Optimal Regional Insurance Provision under Privately Observable Shocks

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Abstract

We study the design and implementation of optimal regional insurance provision against privately observable shocks to the degree of intergenerational externality (DIE) induced by, or the degree of technological progress (DTP) for producing, intergenerational public goods (IPGs). We obtain four main results. First, the intertemporal allocation is not distorted only at the endpoints of type distribution, and insurance is incomplete. Second, if the grant to bottom type is distorted upward, then its debt must be distorted downward, and vice versa; the direction of distortion is qualitatively reversed between top and bottom types. Third, for all but the endpoints, there is a grant scheme, which is nonlinear and monotonic in debt, that decentralizes welfare optimum. For the endpoints, however, the implementation grant scheme is independent of debt. Fourth, when regions differ in DIE in the course of implementation, grant and debt are complementary in insurance provision; when regions differ in DTP, they are complementary with observable output of, but are substitutive with observable expenditure on, the IPGs.

Keywords: Intergovernmental grant; interregional insurance; regional public debt; intergenerational externality; asymmetric information.

JEL Codes: H41; H74; H77; D82.

1 Introduction

Intergovernmental grants implemented by the central government of a federal fiscal system are justified by helping internalize interregional spillovers from local public goods (Oates, 1972) or migrations (e.g., Hercowitz and Pines, 1991; Breuillé and Gary-Bobo, 2007; Dai, Liu and Tian, 2018a), redistribute income between regions (e.g., Cremer and Pestieau, 1997; Raff and Wilson, 1997; Cornes and Silva, 2000; Bordignon, Manasse and Tabellini, 2001), and serve as a risk-sharing device against region-specific shocks (e.g.,

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Persson and Tabellini, 1996a, 1996b; Bucovetsky, 1998; Lockwood, 1999; Cornes and Silva, 2000; Jüßen, 2006). As shown by Sala-i-Martin and Sachs (1992), even policies aimed at redistribution may have an effect on the degree of interregional risk sharing. Indeed, a bunch of empirical evidence suggests that fiscal transfers from the federal budget provide a substantial degree of insurance against regional economic fluctuations in the United States, Canada, Japan, Norway, and among others.[1] Even in the presence of complete markets, Farhi and Werning (2017) provide a rationale for the existence of government intervention in terms of public risk sharing.

In addition, public debt serves through the intertemporal channel as a public contract for sharing risks between generations or over lifecycle within a given region or state.[2] Since present generations are imperfectly altruistic (e.g., Altonji, Hayashi and Kotlikoff, 1992, 1997), the design of optimal public debt with taking into account possible intergenerational conflicts turns out to be a nontrivial task (e.g., Rangel, 2003, 2005; Huber and Runkel, 2008; Dai, Liu and Tian, 2018b).

The following questions thus arise. How would these two options of insurance provision behave when they are jointly designed by the central government? Under decentralized debt decisions, how would the interregional insurance provided by the central government interact with the intergenerational insurance provided locally? More specifically, shall they exhibit complementarity or substitutability in the course of implementation? Indeed, in terms of regional insurance provision per se, it is informative to identify when intergovernmental grant and local debt should be jointly (or separately) used; otherwise, efficiency losses might arise. In particular, identifying the case with policy complementarity justifies in some sense the coexistence of grant and debt in real-world federations; identifying the case with policy substitutability creates a sort of policy flexibility in terms of insurance provision.[3]

To our knowledge, these issues are left unexplored in the literature and we are unable to find any relevant answers. Our goal is, therefore, to address these questions via tackling the optimal design and implementation of risk-sharing contracts consisting of both intergovernmental grants and regional public debt along the space and time dimensions, respectively.

We consider a federal country that consists of a central government and many subnational governments located in geographically decentralized regions. Throughout, the center is in charge of revenue transfers across regions whereas local governments are responsible for collecting taxes used for the provision of local public goods. Each region is populated by a continuum of identical residents who live for one period only. We focus on an economy that lasts for two periods, thus enabling us to incorporate intergenerational concerns into the current environment, while retaining simplicity and tractability. Precisely, the current generation chooses how much debt to pass to the future generation and how much to invest in intergenerational public goods (IPGs), such as basic science, environmental protection and public capital. Initially, we let the center jointly determine

[1]See, e.g., Atkeson and Bayoumi (1993); Asdrubali, Sørensen and Yosha (1996); Mélitz and Zumer (1999); Athanasoulis and van Wincoop (2001); Kalemli-Ozcan, Sørensen and Yosha (2003); Borge and Matsen (2004); Evers (2015).

[2]In an infinite-horizon economy where individuals face uninsurable risks to their human capital accumulation, Gottardi, Kajii and Nakajima (2015) show that the benefits of government debt increase with the magnitude of risks and the degree of risk aversion.

[3]For example, grant and debt can be used simultaneously while targeting different policy goals.

the amount of public debt a region can issue and the transfer it can receive. We then move to the more realistic situation in which regions are allowed to have the autonomy in choosing the level of regional public debt.

Rather than introducing ex ante heterogeneity, regions are assumed to be ex ante identical but are subject to stochastic shocks to either the degree of intergenerational externality induced by, or the degree of technological change for producing, the IPGs. Shocks are assumed to be continuously distributed over the support of closed intervals. While regional heterogeneity in shock realizations creates a natural role for interregional insurance, we particularly argue from the following two respects that these two sorts of shocks indeed enable public debt to perform as a cross-generation risk-sharing device. Firstly, the current generation incurs the cost of IPG investment that generates a positive externality on the future generation. Secondly, although we assume exogenous technological change to keep things simple, it is well-recognized that the progress made in fields like basic science, space exploration and environmental protection benefits from standing on the shoulder of giants, which corresponds to the technological level in the current generation. As is customary in the fiscal federalism literature[4], regional governments are better informed about the shocks than the federal government. As such, intergovernmental grants and regional public debt form the risk-sharing contracts designed by the center with taking into account the fiscal budget balance, participation and truth-telling constraints.

From solving the mechanism design problem facing the center, conducting the comparison with the full-information optimum (or the first-best allocation), and decentralizing the optimal allocation, we obtain the following four main results, regardless of the source of shocks. First, the intertemporal allocation is not distorted only at the bottom and top types, and there is incomplete insurance. Second, if the intergovernmental grant received by the bottom type is distorted upward, then its public debt must be distorted downward, and vice versa; meanwhile, the direction of distortion is qualitatively reversed between the bottom and top types. These two results characterize the asymmetric-information welfare optimum. Third, for all but the bottom and top types, there is an intergovernmental grant scheme, which is a nonlinear, almost everywhere differentiable, and monotonic function of regional public debt, such that the welfare optimum can be truthfully implemented by regionally decentralized decisions. And fourth, for the bottom and top types, the grant scheme that decentralizes the welfare optimum is, however, independent of the regional public debt. The last two results provide the key features of the implementation scheme over the entire type distribution.

Moreover, we obtain the following results concerning the relationship of these two insurance-provision instruments in the course of implementing welfare optimum. When regions differ in the degree of intergenerational externality, they are complementary in insurance provision. The implication is thus that it is socially optimal to use both insurance schemes simultaneously when facing this sort of shocks. When regions differ in the degree of technological progress for producing the IPGs, they are complementary if it is the physical output of public goods that is observable, whereas they are substitutive if it is the regional expenditure on public goods that is observable. Consequently, it is not always socially beneficial to adopt both at once providing the same source of shocks. Given the optimal choice of local public debt, the social desirability of interregional insurance

[4]See, for example, Oates (1972); Bucovetsky, Marchand and Pestieau (1998); Lockwood (1999); Cornes and Silva (2000, 2002); Huber and Runkel (2008).

relies on the informational constraint placed on the central government. The observability of input (of IPGs) essentially differs from that of output in determining whether interregional insurance completes the insurance provided locally. In terms of identifying the effect of informational asymmetry on the implementation of constrained optimal insurance schemes, this finding contributes to the public finance and regional science literature. In addition, if the intergenerational conflict induced by local debt is the dominant issue in an economy, then the case of substitutability shows the social desirability of tightening local borrowing to the lowest possible level while relying on intergovernmental grants enforced by the center. These results characterize the connections between these public risk-sharing schemes and the underlying environment, thus helping us understand how they should be adopted in real-world federations.

Our study is related to the literature that examines theoretically the design of regional insurance provision in a federation, such as Persson and Tabellini (1996a, 1996b), Bucovetsky (1998), Lockwood (1999), and Cornes and Silva (2000). A comparison with these studies reveals three distinctive features of the current study. Firstly, rather than adopting a one-period static setting, we consider a two-period setting which allows for taking into account intergenerational concerns and a natural role for public debt. Secondly, the source of shocks as well as the informational asymmetries between the center and regions is novel, and also proves to be relevant, in terms of optimal insurance provision. And thirdly, we can investigate the *joint design* of two widely-used risk-sharing schemes along two dimensions, namely intergovernmental grants along the interregional dimension and public debt along the intertemporal/intergenerational dimension, and further analyze their interaction in the course of optimal decentralization. For these features, our paper extends and complements the existing literature.

The remainder of the paper is organized as follows. Section 2 describes the environment. Section 3 derives the welfare optimum and discusses its implementation when regions differ in intergenerational externality. Section 4 derives the welfare optimum and discusses its implementation when regions differ in the technological progress for producing local IPGs. Section 5 concludes. Proofs are relegated to Appendix.

