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Calculating the Equilibria of Heterogeneous-Firm Trade Models*

Timothy J. Kehoe

University of Minnesota,
Federal Reserve Bank of Minneapolis,
and National Bureau of Economic Research

Pau S. Pujolas

McMaster University

Kim J. Ruhl

University of Wisconsin
and National Bureau of Economic Research

ABSTRACT

We develop a family of simple algorithms for analytically calculating the interior equilibria of international trade models with monopolistic competition, heterogeneous firms, increasing returns to scale, and a homogeneous outside good. Variants of the methods handle models with costly entry and models with a fixed number of firms, even when countries are of different sizes, firms face heterogeneous fixed costs, and trade costs and the distributions of firm efficiency are different across countries. The methods reduce to inverting an *n*-by-*n* matrix, where *n* is the number of countries.

Keywords: International trade; Computation of equilibrium; Monopolistic competition; Heterogeneous efficiencies; Heterogeneous fixed costs.

JEL Codes: C62, F12, F17.

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1. Introduction

For more than a decade, models with taste-for-variety preferences, increasing returns to scale in production, and firm heterogeneity have proliferated in the international trade literature. Firm heterogeneity is a key feature of these models, and the two most common approaches to incorporating firm heterogeneity are: either there exists an unbounded mass of potential firms who may pay a cost to enter the market (Melitz, 2003), or there exists a fixed mass of firms with already known efficiency levels, ready to operate (Chaney, 2008).

In this paper, we consider models with two sectors. The first sector is characterized by differentiated products, firm heterogeneity, and fixed export costs. This is the sector of interest. The second sector produces a freely traded, homogeneous *outside good*, using a constant returns to scale production function, which is sold in a perfectly competitive market. The outside-good sector pins down the wage, greatly simplifying the model's solution. This two-sector formulation eliminates the terms of trade effects from changes in trade policy, isolating the effects that work through the reallocation of labor in the differentiated-good sector.

In this framework, it is possible that, in equilibrium, a country will only produce the outside good and trade it for differentiated goods. This type of model is not commonly used in the trade literature, and hence, we do not consider it in this paper. Rather, our focus is on the *interior equilibrium*, in which every country produces both the outside good and some differentiated goods. Our central result is a family of simple methods for analytically calculating these interior equilibria that only requires the inversion of an n-by-n matrix, where n is the number of countries.

We begin by building two models of international trade and firm heterogeneity. In the *fixed-firms* model, there is a fixed number of potential firms; in the *costly-entry* model, it is costly for potential firms to enter the market. The models are quite flexible: we allow for heterogeneous fixed production and export costs across firms, heterogeneous trade costs across countries, and differences in country sizes. In both versions, we focus on the case where the distribution of firm efficiencies is Pareto.

We show that, if it exists, the interior equilibrium in each model is the solution to an n-by-n system of equations. In the fixed-firms model, the system of equations only depends on the vector of profits in each country. Moreover, the system of equations is such that we can invoke the Perron-Frobenius theorem (Gantmacher, 1959, McKenzie, 1959) to establish that the vector of profits is positive. In the costly-entry model, the system of equations only depends on the vector of the masses of firms operating in each country. Unfortunately, the system of equations is such that no similar theorem to the Perron-Frobenius one exists, and the solution may be negative. We establish conditions regarding the solution of the model that, if satisfied, imply that the computed equilibrium is the interior equilibrium — and if not satisfied, the interior equilibrium does not exist. Being able to invoke the Perron-Frobenius theorem implies that the fixed-firms model has to satisfy one condition less — namely, that profits are positive.

When might an interior solution fail to exist? Consider a model in which, in a particular country, the outside-good technology is much less productive than the technologies used by differentiated-good firms. In this case, it is possible that the outside-good sector will not produce, and the outside good will be imported for consumption.

The two approaches to introducing firms into the model are very different in nature. In the fixed-firms model, potential firms already know their efficiency level when deciding whether to enter the market. This formulation implies that, in equilibrium, there are positive profits rebated to consumers and the mass of potential firms is a parameter of the model. In the costly-entry model, there are many potential firms ready to enter the market, but they do not know how efficient they will be after entry. To learn their efficiency level, they have to pay a fixed cost. Firms compare the expected profits from an efficiency draw to the fixed cost, and in equilibrium, the two are equalized. Costly entry typically implies that aggregate profits are zero, and the mass of potential firms drawing efficiencies is an equilibrium object.

Both versions of the international trade model are extensions of earlier closed-economy models of firm heterogeneity. The fixed-firms model, first published in Chaney (2008), is an interpretation of the Lucas (1978) span-of-control model (see Kehoe, Pujolas, and Ruhl, 2016). In this interpretation, the fixed costs that firms incur to produce in the fixed-firms model are the entrepreneurs' forgone wages in the span-of-control model. The costly-entry model, first published in Melitz (2003), is a version of the firm structure in Hopenhayn (1992).

In spirit, our paper is closely related to Allen, Arkolakis, and Li (2016), which also proposes a method for finding the solution to a multi-country trade model with rich heterogeneity. The goal of Allen et al. (2016) is to develop a method applicable to a broad class of models, while our focus is on a specific class of models. The benefit of our narrower scope is that our method, compared to theirs, is simpler and easier to implement. In addition, Allen et al. (2016)

do not consider models with an outside sector, so their method does not need to differentiate between interior equilibria and equilibria with corner solutions.

2. The fixed-firms model

In each country $i=1,...,n$, there is a mass of potential firms, μ_i, each of which is endowed with an efficiency level, x, who decide whether to enter a market j. In contrast to the model in Section 3, the potential firms know their efficiency levels when they make their entry decisions.

Each country has a representative consumer that inelastically supplies $\overline{\ell}_i$ units of labor to the firms and, as the owner of the firms, receives the firms' profits, π_i. The consumer in country i chooses consumption of the outside good, c_{0i}, and differentiated goods, $c_i(\omega)$, to solve

$$\begin{aligned} &\max\ (1-\alpha)\log c_{0i} + (\alpha/\rho)\int_0^{m_i} \log c_i(\omega)^\rho \, d\omega \\ &\text{s.t. } p_0 c_{0i} + \int_0^{m_i} p_i(\omega) c_i(\omega)\, d\omega = w_i \overline{\ell}_i + \pi_i \end{aligned} \tag{1}$$

where $\alpha \in (0,1)$ governs the importance of differentiated goods relative to the outside good; $p_0 > 0$ is the outside-good price; $\rho \in (0,1)$ governs the elasticity of substitution between differentiated varieties; $m_i > 0$ is the measure of differentiated varieties consumed in i; $p_i(\omega) > 0$ is the price of differentiated good ω; and $w_i > 0$ is the wage.

