

# The Government Risk Premium Puzzle\*

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## Abstract

The market value of outstanding government debt reflects the expected present discounted value of current and future primary surpluses. When the discount rate is consistent with the term structure of interest rates and equity prices and government spending growth dynamics are estimated from the data, a government risk premium puzzle emerges. Since tax revenues are pro-cyclical while government spending is counter-cyclical, the tax revenue claim has a higher risk premium and a lower value than the spending claim. This makes the value of the surplus claim negative, and implies that the U.S. government should be a creditor rather than a debtor. We resolve this puzzle by postulating a small but persistent component in expected spending growth, and infer it from the market value of the outstanding government bond portfolio. This component offsets the pro-cyclical movements in current surpluses, reducing its risk and increasing its value. The resulting model is used to study the optimal maturity structure of government debt, and to quantify deviations of the observed portfolio from the optimal one.

**JEL codes: debt maturity, fiscal policy, term structure**

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# 1 Introduction

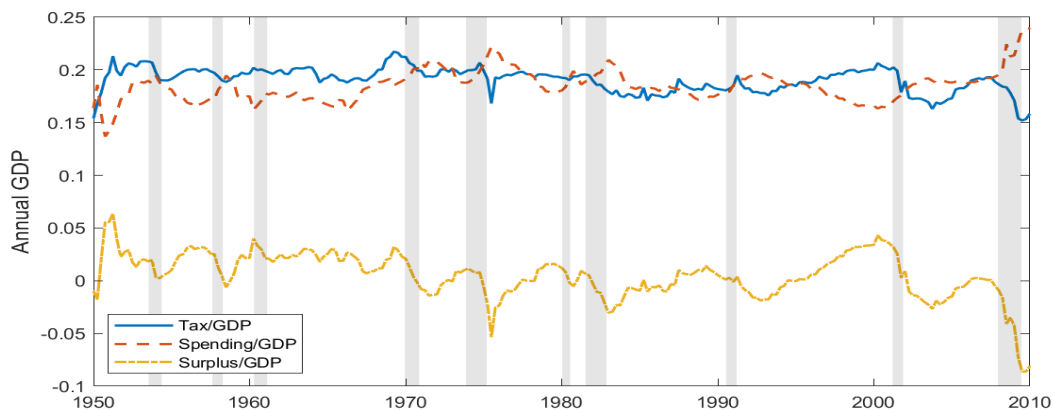
The U.S. government is the largest borrower in the world. As of December 31 2018, outstanding debt held by the public was \$16.1 trillion, or 76.4% of the U.S. annual GDP. In this paper, we take the perspective of the representative Treasury bond investor who buys all new bonds issues and receives all redemptions. When the government runs a budget surplus, this investor receives a payment. In contrast, when the government runs a deficit, this investor finances the deficit by purchasing new government bonds. The government bond portfolio is a claim to current and future surpluses. This portfolio is long tax revenues and short government spending.

Figure 1 plots the ratio of tax revenue to GDP,  $\tau_t$ , government spending to GDP,  $g_t$ , and the primary surplus to GDP,  $\tau_t - g_t$ , from 1947.Q1 to 2017.Q4. The average surplus over this period is just about zero. We also plot the NBER recessions as shaded areas. In recessions, tax revenue tends to decline while government spending tends to rise. Since the tax revenue is pro-cyclical and the government spending is counter-cyclical, the primary surplus is strongly pro-cyclical.

This observation gives rise to a puzzle. As recessions coincide with high marginal utility states of the representative investor, the representative investor requires a high risk premium to hold the claim to future tax revenue and a low risk premium to hold the claim to government spending. The revenue claim is cheap; the spending claim expensive. Therefore, the claim to current and future government surpluses should have a negative present discounted value. But the value of the surplus claim should equal the market

Figure 1: Government Cash Flows

We plot the government tax revenue and the government spending scaled by GDP. (I'll plot NBER recessions.)



value of outstanding debt, which is positive rather than negative. Put in terms of interest rates rather than valuations, the U.S. government's promised payments on its outstanding debt, future surpluses, are highly risky liability and investors should earn a high yield to compensate for this risk. Yet, Treasury market participants seem willing to purchase government debt at low yields.

We propose a resolution to this puzzle by introducing a small, but persistent component in expected government spending growth. If lower expected future spending growth coincides with high current spending or low current revenue growth, then high deficits today will be corrected by tighter fiscal policy going forward. This stabilization mechanism makes the claim to government spending less a worse hedge; its price falls. The lower value of the spending claim raises the value of the surplus claim and can bring it back in line with the observed market value of government debt.

Actual government spending growth time-series data are not very informative about this small, persistent component in expected spending growth. But the market value of government debt is, and helps identify its magnitude. The insight is analogous to that in the long-run risk literature ([Bansal and Yaron, 2004](#)), which models a small but persistent component in expected consumption growth and infers it from bond and stock prices rather than from the time series of realized consumption growth rates. We use the observed time series of the outstanding market value of government debt, alongside with bond and stock prices, to filter out the latent component in expected government spending growth. Reassuringly, the resulting expected spending growth time series predicts future government spending growth. What results is a model that is consistent with the observed dynamics of government revenue and spending growth, asset prices, and the market value of of outstanding government debt.

We use this model to study the optimal maturity structure of government debt. We are guided by the insight of [Bhandari, Evans, Golosov, and Sargent \(2017\)](#) that the optimal debt portfolio choice minimizes the variance of the government's funding needs. This minimization amounts to equalizing the sensitivities of the value of the outstanding bond portfolio to each of the shocks that hit the economy to the sensitivities of the value of the surplus claim to the same shocks. Matching duration, the sensitivity to interest rates, is the simplest example in a world where the only risk is to the level of the term structure. More generally, the positions in bonds of the various maturities can be chosen to fully immunize the government debt portfolio to all shocks.

We show how far off the actual bond portfolio is from such full immunization. At

the end of our sample, the dollar duration of the U.S. government debt portfolio is 10.29 times the annual GDP, whereas the dollar duration of the surplus is 9.69 times GDP. The optimal strategy would require the government to reduce the average duration of its debt portfolio by 0.60 year. While the current government debt portfolio seems thus hedges interest rate risk quite well, the historical deviations have at times been very large. We find larger deviations between the actual and observed exposures to shocks to the slope of the term structure, sometimes called convexity, as well as to macro-economic shocks such as inflation and GDP growth. Current debt maturity policy leaves government bond investors exposed to inflation and GDP shocks.

The rest of the paper is organized as follows. The remainder of the introduction discusses the related literature. Section 2 introduces the government budget constraint. Section 2.2 sets up and solves the asset pricing model. Section ?? shows how we estimate the model. Section 3 formulates the government debt risk puzzle and proposes a resolution. Section 5 studies government debt maturity immunization. Section 6 concludes. The appendix discusses several details of model derivation and estimation.

**Related Literature** Our paper contributes to the literature on the maturity structure of government debt. One view is that the government ought to minimize its expected funding cost by issuing short-term debt when the slope of the yield curve is steep, thus exploiting the failure of the expectations hypothesis. Conversely, the government should issue more long-term debt when the yield curve inverts (see for example [Campbell, 1995](#)). A normative literature on optimal government taxation and debt management offers a different prescription. In this class of dynamic models with distortionary taxation going back to [Lucas and Stokey \(1983\)](#), the government chooses the tax rate optimally to hedge shocks to government spending. If the government can issue state-contingent debt, the optimal tax rate inherits the serial correlation of government spending. To the extent that the government's debt securities do not span all the shocks that hit the economy, maturity choice plays an important role. In a model in which only spending shocks drive the term structure, [Angeletos \(2002\)](#) and [Buera and Nicolini \(2004\)](#) show how the government can choose the maturity of non-state-contingent government debt to mimic the complete markets allocations in [Lucas and Stokey \(1983\)](#), thus creating an explicit role for the maturity structure. In general, the government will not try to replicate the complete markets allocation if variation in interest rates is largely explained by non-spending shocks, as is the case in the data. Market incompleteness imputes more persistence to

the optimal tax rates, as shown by [Aiyagari, Marcet, Sargent, and Seppälä \(2002\)](#). An important shortcoming of the conventional Ramsey analysis is that optimal debt management is derived in a setting that fails to generate realistic asset prices. [Karantounias \(2018\)](#) shows that key optimal tax policy prescriptions change dramatically with [Epstein and Zin \(1989\)](#) preferences. Bond prices becomes more realistic under such preferences ([Piazzesi and Schneider, 2006](#); [Bansal and Shaliastovich, 2013](#)).

Motivated by this observation, we take a pragmatic approach and choose a flexible SDF model that prices the term structure of interest rates precisely. Our asset pricing model builds on [Lustig, Van Nieuwerburgh, and Verdelhan \(2013\)](#), who price a claim to aggregate consumption and study the properties of the price-dividend ratio of this claim, the wealth-consumption ratio. Here we focus on pricing claims to government revenue and spending growth instead. The asset pricing model combines a vector auto-regression model for the state variables as in [Campbell \(1991, 1993, 1996\)](#) with a no-arbitrage model for the (SDF) as in [Duffie and Kan \(1996\)](#); [Dai and Singleton \(2000\)](#); [Ang and Piazzesi \(2003\)](#). [Gupta and Van Nieuwerburgh \(2018\)](#) use a similar framework to evaluate the performance of private equity funds. Our approach takes spending and tax policy as given, rather than being optimally determined. However, both policies are allowed to depend on a rich set of state variables and are estimated from the data. To keep the model tractable, we shut down feedback from tax and spending policy onto the SDF. If the marginal investor in U.S. Treasuries is a foreign official institution or the domestic central bank, demand for Treasuries may not be materially influenced by fiscal policy considerations. Indeed, the share of U.S. Treasury debt held by foreigners has been rising steadily since the 1980s, and the Federal Reserve's holdings ballooned after the financial crisis. Foreign official institutions and the Fed combined have held about two-thirds of U.S. Treasuries over the past twenty years ([Kohn, 2016](#); [Favilukis, Kohn, Ludvigson, and Nieuwerburgh, 2013](#)) and their demand has been characterized as price inelastic.

Our work also connects to the literature on the specialness of U.S. government bonds. [Longstaff \(2004\)](#); [Krishnamurthy and Vissing-Jorgensen \(2012, 2015\)](#); [Nagel \(2016\)](#) find that U.S. government bonds are traded at a premium relative to other risk-free bonds. [Greenwood, Hanson, and Stein \(2015\)](#) study the government debt's optimal maturity in the presence of such premium, and [Valchev \(2017\)](#); [Du, Im, and Schreger \(2018\)](#); [Jiang, Krishnamurthy, and Lustig \(2018\)](#) study this premium in international finance. Our approach takes this premium into account by incorporating Treasury yields in the pricing kernel, and tackles the fundamental question whether the U.S. government fiscal condi-

tion justifies such a premium.

Lastly, our work contributes to the fiscal theory of the price level, which requires a positive present value of government surpluses to determine the price level (Sargent and Wallace, 1984; Leeper, 1991; Woodford, 1994; Sims, 1994; Cochrane, 2001, 2005). Our work demonstrates how the variation in expected future government surpluses affects the discount rate of government cash flows and therefore resolves the government risk premium puzzle. Bianchi and Melosi (2014, 2017, 2018) study different regimes of the fiscal policy and their real effects. For parsimony, our current work focuses on estimating the model with a single regime.

## 2 Setup

### 2.1 The Government Budget Constraint

Let government spending before interest expenses be  $G_t$ , government tax revenue  $T_t$ , and the (primary) surplus,  $S_t = T_t - G_t$ . Denote the ratio of government spending to GDP by  $g_t$ , the ratio of tax revenues to GDP by  $\tau_t$ , and the ratio of the primary surplus to GDP by  $s_t$ . Appendix A shows that iterating on the one-period government budget constraint, today's value of the government debt portfolio inherited from last period equals the expected present discounted value of current and future primary surpluses:

$$\sum_{h=0}^{\infty} P_t^{\$}(h) Q_{t,h}^{\$} + \sum_{h=0}^{\infty} P_t(h) Q_{t,h} = \mathbb{E}_t \left[ \sum_{j=0}^{\infty} M_{t,t+j}^{\$} S_{t+j} \right] \quad (1)$$

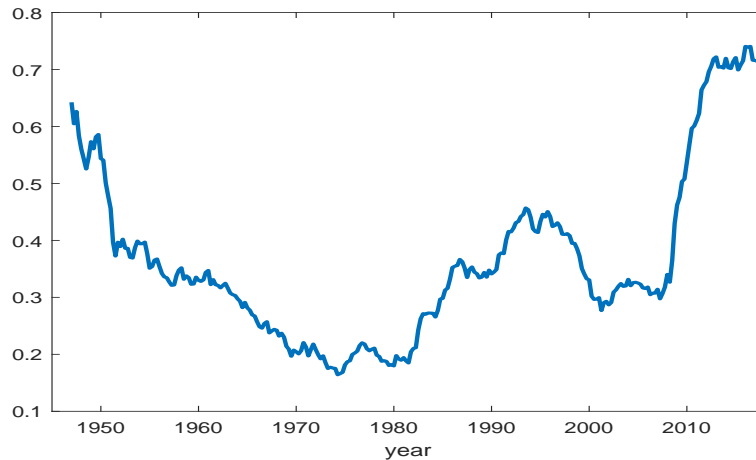
The multi-period stochastic discount factor (SDF)  $M_{t,t+h}^{\$} = \prod_{k=0}^h M_{t,t+k}^{\$}$  is the product of the adjacent one-period SDFs,  $M_{t,t+k}^{\$}$ . The price  $P_t^{\$}(h)$  denotes the price at time  $t$  of a nominal zero-coupon bond that matures at time  $t+h$ , where  $h$  is the maturity of the bond expressed in quarters. Bond prices satisfy  $P_t^{\$}(h) = \mathbb{E}_t \left[ M_{t,t+h}^{\$} \right]$ . By convention  $P_t^{\$}(0) = M_{t,t}^{\$} = M_t^{\$} = 1$  and  $M_{t,t+1}^{\$} = M_{t+1}^{\$}$ . The left-hand side of (1) recognizes that the government issues both nominal and inflation-indexed debt. Real bond prices are denoted without the \$ superscript,  $P_t(h)$ . The government bond portfolio is stripped into zero-coupon bond positions  $Q_{t,h}^{\$}$  and  $Q_{t,h}$ . The quantity  $Q_{t,h}^{\$}$  denotes the outstanding face value at time  $t$  of payments that are due at time  $t+h$ .  $Q_{t,0}^{\$}$  is the amount due today. This outstanding debt reflects all past bond issuance decisions.

Following Hall and Sargent (2011), the strip positions can be constructed from all

coupon-bearing Treasury bonds (all cusips) issued in the past and currently outstanding. This is done separately for nominal and real bonds. Since zero-coupon bond prices are also observable, the left-hand side of (1) is observable.<sup>1</sup> Figure 2 plots the evolution of the market value of outstanding U.S. government debt scaled by GDP.

Figure 2: The Market Value of Outstanding Debt to GDP

The figure plots the ratio of the nominal market value of outstanding government debt divided by nominal GDP. GDP Data are from the Bureau of Economic Analysis. The market value of debt is constructed as follows. We multiply the nominal price (bid / ask average) of each cusip by its total amount outstanding (normalized by the face value), and then sum across all issuance (cusip). The series is quarterly from 1940.Q1 until 2017.Q4. Data Source: CRSP U.S. Treasury Database and BEA.



Equation (1) implies that when the government runs a deficit in a future date and state, it will need to issue new bonds to the investing public. If those dates and states are associated with a high value for the SDF for the representative bond investor, that debt issuance occurs at the wrong time. Investors will need to be induced by low prices to absorb that new debt.

## 2.2 The Asset Pricing Model

Next, we propose a simple no-arbitrage model for the SDF  $\{M_t^s\}$ . We take a stance on the key sources of aggregate risk in the economy, and postulate that their dynamics follow a VAR. The goal of the asset pricing model is to estimate the market prices of risk for these sources of macro-economic risk, such that the model matches observed government bond yields and equity prices. With these market prices of risk in hand, we compute the expected present discounted value of future surpluses, the right-hand side of (1).