2 Environment

We consider a two-period economy of a federation consisting of a federal government (also referred to as the center) and n regions, each of which is inhabited by a representative immobile resident in each period. That is, each resident lives for one period only. The social welfare of region i, for $i = 1, 2, ..., n$, is given by

$$\underbrace{u_1(c_1^i) + g_1(G_1^i)}_{\text{utility of generation 1}} + \underbrace{u_2(c_2^i) + g_2(\theta^i G_1^i + G_2^i)}_{\text{utility of generation 2}}, \tag{1}$$

in which c_1^i and c_2^i are private consumptions, G_1^i and G_2^i are public goods, and $\theta^i \in (0, 1]$ is a parameter measuring the degree of intergenerational externality of the IPG[5], G_1^i. All four functions in (1) are strictly increasing, strictly concave and satisfy the usual Inada conditions.[6]

[5]IPG is a kind of public good produced in generation 1 and still (partially) usable in generation 2 (Rangel, 2005).

[6]Also, note that the preference specification encompasses the special case with $u_2(c_2^R) + g_2(\theta^R G_1^R + G_2^R) \equiv \beta[u_1(c_2^R) + g_1(\theta^R G_1^R + G_2^R)]$, in which $\beta > 0$ is a social discount factor that can be interpreted as

The representative resident of generation t, for $t = 1, 2$, in region i has private budget constraint $c_t^i + \tau_t^i = y_t$, where y_t is the commonly given income across all regions. The lump sum tax τ_t^i is collected by the local government to finance the provision of local public goods. In period 1, it receives a transfer z^i from the center and issues debt b^i. If $z^i < 0$, then the local government has to pay a tax to the center. Debt plus interest has to be repayed in period 2, taking as given the common interest rate $r > 0$.[7] The fiscal budget constraints of region i in periods 1 and 2 can be written as $G_1^i = \tau_1^i + b^i + z^i$ and $\mathcal{G}_2^i \equiv \xi^i G_2^i = \tau_2^i - (1+r)b^i$, respectively, in which the parameter $\xi^i \in (0, 1]$ measures the per unit cost of producing the period-2 public goods in region i. The case of $\xi^i < 1$ captures the effect of technological progress, which as argued by Rangel (2005) is important for IPGs such as infrastructure, space exploration and environmental capital.

For expositional convenience, the region index i is suppressed in the remainder of the set-up. Combining the private budget constraints with the public budget constraints and applying them to equation (1), a region's social welfare maximization problem is given by

$$\max_{c_1, c_2} \quad u_1(c_1) + g_1(y_1 + b + z - c_1) + u_2(c_2) + g_2(\theta(y_1 + b + z - c_1) + \rho[y_2 - b(1 + r) - c_2]), \tag{2}$$

in which $\rho \equiv 1/\xi \geq 1$. Note that in problem (2), choosing c_1 and c_2 is equivalent to choosing τ_1 and τ_2. The first-order conditions are thus written as

$$u_1'(c_1) = g_1'(G_1) + \theta g_2'(\theta G_1 + G_2) \quad \text{and} \quad u_2'(c_2) = \rho g_2'(\theta G_1 + G_2), \tag{3}$$

which represent the Samuelson conditions for the optimal provision of public goods.

We allow regions to differ in two dimensions in terms of privately observable shocks: the degree of intergenerational externality measured by θ and the degree of technological progress measured by ξ (or, equivalently, by ρ). As is well known, it is too difficult to obtain interesting results when there is multidimensional heterogeneity at once. We thus consider these two cases separately.

The random variables in each case are assumed to be continuously distributed in intervals $[\underline{\theta}, \bar{\theta}] \equiv \Theta$ and $[\underline{\xi}, \bar{\xi}] \equiv \Xi$ (or $[\underline{\rho}, \bar{\rho}] \equiv \Upsilon$), and also are identically and independently distributed across regions. We denote by $f = F' > 0$ and F, respectively, the density and distribution functions, which are common knowledge throughout.

3 Welfare Optimum and Implementation when Regions Differ in Intergenerational Externality

We introduce first the problem of the center. Then, we proceed to derive welfare optimum in cases of symmetric and asymmetric information between the center and regions, and discuss the implementation issue.

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a political parameter reflecting the degree to which regional governments take into account the welfare of future generations.

[7] Assuming that there is a common capital market within a federation, there is a unique rental price level of capital such that arbitrage opportunities are eliminated.

3.1 The Problem of the Center

The center is responsible for determining regional debt and cross-region transfers as non-market insurances against shocks on intergenerational externality. Assuming it treats all regions equally and the realization of shocks can be privately observed by each region, it thus maximizes the expectation of the value function (2) of any region, subject to fiscal budget balance, incentive-compatibility and participation constraints.

We follow the mechanism design approach and apply the direct revelation principle. The center offers each region i a contract stipulating the federal transfer and the region's debt conditional only on its report of type θ^i, denoted by $\hat{\theta}^i$, i.e., $b^i = b(\hat{\theta}^i)$ and $z^i = z(\hat{\theta}^i)$. To formulate the constraints facing the mechanism designer, we consider the limiting case with the number of regions being large, i.e., $n \to \infty$. Making use of the weak law of large numbers, the empirical distributions of b^i and z^i across regions approximate the theoretical distributions generated by $b^i = b(\hat{\theta}^i)$, $z^i = z(\hat{\theta}^i)$ and F.

Without loss of any generality, here we let $\xi^i = 1$ for all i to simplify notation, and write the value generated by the maximization problem (2) as $V(b, z, \theta)$. As all regions are ex ante identical, the objective of the center can be written again by the law of large numbers as:

$$EU = \int_{\underline{\theta}}^{\bar{\theta}} V(b(\theta), z(\theta), \theta) f(\theta) d\theta. \tag{4}$$

The truth-telling constraints require that any region with shock realization θ prefer to report θ rather than some θ'; formally

$$V(b(\theta), z(\theta), \theta) \geq V(b(\theta'), z(\theta'), \theta) \;\; \forall \theta' \neq \theta, \;\; \theta', \theta \in \Theta. \tag{5}$$

The participation constraint for any region specifies that it would like to participate in the federation and receive some interregional insurance via the grant system rather than secede. Following the arguments of Lockwood (1999), we impose the ex ante[8] participation constraint:

$$EU \geq \int_{\underline{\theta}}^{\bar{\theta}} \max_{b(\theta)} V(b(\theta), 0, \theta) f(\theta) d\theta, \tag{6}$$

where EU is given by (4). Since the center can always replicate by setting $z \equiv 0$ what any region could get by seceding, we are sure that constraint (6) will never bind and can be safely ignored in the following.

The budget balance constraint for large n reads as

$$\int_{\underline{\theta}}^{\bar{\theta}} z(\theta) f(\theta) d\theta \leq 0, \tag{7}$$

which implies that in aggregate transfers must sum to at most zero.

The problem facing the center is thus to choose $\{b(\theta), z(\theta)\}_{\theta \in \Theta}$ to maximize (4) subject to (5) and (7). As is common in the mechanism design literature, we let b and z be piecewise continuously differentiable functions, and let $b(\theta)$ be everywhere continuous.

[8]Indeed, the ex post participation constraint is usually too strong to be fulfilled.

3.2 Welfare Optimum

As a standard benchmark result to which we can refer, we start our analysis by deriving the full-information (first-best) allocation that maximizes (4) subject to (7) only. We index the first-best optimum by the superscript ^{FB}.

Proposition 3.1 *In the full-information case, the welfare optimum* $\{b^{FB}(\theta), z^{FB}(\theta)\}_{\theta\in\Theta}$ *satisfies:*

(i) The intertemporal rate of substitution between current and future public goods consumption equals intertemporal rate of transformation, namely

$$\frac{g_1'(G_1^{FB}(\theta))}{g_2'(\theta G_1^{FB}(\theta) + G_2^{FB}(\theta))} \quad = \quad 1 + r - \theta \quad \text{for any } \theta \in \Theta.$$

(ii) Full insurance is achievable, namely

$$V_z\left(b^{FB}(\theta), z^{FB}(\theta), \theta\right) \quad = \quad \gamma \quad \text{for any } \theta \in \Theta,$$

in which $\gamma > 0$ denotes the Lagrangian multiplier on the budget constraint (7).

Proof. Straightforward and omitted. ■

Part (i) yields that the intertemporal allocation of any type of regions is not distorted in the first-best optimum. Part (ii) gives the standard insurance condition which states that the consumption of period-2 public goods is the same regardless of the shock realization on the degree of intergenerational spillovers.

We now turn to the more interesting case with asymmetric information between the center and regions. In this case, the realization of the random variable measuring the degree of intergenerational externality is private information so that regions of one type could mimic regions of another type in order to obtain (more) insurance transfers. We now index the second-best allocation by the superscript *.

We shall need the following assumption:

Assumption 3.1 $-\theta G_1 g_2'' \leq g_2'$ *for all* $\theta \in (\underline{\theta}, \bar{\theta})$*, namely the absolute value of the elasticity of generation 2's marginal utility from G_1 is no greater than one for all but the endpoints of the type distribution.*

This is a technical restriction imposed on generation 2's preference on public goods. It is easy to verify that this assumption is satisfied for log and power utility functions.

