{\beta_{c,e}}{\beta_{\epsilon,e}} \frac{y \partial \log w_{e,t^*}}{y - p_{j,t^*}} - \frac{\beta_{c,e}}{\beta_{q,e}} \frac{\partial p_{j,t^*}}{y - p_{j,t^*}} + \frac{\beta_{a,e}}{\beta_{\epsilon,e}} \partial \log a_{t^*}, \qquad y \in (y_{e,j,t^*}, y_{e,j+1,t^*}). \tag{C4}$$

We calculate $N_{e,t^*}/Y_{e,t^*} = (N_{e,t^*}/N_{t^*})/(Y_{e,t^*}/Y_{t^*})$. This methodology allows us to measure $\delta_{h,j}/N_{t^*}$ instead of $\delta_{h,j}$, so we consider changes to the housing stock as a share of the initial population.

C.2 Proof of Lemma 3

We rearrange (39) to produce

$$y_{H,j,t^*} = \widehat{p}_{j-1,t^*} + \frac{\widehat{p}_{j,t^*} - \widehat{p}_{j-1,t^*}}{1 - \left(\frac{y_{L,j,t^*} - \widehat{p}_{j,t^*}}{y_{L,j,t^*} - \widehat{p}_{j-1,t^*}} \right)^{\zeta}} \tag{C5}$$

for each $j \in \{1, ..., 50\}$. We rewrite the term in parentheses as $1 - (\widehat{p}_{j,t^*} - \widehat{p}_{j-1,t^*})/(y_{L,j,t^*} - \widehat{p}_{j-1,t^*})$, which strictly increases from zero to one in $y_{L,j,t^*} \in [\widehat{p}_{j,t^*}, \infty)$. Therefore, as a function of y_{L,j,t^*}, y_{H,j,t^*} strictly increases from p_{j,t^*} to ∞ for $y_{L,j,t^*} \in [\widehat{p}_{j,t^*}, \infty)$. By taking cumulative sums, we rewrite (38)

as

$$\frac{\sum_{j' \in \{j,\dots,50\}} \sum_i g_i \delta_{i,j'}}{\sum_i g_i} = \frac{N_{L,t^*}}{N_{t^*}} \int_{y_{L,j,t^*}}^{\infty} f_{L,t^*}(y)dy + \frac{N_{H,t^*}}{N_{t^*}} \int_{y_{H,j,t^*}(y_{L,j,t^*})}^{\infty} f_{H,t^*}(y)dy \qquad \text{(C6)}$$

for each $j \in \{1,\dots,50\}$, where we use the fact that $y_{e,51,t^*} = \infty$ because the support of the double Pareto-lognormal distribution is unbounded from above. The right side is defined for $y_{L,j,t^*} \geq \widehat{p}_{j,t^*}$ and strictly increases in y_{L,j,t^*}. Because (40) holds, the intermediate value theorem implies the existence of a unique solution for y_{L,j,t^*}, which then delivers a unique y_{H,j,t^*}. If $\zeta = 1$, then $y_{H,j,t^*} = y_{L,j,t^*}$. The solutions to (C6) then strictly increase in j because the left side does.

C.3   Proof of Lemma 4

Define $\rho_e = \beta_{c,e}/\beta_{w,e}$ for $e \in \{L,H\}$. Because $\beta_{q,e}/\beta_{c,e} = (\beta_{w,e} - \beta_{c,e})/\beta_{c,e} = \beta_{w,e}/\beta_{c,e} - 1$, we may rewrite the equation defining ζ as $\zeta = (\rho_L^{-1} - 1)/(\rho_H^{-1} - 1)$, which reduces to $\rho_L = \rho_H(\zeta + (1-\zeta)\rho_H)^{-1}$. We substitute this equation into (45), which takes the form $1 - \alpha = (1 - \bar{e}_\alpha)\rho_L + \bar{e}_\alpha \rho_H$. After multiplying through by $\zeta + (1-\zeta)\rho_H$ and simplifying, we obtain $0 = \bar{e}_\alpha(1-\zeta)\rho_H^2 + (\zeta + (\alpha - \bar{e}_\alpha)(1-\zeta))\rho_H - (1-\alpha)\zeta$. The right side is a quadratic function of ρ_H that changes sign between $\rho_H = 0$, where it equals $-(1-\alpha)\zeta$, and $\rho_H = 1$, where it equals α. Therefore, there exists a unique solution for $\rho_H \in (0,1)$. Because $\zeta + (1-\zeta)\rho_H = \rho_H + \zeta(1-\rho_H) > \rho_H$, the corresponding solution for ρ_L also lies in $(0,1)$.

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