There are two types of firms in this economy: firms producing the outside good, and firms producing differentiated goods. The outside-good market is perfectly competitive, and producers operate a constant returns to scale production function that is identical across countries, $y_0 = \ell_0$. The solution to the outside-good firm's problem is

$$p_0 = w_i\,. \tag{2}$$

We take the price of the outside good to be the numéraire good and set $p_0 = 1$.

Differentiated-good firms are monopolistic competitors. The differentiated-good firm in market i, producing good ω, for sale in market j, operates the production technology $y_{ij} = x(\omega)\ell_{ij}$, where $x(\omega) \in [\underline{x}, \infty)$ is its efficiency, which is independent of the destination market. The firm faces a fixed operating cost $\kappa_{ij}(\omega) \in [\underline{\kappa}, \infty)$ for each market j it serves. The fixed cost is paid in units of labor. A firm producing variety ω is characterized by its efficiency, $x(\omega)$, and its vector of fixed costs, $\{\kappa_{ij}(\omega)\}_{j=1}^{n}$. Production efficiency and the operating fixed costs are independently distributed across firms and markets.

As in Chaney (2008) and much of the literature that followed, we assume that firm efficiencies are drawn from a Pareto distribution with CDF $G_i(x) = 1 - \underline{x}_i x^{-\gamma}$. The parameter $\underline{x}_i > 1$ is the minimum efficiency level and $\gamma > 2$ governs the shape of the distribution. We require that $\gamma > \rho/(1-\rho)$. The fixed operating cost for a firm in country i to sell to country j is drawn from a distribution with CDF $F_{ij}(\kappa)$. The functional form for F_{ij} can be quite general. Below, in equation (12), we show that the only requirement is that it has a well-defined integral.

When a firm from country i sells to country j, it must ship $\tau_{ij} \geq 1$ units of the good in order for one unit to arrive in the destination. We assume that $\tau_{ii} = 1$.

Differentiated-good firms produce with a constant marginal cost, so the maximization problem of a firm can be separated across destination markets. The firm takes as given the residual demand for its good and chooses its price to maximize profits

$$\begin{aligned} &\max\ \pi_{ij}(\omega) = \max\{0, p_{ij}(\omega)c_{ij}(\omega) - w_i \ell_{ij}(\omega) - w_i \kappa_{ij}(\omega)\} \\ &\text{s.t. } x(\omega)\ell_{ij}(\omega) = \tau_{ij} c_{ij}(\omega). \end{aligned} \tag{3}$$

Equilibrium requires that the labor market and the goods markets, clear,

$$\sum_{j=1}^{n} \int_{0}^{m_j} (\ell_{ij}(\omega) + \kappa_{ij}(\omega))\, d\omega + \ell_{0i} = \overline{\ell}_i, \text{ for } i = 1,\ldots,n, \tag{4}$$

$$x(\omega)\ell_{ij}(\omega) = \tau_{ij} c_{ij}(\omega), \text{ for } i = 1,\ldots,n; \omega \in [0, m_i], \tag{5}$$

$$\sum_{i=1}^{n} c_{0i} = \sum_{i=1}^{n} \ell_{0i}. \tag{6}$$

2.1 Equilibrium

We begin by defining an equilibrium and characterizing its properties. We then define and analyze the interior equilibrium of the fixed-firms model.

Definition: *Given a vector of labor endowments* $\{\overline{\ell}_i\}_{i\in n}$, *a vector of potential firms* $\{\mu_i\}_{i\in n}$, *a matrix of iceberg costs* $\{\tau_{ij}\}_{i\in n, j\in n}$, *and distributions of efficiencies and of fixed costs* $\{G_i(x), \{F_{ij}(\kappa)\}_{j\in n}\}_{i\in n}$, ***an equilibrium*** *is prices* $\{\{p_{ij}(\omega)\}_{j\in n;\omega\in m_i}, w_i, p_{0i}\}_{i\in n}$, *and quantities* $\{\{c_{ij}(\omega), \ell_{ij}(\omega), y_{ij}(\omega), \pi_{ij}(\omega)\}_{j\in n;\omega\in m_i}, m_i, y_{0i}, \ell_{0i}, c_{0i}\}_{i\in n}$ *such that:*

1) *the household solves the problem defined in* (1)
2) *the outside-good price satisfies* (2),

3) *differentiated-good firms solve the problem defined in* (3), *and*
4) *feasibility and market clearing,* (4), (5), *and* (6) *hold.*

The solution to the household problem leads to the well-known CES demand functions for differentiated goods,

$$c_{ij}(\omega) = \alpha(w_j \overline{\ell}_j + \pi_j) p_{ij}(\omega)^{\frac{-1}{1-\rho}} P_j^{\frac{\rho}{1-\rho}}, \tag{7}$$

where

$$P_j = \left(\int_0^{m_i} p_{ij}(\omega)^{\frac{-\rho}{1-\rho}} d\omega \right)^{\frac{1-\rho}{-\rho}} \tag{8}$$

is the aggregate price index for differentiated goods.

Firms take the demand for their good as given when they solve their profit maximization problem, which implies that they set prices equal to a markup over marginal cost,

$$p_{ij}(\omega) = \frac{1}{\rho} \frac{w_i}{x(\omega)} \tau_{ij} . \tag{9}$$

For clarity, we have been explicitly including the wage in our expressions. It follows from (2) and from our choice of numeraire, however, that the wage in each country is one. In what follows, we suppress the wage.