Proposition 3.2 *In the asymmetric information case without bunching, the welfare optimum* $\{b^*(\theta), z^*(\theta)\}_{\theta\in\Theta}$ *satisfies:*

(i) Concerning the relationship that governs the intertemporal rate of substitution between current and future public goods consumption and the intertemporal rate of transformation, we have:

$$\frac{g_1'(G_1^*(\theta))}{g_2'(\theta G_1^*(\theta) + G_2^*(\theta))} \begin{cases} = \; 1 + r - \theta & \text{for } \theta \in \{\underline{\theta}, \bar{\theta}\}; \\ < \; 1 + r - \theta & \text{for } \theta \in (\underline{\theta}, \bar{\theta}). \end{cases}$$

(ii) Suppose Assumption 3.1 holds. Let $\mu_1(\theta) > 0$ be the Lagrangian multiplier on the value constraint $v(\theta) \equiv V(b(\theta), z(\theta), \theta)$ of any type-θ region who is reporting truthfully, then we have:

$$V_z\left(b^*(\theta), z^*(\theta), \theta\right) \begin{cases} = \ \gamma/\mu_1(\theta) & \text{for } \theta \in \{\underline{\theta}, \bar{\theta}\}; \\ < \ \gamma/\mu_1(\theta) & \text{for } \theta \in (\underline{\theta}, \bar{\theta}). \end{cases}$$

Proof. See Appendix. ■

The key finding of this proposition is the following. First, the intertemporal allocation under asymmetric information is not distorted only at the two endpoints of type distribution, that is, for the regions of the highest and the lowest possible degrees of intergenerational externality. Second, there is incomplete insurance under asymmetric information.

As the informational friction is what we focus on in this study, it is interesting to identify the effect of asymmetric information between the center and regions on optimal debt and intergovernmental grants policies. To this end, it is worthwhile providing a detailed characterization of the Lagrangian multiplier $\mu_1(\theta)$ after a comparison of Propositions 3.1 and 3.2.

Lemma 3.1 *For the current economic environment, the following statements are true.*

(i) If $\mu_1(\theta)$ is decreasing in θ, then there exists some $\tilde{\theta} \in (\underline{\theta}, \bar{\theta})$ such that $\mu_1(\theta) > 1$ for $\theta \in [\underline{\theta}, \tilde{\theta})$, $\mu_1(\theta) = 1$ for $\theta = \tilde{\theta}$, and $\mu_1(\theta) < 1$ for $\theta \in (\tilde{\theta}, \bar{\theta}]$.

(ii) If $\mu_1(\theta)$ is increasing in θ, then there exists some $\breve{\theta} \in (\underline{\theta}, \bar{\theta})$ such that $\mu_1(\theta) < 1$ for $\theta \in [\underline{\theta}, \breve{\theta})$, $\mu_1(\theta) = 1$ for $\theta = \breve{\theta}$, and $\mu_1(\theta) > 1$ for $\theta \in (\breve{\theta}, \bar{\theta}]$.

Proof. See Appendix. ■

Using Lemma 3.1, the following proposition is established.

Proposition 3.3 *Under Assumption 3.1, the following statements are true.*

(i) If $\mu_1(\theta)$ is decreasing in θ, then (i-a) $z^*(\theta) > z^{FB}(\theta)$ for all $\theta \in [\underline{\theta}, \tilde{\theta}]$; (i-b) $z^*(\bar{\theta}) < z^{FB}(\bar{\theta})$; and (i-c) $b^*(\underline{\theta}) < b^{FB}(\underline{\theta})$ and $b^*(\bar{\theta}) > b^{FB}(\bar{\theta})$.

(ii) If $\mu_1(\theta)$ is increasing in θ, then (ii-a) $z^*(\theta) > z^{FB}(\theta)$ for all $\theta \in [\breve{\theta}, \bar{\theta}]$; (ii-b) $z^*(\underline{\theta}) < z^{FB}(\underline{\theta})$; and (ii-c) $b^*(\underline{\theta}) > b^{FB}(\underline{\theta})$ and $b^*(\bar{\theta}) < b^{FB}(\bar{\theta})$.

Proof. See Appendix. ■

The interpretation of these results is intuitive. The intuition of claim (i-a) is the following. If $\mu_1(\theta)$ is decreasing in θ, then the shadow prices of the value constraint under truth-telling are larger for lower types of regions than for higher types of regions. As a result, it is more likely to be that lower types of regions *extract* the information rent and receive higher transfers under asymmetric information than under full information. Claim (ii-a) can be interpreted in a similar way.

Regardless of whether $\mu_1(\theta)$ is decreasing or increasing in θ, the allocation under asymmetric information is distorted at both ends of support of the distribution of the shock, and importantly, the distortion is qualitatively different at the top and bottom. For example, if the shadow prices of the value constraint under truth-telling are larger for lower types of regions than for higher types of regions, then the top type receives less transfers while issues more debt, and the bottom type receives more transfers while issues less debt than they would exhibit, respectively, under full information.

3.3 Implementation

We have established the welfare optimum under both symmetric and asymmetric information in the previous section, here we proceed to consider how to implement it via regionally-decentralized debt decisions. That is, both regions choose a level of public debt to maximize their regional welfare, taking as given the intergovernmental grants scheme of the center. Formally, the maximization problem of regions of type-θ is

$$\max_{b(\theta)} \ V(b(\theta), z(\theta), \theta)$$

for any given $z(\theta)$. Letting $c_1 = \tilde{\phi}(b(\theta), z(\theta), \theta)$ and $c_2 = \tilde{\psi}(b(\theta), z(\theta), \theta)$ and applying the Envelope Theorem, the first-order condition is thus written as

$$\begin{aligned} & g_1' \left(y_1 + b(\theta) + z(\theta) - \tilde{\phi}(b(\theta), z(\theta), \theta) \right) \\ = \ & (1 + r - \theta) g_2' \left(\theta(y_1 + b(\theta) + z(\theta) - \tilde{\phi}(b(\theta), z(\theta), \theta)) + y_2 - (1 + r)b(\theta) - \tilde{\psi}(b(\theta), z(\theta), \theta) \right), \end{aligned} \tag{8}$$

showing that the intertemporal rate of substitution must be equal to the intertemporal rate of transformation at the regional welfare optimum.

Making use of (8) and Proposition 3.1, we immediately have the following result: The full-information optimum is attained by simply setting $z(\theta) = z^{FB}(\theta)$ for all $\theta \in \Theta$. The reason is that the center can observe the type of each region and also the full-information optimum does not distort the intertemporal allocation desired by each region.

Under asymmetric information, the center should design intergovernmental grants scheme that guarantees incentive compatibility for both regions. It follows from Proposition 3.2 that the intertemporal allocation of regions of all but top and bottom types is distorted, so the asymmetric-information optimum can no longer be implemented by decentralized debt decisions characterized by (8) with the center simply setting $z(\theta) = z^*(\theta)$. Indeed, we have established the following proposition.

Proposition 3.4 *The grant scheme $z^*(b)$ that decentralizes the asymmetric information optimum $\{b^*(\theta), z^*(\theta)\}_{\theta \in \Theta}$ is a nonlinear nondecreasing function of b, almost everywhere differentiable, with the slope*

$$\frac{dz^*}{db} = \Phi(b) \begin{cases} = 0 & \text{for } b \in \{b^*(\underline{\theta}), b^*(\bar{\theta})\}; \\ > 0 & \text{for } b \in (b^*(\underline{\theta}), b^*(\bar{\theta})) \end{cases}$$

in which the expression of $\Phi(b)$ is given in the Appendix.

Proof. See Appendix. ■

If regions of higher degrees of intergenerational spillovers are allowed to issue more debt than regions of lower degrees of intergenerational spillovers, then the grant scheme that decentralizes the asymmetric-information optimum should be designed such that the former regions get strictly more grants than the latter ones. As $V_z > 0$, this grant scheme guarantees that high-type regions with high degrees of intergenerational spillovers shall not mimic low-type regions.

4 Welfare Optimum and Implementation when Regions Differ in Technological Progress

To analyze the optimal regional insurance provision when regions differ in shocks on technological change, we normalize the per unit cost of producing the local public goods in current generation to one, which is public knowledge, while that in the next generation is a random variable and is also the private information of each region. As shown by Maskin and Riley (1985) and mentioned by Lockwood (1999), it in general makes a difference whether the expenditure (or input) or the physical output is observable by the mechanism designer. To make our investigation as complete as possible, we shall discuss both possibilities. For both cases, we derive first the welfare optimum and then proceed to the associated implementation issue.

4.1 The Case with Observable Expenditure on Public Goods

4.1.1 Welfare Optimum

The first-order conditions of problem (2) are now written as

$$u_1'(c_1) = g_1'(G_1) + \theta g_2'(\theta G_1 + \rho \mathcal{G}_2) \text{ and } u_2'(c_2) = \rho g_2'(\theta G_1 + \rho \mathcal{G}_2), \tag{9}$$

and the corresponding regional value function is written as $V(b, z, \rho)$.

Applying the Envelope Theorem, the first-best allocation can be characterized as s-tated in Proposition 4.1. The proof is straightforward and omitted.

Proposition 4.1 *In the full-information case, the welfare optimum* $\{b^{FB}(\rho), z^{FB}(\rho)\}_{\rho \in \Upsilon}$ *satisfies:*

(i) The intertemporal rate of substitution between current and future public goods consumption equals intertemporal rate of transformation, namely

$$\frac{g_1'(G_1^{FB}(\rho))}{g_2'(\theta G_1^{FB}(\rho) + \rho \mathcal{G}_2^{FB}(\rho))} = \rho(1+r) - \theta \text{ for any } \rho \in \Upsilon.$$

(ii) Full insurance is achievable, namely

$$V_z\left(b^{FB}(\rho), z^{FB}(\rho), \rho\right) = \gamma \text{ for any } \rho \in \Upsilon,$$

in which $\gamma > 0$ *denotes the Lagrangian multiplier on the budget constraint* $\int_{\underline{\rho}}^{\bar{\rho}} z(\rho) f(\rho) d\rho \leq 0$.