<sup>1</sup>Since the model fits nominal bond prices very well, as shown below, we can equivalently use model-implied bond prices. Similarly, we can use model-implied prices for real zero-coupon bonds.

We assume that the  $N \times 1$  vector of state variables follows a Gaussian first-order VAR:

$$z_t = \Psi z_{t-1} + \Sigma^{\frac{1}{2}} \varepsilon_t, \quad (2)$$

with shocks  $\varepsilon_t \sim i.i.d. \mathcal{N}(0, I)$  whose variance is the identity matrix. The companion matrix  $\Psi$  is a  $N \times N$  matrix. The vector  $z$  is demeaned. The covariance matrix of the innovations to the state variables is  $\Sigma$ ; the model is homoscedastic. We use a Cholesky decomposition of the covariance matrix,  $\Sigma = \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2} \prime}$ , which has non-zero elements only on and below the diagonal. In this way, we interpret the shock to each state variable as a linear combination of structural shocks  $\varepsilon_t$ , each one of which is orthogonal to the shocks to the state variables that precede it in the VAR. The state vector  $z_t$  contains 8 state variables:

$$z_t = [\pi_t - \pi_0, x_t - x_0, y_t^{\$}(1) - y_0^{\$(1)}, yspr_t^{\$} - yspr_0^{\$(1)}, pd_t - \overline{pd}, \Delta d_t - \mu_d, \Delta \log \tau_t, \Delta \log g_t].$$

The first four variables govern the term structure block. We follow the empirical term structure literature and specify a term structure model that contains two key macro-economic sources of risk, inflation ( $\pi_t$ ) and real GDP growth ( $x_t$ ), as well as two interest rates, the nominal short rate ( $y_t^{\$(1)}$ ) and the yield spread ( $yspr_t^{\$}$ ) defined as the difference between the 5-year and 1-quarter nominal bond yields:  $yspr_t^{\$} = y_t^{\$(20)} - y_t^{\$(1)}$ . This is akin to a model with two observable macro-economic time series and two latent factors. It is well understood that two latent factors are needed to describe the term structure of interest rates since interest rates are not fully spanned by macro-economic time series (Joslin, Pribsch, and Singleton, 2014).

The next two state variables are the log price-dividend ratio and log real dividend growth on the aggregate stock market. Together they imply the time series of returns on stocks.

The last two elements of the VAR are the log change in government revenue to GDP and the log change in government spending to GDP. We discuss these elements in detail below.

We use selector vectors to pick out particular elements of the state vector. For example, the one-month nominal bond yield is  $y_t^{\$(1)} = y_0^{\$(1)} + e'_{yn} z_t$ , where  $y_0^{\$(1)}$  is the unconditional average yield and  $e_{yn}$  is a vector that selects the element of the state vector corresponding to the one-month yield. Similarly, the inflation rate is  $\pi_t = \pi_0 + e'_{\pi} z_t$ . Lowercase letters denote logs.

Motivated by the no-arbitrage term structure literature (Ang and Piazzesi, 2003), we



specify an exponentially affine stochastic discount factor (SDF). The nominal SDF  $M_{t+1}^{\$} = \exp(m_{t+1}^{\$})$  is conditionally log-normal:

$$m_{t+1}^{\$} = -y_t^{\$(1)} - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1}, \quad (3)$$

The real SDF is  $M_{t+1} = \exp(m_{t+1}) = \exp(m_{t+1}^{\$} + \pi_{t+1})$ ; it is also conditionally Gaussian. The innovations in the vector  $\varepsilon_{t+1}$  are associated with a  $N \times 1$  market price of risk vector  $\Lambda_t$  of the affine form:

$$\Lambda_t = \Lambda_0 + \Lambda_1 z_t,$$

The  $N \times 1$  vector  $\Lambda_0$  collects the average prices of risk while the  $N \times N$  matrix  $\Lambda_1$  governs the time variation in risk premia. We specify the restrictions on the market price of risk vector below. Asset pricing in this model amounts to estimating the market prices of risk in  $\Lambda_0$  and  $\Lambda_1$ .

This model offers a simple way to price nominal bonds.

**Proposition 1.** Nominal bond yields of maturity  $h$  are affine in the state vector:

$$y_t^{\$(h)} = -\frac{A^{\$(h)}}{h} - \frac{B^{\$(h)'}}{h} z_t,$$

the scalar  $A^{\$(h)}$  and the vector  $B^{\$(h)}$  follow ordinary difference equations that depend on the properties of the state vector and of the market prices of risk.

The proof is in Appendix B. As shown in Appendix B, a similar formula prices real bonds. We use this affine pricing equation to calculate the real interest rate, real bond risk premia, and inflation risk premia on bonds of various maturities.

Since both the nominal short rate ( $y_t^{\$(1)}$ ) and the slope of the term structure ( $y_t^{\$(20)} - y_t^{\$(1)}$ ) are included in the VAR, the SDF model must price the unconditional mean and the dynamics of the five-year bond yield:

$$-A^{\$(20)}/20 = y_0^{\$(1)} + yspr_0^{\$} = y_0^{\$(20)} \quad (4)$$

$$-B^{\$(20)}/20 = e_{y1} + e_{yspr} \quad (5)$$

We also use a similar formula to price the stock market. First, let  $P_t^d(h)$  denote the price-dividend ratio of the dividend strip with maturity  $h$  (Wachter, 2005; van Binsbergen, Brandt, and Koijen, 2012). Then, the aggregate price-to-dividend ratio can be expressed

as

$$\frac{P_t}{D_t} = \sum_{h=0}^{\infty} P_t^d(h). \quad (6)$$

**Proposition 2.** Log price-dividend ratios on dividend strips are affine in the state vector:

$$p_t^d(h) = \log \left( P_t^d(h) \right) = A^m(h) + B^{m'}(h)z_t.$$

Since we include the log price-dividend ratio on the stock market in the state vector, it is affine in the state vector by assumption; see the left-hand side of (7):

$$\exp \left( \overline{pd} + e'_{pd}z_t \right) = \sum_{h=0}^{\infty} \exp \left( A^m(h) + B^{m'}(h)z_t \right), \quad (7)$$

Equation (7) rewrites the present-value relationship (6), and articulates that it implies a restriction on the coefficients  $A^m(h)$  and  $B^{m'}(h)$ . We impose this restriction in the estimation.

## 2.3 Asset Pricing Results

We estimate the elements of the companion matrix  $\Psi$  by OLS equation by equation, and construct the residual covariance matrix  $\Sigma$ . The state vector  $z_t$  is observed quarterly from 1947.Q1 until 2017.Q4 (284 observations). We impose that lagged spending growth does not affect the state variables that precede it in the VAR:  $\Psi_{(1:7,8)} = 0$  and  $\Sigma_{(1:7,8)}^{\frac{1}{2}} = 0$ . This assumption will buy substantial tractability later on and does not seem overly restrictive. The null hypothesis that  $\Psi_{(1:7,8)} = 0$  cannot be rejected in the data. We then estimate the constant (time-varying) market prices of risk  $\Lambda_0$  ( $\Lambda_1$ ) in order to best fit the prices and expected returns on bonds of various maturities as well as on the aggregate stock market. Appendix C reports the point estimates as well as a detailed discussion of how the market price of risk parameters are identified.

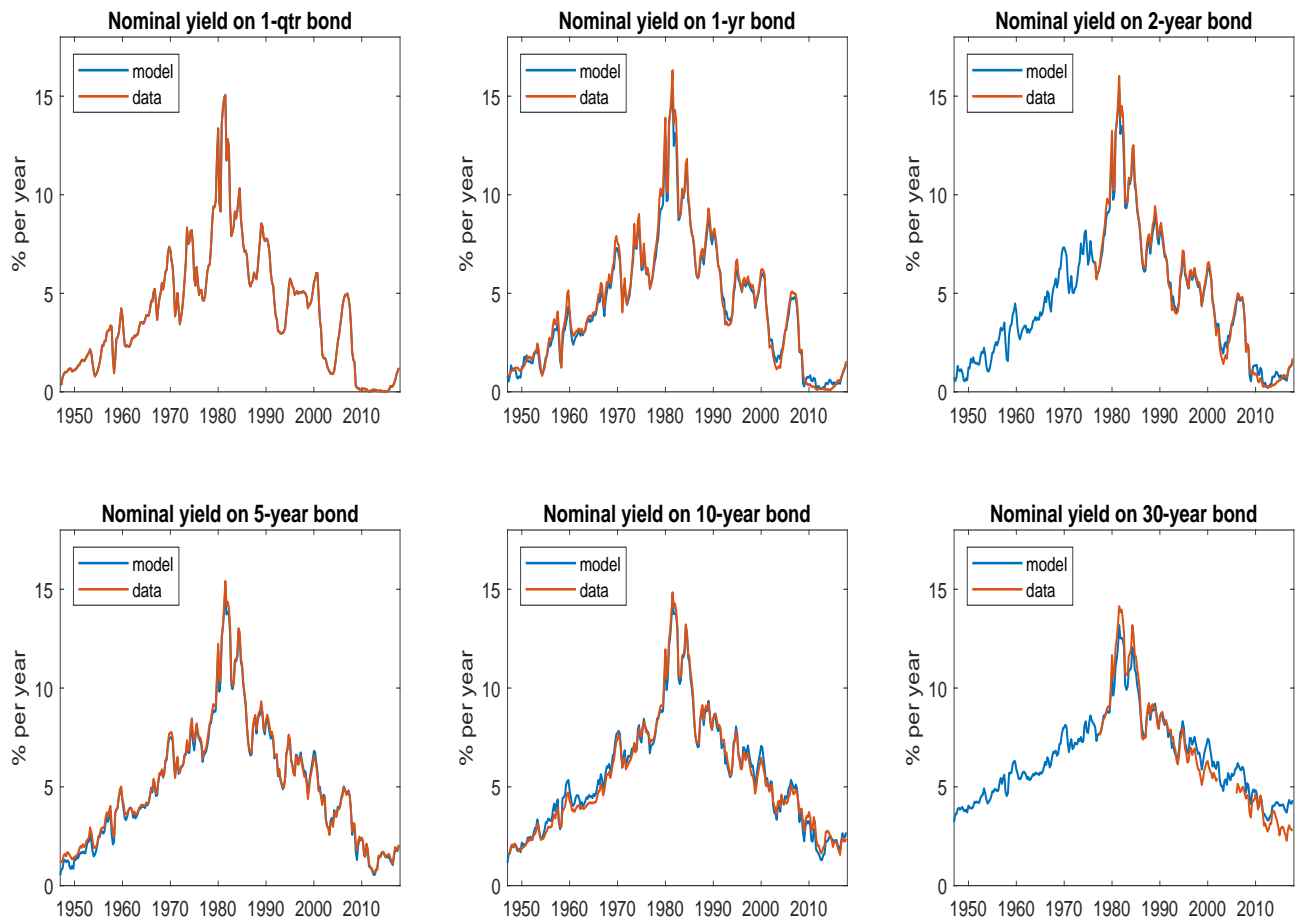
### 2.3.1 Bonds

We use the following moments to estimate the 12 market price of risk parameters that govern the bond block. We include the distance between the observed and model-implied time-series of nominal bond yields for maturities of one quarter, one year, two years, five

years, ten years, and thirty years.<sup>2</sup> We also impose the 7 conditions implied by equations (4) and (5). Since it is part of the VAR, we insist on matching the 5-year bond yield precisely. This gives a total of  $6T+7$  moments. The unconditional market prices of inflation, GDP growth, the level of interest rates, and the slope of the yield curve all have the expected sign. Figure 3 shows that the model matches the time series of bond yields in the data closely.

Figure 3: Dynamics of the Nominal Term Structure of Interest Rates

The figure plots the observed and model-implied 1-, 4-, 8-, 20-, 40-, and 120-quarter nominal bond yields. Data are from FRED and FRASER.



The top panels of Figure 4 show the model’s implications for the average nominal (left panel) and real (right panel) yield curves at longer maturities. These yields are well

<sup>2</sup>We use constant-maturity Treasury (CMT) yield data from FRED. For the 1-year, 5-year, and 10-year bonds, we supplement the time series with data from the Federal Reserve Board’s FRASER archive for the period 1947.Q1-1953.Q1. The 2-year CMT yields are only available in 1976.Q3 and the 30-year CMT yields are available only for 1977.Q2-2002.Q1 and 2006.Q1-2017.Q3. Since our estimation is quarter by quarter, it can handle missing data points.

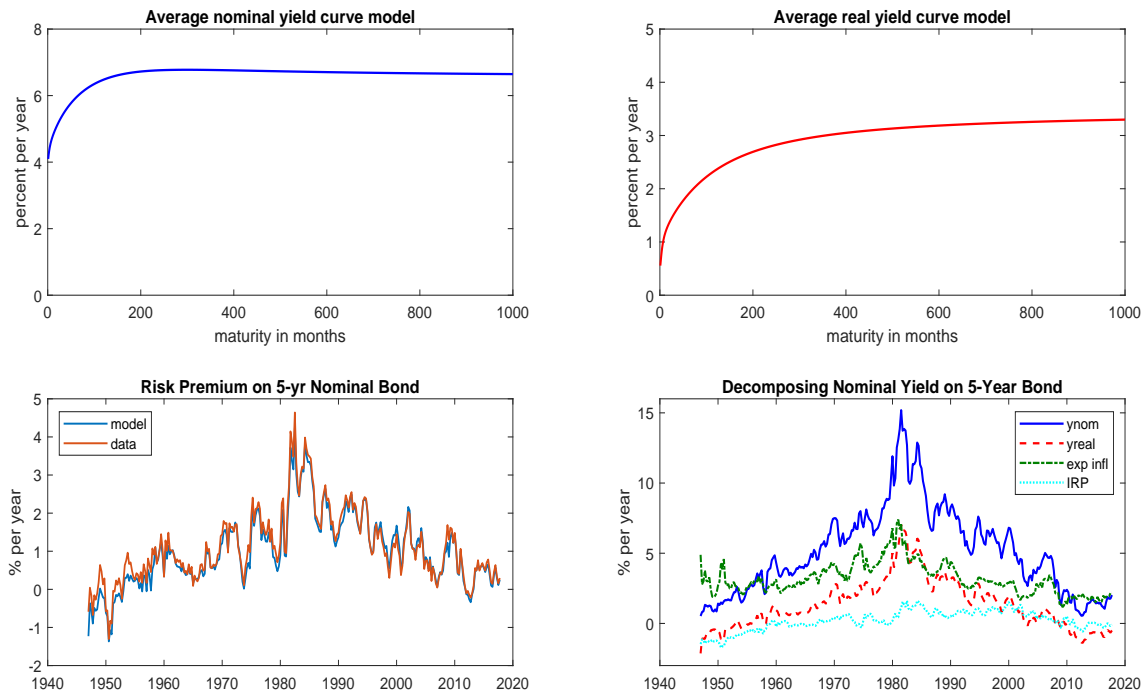
behaved, with very long-run nominal (real) yields stabilizing at around 6.5% (3.5%) per year.<sup>3</sup>

The bottom left panel of Figure 4 shows that the model matches the dynamics of the nominal bond risk premium, defined as the expected excess return on the five-year nominal bond quite well. The bottom right panel shows a decomposition of the nominal bond yield on a five-year bond into the five-year real bond yield, annual expected inflation over the next five years, and the five-year inflation risk premium. On average, the 5.1% nominal bond yield is comprised of a 1.4% real yield, a 3.2% expected inflation rate, and a 0.6% inflation risk premium. The graph shows that the importance of these

<sup>3</sup>We impose conditions that ensure that the nominal and real term structure are well behaved at very long maturities, for which we have no data. Specifically, we impose that average nominal (real) yields of bonds with maturities of 200, 400, 600, 800, 1000, 2000, 3000, and 4000 quarters remain above 6.24% (3.05%) per year, which is the long-run nominal (real) GDP growth rate  $4x_0 + 4\pi_0$  ( $4x_0$ ) observed in our sample. Second, we impose that nominal yields remain above real yields plus 3.15% expected inflation at those same maturities. This imposes that the inflation risk premium remain positive at very long maturities. Third, we impose that the nominal and real term structures of interest rates flatten out, with an average yield difference between 400 and 200 quarter yields that must not exceed 2% per year and between 1000 and 600 quarters that must not exceed 1% per year. These restrictions are satisfied at the optimum.