Since all firms with efficiency x and fixed cost κ_{ij} behave identically, and varieties enter in utility symmetrically, we no longer characterize a good by its label, ω, but by its efficiency-fixed cost pair (x, κ_{ij}). The profit of a firm in country i selling to country j is

$$\pi_{ij}(x, \kappa_{ij}) = (1-\rho)\alpha(\overline{\ell}_j + \pi_j) P_j^{\frac{\rho}{1-\rho}} \left(\frac{\tau_{ij}}{x\rho} \right)^{\frac{-\rho}{1-\rho}} - \kappa_{ij} . \tag{10}$$

We characterize an equilibrium in which there exists, for each country pair, a cut-off function $\hat{x}_{ij}(\kappa_{ij})$ that determines the set of firms in i that are indifferent to selling in market j. The cut-off function is defined as the level of efficiency such that a firm with efficiency $\hat{x}_{ij}$ and fixed cost κ_{ij} makes zero profits from selling in j, $\pi_{ij}(\hat{x}_{ij}, \kappa_{ij}) = 0$. Using (10), we obtain that this condition implies a cut-off of

$$\hat{x}_{ij}(\kappa_{ij}) = \frac{\tau_{ij}}{\rho P_j}((1-\rho)\alpha(\overline{\ell}_j + \pi_j))^{\frac{1-\rho}{-\rho}} \kappa_{ij}^{\frac{1-\rho}{\rho}}. \tag{11}$$

We are interested in parameterizations in which the fixed costs are large enough to ensure that this cut-off function is always greater than the minimum efficiency level, $\hat{x}_{ij}(\underline{\kappa}_{ij}) > \underline{x}_i$. This cut-off function determines the measure of varieties operating in a given country, since

$$m_i = \sum_{j=1}^{n} \int_{\hat{\kappa}_{ij}} \int_{\hat{x}_{ij}(\kappa_{ij})} dG_i(x)\, dF_{ij}(\kappa_{ij}). \tag{12}$$

Notice that the cut-off condition, (11), is expressed in terms of only two endogenous variables: the aggregate price index and the aggregate profits. Substituting the firm's pricing rule, (9), into the aggregate price index, we have

$$P_j = \Lambda\left(\sum_{k=1}^{n} \mu_k \underline{x}_k^{\gamma} \tau_{kj}^{-\gamma} \tilde{\kappa}_{kj}^{\frac{\rho-\gamma(1-\rho)}{\rho}} \right)^{\frac{-1}{\gamma}} (\overline{\ell}_j + \pi_j)^{\frac{-\gamma(1-\rho)+\rho}{\gamma\rho}}, \tag{13}$$

where $\tilde{\kappa}_{ij} = \left(\int_{\underline{\kappa}_{ij}}^{\infty} \kappa^{\frac{\rho-\gamma(1-\rho)}{\rho}} dF_{ij}(\kappa) \right)^{\frac{\rho}{\rho-\gamma(1-\rho)}}$ and $\Lambda = \rho^{-1}\left(1-\rho\gamma^{-1}(1-\rho)^{-1}\right)^{\frac{1}{\gamma}} (\alpha(1-\rho))^{\frac{\rho-\gamma(1-\rho)}{\gamma\rho}}$.

Notice that every endogenous variable in the model either depends on one endogenous variable, aggregate profits, or does not depend on any endogenous variables. Once we have solved for the aggregate profits, it is simple to recover the entire equilibrium. Aggregate profits in country i are the sum of the profits of each firm owned by country i operating in each market j. After substituting (13) into (10) and summing over j, the aggregate profit in country i is

$$\pi_i = \sum_{j=1}^{n} \int_{\hat{\kappa}_{ij}} \int_{\hat{x}_{ij}} \pi_{ij}(x,\kappa_{ij})\, dG_i(x)\, dF_{ij}(\kappa_{ij}) = \frac{\alpha\rho}{\gamma} \sum_{j=1}^{n} \frac{(\overline{\ell}_j + \pi_j)\mu_i \underline{x}_i^{\gamma} \tau_{ij}^{-\gamma} \tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}}}{\sum_{k=1}^{n} \mu_k \underline{x}_k^{\gamma} \tau_{kj}^{-\gamma} \tilde{\kappa}_{kj}^{\frac{\rho-\gamma(1-\rho)}{\rho}}}. \tag{14}$$

This expression for aggregate profit is key for our results. Aggregate profit in country i is a function of the aggregate profit in every country and model parameters. Before we analyze the equilibrium properties implied by (14), we define an interior equilibrium.

Definition: *An **interior equilibrium** is an equilibrium in which:*

1) *The outside good is produced in each country:* $0 < y_{0i}$, *for* $i = 1,...,n$,
2) *For any realization of fixed costs, the measure of firms that choose not to operate is strictly positive:* $\hat{x}_{ij}(\underline{\kappa}_{ij}) > \underline{x}_i$, *for* $i = 1,...,n$ *and* $j = 1,...,n$, *and*

3) *Some of the differentiated good is produced in each country: there exists an* x *and a* j *such that* $y_{ij}(x) > 0 \;\; i = 1,...,n$.

An interior equilibrium is an equilibrium in which, in each country, the outside good and some differentiated goods are produced and some firms do not operate. This equilibrium contrasts with the two corner-solution equilibria that the model may also deliver. The first is an equilibrium in which a set of countries do not produce the outside good. The second is an equilibrium in which every potential differentiated-good firm operates: for at least one market, the cut-off efficiency is the lower bound of the efficiency distribution.

There is, however, one type of corner-solution equilibrium that does not arise in the fixed-firms model: one in which a country does not produce any differentiated goods. This is a result of the specific-factors nature of the model. In the fixed-firms model, countries are endowed with a mass of potential firms that can only be used to produce differentiated goods, so some of this factor is always employed. In contrast, this equilibrium is possible in the costly-entry model that we develop in Section 3.

We are now prepared to compute an interior equilibrium. To do so, arrange the profit functions defined in (14) into a system of n equations and n unknowns, π_i,

$$\pi_i - \lambda \sum_{j=1}^{n} \frac{\theta_{ij}}{\Theta_j} \pi_j = \lambda \sum_{j=1}^{n} \frac{\theta_{ij}}{\Theta_j} \bar{\ell}_j, \;\; for \;\; i = 1,...,n, \tag{15}$$

where $\lambda = \alpha\rho / \gamma$, $\theta_{ij} = \mu_i \underline{x}_i \tau_{ij}^{-\gamma} \tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}}$, and $\Theta_j = \sum_{k=1}^{n} \theta_{kj}$. Notice that the coefficients on profits, $\lambda\theta_{ij} / \Theta_j$, are strictly positive. Writing this system of equations in matrix form, the matrix that pre-multiplies the profit vector consists of the identity matrix minus a matrix whose column sums are less than one. This system of equations satisfies the assumptions needed to apply the Perron-Frobenius theorem (Gantmacher, 1959, McKenzie, 1959), which implies that the solution to (15) is a unique profit vector whose elements are each strictly positive.

Proposition 1: *Any interior equilibrium is the unique solution to* (15).

Proof: See Appendix.

Proposition 1 establishes that an interior equilibrium — if it exists — is the solution to (15) and provides a simple way to compute it. It remains to verify that the equilibrium satisfies the two conditions laid out in the interior equilibrium definition.