Part (i) yields that the intertemporal allocation of any type of regions is not distorted in the first-best optimum. Part (ii) gives the standard insurance condition.

To derive the asymmetric-information optimum, we shall need the following assumption:

Assumption 4.1 $-\rho \mathcal{G}_2 g_2'' \leq g_2'$ *for all* $\rho \in (\underline{\rho}, \bar{\rho})$, *namely the absolute value of the elasticity of marginal utility from* $G_2 = \rho \mathcal{G}_2$ *is no greater than one for all but the endpoints of the type distribution.*

This is a technical restriction that is quite similar to Assumption 3.1. Under Assumption 4.1, the center is thought of as solving the following maximization problem:

$$\max \int_{\underline{\rho}}^{\bar{\rho}} v(\rho) f(\rho) d\rho$$

$$\begin{aligned}
s.t. \quad v(\rho) &= V(b(\rho), z(\rho), \rho); \\
\int_{\underline{\rho}}^{\bar{\rho}} z(\rho) f(\rho) d\rho &\leq 0; \\
\dot{v}(\rho) &= g_2'(\theta\phi(b(\rho), z(\rho), \rho) + \rho\psi(b(\rho), z(\rho), \rho))\, \psi(b(\rho), z(\rho), \rho); \\
\dot{b}(\rho) &\leq 0
\end{aligned} \tag{10}$$

in which $\phi(b(\rho), z(\rho), \rho) = G_1(\rho)$, $\psi(b(\rho), z(\rho), \rho) = \mathcal{G}_2(\rho)$, the first constraint gives the value function of regions of type-ρ when they are truth-telling, the second one is the fiscal budget constraint under pure intergovernmental grants, the third one is the first-order necessary condition for incentive compatibility, and the last one is the second-order sufficient condition for incentive compatibility[9].

Indeed, solving problem (10) gives the following proposition.

Proposition 4.2 *In the asymmetric information case without bunching, the welfare optimum $\{b^*(\rho), z^*(\rho)\}_{\rho\in\Upsilon}$ satisfies:*

(i) Suppose Assumption 4.1 holds. Concerning the relationship that governs the intertemporal rate of substitution between current and future public goods consumption and the intertemporal rate of transformation, we have:

$$\frac{g_1'(G_1^*(\rho))}{g_2'(\theta G_1^*(\rho) + \rho\mathcal{G}_2^*(\rho))} \begin{cases} = \rho(1+r) - \theta & \text{for } \rho \in \{\underline{\rho}, \bar{\rho}\}; \\ > \rho(1+r) - \theta & \text{for } \rho \in (\underline{\rho}, \bar{\rho}). \end{cases}$$

(ii) Let $\mu_1(\rho) > 0$ be the Lagrangian multiplier on the value constraint $v(\rho) \equiv V(b(\rho), z(\rho), \rho)$ of any type-ρ region who is reporting truthfully, then we have:

$$V_z(b^*(\rho), z^*(\rho), \rho) \begin{cases} = \gamma/\mu_1(\rho) & \text{for } \rho \in \{\underline{\rho}, \bar{\rho}\}; \\ > \gamma/\mu_1(\rho) & \text{for } \rho \in (\underline{\rho}, \bar{\rho}). \end{cases}$$

Proof. See Appendix. ■

As shown in Proposition 3.2, the intertemporal allocation under asymmetric information is not distorted only at the two endpoints of type distribution, and also there is incomplete insurance.

Proposition 4.3 *For the current economic environment, the following statements are true.*

(i) If $\mu_1(\rho)$ is decreasing in ρ, then (i-a) there exists some $\breve{\rho} \in (\underline{\rho}, \bar{\rho})$ such that $z^*(\rho) < z^{FB}(\rho)$ for all $\rho \in [\breve{\rho}, \bar{\rho}]$; (i-b) $z^*(\underline{\rho}) > z^{FB}(\underline{\rho})$; and (i-c) $b^*(\underline{\rho}) < b^{FB}(\underline{\rho})$ and $b^*(\bar{\rho}) > b^{FB}(\bar{\rho})$.

[9] The derivation of this monotonicity constraint is given in the proof of Proposition 4.2.

(ii) If $\mu_1(\rho)$ is increasing in ρ, then (ii-a) there exists some $\tilde{\rho} \in (\underline{\rho}, \bar{\rho})$ such that $z^*(\rho) < z^{FB}(\rho)$ for all $\rho \in [\underline{\rho}, \tilde{\rho}]$; (ii-b) $z^*(\bar{\rho}) > z^{FB}(\bar{\rho})$; and (ii-c) $b^*(\underline{\rho}) > b^{FB}(\underline{\rho})$ and $b^*(\bar{\rho}) < b^{FB}(\bar{\rho})$.

Proof. See Appendix. ■

The interpretation of these results is intuitive. The intuition of claim (i-a) is the following. If $\mu_1(\rho)$ is decreasing in ρ, then the shadow prices of the value constraint under truth-telling are larger for lower types of regions than for higher types of regions. As a result, it is more likely to be that higher types of regions *incur* the information rent and receive lower transfers under asymmetric information than under full information. Claim (ii-a) can be interpreted in a similar way.

Regardless of whether $\mu_1(\rho)$ is decreasing or increasing in ρ, the allocation under asymmetric information is distorted at both ends of support of the distribution of the shock, and importantly, the distortion is qualitatively different at the top and bottom. For example, if the shadow prices of the value constraint under truth-telling are larger for lower types of regions than for higher types of regions, then the top type receives less transfers while issues more debt, and the bottom type receives more transfers while issues less debt than they would exhibit, respectively, under full information.

4.1.2 Implementation

To implement the welfare optimum via regionally-decentralized debt decisions, we solve first the maximization problem of regions of type-ρ:

$$\max_{b(\rho)} V(b(\rho), z(\rho), \rho)$$

for any given $z(\rho)$. Letting $c_1 = \tilde{\phi}(b(\rho), z(\rho), \rho)$ and $c_2 = \tilde{\psi}(b(\rho), z(\rho), \rho)$ and applying the Envelope Theorem, the first-order condition is thus written as

$$\begin{aligned} g_1'\left(y_1 + b(\rho) + z(\rho) - \tilde{\phi}(b(\rho), z(\rho), \rho)\right) &= [\rho(1+r) - \theta]\times \\ g_2'\left(\theta(y_1 + b(\rho) + z(\rho) - \tilde{\phi}(b(\rho), z(\rho), \rho)) + \rho[y_2 - (1+r)b(\rho) - \tilde{\psi}(b(\rho), z(\rho), \rho)]\right)&. \end{aligned} \tag{11}$$

Making use of (11) and Proposition 4.1, we immediately have the following result: The full-information optimum is attained by simply setting $z(\rho) = z^{FB}(\rho)$ for all $\rho \in \Upsilon$.

Under asymmetric information, we obtain the following implementation scheme.

Proposition 4.4 *Suppose Assumption 4.1 holds. The grant scheme $z^*(b)$ that decentralizes the asymmetric information optimum $\{b^*(\rho), z^*(\rho)\}_{\rho\in\Upsilon}$ is a nonlinear non-increasing function of b, almost everywhere differentiable, with the slope*

$$\frac{dz^*}{db} = \Gamma(b) \begin{cases} = 0 & \text{for } b \in \{b^*(\underline{\rho}), b^*(\bar{\rho})\}; \\ < 0 & \text{for } b \in (b^*(\bar{\rho}), b^*(\underline{\rho})) \end{cases}$$

in which the expression of $\Gamma(b)$ is given in the Appendix.

Proof. See Appendix. ■

If regions of lower degrees of technological progress (namely smaller ρ) are allowed to issue more debt than regions of higher degrees of technological progress, then the grant scheme that decentralizes the asymmetric-information optimum should be designed such that the former regions get strictly less grants than the latter ones. As $V_z > 0$, this grant scheme guarantees that high-type regions with high degrees of technological progress shall not mimic low-type regions.

4.2 The Case with Observable Physical Output of Public Goods

4.2.1 Welfare Optimum

Now, the value function of regions of type-ξ reads as follows:

$$\begin{aligned} V(b, z, \xi) \;\equiv\; \max_{G_1, G_2} \;& u_1(y_1 + b + z - G_1) + g_1(G_1) \\ & + u_2(y_2 - b(1+r) - \xi G_2) + g_2(\theta G_1 + G_2). \end{aligned} \tag{12}$$

The first-order conditions are thus written as:

$$\begin{aligned} u_1'(y_1 + b + z - G_1) &= g_1'(G_1) + \theta g_2'(\theta G_1 + G_2) \text{ and} \\ \xi u_2'(y_2 - b(1+r) - \xi G_2) &= g_2'(\theta G_1 + G_2). \end{aligned} \tag{13}$$

We still give first the following benchmark result.

Proposition 4.5 *In the full-information case, the welfare optimum $\{b^{FB}(\xi), z^{FB}(\xi)\}_{\xi \in \Xi}$ satisfies:*

(i) The intertemporal rate of substitution between current and future private goods consumption equals intertemporal rate of transformation, namely

$$\frac{u_1'(c_1^{FB}(\xi))}{u_2'(c_2^{FB}(\xi))} = 1 + r \quad \textit{for any } \xi \in \Xi.$$

(ii) Full insurance is achievable, namely

$$u_1'\left(c_1^{FB}(\xi)\right) = \gamma \quad \textit{for any } \xi \in \Xi,$$

in which $\gamma > 0$ denotes the Lagrangian multiplier on the budget constraint $\int_{\underline{\xi}}^{\bar{\xi}} z(\xi) f(\xi) d\xi \leq 0$.