Figure 4: Long-term Yields and Bond Risk Premia

The top panels plot the average bond yield on nominal (left panel) and real (right panel) bonds for maturities ranging from 1 quarter to 1000 quarters. Yields are annualized. The bottom left panel plots the nominal bond risk premium on the five year bond in model and data. The bottom right panel decomposes the model's five-year nominal bond yield into the five-year real bond yield, the five-year inflation risk premium and the five-year real risk premium.



components fluctuates over time.

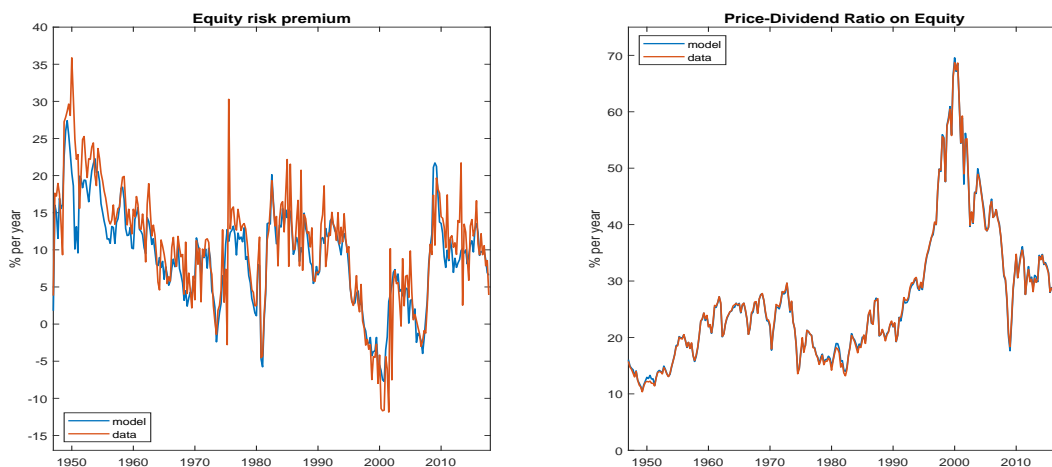
### 2.3.2 Stocks

We allow for non-zero market prices of risk in the sixth element of  $\Lambda_0$  and the first six entries of the sixth row of  $\Lambda_1$ ; the sixth element is the aggregate dividend growth rate on the U.S. stock market. We use the following moments to identify these parameters. First, we include the distance between the observed and model-implied time-series of the price-dividend ratio on the aggregate stock market in each quarter. The model-implied series is constructed from the dividend strips per (7). Second, we impose that the risk premium in the model matches that in the VAR, both in terms of its unconditional average and its dependence on the state variables. This gives a total of  $T+7$  moments.

Figure 5 shows the equity risk premium, the expected excess return, in the left panel and the price-dividend ratio in the right panel. The dynamics of the risk premia in the data are dictated by the VAR. These dynamics are sensible, with low risk premia in the dot-com boom of 1999-2000 and very high risk premia in the Great Financial Crisis of 2008-09. The figure's right panel shows a tight fit for equity price levels. We conclude that the model captures the observed prices and returns on stocks and bonds well.

Figure 5: Equity Risk Premium and Price-Dividend Ratio

The figure plots the observed and model-implied equity risk premium on the overall stock market in the left panel and the price-dividend ratio in the right panel. The quarterly equity risk premium in model and data is multiplied by 400 to express it as an annual percentage number. The price-dividend ratio is the price divided by the annualized dividend.



### 2.3.3 Good deal bounds

Finally, when estimating the market prices of risk, we impose good deal bounds on the standard deviation of the log SDF in the spirit of [Cochrane and Saa-Requejo \(2000\)](#). Specifically, we impose a quadratic penalty for quarterly Sharpe ratios in excess of 1.5.

## 3 The Government Risk Premium Puzzle

### 3.1 Pricing the Surplus Claim

With the VAR dynamics and the SDF in hand, we can calculate the expected present discounted value of the primary surplus:

$$\begin{aligned} \mathbb{E}_t \left[ \sum_{j=0}^{\infty} M_{t,t+j}^{\$} S_{t+j} \right] &= \sum_{j=0}^{\infty} \mathbb{E}_t \left[ M_{t,t+j}^{\$} T_{t+j} \right] - \sum_{j=0}^{\infty} \mathbb{E}_t \left[ M_{t,t+j}^{\$} G_{t+j} \right] \\ &= P_t^{\tau} - P_t^g \end{aligned} \quad (8)$$

where  $P_t^{\tau}$  is the cum-dividend value of a claim to future (nominal) tax revenues and  $P_t^g$  is the cum-dividend value of a claim to future (nominal) government spending.

By construction, nominal tax revenue growth equals:

$$\Delta \log T_{t+1} = \Delta \log \tau_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + (e_{\tau} + e_x + e_{\pi})' z_{t+1}. \quad (9)$$

where we recall that  $\tau_t = T_t/GDP_t$  is the ratio of government revenue to GDP. The dynamics and shock exposures of changes in tax revenues to GDP,  $\Delta \log \tau_{t+1}$ , are captured by the seventh row of the VAR selected by the vector  $e_{\tau}$ .

Nominal government spending growth follows:

$$\Delta \log G_{t+1} = \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + (e_g + e_x + e_{\pi})' z_{t+1}. \quad (10)$$

where  $g_t = G_t/GDP_t$  is the ratio of government spending to GDP. The selector variable  $e_g$  picks out the eighth row of the VAR.

We take an a priori assumption that the tax-to-GDP ratio and the spending-to-GDP ratio have an unconditional average growth rate of 0. Then, tax revenue, government spending and GDP do not arbitrarily drift apart. Thus, long-run nominal tax growth and long-run nominal spending growth both equal long-run nominal GDP growth.

**Proposition 3.** The log price-dividend ratios on the tax claim and the spending claim are affine in the state vector  $z_t$ :

$$\frac{P_t^\tau}{T_t} = 1 + \exp \{ \overline{p^\tau} + B_\tau' z_t \} \quad (11)$$

$$\frac{P_t^g}{G_t} = 1 + \exp \{ \overline{p^g} + B_g' z_t \}. \quad (12)$$

The proof is in Appendix B.4. The vectors  $B_\tau$  and  $B_g$  describe the exposures of the “price-dividend ratios” of the revenue and spending claims to the state variables. They determine the risk premia (expected excess returns corrected for a Jensen term) on the claims:

$$\mathbb{E}_t [r_{t+1}^\tau] - y_t^\$(1) + Jensen = (e_\tau + e_x + e_\pi + \kappa_1^\tau B_\tau)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t) \quad (13)$$

$$\mathbb{E}_t [r_{t+1}^g] - y_t^\$(1) + Jensen = (e_g + e_x + e_\pi + \kappa_1^g B_g)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t) \quad (14)$$

The right-hand side denotes the covariance of the claims’ returns with the SDF. These covariances are crucially driven by the exposure vectors  $B_g$  and  $B_\tau$ .

### 3.2 Pricing the Tax Claim

The top left panel of Figure 6 plots the (cum-dividend) price-dividend ratio on a claim to future tax revenue,  $P_t^\tau / T_t$  in (11). The time-series average of this ratio is 31.5. In other words, the representative agent who is pricing assets in this economy would be willing to pay 31.5 times annual tax revenues for the right to receive all current and future tax revenues. This valuation ratio reflects the risk of tax revenues. Since tax revenues accrue in good times, i.e., low marginal utility times, the tax revenue asset is risky. The annual risk premium on the tax claim is 6.6% per year, a value similar to the historical equity risk premium (10.4% in our model over the same sample). The risk premium reflects mostly compensation for interest rate risk (6.9%), but also for stock market risk (1.75%), offset by slope risk (-1.75%) and GDP risk (-0.3%). The high risk premium translates into a low valuation ratio. Indeed, the average price-dividend ratio of 31.5 is similar to that on the U.S. stock market of 26.8 over the same period.

In addition, the price-dividend ratio of the tax claim displays substantial time-variation. A pronounced V-shape arises from the inverse V-shape of the long-term real interest rate. For example, the bottom right panel of Figure 4 plots the five-year real interest rate in red.

It is high in the middle of the sample and low at the beginning and end of the sample. Intuitively, discounting future tax revenues by a low (high) long-term real rate results in a high (low) valuation ratio.

### 3.3 Pricing the Spending Claim

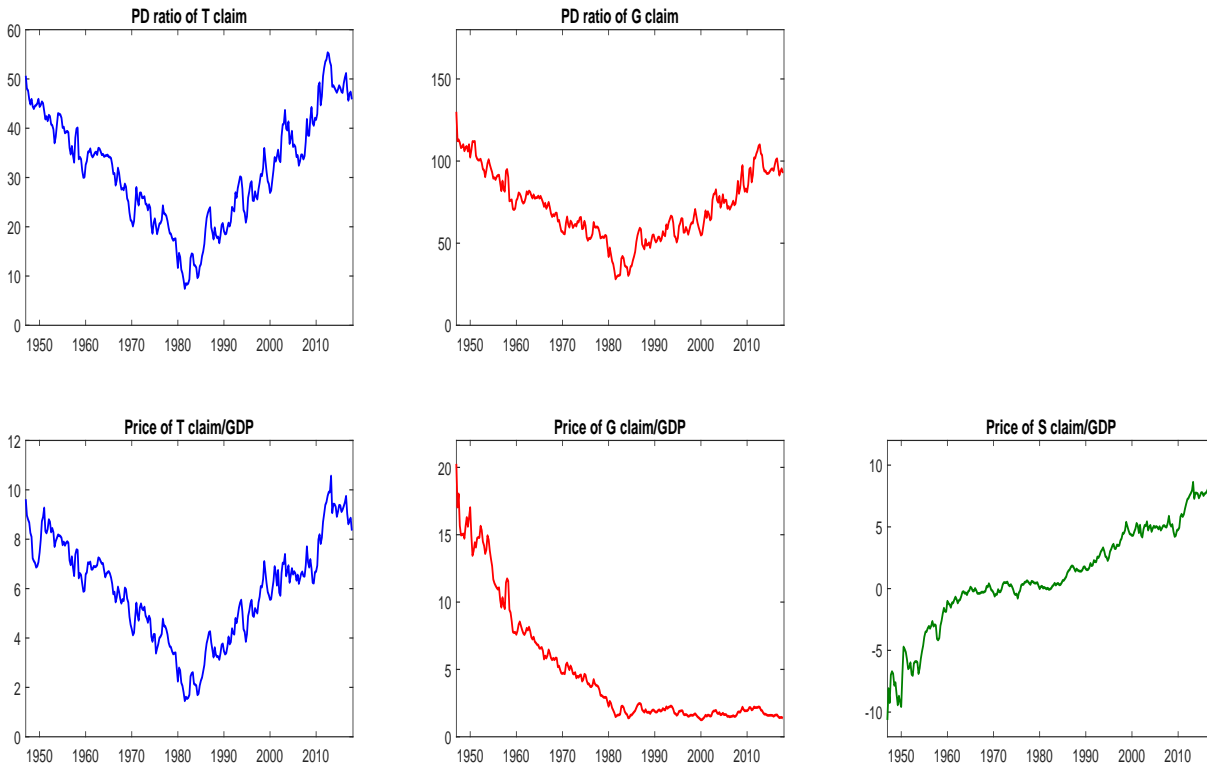
Under the estimated dynamics for spending-to-GDP growth and market prices of risk, the Euler equation for the spending claim has no solution. That is, there is no  $B_g$  vector that solves equation (15), reproduced below:

$$\mathbb{E}_t [r_{t+1}^g] - y_t^g(1) + Jensen = (e_g + e_x + e_\pi + \kappa_1^g B_g)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t)$$

Intuitively, a claim to government revenues is a wonderful asset. It pays out high “dividends” in bad economic times, i.e., high marginal utility states of the world. There-

Figure 6: Government Cash Flows and Prices

The top panels plot the (cum-dividend) price-dividend ratio on the claim to tax revenues (left) and government spending (middle). Both are annualized (divided by 4). The bottom left panel plots the value of a claim to future tax revenue, scaled by GDP. The middle panel plots the value of a claim to future government spending divided by GDP. The bottom right panel plots the value of future government surpluses scaled by GDP.





fore, agents are willing to pay a very high price for such an asset. Put differently, its risk premium and total return are so low that its price-dividend ratio is infinite.

This in turn implies that the value of the surplus claim, which is the difference between the value of the revenue and the value of the spending claim is negative infinity. According to the intertemporal government budget constraint (1), a negative present discounted value of surpluses implies a negative market value of outstanding government debt. This implication makes it puzzling that government debt is worth anything. In our post-war sample, outstanding government debt has been about 75% of GDP, far from negative infinity. This puzzle occurs despite the fact that the average surplus has been just about zero in our sample, because it stems fully from the differential riskiness of the revenue and the spending claims. We coin this the government debt risk puzzle.

One approach to reach a finite valuation for the government spending claim is to assume that spending-GDP growth is not zero, on average, but negative. If we assume that  $\mathbb{E}[\Delta \log g_t] = -0.83\%$  per quarter, then we reduce the value of the spending claim enough to generate an average value of the surplus claim equal to +75% of GDP, the average value of government debt to GDP observed in the data. The second panel of Figure 6 plots the price-dividend ratio on a claim to future government spending,  $\frac{P_t^s}{G_t}$  in (12), under this assumption. The level of the price-dividend ratio on this claim is still very high, despite its declining cash flows. Its time-series mean is 74.6. The reason for this high price is that the claim is still not very risky. The average risk premium is only 0.98% per year. The price-dividend ratio shows the same inverse V-shape dynamics of the price-dividend ratio on the revenue claim.

The left and middle panels in the bottom row of Figure 6 plot the value of the tax revenue and spending claims scaled by GDP. They are computed as the product of the price-dividend ratio (in top panels) and the ratio of tax revenues-to-GDP and spending-to-GDP. Naturally, the price-GDP ratios inherit both the dynamics of the price-dividend ratios and of the cash flows. The low-frequency dynamics in prices reflect the low-frequency dynamics in the real rate. The higher-frequency dynamics in the prices come from the higher-frequency dynamics in revenues and spending, as well as in risk premia. The decline in the ratio of the value of the spending claim to GDP comes from our assumption that spending to GDP falls each quarter by 0.83%. Under this assumption, spending declines from 15.58% of GDP in 1947 to 1.50% in 2017. This is obviously highly counterfactual; recall that Figure 1 plotted the actual dynamics of spending to GDP. The assumption induces growing primary surpluses over time. These surpluses increase the value of the

surplus claim from a large negative number to +0.75 times GDP. The bottom right panel of Figure 6 plots the value of current and future government surpluses scaled by GDP. As in (8), it is the difference between the value of a claim to revenues (left panel) and the value of a claim to spending (middle panel). While the average value of the surplus claim to GDP ratio matches the average market value of government debt to GDP in the data by assumption (that’s how we chose the -0.83% number), the dynamics do not match the observed dynamics of government debt to GDP plotted in Figure 2 at all. In sum, assuming that the government debt to GDP ratio is on a declining trend is both conceptually unappealing and induces counter-factual debt dynamics. In short, it does not constitute a resolution of the government debt risk puzzle.