When will the fixed-firms model admit an interior equilibrium? Unfortunately, there is no simple condition on model parameters that guarantees an interior equilibrium. One approach is to compute the equilibrium associated with the solution from (15) and check to see if the equilibrium is interior. It is possible, however, to construct conditions on the solution to (15), which directly confirm whether the solution is an interior equilibrium. These conditions are summarized in Proposition 2.

Proposition 2: *If, for all countries i, profits satisfy*

$$\pi_i < \frac{\rho}{\gamma-\rho}\bar{\ell}_i \tag{16}$$

and

$$\pi_i < \frac{\gamma}{\alpha(\gamma(1-\rho)-\rho)}\left(\sum_{k=1}^{n}\mu_k\left(\frac{\underline{x}_k}{\underline{x}_j}\right)^{\gamma}\left(\frac{\tau_{ki}}{\tau_{ji}}\right)^{-\gamma}\tilde{\kappa}_{ki}\left(\frac{\tilde{\kappa}_{ki}}{\underline{\kappa}_{ji}}\right)^{\frac{\gamma(1-\rho)}{\rho}}\right)-\bar{\ell}_i \quad j=1,\dots,n \tag{17}$$

then the vector of profits that satisfies equation (15) *is an interior equilibrium.*
Proof: See Appendix.

Besides providing a simple way to determine whether a candidate equilibrium is interior, conditions (16) and (17) provide intuition for why an interior solution may not exist. The first condition means that profits have to be small compared to the total labor endowment. The smaller is $\rho \in (0,1)$, and the larger is γ, the harder this condition is to satisfy. A smaller ρ means that differentiated goods are less substitutable and yield higher profits. Similarly, a higher γ implies there are fewer (successful) differentiated-good firms competing against each other, which yields higher profits. Higher profits in the differentiated-goods sector induces a larger allocation of labor towards the differentiated-goods sector, pushing the economy towards an equilibrium in which no labor is allocated to outside-good production.

The second condition is more complex, but, in general, it requires that household spending is small enough that the household in i will not purchase from firms in j with the minimum efficiency level. To see this, rearrange the conditions, to yield

$$\alpha(\pi_i + \overline{\ell}_i) < \frac{\gamma}{\gamma(1-\rho)-\rho}\left(\sum_{k=1}^{n}\mu_k\left(\frac{\underline{x}_k}{\underline{x}_j}\right)^{\gamma}\left(\frac{\tau_{ki}}{\tau_{ji}}\right)^{-\gamma}\tilde{\kappa}_{ki}\left(\frac{\tilde{\kappa}_{ki}}{\underline{\kappa}_{ji}}\right)^{\frac{\gamma(1-\rho)}{\rho}}\right) \quad j = 1,\ldots,n \ . \qquad (18)$$

The left-hand side of (18) is spending on the differentiated-good sector. The lower is the share of income spent on differentiated goods (α) the easier it will be to satisfy these conditions. The first term on the right-hand side of (18) is increasing in $\rho \in (0,1)$ and decreasing in γ. A larger ρ implies smaller markups and a smaller γ yields a larger mass of relatively efficient firms, both of which make it more difficult for the least-efficient firms to operate.

The second term on the right-hand side of (18) is a type of "multilateral resistance" term, common in multi-country models, that compares the competitiveness of country j in serving i to every other country in the model. The better ($\underline{x}_s$) and larger (μ_s) is the pool of potential firms in other countries, and the lower are trade costs (τ_{si}) from other countries, the larger is the multilateral resistance and the more difficult it will be for the least-efficient firm to operate. The $\tilde{\kappa}_{si}$ summarize the size of the fixed costs facing firms in country s selling to country i. When fixed costs are large, relatively more productive firms serve country i, increasing the difficulty for the least-efficient firm from country j to serve i. Lastly, since κ and x are independently distributed, a smaller minimum fixed cost ($\underline{\kappa}_{ij}$) will make it easier for a least-efficient firm to operate.

3. The costly-entry model

In this section, we study a model that is identical to the one in Section 2, except for the way that new firms are created. After laying out the model, we define an interior equilibrium, and show how to characterize its solution.

The only difference between the model in this section and the model in the previous section is that we no longer assume a fixed mass of potential firms, μ_i. In this model, a potential firm must pay an entry cost, $\phi_i > 0$, denoted in units of labor, to draw an efficiency and a vector of fixed costs. The household's problem, (1), the outside-good firms' problem, (2), the differentiated-good firm's problem conditional on operating, (3), and the market clearing conditions, (4), (5), and (6) remain unchanged.

A potential firm will draw from the efficiency and fixed cost distributions as long as the expected profit from doing so is greater than the entry cost. The entry condition is

$$\sum_{i=1}^{n}\int_{\underline{\kappa}_{ij}}^{\infty}\left(\int_{\hat{x}_{ij}(\kappa)}^{\infty}\left(p_{ij}(x_{ij})c_{ij}(x_{ij})-\frac{c_{ij}(x_{ij})}{x_{ij}}-\kappa_{ij}\right)\gamma\underline{x}_i^{\gamma}x^{-\gamma-1}dx\right)dF_{ij}(\kappa)-w_i\phi_i\geq 0. \tag{19}$$

Note that (19) implies that firms have to pay a cost to enter the market. Some scholars refer to this condition as the free entry condition, but since it is costly for firms to enter, we refer to it as the costly entry condition.

The mass of potential firms that pay ϕ to learn about their potential profitability is μ_i. In the fixed-firms model, μ_i was exogenous; in this model, μ_i is endogenous. In an equilibrium with differentiated-good production in country i, μ_i is strictly positive and (19) will hold with equality. Multiplying (19) by μ_i yields aggregate profits, which are zero,

$$\pi_i = 0. \tag{20}$$

Notice that we have added an endogenous variable to this model relative to the fixed-firms model — the number of potential firms — but, as long as $\mu_i > 0$, aggregate profits are zero. Rather than solve for n aggregate profit levels, in this model, we need to solve for n masses of potential firms.

An interior equilibrium in this environment is the same as in the fixed-firms model: every country produces the outside good and some differentiated goods, and the least-efficient differentiated-good technology is not used.

Next, we demonstrate how to find an interior equilibrium. In the fixed-firms model, we showed that the endogenous variables were simply functions of aggregate profits. In the costly-entry model, following a similar logic, we can show that the endogenous variables are functions of the masses of potential firms, μ_i. Using the same reasoning that we followed in Proposition 1, the equilibrium in the costly-entry model can be summarized as follows.