Under asymmetric information, the center takes into account truth-telling constraints and solves the following program:

$$\max \int_{\underline{\xi}}^{\bar{\xi}} v(\xi) f(\xi) d\xi$$

$$\begin{aligned} s.t. \quad v(\xi) &= V(b(\xi), z(\xi), \xi); \\ \int_{\underline{\xi}}^{\bar{\xi}} z(\xi) f(\xi) d\xi &\leq 0; \\ \dot{v}(\xi) &= -u_2'\left(y_2 - b(\xi)(1+r) - \xi \psi(b(\xi), z(\xi), \xi)\right) \psi(b(\xi), z(\xi), \xi); \\ \dot{b}(\xi) &\leq 0 \end{aligned} \tag{14}$$

in which $\psi(b(\xi), z(\xi), \xi) = G_2(\xi)$ and the constraints can be similarly interpreted as those in program (10).

Moreover, to derive the welfare optimum under asymmetric information, we need the following technical assumption.

Assumption 4.2 *$-G_2 g_2'' \leq g_2'$ for all $\xi \in (\underline{\xi}, \bar{\xi})$, namely the absolute value of the elasticity of marginal utility from G_2 for generation 2 is no greater than one for all but the endpoints of the type distribution.*

Indeed, solving problem (14) gives the following proposition.

Proposition 4.6 *In the asymmetric information case without bunching, the welfare optimum $\{b^*(\xi), z^*(\xi)\}_{\xi \in \Xi}$ satisfies:*

(i) Suppose Assumption 4.2 holds. Concerning the relationship that governs the intertemporal rate of substitution between current and future private goods consumption and the intertemporal rate of transformation, we have:

$$\frac{u_1'(c_1^*(\xi))}{u_2'(c_2^*(\xi))} \begin{cases} = & 1+r \quad \text{for } \xi \in \{\underline{\xi}, \bar{\xi}\}; \\ < & 1+r \quad \text{for } \xi \in (\underline{\xi}, \bar{\xi}). \end{cases}$$

(ii) Let $\mu_1(\xi) > 0$ be the Lagrangian multiplier on the value constraint $v(\xi) \equiv V(b(\xi), z(\xi), \xi)$ of any type-ξ region who is reporting truthfully, then we have:

$$u_1'\left(c_1^*(\xi)\right) \begin{cases} = & \gamma/\mu_1(\xi) \quad \text{for } \xi \in \{\underline{\xi}, \bar{\xi}\}; \\ < & \gamma/\mu_1(\xi) \quad \text{for } \xi \in (\underline{\xi}, \bar{\xi}). \end{cases}$$

Proof. See Appendix. ■

As before, due to the informational constraint, the intertemporal allocation is not distorted only at the endpoints of shock distribution and there is incomplete insurance.

Also, a comparison of the first-best allocation and the asymmetric-information optimum leads to the following proposition.

Proposition 4.7 *For the current economic environment, the following statements are true.*

(i) If $\mu_1(\xi)$ is decreasing in ξ, then (i-a) there exists some $\tilde{\xi} \in (\underline{\xi}, \bar{\xi})$ such that $z^*(\xi) > z^{FB}(\xi)$ for all $\xi \in [\underline{\xi}, \tilde{\xi}]$; (i-b) $z^*(\bar{\xi}) < z^{FB}(\bar{\xi})$; and (i-c) $b^*(\underline{\xi}) < b^{FB}(\underline{\xi})$ and $b^*(\bar{\xi}) > b^{FB}(\bar{\xi})$ whenever $g_1''/g_2'' \leq \rho\theta(1+r)$.

(ii) If $\mu_1(\xi)$ is increasing in ξ, then (ii-a) there exists some $\breve{\xi} \in (\underline{\xi}, \bar{\xi})$ such that $z^*(\xi) > z^{FB}(\xi)$ for all $\xi \in [\breve{\xi}, \bar{\xi}]$; (ii-b) $z^*(\underline{\xi}) < z^{FB}(\underline{\xi})$; and (ii-c) $b^*(\underline{\xi}) > b^{FB}(\underline{\xi})$ and $b^*(\bar{\xi}) < b^{FB}(\bar{\xi})$ whenever $g_1''/g_2'' \leq \rho\theta(1+r)$.

Proof. See Appendix. ■

The intuition of claim (i-a) is the following. If $\mu_1(\xi)$ is decreasing in ξ, then the shadow prices of the value constraint under truth-telling are larger for lower types of regions than for higher types of regions. As a result, it is more likely to be that lower types of regions *extract* the information rent and receive larger grants under asymmetric information than under full information. Claim (ii-a) can be interpreted in a similar way.

Regardless of whether $\mu_1(\xi)$ is decreasing or increasing in ξ, the allocation under asymmetric information is distorted at both ends of support of the distribution of the shock, and importantly, the distortion is qualitatively different at the top and bottom. For example, in the case of part (i), the top type receives smaller grants and issues more debt, whereas the bottom type receives larger grants and issues less debt than they would exhibit, respectively, under full information.

In particular, if $g_1(\cdot) = \ln G_1$ and $g_2(\cdot) = \ln(\theta G_1 + \xi G_2)$, then $g_1''/g_2'' \leq \rho\theta(1+r)$ implies that $G_2/G_1 \leq [\sqrt{\rho\theta(1+r)} - \theta]\rho$ with $\sqrt{\rho\theta(1+r)} > \theta$; if $g_1(\cdot) = G_1^\alpha$ and $g_2(\cdot) = (\theta G_1 + \xi G_2)^\alpha$ for some parameter $\alpha \in (0,1)$, then $g_1''/g_2'' \leq \rho\theta(1+r)$ implies that $G_2/G_1 \leq \{[\rho\theta(1+r)]^{1/(2-\alpha)} - \theta\}\rho$ with $[\rho\theta(1+r)]^{1/(2-\alpha)} > \theta$. That is, under log or power utility functions of public goods consumption, the sufficient condition for claims (i-c) and (ii-c) to hold is that the growth rate of the amount of local public goods produced must be bounded above.

4.2.2 Implementation

To implement the welfare optimum via regionally-decentralized debt decisions, we solve first the maximization problem of regions of type-ξ:

$$\max_{b(\xi)} V(b(\xi), z(\xi), \xi)$$

for any given $z(\xi)$. Letting $G_1 = \phi(b(\xi), z(\xi), \xi)$ and $G_2 = \psi(b(\xi), z(\xi), \xi)$ and applying the Envelope Theorem, the first-order condition is thus written as

$$\begin{aligned} & u_1'\left(y_1 + b(\xi) + z(\xi) - \phi(b(\xi), z(\xi), \xi)\right) \\ = & (1+r)u_2'\left(y_2 - (1+r)b(\xi) - \xi\psi(b(\xi), z(\xi), \xi)\right). \end{aligned} \tag{15}$$

Making use of (15) and Proposition 4.5, we immediately have the following result: The full-information optimum is attained by simply setting $z(\xi) = z^{FB}(\xi)$ for all $\xi \in \Xi$.

Under asymmetric information, we obtain the following implementation scheme.

Proposition 4.8 *Suppose Assumption 4.2 holds. The grant scheme $z^*(b)$ that decentralizes the asymmetric information optimum $\{b^*(\xi), z^*(\xi)\}_{\xi\in\Xi}$ is a nonlinear nondecreasing function of b, almost everywhere differentiable, with the slope*

$$\frac{dz^*}{db} = \Psi(b) \begin{cases} = 0 & \text{for } b \in \{b^*(\underline{\xi}), b^*(\bar{\xi})\}; \\ > 0 & \text{for } b \in (b^*(\bar{\xi}), b^*(\underline{\xi})) \end{cases}$$

in which the expression of $\Psi(b)$ is given in the Appendix.

Proof. See Appendix. ■

If regions of higher degrees of technological progress (namely smaller ξ) are allowed to issue more debt than regions of lower degrees of technological progress, then the grant scheme that decentralizes the asymmetric-information optimum should be designed such that the former regions get strictly more grants than the latter ones. As $V_z > 0$, this grant scheme guarantees that high-type regions with high degrees of technological progress shall not mimic low-type regions.

5 Conclusion

This paper aims to study theoretically the design and implementation of optimal insurance provision to sub-national jurisdictions under regionally privately observable shocks. We consider two possible shocks to the regional economy, one is to the degree of intergenerational spillovers induced by IPGs and the other is to the degree of technological progress for producing the local public goods. Indeed, as the informational asymmetries, e.g., the degree of interregional externality and the cost of public goods provision, usually assumed in the literature, we believe that these two sources of shocks be as well likely to be the private information of local regions.

Importantly, we focus on the joint design of two widely-adopted public risk-sharing schemes, namely intergovernmental grants that provide *cross-region insurance* along the space dimension and public debt that provides *cross-generation insurance* along the time dimension. To the best of our knowledge, this paper is the first attempt towards the joint design of these two risk-sharing schemes and the formal analysis of their interaction in the course of implementing welfare optimum in the literature of regional insurance provision within federations.