## 4 Resolving the Puzzle

### 4.1 The Long-Run Growth Rate of Government Spending

We assume that the expected growth rate in the government spending-to-GDP ratio contains a persistent component  $\zeta_t$ . The state variable  $\zeta_t$  is mean-zero and is added as the 9th and last element of the VAR. Its dynamics are given by equation (16):

$$\mathbb{E}_t [\Delta \log g_{t+1}] = \Psi_{(8,1:7)} z_t + \Psi_{(8,8)} \Delta \log g_t + \zeta_t, \quad (15)$$

$$\zeta_{t+1} = \Psi_{(9,7)} \Delta \log \tau_t + \Psi_{(9,8)} (\Delta \log g_t - \zeta_0) + \Psi_{(9,9)} \zeta_t + \Sigma_{(9,\cdot)}^{\frac{1}{2}} \varepsilon_{t+1} \quad (16)$$

Analogously to the modeling of expected consumption growth in the long-run risk model, this component is small, persistent, and unobserved to the econometrician but known to the agents inside the model. The high persistence is reflected in a value for the coefficient  $\Psi_{(9,9)}$  close to 1. By “small,” we mean that  $\zeta_{t+1}$  has low innovation variance relative to the variance of  $\Delta \log g_{t+1}$ .

If the coefficients  $\Psi_{(9,7)} > 0$  and  $\Psi_{(9,8)} < 0$ , a lower-than-usual tax growth and a higher-than-usual spending growth (relative to GDP growth) in the current period  $t$  result in a lower value for  $\zeta_{t+1}$  in the next period. This reduction in expected future spending stabilizes the debt because it connects current deficits to higher expected future surpluses. In the presence of such stabilizing forces, the claim to current and future government spending becomes less counter-cyclical and therefore has a higher discount rate. As long as investors care sufficiently about such long-run states of the world, the value of the spending claim will be substantially reduced.

To keep the problem tractable, we assume that the last three structural shocks of the augmented VAR,  $(\varepsilon_7, \varepsilon_8, \varepsilon_9)$ , are not priced. The assumption does not imply that shocks to government revenue or spending growth are not priced. It only assumes that the innovations to tax revenue-to-GDP growth and spending-to-GDP growth that are orthogonal to the first six shocks that precede it in the VAR are not priced. Likewise, only the innovations to  $\zeta_t$  that are orthogonal to the innovations in the first six state variables are not priced. This assumption greatly simplifies the estimation since it allows us to separate the problem of estimating the market prices of risk  $\{\Lambda_t\}$  from the problem of valuing the claims to  $\{G_t\}$  and  $\{T_t\}$ .

Because the first seven state variables in  $z_t$  and the pricing kernel do not depend on the dynamics of government spending growth ( $\Delta \log g_t$  and  $\zeta_t$ ), we can separate the estimation of the first seven equations of the VAR system from the estimation of the last two equations. In other words, the first seven rows of  $\Psi$  and  $\Sigma^{\frac{1}{2}}$  are the same as in the model without latent component in spending growth. Likewise, the market price of risk estimates  $\Lambda_0$  and  $\Lambda_1$  are identical.

## 4.2 The Kalman Filter

Next, we estimate the process for government spending growth, the last two equations of the VAR given by (15) and (16). Since the persistent component in expected spending-to-GDP growth  $\zeta_t$  is latent and our setup is conditionally Gaussian, we use the Kalman filter. The main insight we exploit is that the market value of government debt should be highly informative about the latent process  $\zeta_t$ , since the latter has a major impact on the value of a claim to government spending. Combining equations (1), (8), (11), and (12), we get:

$$\sum_{h=0}^{\infty} P_t^{\$}(h) Q_{t,h} = T_t (1 + \exp \{ \overline{p\tau} + B'_{\tau} z_t \}) - G_t (1 + \exp \{ \overline{p\bar{g}} + B'_g z_t \}) \quad (17)$$

Because the  $\Delta \log \tau_t$  process and the market prices of risk do not depend on  $\Delta \log g_t$  and  $\zeta_t$ , the value of tax claim does not depend on those two states:  $B_{\tau}(8) = B_{\tau}(9) = 0$ . Only the valuation ratio of the spending claim depends on  $\zeta_t$ . Given the estimates obtained in the first two steps of the estimation procedure, we can use (17) to back out a time series  $\hat{\zeta}_t$ :

$$\hat{\zeta}_t = \frac{1}{B_g(9)} \left( \log \left( \frac{T_t (1 + \exp \{ \overline{p\tau} + B'_{\tau} z_t \}) - \sum_{h=0}^{\infty} P_{t,h}^{\$} Q_{t,h}}{G_t} - 1 \right) - \overline{p\bar{g}} - B_g(1, \dots, 8)' z_t(1, \dots, 8) \right).$$

Notice that  $\hat{\zeta}_t$  in the equation above depends on  $B_g$ , which itself depends on  $\zeta_t$ . We treat  $\hat{\zeta}_t$  as a noisy signal of the true latent variable  $\zeta_t$ . The observation equation in the Kalman filtering problem is:

$$\hat{\zeta}_t = \zeta_t + \omega \varepsilon_t^{obs}, \quad (18)$$

where  $\varepsilon_t^{obs}$  is an i.i.d. standard normally distributed random variable with standard deviation  $\omega \geq 0$ . The state evolution equation in the Kalman filtering problem is (16). The 29 parameters to estimate are  $\Theta = \left( \Psi_{(8,1:8)}, \Psi_{(9,7:9)}, \Sigma_{(8,1:8)}^{\frac{1}{2}}, \Sigma_{(9,1:9)}^{\frac{1}{2}}, \omega \right)$ . Appendix D describes the estimation in more detail.

The Kalman Filter results in the following point estimates:

$$\Psi_{(8:9,1:9)} = \begin{bmatrix} -0.116 & -0.892 & -0.032 & -0.924 & -0.003 & -0.016 & -0.123 & -0.033 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.041 & -0.027 & 1.007 \end{bmatrix}$$

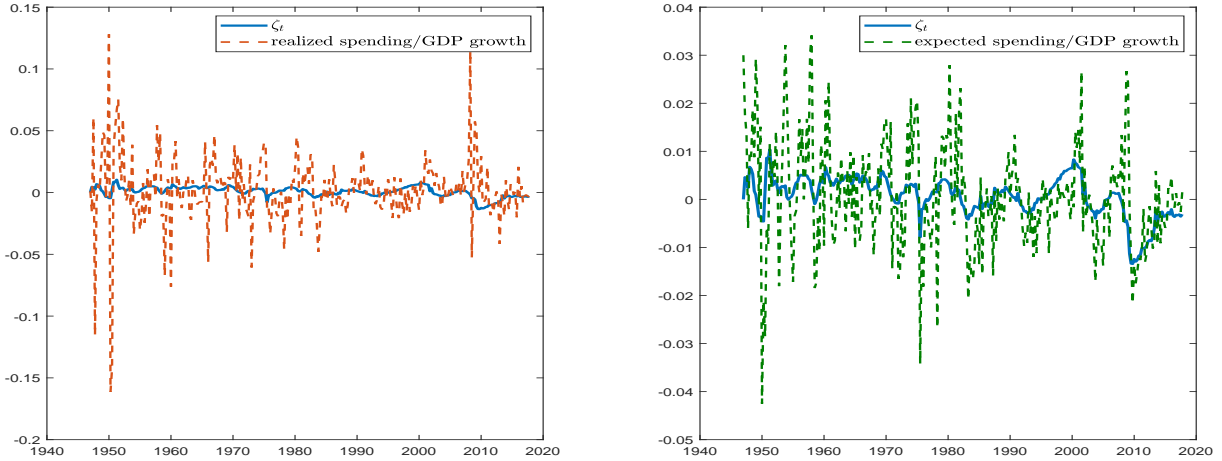
$$100 \times \Sigma_{(8:9,1:9)}^{\frac{1}{2}} = \begin{bmatrix} -0.39 & -0.88 & -0.14 & 0.13 & 0.12 & -0.30 & -0.08 & 2.36 & 0 \\ -0.000 & -0.004 & -0.002 & 0.001 & 0.001 & -0.003 & -0.019 & -0.005 & 0.009 \end{bmatrix}$$

High spending growth in the current period results in a lower value for latent expected future spending growth  $\zeta_{t+1}$ :  $\Psi_{(9,8)} = -0.027$ . Also, when tax revenue is low, future expected spending growth is low:  $\Psi_{(9,8)} = 0.041$ . These are key stabilizing forces which render the spending claim more risky and hence much less valuable. We also estimate strong persistence in  $\zeta_t$ :  $\Psi_{(9,9)} = 1.007$ . We verify that the VAR is stationary; all eigenvalues are inside the unit circle. The error covariance matrix  $\Sigma^{\frac{1}{2}}$  also affects the riskiness of the spending claim in important ways. Positive shocks to current spending growth coincide with negative shocks to future spending growth ( $\Sigma_{(9,8)}^{\frac{1}{2}} < 0$ ). The (orthogonal) innovation volatility of  $\zeta_t$  is small compared to that of  $\Delta \log g_t$ : 0.009% versus 2.36% per quarter.

The thick line in Figure 7 plots the filtered time series for  $\zeta_t$  evaluated at the parameter point estimates. The left panel includes realized spending-to-GDP growth  $\Delta \log g_{t+1}$  and illustrates that the latent component  $\zeta_t$  is small. The right panel zooms in on the latent time series -note the different scale on the vertical axis- and shows that latent spending-GDP growth is close to zero on average. The right panel also plots expected spending growth  $\mathbb{E}_t[\Delta \log g_{t+1}]$ , see equation (15). The latter contains the low-frequency component  $\zeta_t$  but also the dependence of expected spending-GDP growth on the first eight variables in the VAR. The latter adds a business-cycle frequency component to expected spending

Figure 7: Realized and Expected Spending Growth

The figure plots the filtered time series of  $\zeta_t$  evaluated at the parameter point estimates. The left panel compares the filtered time series  $\zeta_t$  with the realized spending-to-GDP growth  $\Delta \log g_{t+1}$ . The right panel compares the filtered time series  $\zeta_t$  with the expected spending-to-GDP growth  $\mathbb{E}_t[\Delta \log g_{t+1}]$ .



growth. In expansions, the business cycle component strongly reduces expected spending growth while in recessions that cyclical component increases expected spending growth. This pattern reflects the automatic stabilizers in fiscal policy. The low frequency component shows a slow decline over the post-war period, with the exception of an upward drift in the 1990s. Since 2000, the persistent component of expected spending-GDP growth has been consistently negative and generally declining. In sum, investors in U.S. Treasuries must be expecting the U.S. government to reduce spending in the long-run for debt to be priced as highly as it is today.

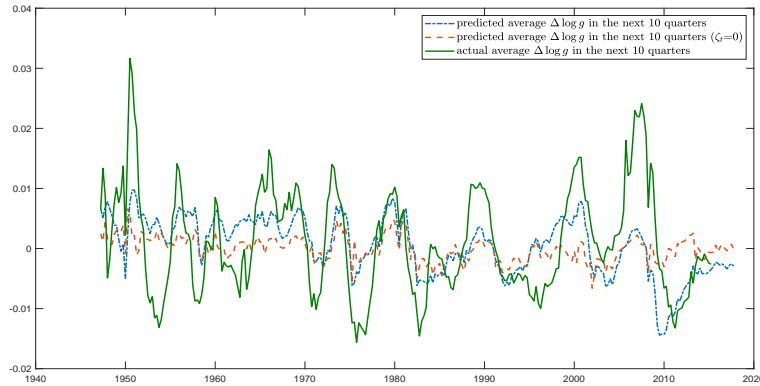
We can also test whether the latent component implied in the market value of outstanding Treasury debt forecasts future government spending growth. Figure 8 shows that it does. It plots the actual average of the spending-to-GDP growth in the next 10 quarters alongside predicted average  $\Delta \log g_{t+k}$  in next 10 quarters (the blue dotted line) with the filtered time series of  $\zeta_t$ . The two series have a positive correlation of 51%. The correlation of the actual average spending-to-GDP growth with the predicted average  $\Delta \log g_{t+k}$  in the next 10 quarters assuming that  $\zeta_t = 0$  (the red dashed line) is only 39%.

### 4.3 Pricing Implications

In the presence of the long-run growth rate  $\zeta_t$ , the risk properties of the claim to spending growth change dramatically. While current spending takes place in high SDF states, high current spending now also entails lower future spending. Such lower future spending-

Figure 8: Predicting Long-Term Spending-to-GDP Growth

The figure plots the predicted average of the spending-to-GDP growth  $\mathbb{E}_t[\Delta \log g_{t+1}]$  over the next 10 quarters (with the filtered time series  $\zeta_t$  or with  $\zeta_t = 0$ ) and the actual average of the spending-to-GDP growth  $\Delta \log g_{t+1}$  over the next 10 quarters.



to-GDP growth states are also high SDF states. This is because such states are low GDP growth states and the market price of GDP growth risk is positive. Compared to the  $\zeta_t = 0$  case, the claim to spending is now much riskier. Consequently, its valuation ratio of the spending claim is much lower. The middle panels of Figure 9 show the price of the spending claim scaled by current spending (top) and GDP (bottom) in the benchmark model. Compared with Figure 6, we see dramatically lowered values. The revenue claim plotted in the left panels is unaffected. The bottom right panel shows the value of the surplus claim scaled by GDP, obtained by subtracting the spending claim's value from the revenue claim's value. By virtue of the estimation approach, it matches the outstanding value of debt very closely. All the while, the approach keeps the conceptually desirable assumption that  $\mathbb{E}[\Delta \log g_t] = 0$ .

## 5 Application: Government Risk Management

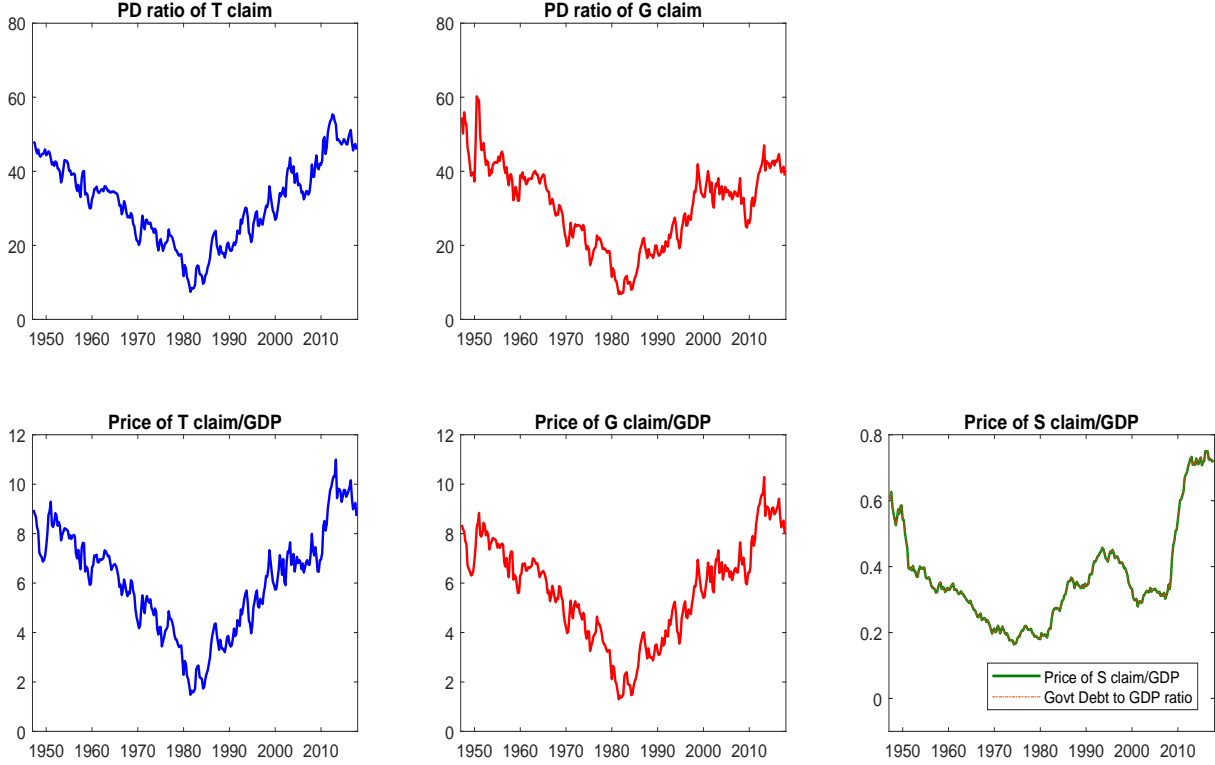
### 5.1 Hedging against Nominal Yield Shocks

Bhandari, Evans, Golosov, and Sargent (2017) shows that in a broad class of models, the optimal maturity choice for government debt amounts to minimizing the variance of government funding needs. The funding need at  $t + 1$  is defined as the time  $t + 1$  value of its time- $t$  bond portfolio minus the expected discounted value of its future surpluses, holding fixed the tax policy:<sup>4</sup>

<sup>4</sup>We interpret a fixed tax policy as tax revenues following the same stochastic process as in the past, i.e., as implied by the VAR.