Proposition 3: *Any interior equilibrium is the solution to this system of equations:*

$$\frac{\alpha\rho}{\gamma}\sum_{j=1}^{n}\frac{\overline{\ell}_j \underline{x}_i^{\gamma}\tau_{ij}^{-\gamma}\tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}}}{\sum_{k=1}^{n}\mu_k \underline{x}_k^{\gamma}\tau_{kj}^{-\gamma}\tilde{\kappa}_{kj}^{\frac{\rho-\gamma(1-\rho)}{\rho}}}=\phi_i \text{ for } i=1,\ldots,n\,. \tag{21}$$

Proof: See Appendix.

In contrast to the system of equations in Proposition 1, the system of equations in (21) is nonlinear. It is also the case that some of the μ_i may be negative — so there is no theorem that we can invoke to guarantee that some differentiated goods are produced in equilibrium. This means that the costly-entry model can fail to produce an interior equilibrium for different reasons than the fixed-firms model.

Both models may admit an equilibrium in which none of the outside good is produced. The two models differ in the behavior of the differentiated-goods sector. In the fixed-firms model, some differentiated goods are always produced, but it is possible that the least-efficient technology is operated. In the costly-entry model, the least-efficient firm never operates, but it is possible that no differentiated-good firm operates. In an equilibrium where some differentiated-goods are produced, it must be that aggregate profit is zero. If aggregate profit is zero, then some differentiated-good firms must earn negative profits, and by the monotonicity of profits in x, the least-efficient technology will not operate.

When will the costly-entry model admit an interior equilibrium? As we did in Proposition 2, we can construct conditions on the vector of potential firms that need to be satisfied for the equilibrium to be interior.

Proposition 4: *If, for all countries i, the masses of potential firms satisfy*

$$0<\mu_i<\frac{\rho}{(\gamma-\rho)\phi}\overline{\ell}_i \tag{22}$$

and

$$\mu_i>\overline{\ell}_j\frac{\alpha(\gamma(1-\rho)-\rho)}{\gamma\underline{\kappa}_{ij}^{\frac{\gamma(1-\rho)}{\rho}}\tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}}}-\sum_{k\neq i}\mu_k\left(\frac{\underline{x}_k}{\underline{x}_i}\right)^{\gamma}\left(\frac{\tau_{kj}}{\tau_{ij}}\right)^{-\gamma}\left(\frac{\tilde{\kappa}_{kj}}{\tilde{\kappa}_{ij}}\right)^{\frac{\rho-\gamma(1-\rho)}{\rho}}\quad j=1,\ldots,n \tag{23}$$

then the associated equilibrium is interior.

Proof: See Appendix.

Proposition 4 requires that the mass of firms must be positive, as otherwise the equilibrium cannot be interior — a restriction that does not appear in the fixed-firms model. The additional two conditions in Proposition 4 are the equivalent of the two conditions in Proposition 2.

Condition (22) is very similar to condition (16). To see this, recall that aggregate profit gross of entry costs in the costly-entry model is equal to the value of all of the entry costs paid, $\mu_i \phi_i$. In both models, gross profits must be less than $\rho / (\gamma - \rho) \ell_i$.

Condition (23) is less intuitive than its counterpart in Proposition 2, but has similar properties. The second term in the right-hand side is similar to a multilateral resistance term. A larger (μ_k) and more efficient $(\underline{x}_k)$ pool of potential firms in the rest of the world, lower variable trade costs (τ_{ki}), and higher fixed costs $(\tilde{\kappa}_{si})$ imply greater competition in market i, making it less likely that the least-efficient technology is profitable. The first term on the right-hand side of (23) is similarly affected by fixed costs and the pool of potential firms. This term also depends on the importance of differentiated goods in utility (α) and markups (ρ).

4. Concluding remarks

We provide a simple method for solving the ubiquitous two-sector, multicountry, heterogeneous-firm trade model. The method finds the interior equilibrium of the model, in which both the homogeneous outside good and some differentiated goods are produced. The models can have other equilibria, but the interior equilibrium is the one most often studied in the literature. The method is easy to employ — it requires the solution of a simple system of equations — and provides useful intuition as to why an interior equilibrium may fail to exist.

References

Allen, T., C. Arkolakis, and X. Li (2016), "On the Existence and Uniqueness of Trade Equilibria," unpublished manuscript, Dartmouth College.

Chaney, T. (2008), "Distorted Gravity: The Intensive and Extensive Margins of International Trade," *American Economic Review*, 98(4), 1707–1721.

Gantmacher, F. R. (1959), *The Theory of Matrices*, Vol. II, Chelsea Publishing Company, New York, NY.

Hopenhayn, H. A. (1992), "Entry, Exit, and Firm Dynamics in Long Run Equilibrium," *Econometrica*, 60(5), 1127–50.

Kehoe, T. J., P. S. Pujolas and K. J. Ruhl (2016), "The Opportunity Costs of Entrepreneurs in International Trade," *Economics Letters*, 146, 1–3.

Lucas, R. E. (1978), "On the Size Distribution of Business Firms," *Bell Journal of Economics*, 9(2), 508–523.

McKenzie, L. W. (1959), "Matrices with Dominant Diagonals and Economic Theory," in K. Arrow, S. Karlin and P. Suppes, eds., *Mathematical Methods in the Social Sciences*, Stanford University Press, Stanford, CA.

Melitz, M. J. (2003), "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity," *Econometrica*, 71(6), 1695–1725.

Appendix A

In this appendix, we derive all the algebra required for the results in Section 2. The expression for the ideal price index, equation (8), when writing it as a function of efficiency levels and operating costs, turns into

$$P_j = \left(\sum_{k=1}^{n} \mu_k \int_{\underline{\kappa}_{kj}}^{\infty} \left(\int_{\hat{x}_{kj}(\kappa)}^{\infty} p_{kj}(x)^{\frac{-\rho}{1-\rho}} \gamma \underline{x}_k^{\gamma} x^{-\gamma-1} dx \right) dF_{kj}(\kappa) \right)^{\frac{1-\rho}{-\rho}} . \tag{24}$$