In the welfare optimum, we have the following four predictions. First, the informational asymmetries considered here preclude the availability of complete public insurance under risk-averse individual preferences. This is consistent with the empirical evidence that public insurance schemes need to be complemented by private ones. Second, when regions differ in the degree of intergenerational externality, optimal regional debt is non-decreasing in shocks, i.e., for regions facing better shocks in the sense of realizing higher degrees of intergenerational spillovers, the debt issued in the current generation and to be repayed in the future generation should be larger. Third, when regions differ in the degree of technological progress, public debt is smaller for regions of bigger technological progress if only the expenditure on public goods is observable, whereas it is smaller for regions of smaller technological progress if only the physical output of public goods is observable. In consequence, it is indeed worthwhile distinguishing the case of observable input to that of observable output in terms of the optimal design of intergenerational insurance. Fourth, for the top and bottom types of regions, there is a substitutive relationship between interregional insurance and intergenerational insurance, regardless of the source of shocks.

To implement truthfully the welfare optimum via decentralized debt decisions, we have the following three predictions. First, for the top and bottom types of regions, the intergovernmental grant scheme that decentralizes the asymmetric information optimum turns out to be independent of regional public debt, regardless of the source of shocks and whether the expenditure on or the output of public goods is observable when regions differ in technological progress. Second, for all other types of regions when they differ in the degree of intergenerational externality, regional debt is complementary to the grant scheme that decentralizes the welfare optimum. That is, in the course of implementing welfare optimum, intergenerational insurance and interregional insurance exhibit complementarity. Third, for all other types of regions when they differ in the degree of technological progress, regional debt is complementary to the grant scheme that decentralizes the welfare optimum if only the physical output of public goods is observable; otherwise, it is substitutive to the grant scheme. Therefore, it is again worthwhile distinguishing the case of observable input to that of observable output in terms of implementation.

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Appendix: Proofs

Proof of Proposition 3.2. We shall complete the proof in four steps.

Step 1: We define the value function to a type-θ region when it is truth-telling as

$$v(\theta) \equiv V(b(\theta), z(\theta), \theta). \tag{16}$$

Applying Envelope Theorem to (2), we get the following first-order necessary condition for the truth-telling constraints (5) to be satisfied:

$$\dot{v}(\theta) = g_2'(\theta\phi(b(\theta), z(\theta), \theta) + \psi(b(\theta), z(\theta), \theta))\phi(b(\theta), z(\theta), \theta), \tag{17}$$

in which $G_1(\theta) \equiv \phi(b(\theta), z(\theta), \theta)$ and $G_2(\theta) \equiv \psi(b(\theta), z(\theta), \theta)$.

We now derive the second-order sufficient condition for incentive compatibility. After some algebra, the local second-order condition of (5) can be written as

$$\dot{b}(\theta) \cdot V_z(b(\theta), z(\theta), \theta) \cdot \frac{\partial}{\partial\tilde{\theta}}\left(\frac{V_b(b(\theta), z(\theta), \tilde{\theta})}{V_z(b(\theta), z(\theta), \tilde{\theta})}\right)\Bigg|_{\tilde{\theta}=\theta} \geq 0.$$

Noting that $V_z(\cdot) = g_1' + \theta g_2' > 0$ and the Spence-Mirrlees property reads as

$$\frac{\partial}{\partial\theta}\left(\frac{V_b}{V_z}\right) = \frac{(1+r)[(g_2')^2 - G_1 g_1' g_2'']}{(g_1' + \theta g_2')^2} > 0,$$

we thus must have

$$\dot{b}(\theta) \geq 0, \tag{18}$$

which gives the desired monotonicity constraint. It is easy to verify that the local second-order condition also implies global optimality of the truth-telling strategy with the help of the above Spence-Mirrlees property.

We can equivalently rewrite (18) as

$$\dot{b}(\theta) = \beta(\theta), \quad \beta(\theta) \geq 0. \tag{19}$$

The problem of the center is therefore to choose piecewise continuous control variables $b(\theta)$ and $z(\theta)$ to maximize

$$\int_{\underline{\theta}}^{\bar{\theta}} v(\theta)f(\theta)d\theta$$

subject to constraints (7), (16), (17) and (19).

Step 2: To solve the optimal control problem with integral and inequality constraints, we write the generalized Hamiltonian as:

$$\begin{aligned}\mathcal{H} &= v(\theta)f(\theta) + \mu_1(\theta)[V(b(\theta), z(\theta), \theta) - v(\theta)]f(\theta) + \mu_2(\theta)\beta(\theta) - \gamma z(\theta)f(\theta)\\ &\quad + \eta_1(\theta)g_2'(\theta\phi(b(\theta), z(\theta), \theta) + \psi(b(\theta), z(\theta), \theta))\phi(b(\theta), z(\theta), \theta) + \eta_2(\theta)\beta(\theta),\end{aligned}$$

where $\mu_1(\theta)$, $\mu_2(\theta)$ and γ are non-negative Lagrangian multipliers, and $\eta_1(\theta)$ and $\eta_2(\theta)$ are co-state variables. The first-order necessary conditions for a solution to the optimal control problem can now be stated as the state equations (17) and (19), plus

$$\mathcal{H}_z = \mu_1(\theta)V_z(b(\theta), z(\theta), \theta)f(\theta) - \gamma f(\theta) + \eta_1(\theta)[g_2''(\theta\phi_z + \psi_z)\phi + g_2'\phi_z] = 0, \tag{20}$$

$$\mathcal{H}_\beta \;=\; \mu_2(\theta) + \eta_2(\theta) = 0, \tag{21}$$

and

$$\dot{\eta}_1(\theta) \;=\; -\mathcal{H}_v \;=\; [\mu_1(\theta) - 1]f(\theta), \tag{22}$$

$$\dot{\eta}_2(\theta) \;=\; -\mathcal{H}_b \;=\; -\mu_1(\theta)[g_1' - (1+r-\theta)g_2']f(\theta) - \eta_1(\theta)[g_2''(\theta\phi_b + \psi_b)\phi + g_2'\phi_b]. \tag{23}$$

In addition, we have the following transversality conditions:

$$\eta_1(\theta) \;=\; \eta_2(\theta) \;=\; 0 \;\text{ for } \forall\theta \in \{\underline{\theta}, \bar{\theta}\}. \tag{24}$$

Step 3: Using (3) and the assumption that $\rho = 1$, we can write private consumptions as functions of debt, transfers and the degree of intergenerational spillovers: $c_1 \equiv \tilde{\phi}(b, z, \theta)$ and $c_2 \equiv \tilde{\psi}(b, z, \theta)$. Applying Implicit Function Theorem to (3), we have these partial derivatives:

$$\tilde{\phi}_b(b, z, \theta) = \frac{g_1''(u_2'' + g_2'') - (1+r-\theta)\theta u_2'' g_2''}{\Sigma},\; \tilde{\psi}_b(b, z, \theta) = \frac{\theta u_1'' g_2'' - (1+r)(u_1'' + g_1'')g_2''}{\Sigma}; \tag{25}$$

and

$$\tilde{\phi}_z(b, z, \theta) = \frac{g_1''(u_2'' + g_2'') + \theta^2 u_2'' g_2''}{\Sigma},\; \tilde{\psi}_z(b, z, \theta) = \frac{\theta u_1'' g_2''}{\Sigma}; \tag{26}$$

with $\Sigma \equiv (u_1'' + g_1'')(u_2'' + g_2'') + \theta^2 u_2'' g_2'' > 0$. Using $\phi(b, z, \theta) = y_1 + b + z - \tilde{\phi}(b, z, \theta)$, $\psi(b, z, \theta) = y_2 - b(1+r) - \tilde{\psi}(b, z, \theta)$, (25) and (26), we obtain

$$\begin{aligned} \phi_b \;&=\; \frac{u_1''(u_2'' + g_2'') + \theta^2 u_2'' g_2'' + (1+r-\theta)\theta u_2'' g_2''}{\Sigma} > 0, \\ \psi_b \;&=\; -\frac{\theta u_1'' g_2'' + (1+r)[(u_1'' + g_1'')u_2'' + \theta^2 u_2'' g_2'']}{\Sigma} < 0; \end{aligned} \tag{27}$$

and

$$\phi_z = \frac{u_1''(u_2'' + g_2'')}{\Sigma} > 0,\; \psi_z = -\frac{\theta u_1'' g_2''}{\Sigma} < 0. \tag{28}$$

Using (27) and (28), we get

$$\theta\phi_z + \psi_z = \frac{\theta u_1'' u_2''}{\Sigma} > 0,\; \theta\phi_b + \psi_b = -\frac{(1+r-\theta)u_1'' u_2'' + (1+r)g_1'' u_2''}{\Sigma} < 0. \tag{29}$$

Using (29) and (28) gives

$$g_2''(\theta\phi_z + \psi_z)\phi + g_2'\phi_z = \frac{(\theta G_1 g_2'' + g_2')u_1'' u_2'' + u_1'' g_2'' g_2'}{\Sigma} > 0 \tag{30}$$

under Assumption 3.1. Also, it is immediate from (29) and (27) that

$$g_2''(\theta\phi_b + \psi_b)\phi + g_2'\phi_b > 0. \tag{31}$$

Step 4: Since we are interested in the case without bunching, the monotonicity constraint (18) must be $\dot{b}(\theta) > 0$, and hence $\mu_2(\theta) = 0$ for all $\theta \in \Theta$ based on the complementary slackness conditions. By (21), we must have $\eta_2(\theta) = 0$ everywhere, yielding $\dot{\eta}_2 \equiv 0$. Consequently, we get from (23) and (31) that

$$\mu_1(\theta)[g_1' - (1+r-\theta)g_2']f(\theta) \;=\; \underbrace{-\eta_1(\theta)[g_2''(\theta\phi_b + \psi_b)\phi + g_2'\phi_b]}_{\le 0},$$

by which combined with (24) and $\mu_1(\theta)f(\theta) > 0$ for all $\theta \in \Theta$ we have established the result in part (i).