Figure 9: Government Cash Flows and Prices with Latent Spending Growth

This graph is for the model that estimates  $\zeta_t$  from the data, and in particular from the observed market value of outstanding government debt. The top panels plot the (cum-dividend) price-dividend ratio on the claim to tax revenues (left) and government spending (middle). Both are annualized (divided by 4). The bottom left panel plots the value of a claim to future tax revenue, scaled by GDP. The middle panel plots the value of a claim to future government spending divided by GDP. The bottom right panel plots the value of future government surpluses scaled by GDP.



$$\begin{aligned}
 Need_{t+1} &= \sum_{h=0}^{\infty} P_{t+1}^{\$}(h) Q_{t+1,h}^{\$} + \sum_{h=0}^{\infty} P_{t+1}(h) Q_{t+1,h} - \mathbb{E}_{t+1} \left[ \sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} S_{t+1+j} \right] \quad (19) \\
 &= \sum_{h=0}^{\infty} P_{t+1}^{\$}(h) Q_{t+1,h}^{\$} + \sum_{h=0}^{\infty} P_{t+1}(h) Q_{t+1,h} - P_{t+1}^{\tau} + P_{t+1}^g
 \end{aligned}$$

The equation clarifies that the government can issue both nominal and real bonds, and we let  $Q_{t+1,h}^{\$}$  ( $Q_{t+1,h}$ ) denote the outstanding nominal (real) bond amounts at  $t + 1$ , chosen at time  $t$ . When  $Need_{t+1} > 0$ , the government faces a funding shortfall and has to adjust taxes, which is distortionary. At time  $t$ , the government optimally chooses its debt issuance policy along the maturity curve to minimize the variance of its funding needs:

$$\min_{\{Q_{t+1,h}\}} Var_t [Need_{t+1}] \quad (20)$$

If the only shock in the model is the interest rate shock, the government can set  $Q_{t+1,h} = S_{t+1+h}$ ,  $h = 1, \dots, \infty$  for each maturity  $h$  and reduce the variance of its funding needs to zero. The government debt portfolio is fully immunized against interest rate shocks. This is essentially the optimization problem faced by a defined benefits plan. However, in the presence of other shocks that affect the SDF and government spending, the choice of the optimal government bond portfolio is more complicated. To develop some intuition, we start by considering a portfolio that is locally hedged against interest rate shocks, defined below. Assume for simplicity that the government only issues nominal bonds.

**Definition 1.** A portfolio of government nominal bonds  $\{Q_{t+1,h}^{\$,*}\}$  is locally hedged against a change in short-term interest rates if the value of the surplus claim adjusts by the same amount as the value of the outstanding bond portfolio in response to an interest rate shock:

$$-\sum_{h=0}^{\infty} Q_{t+1,h}^{\$,*} \frac{\partial P_{t+1,h}^{\$}}{\partial y_{t+1}^{\$}(1)} = -\frac{\partial \mathbb{E}_{t+1} \left[ \sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} S_{t+1+j} \right]}{\partial y_{t+1}^{\$}(1)} \quad (21)$$

In other words, the government surplus claim and the government bond portfolio have the same dollar duration. Interest rate hedging can also be expressed in terms of modified duration, which is an elasticity:

$$-\frac{\sum_{h=0}^{\infty} P_{t,h}^{\$} Q_{t+1,h}^{\$,*} \frac{\partial \log P_{t+1,h}^{\$}}{\partial y_{t+1}^{\$}(1)}}{\sum_{h=0}^{\infty} P_{t,h}^{\$} Q_{t+1,h}^*} = -\frac{\partial \log \mathbb{E}_{t+1} \left[ \sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} S_{t+1+j} \right]}{\partial y_{t+1}^{\$}(1)} \quad (22)$$

A modified duration of  $h$  means the interest rate exposure of the claim is the same as that of a  $h$ -year zero-coupon bond. The modified durations of the tax claim and the spending claim tells us the percentage decline in value for a one percentage point increase in rates:

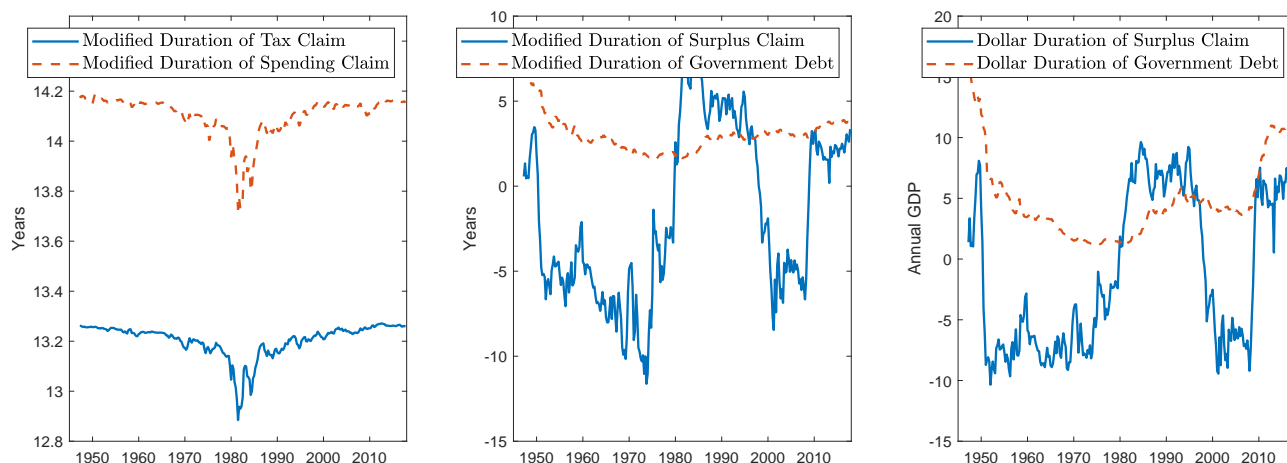
$$-\frac{\partial \log \mathbb{E}_{t+1} \left[ \sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} T_{t+1+j} \right]}{\partial y_{t+1}^{\$}(1)} \quad \text{and} \quad -\frac{\partial \log \mathbb{E}_{t+1} \left[ \sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} G_{t+1+j} \right]}{\partial y_{t+1}^{\$}(1)} \quad (23)$$

The left panel of Figure 10 shows the modified durations of the government tax claim and spending claim. The modified duration of the spending claim is about 14, higher than that of the tax claim, which is about 13. The spending claim has more interest rate risk than the tax revenue claim. The modified duration of the government surplus claim is the difference between the durations of the revenue and spending claims, each weighted



Figure 10: The Durations of the Government Bond Portfolio and Government Surpluses

The figure plots the durations of the government bond portfolio and those of government surpluses. The modified durations are in the unit of years, and the dollar durations are normalized by annual U.S. GDP. The series is quarterly from 1940.Q1 until 2017.Q4. Data Source: CRSP U.S. Treasury Database and BEA.



by the respective values of the claims. It is plotted in the middle panel, alongside the modified duration of the observed government bond portfolio, constructed from all outstanding (nominal and real) bonds. Since the dollar values of the tax and spending claims fluctuate, the modified duration of the surplus claim can be either positive or negative. In comparison, the modified duration of the government bond portfolio is always positive. At the end of the sample, the modified duration is 3.7 years. An increase in the nominal yield by 1% point will decrease the value of the government bond portfolio by approximately 3.7%. A small adjustment in the average maturity of government debt of 0.6 years would suffice to align the durations.

The right panel reports the dollar durations of the government bond portfolio and the government surpluses, weighted by annual GDP. A dollar duration of 1 means a 1% point increase in the nominal yield will decrease the dollar value by 1% of annual GDP. The graph shows large deviations of the observed bond portfolio duration from that of the surplus. The correlation between the two series is 36%. More recently, the two durations are much closer.

## 5.2 Hedging against All Shocks

In our model, as well as in the real world, shocks beyond interest rate shocks that change the value of government debt and the surplus claim. Under the same objective of min-

imizing the variability of funding needs, the government's bond portfolio should also hedge those shocks.

**Definition 2.** A government bond portfolio  $\{Q_{t+1,h}^{\$,*}\}$  is fully immunized against all shocks to the state variables if the changes in value of the outstanding bond portfolio equal the changes in the value of the surplus claim for each shock:

$$-\sum_{h=0}^{\infty} Q_{t+1,h}^{\$,*} \frac{\partial P_{t+1,h}^{\$}}{\partial z_{t+1}} = -\frac{\partial \mathbb{E}_{t+1} \left[ \sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} S_{t+1+j} \right]}{\partial z_{t+1}} \quad (24)$$

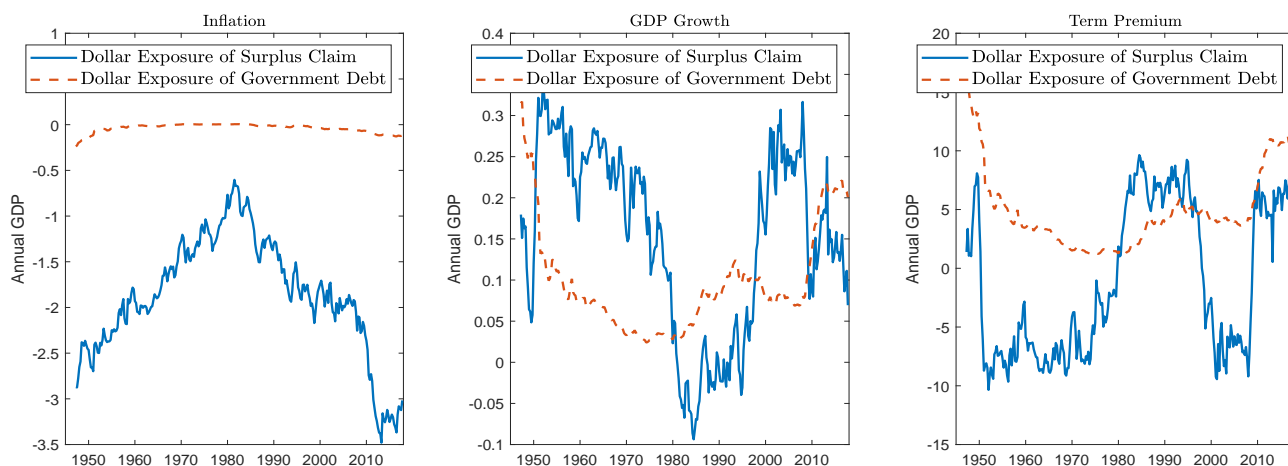
Equation (24) describes a system of  $N$  equations, one for each of the state variables. Appendix E contains the formulas. Since the short-term nominal yield one of the state variables, equation (21) is a special case of (24). Given  $N$  bond maturities, the government can fully immunize its debt portfolio against all economic shocks. Our main objective here is to quantify how far the existing government debt portfolio is from full immunization.

Figure 11 reports the dollar exposures of the government surplus claim and the government debt portfolio with respect to inflation  $\pi_t$ , GDP growth  $x_t$ , and the term spread  $yspr_t^{\$}$ . Overall, the government debt portfolio does not hedge changes in these variables. There is less state-contingency in the data than in the model.

To measure how much the government debt portfolio deviates from full immunization

Figure 11: Dollar Exposure of the Government Bond Portfolio and Surplus Claim

The figure plots the dollar exposures of the government bond portfolio and government surpluses with respect to inflation  $\pi_t$ , GDP growth rate  $x_t$ , and the term premium  $yspr_t^{\$}$ , scaled by annual U.S. GDP. The series is quarterly from 1940.Q1 until 2017.Q4. Data Source: CRSP U.S. Treasury Database and BEA.

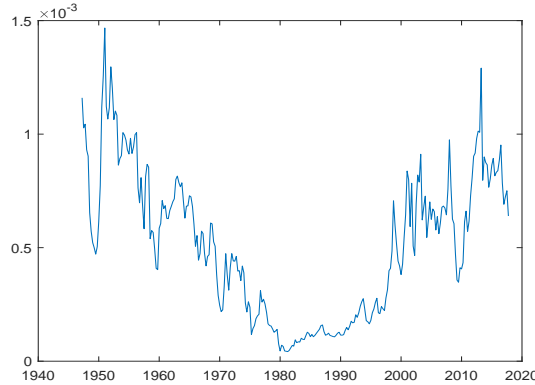


against all shocks, we express the conditional standard deviation of the funding needs as

$$\begin{aligned}
 (Var_t [Need_{t+1}])^{1/2} &= \left( Var_t \left[ \sum_{h=0}^{\infty} P_{t+1}^{\$}(h) Q_{t+1,h}^{\$} + \sum_{h=0}^{\infty} P_{t+1}(h) Q_{t+1,h} - P_{t+1}^{\tau} + P_{t+1}^{\$} \right] \right)^{1/2} \\
 &= \left( Var_t \left[ \left( \sum_{h=0}^{\infty} Q_{t+1,h}^* \frac{\partial P_{t+1,h}^{\$}}{\partial z_{t+1}} - \frac{\partial \mathbb{E}_{t+1} \left[ \sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} S_{t+1+j} \right]}{\partial z_{t+1}} \right) \Sigma^{\frac{1}{2}} \varepsilon_{t+1} \right] \right)^{1/2}
 \end{aligned} \tag{25}$$

Figure 12 reports this standard deviation. In 2017, it is about 6 basis points, suggesting that the government will face a funding deficit or surplus of 6 basis points of the annual GDP in the next quarter if the shocks move by one standard deviation. The movement in this standard deviation closely tracks the magnitude of  $P^{\tau}$  and  $P^{\$}$ : In the 1980s,  $P^{\tau}$  and  $P^{\$}$  are very small because of large interest rates, and accordingly the value of government surpluses does not vary greatly. More recently, the deviations from full immunization have increased substantially.

Figure 12: Conditional Standard Deviation of Funding Needs



## 6 Conclusion

Governments must tap debt markets at inopportune times since deficits tend to occur in recessions, times when bond investors face high marginal utility. The positive covariance of deficits with the SDF makes the portfolio of outstanding government debt risky. The low yields that the government pays are puzzling. We propose a resolution of this puzzle. Market values of outstanding government debt imply that bond investors expect high current government spending to be followed by persistently lower spending.