Using the CES demand for a differentiated good, equation (7), we can write down the monopolistic competitor's maximization problem (conditional on operating):

$$\max \quad \frac{\alpha\left(\overline{\ell}_j + \pi_j\right)}{p_{ij}(x)^{\frac{1}{1-\rho}} P_j^{\frac{-\rho}{1-\rho}}} p_{ij}(x) - \left(\frac{\tau_{ij}\alpha\left(\overline{\ell}_j + \pi_j\right)}{x p_{ij}(x)^{\frac{1}{1-\rho}} P_j^{\frac{-\rho}{1-\rho}}} + \kappa_{ij} \right), \tag{25}$$

which implies the well-known solution of price equal to the constant mark-up over the marginal cost — equation (9). In that equation, note that the pricing decision is independent of κ_{ij}. Equation (25) equal to zero, and using the pricing equation can be used to compute the cut-off function, $\hat{x}_{ij}\left(\kappa_{ij}\right)$. This function gives all the pairs efficiency-fixed cost that determine whether firms are willing to operate (larger efficiency or smaller fixed cost) or not (vice versa)

$$\hat{x}_{ij}\left(\kappa_{ij}\right) = \frac{\tau_{ij}}{\rho P_j} \left((1-\rho)\alpha\left(\overline{\ell}_j + \pi_j\right) \right)^{\frac{1-\rho}{-\rho}} \kappa_{ij}^{\frac{1-\rho}{\rho}} . \tag{26}$$

For what follows, it is convenient to define $\overline{x}_{ij}$

$$\overline{x}_{ij}^{\frac{\rho}{1-\rho}} = \int_{\underline{\kappa}_{ij}}^{\infty} \left(\int_{\hat{x}_{ij}(\kappa)}^{\infty} x^{\frac{\rho}{1-\rho}} \gamma \underline{x}_i^{\gamma} x^{-\gamma-1} dx \right) dF_{ij}(\kappa), \tag{27}$$

which requires parameters to satisfy $\rho < \gamma(1-\rho)$ for the integral to be well defined. Note that the inner integral in equation (27) is

$$\int_{\hat{x}_{ij}(\kappa)}^{\infty} x^{\frac{\rho}{1-\rho}} \gamma \underline{x}_i^{\gamma} x^{-\gamma-1} dx = \gamma \underline{x}_i^{\gamma} \int_{\hat{x}_{ij}(\kappa)}^{\infty} x^{\frac{\rho}{1-\rho}-\gamma-1} dx = \frac{\underline{x}_i^{\gamma}\gamma(1-\rho)}{\gamma(1-\rho)-\rho} \hat{x}_{ij}(\kappa)^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} . \tag{28}$$

Consequently,

$$\overline{x}_{ij}^{\frac{\rho}{1-\rho}} = \frac{\underline{x}_i^{\gamma}\gamma(1-\rho)}{\gamma(1-\rho)-\rho} \left(\frac{\tau_{ij}}{\rho P_j} \right)^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \left(\alpha(1-\rho)\left(\overline{\ell}_j + \pi_j\right) \right)^{\frac{\gamma(1-\rho)-\rho}{\rho}} \int_{\underline{\kappa}_{ij}}^{\infty} \kappa^{\frac{\rho-\gamma(1-\rho)}{\rho}} dF_{ij}(\kappa). \tag{29}$$

Again, for clarity of exposition is convenient to define

$$\tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}} = \int_{\underline{\kappa}_{ij}}^{\infty} \kappa^{\frac{\rho-\gamma(1-\rho)}{\rho}} dF_{ij}(\kappa). \tag{30}$$

We can then re-write the ideal price index, P_j, as

$$P_j = \frac{1}{\rho}\left(\frac{\gamma(1-\rho)-\rho}{\gamma(1-\rho)}\right)^{\frac{1}{\gamma}} \left(\sum_{k=1}^{n} \mu_k \underline{x}_k^{\gamma} \tau_{kj}^{-\gamma} \tilde{\kappa}_{kj}^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right)^{-\frac{1}{\gamma}} \left(\alpha(1-\rho)\left(\overline{\ell}_j+\pi_j\right)\right)^{\frac{\rho-\gamma(1-\rho)}{\gamma\rho}}, \tag{31}$$

implying that

$$\begin{aligned}\overline{x}_{ij} &= \underline{x}_i^{\gamma\frac{1-\rho}{\rho}} \left(\frac{\gamma(1-\rho)}{\gamma(1-\rho)-\rho}\right)^{\frac{1}{\gamma}} \left(\sum_{k=1}^{n} \mu_k \underline{x}_k^{\gamma} \tau_{kj}^{-\gamma} \tilde{\kappa}_{kj} \left(\frac{\tilde{\kappa}_{ij}}{\tilde{\kappa}_{kj}}\right)^{\frac{\gamma(1-\rho)}{\rho}}\right)^{\frac{\rho-\gamma(1-\rho)}{\gamma\rho}} \\ &\times\left(\alpha(1-\rho)\tau_{ij}^{-\gamma}\left(\overline{\ell}_j+\pi_j\right)\right)^{\frac{\gamma(1-\rho)-\rho}{\gamma\rho}}\end{aligned}. \tag{32}$$

Note that profits from selling to a given destination are given by

$$\pi_{ij} = \mu_i \int_{\underline{\kappa}_{ij}}^{\infty} \left(\int_{\hat{x}_{ij}(\kappa)}^{\infty} \left(\frac{1-\rho}{\rho^{\frac{-\rho}{1-\rho}}} \frac{\tau_{ij}\alpha\left(\overline{\ell}_j+\pi_j\right)}{\tau_{ij}^{\frac{1}{1-\rho}} P_j^{\frac{-\rho}{1-\rho}}} x^{\frac{\rho}{1-\rho}} - \kappa_{ij} \right) \gamma \underline{x}_i^{\gamma} x^{-\gamma-1} dx \right) dF_{ij}(\kappa), \tag{33}$$

which is equivalent to

$$\pi_{ij} = \mu_i \frac{1-\rho}{\rho^{\frac{-\rho}{1-\rho}}} \frac{\tau_{ij}\alpha\left(\overline{\ell}_j+\pi_j\right)}{\tau_{ij}^{\frac{1}{1-\rho}} P_j^{\frac{-\rho}{1-\rho}}} \overline{x}_{ij}^{\frac{\rho}{1-\rho}} - \mu_i \int_{\underline{\kappa}_{ij}}^{\infty} \kappa_{ij} \underline{x}_i^{\gamma} \hat{x}_{ij}(\kappa)^{-\gamma} dF_{ij}(\kappa), \tag{34}$$

where most terms in the two parts of the equation coincide after plugging in the expressions for $\overline{x}_{ij}$ and $\hat{x}_{ij}(\kappa)$,