Moreover, using (20) and (30) gives rise to

$$\mu_1(\theta)V_z(b(\theta), z(\theta), \theta)f(\theta) - \gamma f(\theta) \;=\; \underbrace{-\eta_1(\theta)[g_2''(\theta\phi_z + \psi_z)\phi + g_2'\phi_z]}_{\leq 0}$$

under Assumption 3.1. This combined with (24) and $\mu_1(\theta)f(\theta) > 0$ for all $\theta \in \Theta$ completes the proof of part (ii). ■

Proof of Lemma 3.1. It follows from (22) and (24) that

$$\int_{\underline{\theta}}^{\bar{\theta}} [\mu_1(\theta) - 1]f(\theta)d\theta \;=\; \eta_1(\bar{\theta}) - \eta_1(\underline{\theta}) \;=\; 0. \tag{32}$$

By (20), $\mu_1(\theta)$ must be everywhere continuous. Therefore, if $\mu_1(\theta)$ is decreasing in θ, then (32) implies that $\mu_1(\theta) - 1$ is first positive and then negative as θ increases, and that the application of Intermediate Value Theorem yields that there must be some $\tilde{\theta} \in (\underline{\theta}, \bar{\theta})$ such that $\mu_1(\tilde{\theta}) = 1$, as desired in part (i). The proof of part (ii) is similar. ■

Proof of Proposition 3.3. Here we just need to show the proof of part (i) because that of part (ii) is similar. We have by applying $\rho = 1$ and Envelope Theorem to (2) that $V_z = g_1'(\phi(b, z, \theta)) + \theta g_2'(\theta\phi(b, z, \theta) + \psi(b, z, \theta))$. Using this, (28) and (29) gives

$$V_{zz}(b, z, \theta) \;=\; g_1''\phi_z + \theta g_2''(\theta\phi_z + \psi_z) < 0,$$

which combined with Lemma 3.1 produces the the desired results (i-a) and (i-b).

We now proceed to prove result (i-c). It follows from both Propositions 3.1 and 3.2 that the optimal debt policy is a solution to the equation

$$g_1'(\phi(b, z, \theta)) \;=\; (1 + r - \theta)g_2'(\theta\phi(b, z, \theta) + \psi(b, z, \theta)) \tag{33}$$

for any $\theta \in \{\underline{\theta}, \bar{\theta}\}$. Differentiating both sides of equation (33) with respect to z and rearranging the algebra reveal that

$$[g_1''\phi_b - (1 + r - \theta)g_2''(\theta\phi_b + \psi_b)]\frac{db}{dz} = [(1 + r - \theta)\theta g_2'' - g_1'']\phi_z + (1 + r - \theta)g_2''\psi_z.$$

Using (27) and (29) shows that $g_1''\phi_b - (1 + r - \theta)g_2''(\theta\phi_b + \psi_b) < 0$. Differentiating both sides of equation (33) with respect to G_1 reveals that $(1+r-\theta)\theta g_2'' = g_1''$. Moreover, using (28) leads us to that $(1 + r - \theta)g_2''\psi_z > 0$. In consequence, we must have $db/dz < 0$ for any $\theta \in \{\underline{\theta}, \bar{\theta}\}$. This combined with results (i-a) and (i-b) completes the proof. ■

Proof of Proposition 3.4. By (5) and applying Envelope Theorem to (2), the first-order condition for incentive compatibility can be written as:

$$(g_1' + \theta g_2')\,\frac{dz}{d\theta} \;=\; [(1 + r - \theta)g_2' - g_1']\,\frac{db}{d\theta},$$

by which we arrive at

$$\frac{dz}{db} \;=\; \frac{dz}{d\theta}\frac{d\theta}{db} \;=\; \frac{(1 + r - \theta)g_2' - g_1'}{g_1' + \theta g_2'}. \tag{34}$$

It follows from (20) and (23) that

$$g_1' + \theta g_2' \;=\; \frac{\gamma}{\mu_1(\theta)} - \frac{\eta_1(\theta)}{\mu_1(\theta)f(\theta)}[g_2''(\theta\phi_z + \psi_z)\phi + g_2'\phi_z] \tag{35}$$

and

$$(1 + r - \theta)g_2' - g_1' \;=\; \frac{\eta_1(\theta)}{\mu_1(\theta)f(\theta)}[g_2''(\theta\phi_b + \psi_b)\phi + g_2'\phi_b] \tag{36}$$

whenever there is no bunching. Plugging (35) and (36) in (34) results in

$$\frac{dz}{db} \;=\; \frac{\eta_1(\theta(b))[g_2''(\theta(b)\phi_b + \psi_b)\phi + g_2'\phi_b]}{\gamma f(\theta(b)) - \eta_1(\theta(b))[g_2''(\theta(b)\phi_z + \psi_z)\phi + g_2'\phi_z]} \;\equiv\; \Phi(b)$$

in which $\theta(b)$ is the inverse of $b(\theta)$, which exists given that $\dot{b}(\theta) > 0$. As is obvious, $\Phi(b)$ satisfies the required property. ■

Proof of Proposition 4.2. We shall complete the proof in three steps.

Step 1: Applying Envelope Theorem to the value function $V(b, z, \rho)$ and simplifying the algebra, we obtain the Spence-Mirrlees property:

$$\frac{\partial}{\partial \rho}\left[\frac{V_b(b, z, \rho)}{V_z(b, z, \rho)}\right] = -(1 + r)\frac{\theta(g_2')^2 + g_1'[g_2' + \rho\mathcal{G}_2 g_2'']}{(g_1' + \theta g_2')^2} \;<\; 0$$

under Assumption 4.1. Noting that $V_z(\cdot) = g_1' + \theta g_2' > 0$, the second-order condition for incentive compatibility can be written as

$$\dot{b}(\rho) \cdot V_z(b(\rho), z(\rho), \rho) \cdot \frac{\partial}{\partial \tilde{\rho}}\left(\frac{V_b(b(\rho), z(\rho), \tilde{\rho})}{V_z(b(\rho), z(\rho), \tilde{\rho})}\right)\Bigg|_{\tilde{\rho}=\rho} \geq 0,$$

which leads to $\dot{b}(\rho) \leq 0$ under Assumption 4.1, as desired in (10). Let's equivalently rewrite this monotonicity constraint as $\dot{b}(\rho) = \beta(\rho)$ and $\beta(\rho) \leq 0$, then the Hamiltonian of the optimal control problem (10) is given by

$$\begin{aligned}\mathcal{H} \;&= v(\rho)f(\rho) + \mu_1(\rho)[V(b(\rho), z(\rho), \rho) - v(\rho)]f(\rho) - \mu_2(\rho)\beta(\rho) - \gamma z(\rho)f(\rho) \\ &\quad + \eta_1(\rho)g_2'(\theta\phi(b(\rho), z(\rho), \rho) + \rho\psi(b(\rho), z(\rho), \rho))\psi(b(\rho), z(\rho), \rho) + \eta_2(\rho)\beta(\rho),\end{aligned}$$

where $\phi(b(\rho), z(\rho), \rho) = G_1(\rho)$, $\psi(b(\rho), z(\rho), \rho) = \mathcal{G}_2(\rho)$, $\mu_1(\rho)$, $\mu_2(\rho)$ and γ are non-negative Lagrangian multipliers, and $\eta_1(\rho)$ and $\eta_2(\rho)$ are co-state variables. The first-order necessary conditions are given by

$$\mathcal{H}_z \;=\; \mu_1(\rho)V_z(b(\rho), z(\rho), \rho)f(\rho) - \gamma f(\rho) + \eta_1(\rho)[g_2''(\theta\phi_z + \rho\psi_z)\psi + g_2'\psi_z] = 0, \tag{37}$$

$$\mathcal{H}_\beta \;=\; -\mu_2(\rho) + \eta_2(\rho) = 0, \tag{38}$$

and

$$\dot{\eta}_1(\rho) \;=\; -\mathcal{H}_v \;=\; [\mu_1(\rho) - 1]f(\rho), \tag{39}$$

$$\dot{\eta}_2(\rho) \;=\; -\mathcal{H}_b \;=\; -\mu_1(\rho)\{g_1' - [\rho(1+r) - \theta]g_2'\}f(\rho) - \eta_1(\rho)[g_2''(\theta\phi_b + \rho\psi_b)\psi + g_2'\psi_b]. \tag{40}$$

In addition, we have the following transversality conditions:

$$\eta_1(\rho) \;=\; \eta_2(\rho) \;=\; 0 \;\text{ for } \forall\rho \in \{\underline{\rho}, \bar{\rho}\}. \tag{41}$$

Step 2: Applying Implicit Function Theorem to (9) gives rise to:

$$\phi_b = \frac{u_1''(u_2''+\rho^2 g_2'')+\rho\theta(1+r)u_2''g_2''}{M} > 0, \ \phi_z = \frac{u_1''(u_1''+\rho^2 g_2'')}{M} > 0; \tag{42}$$

and

$$\psi_b = -\frac{(1+r)[(u_1''+g_1'')u_2''+\theta^2 u_2''g_2'']+\rho\theta u_1''g_2''}{M} < 0, \quad \psi_z = -\frac{\rho\theta u_1''g_2''}{M} < 0 \tag{43}$$

in which $M \equiv (u_1''+g_1'')(u_2''+\rho^2 g_2'')+\theta^2 u_2''g_2'' > 0$. Making use of (42) and (43), we have

$$g_2''(\theta\phi_z+\rho\psi_z)\psi + g_2'\psi_z < 0 \tag{44}$$

given that

$$\theta\phi_z+\rho\psi_z = \frac{\theta u_1''u_2''}{M} > 0. \tag{45}$$

In addition, we get by (42), (43) and

$$\theta\phi_b+\rho\psi_b = -\frac{[\rho(1+r)-\theta]u_1''u_2''+\rho(1+r)u_2''g_1''}{M} < 0 \tag{46}$$

that

$$\begin{aligned} & g_2''(\theta\phi_b+\rho\psi_b)\psi + g_2'\psi_b \\ = & -\frac{[\rho\mathcal{G}_2 g_2''+g_2'](1+r)(u_1''+g_1'')u_2''-\theta\mathcal{G}_2 u_1''u_2''g_2''+[(1+r)\theta u_2''+\rho u_1'']\theta g_2'g_2''}{M} < 0 \end{aligned} \tag{47}$$

under Assumption 4.1.