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## A Government Budget Constraint

All objects in this appendix are in nominal terms but we drop the superscript  $\$$  for ease of notation. The government faces the following one-period budget constraint:

$$G_t - T_t + B_{t-1}^1 = \sum_{k=1}^K (B_t^k - B_{t-1}^{k+1}) P_t^k,$$

where  $G_t$  is total nominal government spending,  $T_t$  is total nominal government revenue,  $B_t^k$  is the number of nominal zero-coupon bonds of maturity  $k$  outstanding in period  $t$  each promising to pay back \$1 at time  $t + k$ , and  $P_t^k$  is today's price for a  $k$ -period zero-coupon bond with \$1 face value. A unit of  $k + 1$ -period bonds issued at  $t - 1$  becomes a unit of  $k$ -period bonds in period  $t$ . That is, the stock of bonds evolves of each maturity evolves according to  $B_t^k = B_{t-1}^{k+1} + \Delta B_t^k$ . Note that this notation can easily handle coupon-bearing bonds. For any bond with deterministic cash-flow sequence, we can write the price (present value) of the bond as the sum of the present values of each of its coupons. The left-hand side of the budget constraint denotes new financing needs in the current period, due to primary deficit  $G - T$  and one-period debt from last period that is now maturing. The right hand side shows that the money is raised by issuing new bonds of various maturities. Alternatively, we can write the budget constraint as total expenses equalling total income:

$$G_t + B_{t-1}^1 + \sum_{k=1}^K B_{t-1}^{k+1} P_t^k = T_t + \sum_{k=1}^K B_t^k P_t^k,$$

We can now iterate the budget constraint forward. The period  $t$  constraint is given by:

$$T_t - G_t = B_{t-1}^1 - B_t^1 P_t^1 + B_{t-1}^2 P_t^1 - B_t^2 P_t^2 + B_{t-1}^3 P_t^2 - B_t^3 P_t^3 + \dots - B_t^K P_t^K + B_{t-1}^{K+1} P_t^K.$$

Consider the period- $t + 1$  constraint, multiplied by  $M_{t+1}$ , and take expectations conditional on time  $t$ :

$$\begin{aligned} E_t [M_{t+1}(T_{t+1} - G_{t+1})] &= B_t^1 P_t^1 - E_t [B_{t+1}^1 M_{t+1} P_{t+1}^1] + B_t^2 P_t^2 - E_t [B_{t+1}^2 M_{t+1} P_{t+1}^2] + B_t^3 P_t^3 \\ &\quad - E_t [B_{t+1}^3 M_{t+1} P_{t+1}^3] + \dots + B_t^K P_t^K - E_t [B_{t+1}^K M_{t+1} P_{t+1}^K] + B_t^{K+1} P_t^{K+1}, \end{aligned}$$

where we use the asset pricing equations  $E_t [M_{t+1}] = P_t^1$ ,  $E_t [M_{t+1} P_{t+1}^1] = P_t^2$ ,  $\dots$ ,  $E_t [M_{t+1} P_{t+1}^{K-1}] = P_t^K$ , and  $E_t [M_{t+1} P_{t+1}^K] = P_t^{K+1}$ .



Consider the period  $t + 2$  constraint, multiplied by  $M_{t+1}M_{t+2}$  and take time- $t$  expectations:

$$\begin{aligned} E_t [M_{t+1}M_{t+1}(T_{t+2} - G_{t+2})] &= E_t[B_{t+1}^1 M_{t+1}P_{t+1}^1] - E_t[B_{t+2}^1 M_{t+1}M_{t+2}P_{t+2}^1] + E_t[B_{t+1}^2 M_{t+1}P_{t+1}^2] \\ &\quad - E_t[B_{t+2}^2 M_{t+1}M_{t+2}P_{t+2}^2] + E_t[B_{t+1}^3 M_{t+1}P_{t+1}^3] - \dots \\ &\quad + E_t[B_{t+1}^K M_{t+1}P_{t+1}^K] - E_t[B_{t+2}^K M_{t+1}M_{t+2}P_{t+2}^K] + E_t[B_{t+1}^{K+1} M_{t+1}P_{t+1}^{K+1}], \end{aligned}$$

where we used the law of iterated expectations and  $E_{t+1}[M_{t+2}] = P_{t+1}^1$ ,  $E_{t+1}[M_{t+2}P_{t+2}^1] = P_{t+1}^2$ , etc.

Similarly consider the one-period government budget constraints at times  $t + 3, t + 4$ , etc. Then add up all one-period budget constraints. Note how the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints.

$$E_t \left[ \sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j}) \right] = \sum_{k=0}^K B_{t-1}^{k+1} P_t^k + E_t \left[ \sum_{j=0}^{\infty} M_{t,t+j} B_{t+j}^{K+1} P_{t+j}^{K+1} \right]$$

where we used the cumulate SDF notation  $M_{t,t+j} = \prod_{i=0}^j M_{t+i}$  and by convention  $M_{t,t} = M_t = 1$  and  $P_t^0 = 1$ . We obtain that the expected present-discounted value of the primary surplus stream  $\{T_{t+j} - G_{t+j}\}$  at time  $t$  equals the market value of the outstanding debt inherited from the previous period plus the market value of the longest maturity debt issued (today and) in the future.

Consider the transversality condition, which takes the limit w.r.t. the maximum bond maturity  $K$ :

$$\lim_{K \rightarrow \infty} E_t \left[ \sum_{j=0}^{\infty} M_{t,t+j} B_{t+j}^{K+1} P_{t+j}^{K+1} \right] = 0$$

which says that there can be no new resources raised from issuing infinite maturity bonds.

If the longest-maturity  $M$  becomes large and if the transversality condition is satisfied, we get the result that the outstanding debt today reflects the expected present-discounted value of the current and all future primary surpluses:

$$E_t \left[ \sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j}) \right] = \sum_{k=0}^{\infty} B_{t-1}^{k+1} P_t^k.$$

This is equation (1) in the main text.

## B Asset Pricing Model

### B.1 Risk-free rate

The real short yield  $y_t(1)$ , or risk-free rate, satisfies  $E_t[\exp\{m_{t+1} + y_t(1)\}] = 1$ . Solving out this Euler equation, we get:

$$\begin{aligned} y_t(1) &= y_t^\$(1) - E_t[\pi_{t+1}] - \frac{1}{2}e'_\pi \Sigma e_\pi + e'_\pi \Sigma^{\frac{1}{2}} \Lambda_t \\ &= y_0(1) + \left[ e'_{yn} - e'_\pi \Psi + e'_\pi \Sigma^{\frac{1}{2}} \Lambda_1 \right] z_t. \end{aligned} \quad (\text{A.1})$$

$$y_0(1) \equiv y_0^\$(1) - \pi_0 - \frac{1}{2}e'_\pi \Sigma e_\pi + e'_\pi \Sigma^{\frac{1}{2}} \Lambda_0. \quad (\text{A.2})$$

where we used the expression for the real SDF

$$\begin{aligned} m_{t+1} &= m_{t+1}^\$ + \pi_{t+1} \\ &= -y_t^\$(1) - \frac{1}{2}\Lambda'_t \Lambda_t - \Lambda'_t \varepsilon_{t+1} + \pi_0 + e'_\pi \Psi z_t + e'_\pi \Sigma^{\frac{1}{2}} \varepsilon_{t+1} \\ &= -y_t(1) - \frac{1}{2}e'_\pi \Sigma e_\pi + e'_\pi \Sigma^{\frac{1}{2}} \Lambda_t - \frac{1}{2}\Lambda'_t \Lambda_t - \left( \Lambda'_t - e'_\pi \Sigma^{\frac{1}{2}} \right) \varepsilon_{t+1} \end{aligned}$$

The real short yield is the nominal short yield minus expected inflation minus a Jensen adjustment minus the inflation risk premium.

### B.2 Nominal and real term structure

**Proposition 4.** Nominal bond yields are affine in the state vector:

$$y_t^\$(h) = -\frac{A^\$(h)}{h} - \frac{B^\$(h)'}{h} z_t,$$

where the coefficients  $A^\$(h)$  and  $B^\$(h)$  satisfy the following recursions:

$$A^\$(h+1) = -y_0^\$(1) + A^\$(h) + \frac{1}{2} \left( B^\$(h) \right)' \Sigma \left( B^\$(h) \right) - \left( B^\$(h) \right)' \Sigma^{\frac{1}{2}} \Lambda_0, \quad (\text{A.3})$$

$$\left( B^\$(h+1) \right)' = \left( B^\$(h) \right)' \Psi - e'_{yn} - \left( B^\$(h) \right)' \Sigma^{\frac{1}{2}} \Lambda_1, \quad (\text{A.4})$$

initialized at  $A^\$(0) = 0$  and  $B^\$(0) = \mathbf{0}$ .

*Proof.* We conjecture that the  $t+1$ -price of a  $\tau$ -period bond is exponentially affine in the

state:

$$\log(P_{t+1}^{\$}(h)) = A^{\$}(h) + \left(B^{\$}(h)\right)' z_{t+1}$$

and solve for the coefficients  $A^{\$}(h+1)$  and  $B^{\$}(h+1)$  in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned} P_t^{\$}(h+1) &= E_t[\exp\{m_{t+1}^{\$} + \log(P_{t+1}^{\$}(h))\}] \\ &= E_t[\exp\{-y_t^{\$}(1) - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + A^{\$}(h) + \left(B^{\$}(h)\right)' z_{t+1}\}] \\ &= \exp\{-y_0^{\$}(1) - e'_{yn}z_t - \frac{1}{2}\Lambda_t'\Lambda_t + A^{\$}(h) + \left(B^{\$}(h)\right)' \Psi_{z_t}\} \times \\ &\quad E_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}}\varepsilon_{t+1}\}\right]. \end{aligned}$$

We use the log-normality of  $\varepsilon_{t+1}$  and substitute for the affine expression for  $\Lambda_t$  to get:

$$\begin{aligned} P_t^{\$}(h+1) &= \exp\left\{-y_0^{\$}(1) - e'_{yn}z_t + A^{\$}(h) + \left(B^{\$}(h)\right)' \Psi_{z_t} + \frac{1}{2}\left(B^{\$}(h)\right)' \Sigma \left(B^{\$}(h)\right) \right. \\ &\quad \left. - \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}}(\Lambda_0 + \Lambda_1 z_t)\right\}. \end{aligned}$$

Taking logs and collecting terms, we obtain a linear equation for  $\log(p_t(h+1))$ :

$$\log\left(P_t^{\$}(h+1)\right) = A^{\$}(h+1) + \left(B^{\$}(h+1)\right)' z_t,$$

where  $A^{\$}(h+1)$  satisfies (A.3) and  $B^{\$}(h+1)$  satisfies (A.4). The relationship between log bond prices and bond yields is given by  $-\log\left(P_t^{\$}(h)\right) / \tau = y_t^{\$}(h)$ .  $\square$

Define the one-period return on a nominal zero-coupon bond as:

$$r_{t+1}^{b,\$}(h) = \log\left(P_{t+1}^{\$}(h)\right) - \log\left(P_t^{\$}(h+1)\right)$$

The nominal bond risk premium on a bond of maturity  $\tau$  is the expected excess return corrected for a Jensen term, and equals negative the conditional covariance between that bond return and the nominal SDF:

$$\begin{aligned} E_t\left[r_{t+1}^{b,\$}(h)\right] - y_t^{\$}(1) + \frac{1}{2}V_t\left[r_{t+1}^{b,\$}(h)\right] &= -Cov_t\left[m_{t+1}^{\$}, r_{t+1}^{b,\$}(h)\right] \\ &= \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}}\Lambda_t \end{aligned}$$

Real bond yields,  $y_t(h)$ , denoted without the \$ superscript, are affine as well with coefficients that follow similar recursions:

$$A(h+1) = -y_0(1) + A(h) + \frac{1}{2} (B(h))' \Sigma (B(h)) - (B(h))' \Sigma^{\frac{1}{2}} \left( \Lambda_0 - \Sigma^{\frac{1}{2}} e_\pi \right) \quad (\text{A.5})$$

$$(B(h+1))' = -e'_{yn} + (e_\pi + B(h))' \left( \Psi - \Sigma^{\frac{1}{2}} \Lambda_1 \right). \quad (\text{A.6})$$

For  $\tau = 1$ , we recover the expression for the risk-free rate in (A.1)-(A.2).

## B.3 Stocks

### B.3.1 Aggregate Stock Market

We define the real return on the aggregate stock market as  $R_{t+1}^m = \frac{P_{t+1}^m + D_{t+1}^m}{P_t^m}$ , where  $P_t^m$  is the ex-dividend price on the equity market. A log-linearization delivers:

$$r_{t+1}^m = \kappa_0^m + \Delta d_{t+1}^m + \kappa_1^m p d_{t+1}^m - p d_t^m. \quad (\text{A.7})$$

The unconditional mean log real stock return is  $r_0^m = E[r_t^m]$ , the unconditional mean dividend growth rate is  $\mu^m = E[\Delta d_{t+1}^m]$ , and  $\overline{p d^m} = E[p d_t^m]$  is the unconditional average log price-dividend ratio on equity. The linearization constants  $\kappa_0^m$  and  $\kappa_1^m$  are defined as:

$$\kappa_1^m = \frac{e^{\overline{p d^m}}}{e^{\overline{p d^m}} + 1} < 1 \quad \text{and} \quad \kappa_0^m = \log \left( e^{\overline{p d^m}} + 1 \right) - \frac{e^{\overline{p d^m}}}{e^{\overline{p d^m}} + 1} \overline{p d^m}. \quad (\text{A.8})$$

Our state vector  $z$  contains the (demeaned) log real dividend growth rate on the stock market,  $\Delta d_{t+1}^m - \mu^m$ , and the (demeaned) log price-dividend ratio  $p d_t^m - \overline{p d^m}$ .

$$\begin{aligned} p d_t^m(h) &= \overline{p d^m} + e'_{pd} z_t, \\ \Delta d_t^m &= \mu^m + e'_{divm} z_t, \end{aligned}$$

where  $e'_{pd}$  ( $e'_{divm}$ ) is a selector vector that has a one in the fifth (sixth) entry, since the log pd ratio (log dividend growth rate) is the fifth (sixth) element of the VAR.

We define the log return on the stock market so that the return equation holds exactly, given the time series for  $\{\Delta d_t^m, p d_t^m\}$ . Rewriting (A.7):

$$\begin{aligned} r_{t+1}^m - r_0^m &= \left[ (e'_{divm} + \kappa_1^m e'_{pd})' \Psi - e'_{pd} \right] z_t + (e'_{divm} + \kappa_1^m e'_{pd})' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}. \\ r_0^m &= \mu^m + \kappa_0^m - \overline{p d^m} (1 - \kappa_1^m). \end{aligned}$$

The equity risk premium is the expected excess return on the stock market corrected for a Jensen term. By the Euler equation, it equals minus the conditional covariance between the log SDF and the log return:

$$\begin{aligned}
1 &= E_t \left[ M_{t+1} \frac{P_{t+1}^m + D_{t+1}^m}{P_t^m} \right] = E_t \left[ \exp \{ m_{t+1}^{\$} + \pi_{t+1} + r_{t+1}^m \} \right] \\
&= E_t \left[ \exp \left\{ -y_{t,1}^{\$} - \frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1} + \pi_0 + e_{\pi}' z_{t+1} + r_0^m + (e_{divm} + \kappa_1^m e_{pd})' z_{t+1} - e_{pd}' z_t \right\} \right] \\
&= \exp \left\{ -y_0^{\$}(1) - \frac{1}{2} \Lambda_t' \Lambda_t + \pi_0 + r_0^m + \left[ (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Psi - e_{pd}' - e_{yn}' \right] z_t \right\} \\
&\quad \times E_t \left[ \exp \{ -\Lambda_t' \varepsilon_{t+1} + (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Sigma^{\frac{1}{2}} \varepsilon_{t+1} \} \right] \\
&= \exp \left\{ r_0^m + \pi_0 - y_0^{\$}(1) + \left[ (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Psi - e_{pd}' - e_{yn}' \right] z_t \right\} \\
&\quad \times \exp \left\{ \frac{1}{2} (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Sigma (e_{divm} + \kappa_1^m e_{pd} + e_{\pi}) - (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Sigma^{\frac{1}{2}} \Lambda_t \right\}
\end{aligned}$$

Taking logs on both sides delivers:

$$\begin{aligned}
r_0^m + \pi_0 - y_0^{\$}(1) + \left[ (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Psi - e_{pd}' - e_{yn}' \right] z_t & \tag{A.9} \\
+ \frac{1}{2} (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Sigma (e_{divm} + \kappa_1^m e_{pd} + e_{\pi}) &= (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Sigma^{\frac{1}{2}} \Lambda_t \\
E_t \left[ r_{t+1}^{m,\$} \right] - y_{t,1}^{\$} + \frac{1}{2} V_t \left[ r_{t+1}^{m,\$} \right] &= -Cov_t \left[ m_{t+1}^{\$}, r_{t+1}^{m,\$} \right]
\end{aligned}$$

The left-hand side is the expected excess return on the stock market, corrected for a Jensen term, while the right-hand side is the negative of the conditional covariance between the (nominal) log stock return and the nominal log SDF. We refer to the left-hand side as the equity risk premium in the data, since it is implied directly by the VAR. We refer to the right-hand side as the equity risk premium in the model, since it requires knowledge of the market prices of risk.