$$\pi_{ij} = \mu_i \underline{x}_i^{\gamma} \tau_{ij}^{-\gamma} \left(\gamma^{-1}\rho\alpha\left(\overline{\ell}_j+\pi_j\right)\right) \tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}} \left(\sum_{k=1}^{n} \mu_k \underline{x}_k^{\gamma} \tau_{kj}^{-\gamma} \tilde{\kappa}_{kj}^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right)^{-1}, \tag{35}$$

which implies that aggregate profits in a given country from selling to all the other countries is simply

$$\pi_i = \frac{\rho\alpha}{\gamma} \sum_{j=1}^{n} \frac{\mu_i \underline{x}_i^{\gamma} \tau_{ij}^{-\gamma} \tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}}}{\sum_{k=1}^{n} \mu_k \underline{x}_k^{\gamma} \tau_{kj}^{-\gamma} \tilde{\kappa}_{kj}^{\frac{\rho-\gamma(1-\rho)}{\rho}}} \left(\overline{\ell}_j+\pi_j\right). \tag{36}$$

This expression is very useful to proof Proposition 1 in the paper.

Proof of Proposition 1

The system of equations in (36) consists of n linear equations, one for each country, in n unknowns — the π_i. Setting it up in matricial form and rearranging terms yield

$$\pi_i - \sum_{j=1}^{n} \lambda \frac{\theta_{ij}}{\Theta_j} \pi_j = \sum_{j=1}^{n} \lambda \frac{\theta_{ij}}{\Theta_j} \overline{\ell}_j \text{, for } i = 1,\ldots,n \ , \tag{37}$$

where $\lambda = \frac{\alpha\rho}{\gamma}$, $\theta_{ij} = \mu_i \underline{x}_i \tau_{ij}^{-\gamma} \tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}}$ and $\Theta_j = \sum_{k=1}^{n} \theta_{kj}$. If we define

$$b_i = \sum_{j=1}^{n} \lambda \frac{\theta_{ij}}{\Theta_j} \overline{\ell}_j , \tag{38}$$

$$d_{ij} = \lambda \frac{\theta_{ij}}{\Theta_j} , \tag{39}$$

we can rewrite the system in **Error! Reference source not found.** as

$$[I - D]\pi = b \ , \tag{40}$$

which facilitates a simple proof of uniqueness: the column sums of D are $\lambda < 1$ and the vector b is strictly positive, so the Perron-Frobenius theorem tells us that the vector π which solves **Error! Reference source not found.** is strictly positive. In fact, $[I - D]^{-1}$ is strictly positive. □

Proposition 1 is very useful to establish that the equilibrium in this model is always such that heterogeneous efficiency firms always produce, and hence aggregate profits are always positive. However, the equilibrium of the model need not be interior. For this to be the case, it has to satisfy that some firms do not produce, that the homogeneous good is always produced, and that it does not always use up all the labor in the economy. The result from Proposition 1 already informs us that the last condition is always respected. Hence, for an equilibrium to be interior, we only need the first two parts to hold. Next we formalize these two conditions.

Proof of Proposition 2:

The first condition for the equilibrium be interior is given by

$$y_{0i} > 0, \forall i \Leftrightarrow l_{01} > 0 \Leftrightarrow \ell_i^d < \overline{\ell}_i \ . \tag{41}$$

Where ℓ_i^d is total labor devoted to the production of the heterogeneous goods.

We now calculate the amount of labor used to produce differentiated products in country i:

$$\ell_i^d = \mu_i \sum_{j=1}^{n} \int_{\underline{\kappa}_{ij}}^{\infty} \int_{\hat{x}_{ij}(\kappa)}^{\infty} \left(\frac{\tau_{ij} c_{ij}(x)}{x} + \kappa_{ij} \right) dF(x) dF_{ij}(\kappa) \ . \tag{42}$$

which implies that

$$\ell_i^d = \alpha \left(\frac{\gamma - \rho}{\gamma} \right) \sum_{j=1}^{n} \frac{\mu_i \tau_{ij}^{-\gamma} \underline{x}_i^{\gamma} \tilde{\kappa}_{ij}^{\frac{\rho - \gamma(1-\rho)}{\rho}}}{\sum_{k=1}^{n} \mu_k \underline{x}_k^{\gamma} \tau_{kj}^{-\gamma} \tilde{\kappa}_{kj}^{\frac{\rho - \gamma(1-\rho)}{\rho}}} \left(\overline{\ell}_j + \pi_j \right) \ . \tag{43}$$

$$\Leftrightarrow \ell_i^d = (\gamma - \rho) \frac{\pi_i}{\rho}$$

Hence, the equilibrium is interior if the following inequality is respected.

$$\pi_i < \overline{\ell}_i \frac{\rho}{\gamma - \rho} \ . \tag{44}$$

The second condition for the equilibrium be interior is given by

$$\hat{x}_{ij}\left(\kappa_{ij}\right) > \underline{x}_i \, . \tag{45}$$

which, by definition is

$$\frac{\tau_{ij}}{\rho P_j} ((1-\rho)\alpha(\overline{\ell}_j + \pi_j))^{\frac{1-\rho}{-\rho}} \underline{\kappa}_{ij}^{\frac{1-\rho}{\rho}} > \underline{x}_i \, . \tag{46}$$

using the definition of the aggregate price index, this is

$$\frac{\tau_{ij}}{\rho} [(1-\rho)\alpha]^{\frac{1-\rho}{-\rho}} (\overline{\ell}_j + \pi_j)^{\frac{1-\rho}{-\rho}} \underline{\kappa}_{ij}^{\frac{1-\rho}{\rho}} > \underline{x}_i \Lambda \left(\sum_{k=1}^{n} \mu_k \underline{x}_k^{\gamma} \tau_{kj}^{-\gamma} \tilde{\kappa}_{kj}^{\frac{\rho - \gamma(1-\rho)}{\rho}} \right)^{\frac{-1}{\gamma}} (\overline{\ell}_j + \pi_j)^{\frac{-\gamma(1-\rho)+\rho}{\gamma\rho}} \ . \tag{47}$$

After a little algebra this expression becomes

$$\frac{\gamma}{\alpha(\gamma(1-\rho)-\rho)} \sum_{k=1}^{n} \mu_k \left(\frac{\underline{x}_k}{\underline{x}_i} \right)^{\gamma} \left(\frac{\tau_{kj}}{\tau_{ij}} \right)^{-\gamma} \underline{\kappa}_{ij}^{\frac{\gamma(1-\rho)}{\rho}} \tilde{\kappa}_{kj}^{\frac{\rho - \gamma(1-\rho)}{\rho}} > \overline{\ell}_j + \pi_j \, . \tag{48}$$

and rearranging delivers the desired result. □

Appendix B

In this appendix, we derive all the algebra required for the results in Section 3. Note that most algebra is identical to the one derived in the previous appendix, except that profits are zero, and the number of firms becomes a variable of the model. In Proposition 3, we solve for the mass of firms.