Step 3: Since we focus on the case without bunching, we must have $\mu_2(\rho)=0$ for all $\rho\in\Upsilon$. By (38), we have $\eta_2(\rho)=0$ everywhere, implying that $\dot{\eta}_2\equiv 0$. Applying this, (41) and (47) to (40) yields the desired assertion in part (i). Finally, applying (44), (41) and $\mu_1(\rho)f(\rho)>0$ to (37) produces the desired assertion in part (ii). ■

Proof of Proposition 4.3. Using (39), the proof is quite similar to that of Proposition 3.3. Here we just need to show the following. Firstly, using (42) and (45) reveals that $V_{zz}=g_1''\phi_z+\theta g_2''(\theta\phi_z+\rho\psi_z)<0$ for all $\rho\in\Upsilon$. Secondly, by differentiating both sides of equation $g_1'=[\rho(1+r)-\theta]g_2'$ with respect to z, we obtain

$$\underbrace{[g_1''\phi_b-[\rho(1+r)-\theta]g_2''(\theta\phi_b+\rho\psi_b)]}_{<0}\frac{db}{dz} = [\rho(1+r)-\theta]g_2''(\theta\phi_z+\rho\psi_z)-g_1''\phi_z$$

under (42) and (46). As we get from (42) and (45) that

$$[\rho(1+r)-\theta]g_2''(\theta\phi_z+\rho\psi_z)-g_1''\phi_z = -\frac{\rho^2 u_1''g_1''g_2''}{M} > 0,$$

we thus have $db/dz<0$ at the welfare optimum. ■

Proof of Proposition 4.4. The key for a grant scheme to decentralize the asymmetric-information optimum is that it takes into account the incentive-compatibility constraint. First, making use of the first-order necessary condition for incentive compatibility, we get

$$\frac{dz}{db} = \frac{dz}{d\rho}\frac{d\rho}{db} = -\frac{V_b}{V_z}.$$

As the monotonicity constraint is assumed to be not binding, we get from (37) and (40) that

$$\frac{dz}{db} = \frac{\eta_1(\rho(b))\,[g_2''(\theta\phi_b + \rho(b)\psi_b)\psi + g_2'\psi_b]}{\gamma f(\rho(b)) - \eta_1(\rho(b))\,[g_2''(\theta\phi_z + \rho(b)\psi_z)\psi + g_2'\psi_z]} \equiv \Gamma(b),$$

in which $\rho(b)$ denotes the inverse of $b(\rho)$. Secondly, making use of (41), (44) and (47), the proof is immediately complete. ■

Proof of Proposition 4.6. We shall complete the proof in two steps.

Step 1: As before, the Hamiltonian of the optimal control problem (14) is given by

$$\begin{aligned}\mathcal{H} &= v(\xi)f(\xi) + \mu_1(\xi)[V(b(\xi), z(\xi), \xi) - v(\xi)]f(\xi) - \mu_2(\xi)\beta(\xi) - \gamma z(\xi)f(\xi)\\ &\quad - \eta_1(\xi)u_2'\left(y_2 - (1+r)b(\xi) - \xi\psi(b(\xi), z(\xi), \xi)\right)\psi(b(\xi), z(\xi), \xi) + \eta_2(\xi)\beta(\xi).\end{aligned}$$

The first-order necessary conditions are given by

$$\mathcal{H}_z \;=\; \mu_1(\xi)V_z(b(\xi), z(\xi), \xi)f(\xi) - \gamma f(\xi) - \eta_1(\xi)(-\xi u_2''\psi_z\psi + u_2'\psi_z) = 0, \tag{48}$$

$$\mathcal{H}_\beta \;=\; -\mu_2(\xi) + \eta_2(\xi) = 0, \tag{49}$$

and

$$\dot{\eta}_1(\xi) \;=\; -\mathcal{H}_v \;=\; [\mu_1(\xi) - 1]f(\xi), \tag{50}$$

$$\dot{\eta}_2(\xi) \;=\; -\mathcal{H}_b \;=\; -\mu_1(\xi)[u_1' - (1+r)u_2']f(\xi) + \eta_1(\xi)[-(1+r)u_2''\psi - \xi u_2''\psi_b\psi + u_2'\psi_b]. \tag{51}$$

In addition, we have the following transversality conditions:

$$\eta_1(\xi) \;=\; \eta_2(\xi) \;=\; 0 \quad \text{for } \forall\xi \in \{\underline{\xi}, \bar{\xi}\}. \tag{52}$$

Step 2: Applying Implicit Function Theorem to (13) gives rise to:

$$\phi_b \;=\; \frac{u_1''(\xi^2u_2'' + g_2'') + \xi\theta(1+r)u_2''g_2''}{Q} > 0, \;\; \phi_z \;=\; \frac{u_1''(\xi^2u_2'' + g_2'')}{Q} > 0; \tag{53}$$

and

$$\psi_b \;=\; -\frac{\xi(1+r)(u_1'' + g_1'' + \theta^2g_2'')u_2'' + \theta u_1''g_2''}{Q} < 0, \quad \psi_z \;=\; -\frac{\theta u_1''g_2''}{Q} < 0 \tag{54}$$

in which $Q \equiv (u_1'' + g_1'')(\xi^2u_2'' + g_2'') + \xi^2\theta^2u_2''g_2'' > 0$. Now, applying (52) and (54) to (48) gives the desired assertion in part (ii). Moreover, using (54) again reveals that

$$\begin{aligned}&-(1+r)u_2''\psi - \xi u_2''\psi_b\psi + u_2'\psi_b \;=\\ &-\frac{(1+r)(u_1'' + g_1'')u_2''(g_2''\psi + g_2') + \theta u_1''g_2''(u_2' - \xi\psi u_2'') + \theta^2(1+r)u_1''u_2''g_2'}{Q} < 0\end{aligned} \tag{55}$$

under Assumption 4.2. In the case of no bunching, applying (49), (52) and (55) to (51) produces the desired assertion in part (i). ■

Proof of Proposition 4.7. The proof follows from using (50), (52) and Propositions 4.5 and 4.6. Here we just give by using (53), (54) and $u_1' = (1+r)u_2'$ evaluated at the welfare optimum that

$$V_{zz} = u_1''(1-\phi_z) = \frac{u_1''g_1''(\xi^2 u_2'' + g_2'') + \theta^2\xi^2 u_1''u_2''g_2''}{Q} < 0$$

and

$$[(1-\phi_b)u_1'' + (1+r)(1+r+\xi\psi_b)u_2'']\frac{db}{dz} \;=\; -\xi(1+r)u_2''\psi_z - (1-\phi_z)u_1''$$

in which

$$\begin{aligned} & (1-\phi_b)u_1'' + (1+r)(1+r+\xi\psi_b)u_2'' \\ = \;& \frac{u_1''g_1''(\xi^2 u_2'' + g_2'') + (1+r-\theta\xi)^2 u_1''u_2''g_2'' + (1+r)^2 u_2''g_1''g_2''}{Q} < 0 \end{aligned}$$

and

$$\begin{aligned} & -\xi(1+r)u_2''\psi_z - (1-\phi_z)u_1'' \\ = \;& -\frac{\xi u_1''u_2''[\xi g_1'' - (1+r)\theta g_2''] + \theta^2\xi^2 u_1''u_2''g_2'' + u_1''g_1''g_2''}{Q} > 0 \end{aligned}$$

whenever $g_1'' \leq \rho\theta(1+r)g_2''$ holds. ■

Proof of Proposition 4.8. As we focus on the case of no bunching, we have by using the first-order necessary condition for incentive compatibility, (48) and (51) that

$$\frac{dz^*}{db} \;=\; \frac{-\eta_1(\xi(b))[-(1+r)u_2''\psi - \xi(b)u_2''\psi_b\psi + u_2'\psi_b]}{\gamma f(\xi(b)) + \eta_1(\xi(b))[-\xi(b)u_2''\psi_z\psi + u_2'\psi_z]} \;\equiv\; \Psi(b),$$

in which $\xi(b)$ denotes the inverse of $b(\xi)$ under the assumption of $\dot{b}(\xi) < 0$. By using (55) and (52), we see that $\Psi(b)$ satisfies the required property. ■