Note that we can obtain the same expression using the log real SDF and log real stock return:

$$\begin{aligned}
E_t \left[ r_{t+1}^m \right] - y_{t,1} + \frac{1}{2} V_t \left[ r_{t+1}^m \right] &= -Cov_t \left[ m_{t+1}, r_{t+1}^m \right] \\
r_0^m - y_0(1) + \left[ (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Psi - e_{pd}' - e_{yn}' - e_{\pi}' \Sigma^{1/2} \Lambda_1 \right] z_t & \\
+ \frac{1}{2} (e_{divm} + \kappa_1^m e_{pd})' \Sigma (e_{divm} + \kappa_1^m e_{pd}) &= (e_{divm} + \kappa_1^m e_{pd})' \Sigma^{1/2} (\Lambda_t - (\Sigma^{1/2})' e_{\pi})
\end{aligned}$$

We combine the terms in  $\Lambda_0$  and  $\Lambda_1$  on the right-hand side and plug in for  $y_0(1)$  from (A.2) to get:

$$r_0^m + \pi_0 - y_{0,1}^{\$} + \frac{1}{2}e'_{\pi}\Sigma e_{\pi} + \frac{1}{2}(e_{divm} + \kappa_1^m e_{pd})'\Sigma(e_{divm} + \kappa_1^m e_{pd}) + e'_{\pi}\Sigma(e_{divm} + \kappa_1^m e_{pd}) + \left[ (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})'\Psi - e'_{pd} - e'_{yn} \right] z_t = (e_{divm} + \kappa_1^m e_{pd})'\Sigma^{1/2}\Lambda_t + e'_{\pi}\Sigma^{\frac{1}{2}}\Lambda_0 + e'_{\pi}\Sigma^{1/2}\Lambda_1 z_t$$

This recovers equation (A.9).

### B.3.2 Dividend Strips

**Proposition 5.** Log price-dividend ratios on dividend strips are affine in the state vector:

$$p_t^d(h) = \log\left(P_t^d(h)\right) = A^m(h) + B^{m'}(h)z_t,$$

where the coefficients  $A^m(h)$  and  $B^{m'}(h)$  follow recursions:

$$A^m(h+1) = A^m(h) + \mu_m - y_0(1) + \frac{1}{2}(e_{divm} + B^m(h))'\Sigma(e_{divm} + B^m(h)) - (e_{divm} + B^m(h))'\Sigma^{\frac{1}{2}}\left(\Lambda_0 - \Sigma^{\frac{1}{2}}e_{\pi}\right), \quad (\text{A.10})$$

$$B^{m'}(h+1) = (e_{divm} + e_{\pi} + B^m(h))'\Psi - e'_{yn} - (e_{divm} + e_{\pi} + B^m(h))'\Sigma^{\frac{1}{2}}\Lambda_1, \quad (\text{A.11})$$

initialized at  $A_0^m = 0$  and  $B_0^m = 0$ .

*Proof.* We conjecture the affine structure and solve for the coefficients  $A^m(h+1)$  and  $B^{m'}(h+1)$  in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned} P_t^d(h+1) &= \mathbb{E}_t \left[ M_{t+1} P_{t+1}^d(h) \frac{D_{t+1}^m}{D_t^m} \right] = E_t \left[ \exp\{m_{t+1}^{\$} + \pi_{t+1} + \Delta d_{t+1}^m + p_{t+1}^d(h)\} \right] \\ &= \mathbb{E}_t \left[ \exp\left\{-y_{t,1}^{\$} - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + \pi_0 + e'_{\pi}z_{t+1} + \mu^m + e'_{divm}z_{t+1} + A^m(h) + B(h)^{m'}z_{t+1}\right\} \right] \\ &= \exp\{-y_0^{\$}(1) - e'_{yn}z_t - \frac{1}{2}\Lambda_t'\Lambda_t + \pi_0 + e'_{\pi}\Psi z_t + \mu_m + e'_{divm}\Psi z_t + A^m(h) + B(h)^{m'}\Psi z_t\} \\ &\quad \times \mathbb{E}_t \left[ \exp\{-\Lambda_t'\varepsilon_{t+1} + (e_{divm} + e_{\pi} + B^m(h))'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\} \right]. \end{aligned}$$

We use the log-normality of  $\varepsilon_{t+1}$  and substitute for the affine expression for  $\Lambda_t$  to get:

$$P_t^d(h+1) = \exp\{-y_0^{\$}(1) + \pi_0 + \mu_m + A^m(h) + [(e_{divm} + e_{\pi} + B^m(h))'\Psi - e'_{yn}]z_t\}$$

$$\begin{aligned}
& + \frac{1}{2} (e_{divm} + e_\pi + B^m(h))' \Sigma (e_{divm} + e_\pi + B^m(h)) \\
& - (e_{divm} + e_\pi + B^m(h))' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t) \}
\end{aligned}$$

Taking logs and collecting terms, we obtain a log-linear expression for  $p_t^d(h+1)$ :

$$p_t^d(h+1) = A^m(h+1) + B^{m'}(h+1)z_t,$$

where:

$$\begin{aligned}
A^m(h+1) &= A^m(h) + \mu_m - y_0^\$(1) + \pi_0 + \frac{1}{2} (e_{divm} + e_\pi + B^m(h))' \Sigma (e_{divm} + e_\pi + B^m(h)) \\
&\quad - (e_{divm} + e_\pi + B^m(h))' \Sigma^{\frac{1}{2}} \Lambda_0, \\
B^{m'}(h+1) &= (e_{divm} + e_\pi + B^m(h))' \Psi - e'_{yn} - (e_{divm} + e_\pi + B^m(h))' \Sigma^{\frac{1}{2}} \Lambda_1.
\end{aligned}$$

We recover the recursions in (A.10) and (A.11) after using equation (A.2).  $\square$

We define the dividend strip risk premium as:

$$\begin{aligned}
\mathbb{E}_t \left[ r_{t+1}^{d,\$}(h) \right] - y_{t,1}^\$ + \frac{1}{2} V_t \left[ r_{t+1}^{d,\$}(h) \right] &= -Cov_t \left[ m_{t+1}^\$, r_{t+1}^{d,\$}(h) \right] \\
&= (e_{divm} + e_\pi + B^m(h))' \Sigma^{\frac{1}{2}} \Lambda_t
\end{aligned}$$

## B.4 Claim to Future Government Spending and Tax Revenues

This appendix computes  $P_t^r$ , the value of a claim to future tax revenues, and  $P_t^g$ , the value of a claim to future government spending.

### B.4.1 Spending Claim

Nominal government spending growth equals

$$\Delta \log G_{t+1} = \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \zeta_0 + (e_g + e_x + e_\pi)' z_{t+1}. \quad (\text{A.12})$$

We define the (nominal) return on the claim as  $R_{t+1}^g = \frac{P_{t+1}^g}{P_t^g - G_t} = \frac{P_{t+1}^{g,ex} + G_{t+1}}{P_t^{g,ex}}$ , where  $P_t^g$  is the cum-dividend price on the spending claim and  $P_t^{g,ex}$  is the ex-dividend price. A log-linearization of the return expression delivers:

$$r_{t+1}^g = \kappa_0^g + \Delta \log G_{t+1} + \kappa_1^g p g_{t+1} - p g_t. \quad (\text{A.13})$$

where  $pg_t \equiv \log\left(\frac{P_t^{s,ex}}{G_t}\right) = \log\left(\frac{P_t^s}{G_t} - 1\right)$ . The unconditional mean log real stock return is  $r_0^s = E[r_t^s]$ . Let  $\bar{p}\bar{g} = E[pg_t]$  be the unconditional average log ex-dividend price-dividend ratio on the spending claim. The linearization constants  $\kappa_0^s$  and  $\kappa_1^s$  are defined as:

$$\kappa_1^s = \frac{e^{\bar{p}\bar{g}}}{e^{\bar{p}\bar{g}} + 1} < 1 \quad \text{and} \quad \kappa_0^s = \log\left(e^{\bar{p}\bar{g}} + 1\right) - \frac{e^{\bar{p}\bar{g}}}{e^{\bar{p}\bar{g}} + 1} \bar{p}\bar{g}. \quad (\text{A.14})$$

We define the return so that equation (A.13) holds exactly, rather than as an approximation. The unconditional expected return is:

$$r_0^s = x_0 + \pi_0 + \kappa_0^s - \bar{p}\bar{g}(1 - \kappa_1^s).$$

We conjecture that the log ex-dividend price-dividend ratio on the spending claim is affine in the state vector and verify the conjecture by solving the Euler equation for the claim.

$$pg_t = \bar{p}\bar{g} + B'_g z_t \quad (\text{A.15})$$

This allows us to write the return as:

$$r_{t+1}^s = r_0^s + (e_g + e_x + e_\pi + \kappa_1^s B_g)' z_{t+1} - B'_g z_t. \quad (\text{A.16})$$

*Proof.* Starting from the Euler equation:

$$\begin{aligned} 1 &= E_t \left[ \exp\{m_{t+1}^s + r_{t+1}^s\} \right] \\ &= \exp\{-y_0^s(1) - e'_{yn} z_t - \frac{1}{2} \Lambda'_t \Lambda_t + r_0^s + [(e_g + e_x + e_\pi + \kappa_1^s B_g)' \Psi - B'_g] z_t\} \\ &\quad \times E_t \left[ \exp\{-\Lambda'_t \varepsilon_{t+1} + (e_g + e_x + e_\pi + \kappa_1^s B_g)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\} \right]. \end{aligned}$$

We use the log-normality of  $\varepsilon_{t+1}$  and substitute for the affine expression for  $\Lambda_t$  to get:

$$\begin{aligned} 1 &= \exp\{r_0^s - y_0^s(1) + [(e_g + e_x + e_\pi + \kappa_1^s B_g)' \Psi - B'_g - e'_{yn}] z_t \\ &\quad + \underbrace{\frac{1}{2} (e_g + e_x + e_\pi + \kappa_1^s B_g)' \Sigma (e_g + e_x + e_\pi + \kappa_1^s B_g)}_{\text{Jensen}} \\ &\quad - (e_g + e_x + e_\pi + \kappa_1^s B_g)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t)\} \end{aligned}$$



Taking logs and collecting terms, we obtain the following system of equations:

$$r_0^g - y_0^g(1) + Jensen = (e_g + e_x + e_\pi + \kappa_1^g B_g)' \Sigma^{\frac{1}{2}} \Lambda_0 \quad (\text{A.17})$$

and

$$(e_g + e_x + e_\pi + \kappa_1^g B_g)' \Psi - B_g' - e_{yn}' = (e_g + e_x + e_\pi + \kappa_1^g B_g)' \Sigma^{\frac{1}{2}} \Lambda_1 \quad (\text{A.18})$$

Given  $\Lambda_1$ , the system of  $N$  equations (A.18) can be solved for the vector  $B_g$ :

$$B_g = \left( I - \kappa_1^g \left( \Psi - \Sigma^{\frac{1}{2}} \Lambda_1 \right)' \right)^{-1} \left[ \left( \Psi - \Sigma^{\frac{1}{2}} \Lambda_1 \right)' (e_g + e_x + e_\pi) - e_{yn}' \right] \quad (\text{A.19})$$

Plugging (A.19) into equation (A.17) delivers one (non-linear) equation that can be solved numerically for  $\bar{p}g$ .

The left-hand side of this equation is the unconditional expected excess log return with Jensen adjustment. The right hand side is the unconditional covariance of the log SDF with the log return. This equation describes the unconditional risk premium on the claim to government spending. Equation (A.18) describes the time-varying component of the government spending risk premium.  $\square$

## B.4.2 Revenue Claim

Nominal government revenue growth equals

$$\Delta \log T_{t+1} = \Delta \log \tau_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + (e_\tau + e_x + e_\pi)' z_{t+1}. \quad (\text{A.20})$$

where  $\tau_t = T_t/GDP_t$  is the ratio of government revenue to GDP. Note that this ratio is assumed to have a long-run growth rate of zero. This imposes cointegration between government revenue and GDP. The growth ratio in this ratio can only temporarily deviate from zero.

The derivation of the return and (ex-dividend) price-dividend ratio of the claim to government revenue is exactly analogous to that for the spending claim:

$$p\tau_t \equiv \log \left( \frac{P_t^{\tau,ex}}{T_t} \right) = \log \left( \frac{P_t^\tau}{T_t} - 1 \right) = \bar{p}\tau + B_\tau' z_t \quad (\text{A.21})$$

$$r_{t+1}^\tau = r_0^\tau + (e_\tau + e_x + e_\pi + \kappa_1^g B_\tau)' z_{t+1} - B_\tau' z_t. \quad (\text{A.22})$$

with

$$r_0^\tau = x_0 + \pi_0 + \kappa_0^\tau - \bar{p}\tau(1 - \kappa_1^\tau).$$

and

$$\kappa_1^\tau = \frac{e^{\bar{p}\tau}}{e^{\bar{p}\tau} + 1} < 1 \quad \text{and} \quad \kappa_0^\tau = \log(e^{\bar{p}\tau} + 1) - \frac{e^{\bar{p}\tau}}{e^{\bar{p}\tau} + 1} \bar{p}\tau. \quad (\text{A.23})$$

The unknown coefficients  $\bar{p}\tau$  and  $B_\tau$  solve the following system of equations:

$$r_0^\tau - y_0^\$(1) + Jensen = (e_\tau + e_x + e_\pi + \kappa_1^\tau B_\tau)' \Sigma^{\frac{1}{2}} \Lambda_0 \quad (\text{A.24})$$

and

$$(e_\tau + e_x + e_\pi + \kappa_1^\tau B_\tau)' \Psi - B_\tau' - e'_{yn} = (e_\tau + e_x + e_\pi + \kappa_1^\tau B_\tau)' \Sigma^{\frac{1}{2}} \Lambda_1 \quad (\text{A.25})$$

## C Coefficient Estimates

### C.1 VAR Model Without Latent Component in Spending Growth

The companion matrix  $\Psi$  is given in Table A.1. It is estimated by OLS, equation by equation. The t-statistics are reported below the point estimates (in italics). The first four-by-four block of the companion matrix governs the bond market dynamics. It shows substantial diagonal elements (persistence) as well as several non-zero off-diagonal elements. For example, lagged GDP growth and lagged short rates predict the inflation rate, the lagged slope of the yield curve predicts GDP growth, and lagged GDP growth predicts the slope in turn. Lagged revenue-to-GDP growth has little effect on the dynamics of inflation, GDP growth, the short-term interest rate, or the slope of the term structure. The same is true for lagged government spending-to-GDP growth  $\Delta \log g_{t-1}$ . This implies that the zero restrictions we impose in the eight column of rows 1-7 are not very strong.

The next two rows give the dynamics of the stock market variables. As expected, the pd ratio is highly persistent. Lagged inflation and dividend growth predict the valuation ratio with a negative sign. Dividend growth has a quarterly persistence of 0.437, exceeding that of GDP growth of 0.329. Lagged inflation, pd ratio, and government revenue-to-GDP growth predict dividend growth.