Proof of Proposition 3

To calculate μ_i, we solve for the condition that equalizes expected profits to the cost of getting an efficiency draw

$$\sum_{i=1}^{n}\int_{\underline{\kappa}_{ij}}^{\infty}\left(\int_{\hat{x}_{ij}(\kappa)}^{\infty}\left(p_{ij}(x_{ij})c_{ij}(x_{ij})-\frac{c_{ij}(x_{ij})}{x_{ij}}-\kappa_{ij}\right)\gamma \underline{x}_i^{\gamma}x^{-\gamma-1}dx\right)dF_{ij}(\kappa)=\phi_i\ , \tag{49}$$

Rearranging, this equation becomes

$$\sum_{j=1}^{n}\frac{\overline{\ell}_j\tau_{ij}^{-\gamma}\tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}}}{\sum_{k=1}^{n}\mu_k\underline{x}_k\tau_{kj}^{-\gamma}\tilde{\kappa}_{kj}^{\frac{\rho-\gamma(1-\rho)}{\rho}}}=\frac{\gamma\phi_i}{\underline{x}_i\alpha\rho}. \tag{50}$$

Notice that this is a system of n nonlinear equations in μ_i, for $i=1,...,n$. Let

$$\lambda_j=\frac{\overline{\ell}_j}{\sum_{k=1}^{n}\mu_k\underline{x}_k\tau_{kj}^{-\gamma}\tilde{\kappa}_{kj}^{\frac{\rho-\gamma(1-\rho)}{\rho}}}. \tag{51}$$

We can solve the system of linear equations

$$\sum_{j=1}^{n}\tau_{ij}^{-\gamma}\tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}}\lambda_j=\frac{\phi_i\gamma}{\underline{x}_i\rho\alpha}\ , \tag{52}$$

to obtain λ_j, for $j=1,...,n$. Then we can solve

$$\sum_{k=1}^{n}\mu_k\underline{x}_k\tau_{kj}^{-\gamma}\tilde{\kappa}_{kj}^{\frac{\rho-\gamma(1-\rho)}{\rho}}=\frac{\overline{\ell}_j}{\lambda_j}\ , \tag{53}$$

to obtain μ_i, for $i=1,...,n$. Notice that we only need to invert one matrix, D, where

$$d_{ij}=\tau_{ij}^{-\gamma}\tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}}. \tag{54}$$

The first set of equations has the form

$$D\lambda = \begin{bmatrix} \frac{\phi_1 \gamma}{\underline{x}_1 \rho \alpha} \\ \vdots \\ \frac{\phi_n \gamma}{\underline{x}_n \rho \alpha} \end{bmatrix}, \tag{55}$$

and the second has the form

$$D'\mu = \begin{bmatrix} \frac{\overline{\ell}_1}{\lambda_1} \\ \vdots \\ \frac{\overline{\ell}_n}{\lambda_n} \end{bmatrix}. \tag{56}$$

Recall that $(D')^{-1} = (D^{-1})'$.

Notice that unlike the case with a fixed distribution of potential firms, there is no reason for either λ or μ to be strictly positive. □

Proposition 3 is to the model with costly entry the same as Proposition 1 to the model with fixed firms. There is one key difference between the two Propositions: in this one we cannot invoke a Theorem similar to the Perron-Frobenious one that guarantees the mass of firms operating is positive in each country. This difference is important when it comes to how many conditions need to be satisfied for the equilibrium be interior. An equilibrium is interior if it has a positive of measure of firms in each country operating in the differentiated good industry, with some firms not producing; if the homogeneous good is always produced, and if the homogeneous good does not use up all the labor in the economy. Similar to before (when positive profits implied that the homogeneous good could not use all the labor in the country), we now have that if there is a positive measure of heterogeneous firms, then the homogeneous good sector cannot use up all the labor available. However, unlike before, we cannot be sure that the measure of firms will be positive for sure. Hence, we need to check whether the measure of firms is positive, once the solution is computed. We formalize the three conditions in the following Proposition.

<u>Proof of Proposition 4:</u>

The first condition for the equilibrium be interior is that the mass of firms is positive in each country, namely, $\mu_i > 0$.

Moreover, as in the model with fixed firms, we need that

$$y_{0i} > 0, \forall i \Leftrightarrow l_{01} > 0 \Leftrightarrow \ell_i^d < \overline{\ell}_i \ . \tag{57}$$

Where ℓ_i^d stands for the total labor devoted to the production of the heterogeneous good.

The amount of labor used to produce differentiated products in country i,

$$\ell_i^d = \mu_i \sum_{j=1}^{n} \int_{\underline{\kappa}_{ij}}^{\infty} \int_{\hat{x}_{ij}(\kappa)}^{\infty} \left(\frac{\tau_{ij} c_{ij}(x)}{x} + \kappa_{ij} \right) dF(x) dF_{ij}(\kappa) , \tag{58}$$

translates to having a measure of firms that is

$$\mu_i < \frac{\rho}{(\gamma - \rho)} \frac{\overline{\ell}_i}{\phi} \ . \tag{59}$$

Last, the equilibrium is interior if

$$\hat{x}_{ij}\left(\kappa_{ij}\right) > \underline{x}_i . \tag{60}$$

which after some algebra (which mimics the one from Proposition 2) becomes

$$\mu_i > \overline{\ell}_j \frac{\alpha(\gamma(1-\rho)-\rho)}{\gamma \underline{\kappa}_{ij}^{\frac{\gamma(1-\rho)}{\rho}} \tilde{\kappa}_{ij}^{\frac{\rho-\gamma(1-\rho)}{\rho}}} - \sum_{k \neq i} \mu_k \left(\frac{\underline{x}_k}{\underline{x}_i} \right)^{\gamma} \left(\frac{\tau_{kj}}{\tau_{ij}} \right)^{-\gamma} \left(\frac{\tilde{\kappa}_{kj}}{\tilde{\kappa}_{ij}} \right)^{\frac{\rho-\gamma(1-\rho)}{\rho}} . \tag{61}$$

□