The seventh row shows that lagged GDP growth predicts government tax revenue-to-GDP growth. In terms of high growth, government revenues rise as a share of GDP. This is the pro-cyclicality of tax revenues emphasized earlier. Overall, there is only modest

predictability for revenue-to-GDP growth compared to dividend or GDP growth. The eighth row shows that government spending-to-GDP growth is predicted by lag GDP growth and by the lagged slope of the term structure. Since both signs are negative, this indicates the counter-cyclical nature of spending growth. Lagged tax revenue-to-GDP growth also forecasts lower future spending growth. There is very little own-persistence in spending growth after controlling for the other lagged state variables.

Table A.1: VAR Estimates  $\Psi$

	$\pi_{t-1}$	$x_{t-1}$	$y_{t-1}^{\$}(1)$	$yspr_{t-1}^{\$}$	$pd_{t-1}$	$\Delta \log d_{t-1}$	$\Delta \log \tau_{t-1}$	$\Delta \log g_{t-1}$	$R^2$
$\pi_t$	0.542	0.063	0.195	-0.181	-0.0009	0.020	0.003	0	64.5
	13.22	2.63	5.88	1.74	1.56	2.07	0.39	–	
$x_t$	-0.007	0.329	-0.096	0.575	-0.0008	0.009	0.028	0	16.6
	0.07	5.74	1.21	2.30	0.57	0.36	1.34	–	
$y_t^{\$(1)}$	0.045	0.042	0.960	0.032	0.0001	0.003	0.000	0	95.7
	2.49	4.01	66.08	0.70	0.59	0.64	0.03	–	
$yspr_t^{\$}$	-0.015	-0.019	0.018	0.860	-0.0001	-0.003	0.000	0	74.5
	1.19	2.69	1.77	27.23	0.42	0.92	0.18	–	
$pd_t$	-2.076	-0.449	0.363	3.213	0.9636	-0.469	0.191	0	96.2
	2.32	0.86	0.50	1.41	76.56	2.21	1.02	–	
$\Delta \log d_t$	0.530	0.178	-0.258	-0.350	0.0023	0.437	0.131	0	28.8
	2.36	1.36	1.42	0.61	0.74	8.23	2.78	–	
$\Delta \log \tau_t$	-0.473	0.573	0.000	-0.579	-0.0004	0.091	-0.106	0	6.9
	1.61	3.36	0.00	0.78	0.10	1.31	1.73	–	
$\Delta \log g_t$	-0.06	-0.81	0.08	-1.66	-0.00	-0.01	-0.12	-0.03	13.2
	0.21	4.35	0.35	2.24	0.76	0.14	2.03	0.49	

The Cholesky decomposition of the residual variance-covariance matrix,  $\Sigma^{\frac{1}{2}}$ , multiplied by 100 for readability is given by:

$$100 \times \Sigma^{\frac{1}{2}} = \begin{bmatrix} 0.36 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.02 & 0.87 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.02 & 0.04 & 0.15 & 0 & 0 & 0 & 0 & 0 \\ -0.00 & -0.01 & -0.07 & 0.08 & 0 & 0 & 0 & 0 \\ -0.71 & 0.77 & -1.09 & -0.24 & 7.78 & 0 & 0 & 0 \\ 0.09 & 0.09 & 0.03 & -0.16 & -0.33 & 1.95 & 0 & 0 \\ 0.56 & 0.63 & 0.09 & -0.05 & 0.16 & 0.13 & 2.44 & 0 \\ -0.39 & -0.88 & -0.14 & 0.13 & 0.12 & -0.30 & -0.08 & 2.36 \end{bmatrix}$$

## C.2 Market Prices of Risk

### C.2.1 Parameter Estimates

The constant market price of risk vector is estimated at:

$$\Lambda'_0 = [-0.24, 0.25, -0.46, 0.08, 0, 0.81, 0, 0, 0]$$

The time-varying market price of risk matrix is estimated at:

$$\Lambda_1 = \begin{bmatrix} 19.14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 24.10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -25.87 & -181.45 & 0 & 0 & 0 & 0 \\ 31.27 & -6.37 & -0.49 & -5.64 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -37.12 & -22.25 & -43.03 & 36.89 & -1.87 & -3.27 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

### C.2.2 Identification

We allow for first four elements of  $\Lambda_0$  to be non-zero. The market price of the first shock, the inflation shock, will partly capture a shock to expected inflation given the persistence of inflation. Movements in expected inflation are a key determinants of parallel shifts in the term structure of interest rates. i.e., they are the main driver of level of the term structure. Since inflation is usually bad news to the representative agent, we expect a negative price of risk for this shock. Shocks to GDP growth affect the slope of the term structure. They affect long rates more than short rates. We expect a positive price of risk since positive innovations to GDP growth are good news. The third risk price  $\Lambda_0(3)$  is the price of risk for a shock to the interest rate that is orthogonal to inflation and GDP growth shocks. As in the classic term structure models of Vasicek and Cox, Ingersoll, and Ross, we expect this risk price to be negative. We expect the shock to the yield spread that is orthogonal to the preceding three shocks to carry a positive risk price  $\Lambda_0(4)$ , as positive slopes indicate improving economic conditions. This risk price helps the model match the average slope of the term structure.

We allow for eight non-zero elements in the first four rows (term structure block) of  $\Lambda_1$ , which describes the dynamics in the risk prices. We let the price of inflation risk

depend on the level of inflation to capture that periods like the late 1970s, early 1980s may have had elevated inflation risk. We let the price of GDP growth risk depend on the level of GDP growth. We let the price of short rate risk depend on the short rate as well as the term spread. The first dependence is a feature of the Vasicek and Cox, Ingersoll model, for example. The second dependence captures that the slope of the term structure predicts higher future returns on bonds (Capmbell and Shiller). Finally, we need four non-zero elements in the fourth row of  $\Lambda_1$  in order to allow the model to closely match the dynamics of the slope of the term structure, which is one of the variables included in the VAR. The dynamics of the five-year bond yield must satisfy (5). Given the first three rows of  $\Lambda_1$ , satisfying these conditions requires that the first four elements of the fourth row of  $\Lambda_1$  all be no-zero.

## D Kalman Filter

We have the estimates for  $\Psi_{(1:7,1:7)}$ ,  $\Sigma_{(1:7,1:7)}^{\frac{1}{2}}$ ,  $\Lambda_0$ , and  $\Lambda_1$  from the first two steps of the estimation procedure. With those, we can solve for we can solve for  $\overline{p\tau}$  and  $B_\tau$ .

Recall from the main text that we can use (17) to back out a time series  $\hat{\zeta}_t$ :

$$\begin{aligned} \hat{\zeta}_t = & \frac{1}{B_g(9)} \left( \log \left( \frac{T_t(1 + \exp \{ \overline{p\tau} + B'_\tau z_t \}) - \sum_{h=0}^{\infty} P_{t,h}^\$ Q_{t,h}}{G_t} - 1 \right) \right. \\ & \left. - \overline{p\bar{g}} - B_g(1, \dots, 8)' z_t(1, \dots, 8) \right) \end{aligned} \quad (\text{A.26})$$

We treat  $\hat{\zeta}_t$  as a noisy signal of the true latent variable  $\zeta_t$ . The observation equation in the Kalman filtering problem is:

$$\hat{\zeta}_t = \zeta_t + \omega \varepsilon_t^{obs}, \quad (\text{A.27})$$

where  $\varepsilon_t^{obs}$  is an i.i.d. standard normal variable. The state evolution equation in the Kalman filtering problem is:

$$\zeta_{t+1} = \Psi_{(9,7)} \Delta \log \tau_t + \Psi_{(9,8)} \Delta \log g_t + \Psi_{(9,9)} \zeta_t + \Sigma_{(9,\cdot)}^{\frac{1}{2}} \varepsilon_{t+1} \quad (\text{A.28})$$

where  $\Delta \log \tau_t$  and  $\Delta \log g_t$  are observed variables. The parameters to estimate are  $\Theta = (\Psi_{(8,1:8)}, \Psi_{(9,7:9)}, \Sigma_{(8,1:8)}^{\frac{1}{2}}, \Sigma_{(9,1:9)}^{\frac{1}{2}}, \omega)$ .

The Kalman Filter allows us to back out the a priori state estimate  $\zeta_{t|t-1}$ , the a posterior state estimate  $\zeta_{t|t}$ , and the a posterior variance estimate  $P_{t|t}$ . The former is the best prediction of  $\zeta_t$  condition on information at  $t - 1$ , and the latter is the best prediction of  $\zeta_t$  condition on information at  $t$ .

Then, we form the residuals. For the first 7 variables, their residuals are from the VAR

$$\mathbf{e}_t(1, \dots, 7) = \mathbf{z}_t(1, \dots, 7) - \mathbf{\Psi}(1, \dots, 7; 1, \dots, 7) \mathbf{z}_{t-1}(1, \dots, 7) \quad (\text{A.29})$$

For the last two variables  $\Delta \log g_t$  and  $\zeta_t$ , their residuals depend on the a priori state estimate:

$$\begin{aligned} \mathbf{e}_t(8) &= \Delta \log g_t - \mathbf{\Psi}(8; 1, \dots, 7) \mathbf{z}_{t-1}(1, \dots, 7) - \mathbf{\Psi}(8; 8) \Delta \log g_{t-1} - \zeta_{t|t-1}, \\ \mathbf{e}_t(9) &= \hat{\zeta}_t - \zeta_{t|t-1} \\ &= \hat{\zeta}_t - \mathbf{\Psi}_{(9,9)} \zeta_{t-1|t-1} - \mathbf{\Psi}_{(9,7)} \Delta \log \tau_{t-1} - \mathbf{\Psi}_{(9,8)} \Delta \log g_{t-1}. \end{aligned}$$

where in the last equation for  $\mathbf{e}_t(9)$  we combined the observation equation (A.27) with the state equation (A.28).

We search over the parameter space to maximize the log likelihood of the residual, defined as

$$\mathcal{L} = \sum_t -\frac{1}{2} \log |\mathbf{\Omega}_t| - \frac{1}{2} \mathbf{e}_t' \mathbf{\Omega}_t^{-1} \mathbf{e}_t. \quad (\text{A.30})$$

The residual vector  $\mathbf{e}_t$  is a combination of the shocks in state transition and the observation noise. By the standard Kalman Filter formula, the covariance matrix  $\mathbf{\Omega}_t$  is the sum of the covariance matrix in the state transition equation and the variance in the observation equation:

$$\mathbf{\Omega}_t = \mathbf{\Sigma} + \begin{pmatrix} 8 \text{ by } 9, \text{ all } 0 \\ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \omega^2 + \mathbf{\Psi}_{(9,9)}^2 P_{t-1|t-1}. \end{pmatrix} \quad (\text{A.31})$$

This procedure evaluates the joint distribution of all shocks  $\varepsilon_{t+1}$ , not just the shock to  $\zeta_{t+1}$ . To attain a high likelihood, the shock to  $\zeta_{t+1}$  should be correlated with the other shocks in a way consistent with the covariance matrix  $\mathbf{\Sigma}$ . Accordingly, we are running a big optimization over the parameter space  $\Theta$ .

## E Immunization

To immunize against all shocks, we construct a replicating bond portfolio for the surplus claim. For the G-claim, we can approximate the change in the valuation of the spending claim as:

$$\begin{aligned}
P_{t+1}^g - P_t^g &\approx G_t \Delta \log G_{t+1} + G_t \exp \left\{ \overline{p\bar{g}} + B'_g z_t \right\} (\Delta \log G_{t+1} + B'_g \Delta z_{t+1}) \\
&= G_t (x_0 + \pi_0 + \bar{\zeta} + (e_g + e_x + e_\pi)' (\Psi z_t + \Sigma^{\frac{1}{2}} \varepsilon_{t+1})) + G_t \exp \left\{ \overline{p\bar{g}} + B'_g z_t \right\} \\
&\quad \times \left( x_0 + \pi_0 + \bar{\zeta} + \left[ (e_g + e_x + e_\pi)' \Psi - B'_g (I - \Psi) \right] z_t + (B_g + e_g + e_x + e_\pi)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1} \right) \\
&= P_t^g (x_0 + \pi_0 + \bar{\zeta} + (e_g + e_x + e_\pi)' (\Psi z_t + \Sigma^{\frac{1}{2}} \varepsilon_{t+1})) \\
&\quad + G_t \exp \left\{ \overline{p\bar{g}} + B'_g z_t \right\} \left( B'_g (\Psi - I) z_t + B'_g \Sigma^{\frac{1}{2}} \varepsilon_{t+1} \right).
\end{aligned}$$

Similarly, we can approximate the change in the valuation of the T-claim, assuming a constant tax revenue-to-GDP ratio, as:

$$\begin{aligned}
P_{t+1}^\tau - P_t^\tau &\approx P_t^\tau (x_0 + \pi_0 + \tau_0 + (e_\tau + e_x + e_\pi)' (\Psi z_t + \Sigma^{\frac{1}{2}} \varepsilon_{t+1})) \\
&\quad + T_t \exp \left\{ \overline{p\bar{\tau}} + B'_\tau z_t \right\} \left( B'_\tau (\Psi - I) z_t + B'_\tau \Sigma^{\frac{1}{2}} \varepsilon_{t+1} \right).
\end{aligned}$$

After collecting terms, we can state the change in the valuation of the surplus claim as:

$$(P_{t+1}^\tau - P_t^\tau) - (P_{t+1}^g - P_t^g) \approx a_t^s + b_t^s \Sigma^{\frac{1}{2}} \varepsilon_{t+1}$$

Next, we compute the sensitivity of the nominal bond price to the state variables, for a generic bond of maturity  $k$  quarters:

$$\log P_{t+1}^\$(k) - \log P_t^\$(k+1) = A^\$(k) - A^\$(k+1) + \left[ (B^\$(k))' \Psi - (B^\$(k+1))' \right] z_t + (B^\$(k))' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}.$$

Hence, we can approximate the change in the price of the bond as:

$$\begin{aligned}
\left( P_{t+1}^\$(k) - P_t^\$(k+1) \right) &\approx P_t^\$(k+1) \left( A^\$(k) - A^\$(k+1) \right) + P_t^\$(k+1) \left[ \left( (B^\$(k))' \Psi - (B^\$(k+1))' \right) z_t \right. \\
&\quad \left. + P_t^\$(k+1) (B^\$(k))' \Sigma^{\frac{1}{2}} \varepsilon_{t+1} \right].
\end{aligned}$$

We can state the latter, collecting terms, as:

$$\left( P_{t+1}^\$(k) - P_t^\$(k+1) \right) = P_t^\$(k+1) a_t^\$(k) + P_t^\$(k+1) B^\$(k) \Sigma^{\frac{1}{2}} \varepsilon_{t+1}$$

where

$$a_t^\$(k) = \left( A^\$(k) - A^\$(k+1) \right) + \left[ ((B^\$(k))' \Psi) - (B^\$(k+1))' \right] z_t$$

The sensitivity of real bonds takes the exact same expression, except without the dollar superscripts:

$$(P_{t+1}(k) - P_t(k+1)) = P_t(k+1)a_t(k) + P_t(k+1)B(k)\Sigma^{\frac{1}{2}}\varepsilon_{t+1}$$

where

$$a_t(k) = (A(k) - A(k+1)) + [((B(k))' \Psi) - (B(k+1))'] z_t$$

Let  $Q_{t,k}^\$$  ( $Q_{t,k}$ ) denote the observed position in the  $k$ -quarter nominal (real) zero coupon bond in the data. If the government is immunizing the risk exposure of its funding shock according to BEGS, the dollar exposure of the government bond portfolio to each shock should equal the dollar exposure of the surplus claim:

$$\begin{aligned} \sum_{k=0}^K Q_{t,k+1}^\$ P_t^\$(k+1) a_t^\$(k) + \sum_{k=0}^K Q_{t,k+1} P_t(k+1) a_t(k) &= a_t^s \\ \sum_{k=0}^K Q_{t,k+1}^\$ P_t^\$(k+1) B^\$(k) + \sum_{k=0}^K Q_{t,k+1} P_t(k+1) B(k) &= b_t^s \end{aligned}$$

We quantify the differences between the left-hand side and the right-hand side.