Risk externalities

Felipe S. Iachan and Dejanir Silva*

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(Preliminary – most recent version here). Current analysis of macro-prudential policy has largely focused on excessive leverage and fire sales, while abstracting from firm-level risk and portfolio decisions. We show that these matter for the design of optimal interventions. We study a macroeconomic model in which firms make investment decisions in the presence of both aggregate and firm-specific risk. Financial constraints limit access to external funds and tie each firm's output to its net worth and a firm-specific return on investment. The laissez-faire outcome is constrained inefficient, as individual decisions fail to internalize their consequences on the distribution of risky returns over other agents (a *risk externality*). Firms are, at the same time, overexposed to aggregate risk and underexposed to idiosyncratic risk. In a quantitative exploration, interventions are shown to be counter-cyclical in their magnitude and to lead to large increases in aggregate TFP and output.

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*Felipe Iachan: FGV EPGE – felipe.iachan@fgv.br; Dejanir Silva: University of Illinois, Urbana-Champaign – dejanir@illinois.edu.
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1 Introduction

A growing literature in Macroeconomics and Finance emphacizes the role of microeconomic uncertainty. From input-output linkages explaining how idiosyncratic shocks might propagate throughout the economy, to the consequences of state-varying idiosyncratic uncertainty on aggregate investment, and the roles of higher moment shocks along the business cycle, these contributions have reshaped our understanding of aggregate fluctuations. We show that, in the presence of uninsurable firm-level risk, equilibrium investment and risk-taking decisions are inefficient. We derive new lessons for macro-prudential policy in which the cross-sectional distribution of risk plays a central role.

We study the allocation of investment across portfolios of projects and firms that is limited by financial constraints. Our main contributions are to theoretically identify and provide a first empirical quantification of a novel pecuniary externality associated with firms risk-taking decisions, which we call *risk externalities*.[1] We find that the economy is subject to *under*-investment and there is inefficient risk-taking. We show that firms are, at the same time, over-exposed to aggregate risk and underexposed to idiosyncratic risk. This is in contrast to standard *collateral* or *fire-sales* externalities, where the economy is subject to over-investment.[2] Hence, the inefficiencies in our economy do not come from excessive leverage and ex ante overinvestment, but from investing in the wrong portfolio of projects ex ante and reallocating resources too little ex post.

In order to isolate the new externality, we consider an environment that abstracts from the standard fire-sale externality. Even though firms are subject to collateral constraints, prices do not directly enter into the constraint. As a consequence, there is no room for a plain intervention that attempts to relax borrowing constraints by manipulating asset prices. In contrast, our result relies on the assumption that there is return risk ex-ante and firm heterogeneity ex-post. The externality comes from the fact that agents do not internalize the impact on their actions on the distribution of risk borne by other agents.

The economic environment can be described as follows. The economy lasts for three dates and it is populated by a continuum of risk-averse entrepreneurs, workers, financial intermediaries, and a government. In the first period, entrepreneurs are identical and must allocate an initial endowment between consumption and the purchase of contingent securities that pay off based on both the aggregate state of the economy and the idiosyncratic state of the firm. These allocation decisions can be distorted by portfolio taxes. The assumption of complete markets allows us to show that the inefficiency in risk taking is not simply a result of market incompleteness and is later relaxed. In the interim period, entrepreneurs learn their productivity and combine capital and labor to produce final goods, which are consumed in the final period. The amount of capital a entrepreneur can operate is a multiple of her net worth, a standard form of borrowing constraint that is typically motivated with some form of limited commitment to debt repayment. Importantly, prices do not enter directly the constraint.

Ex-post heterogeneity creates the scope for a market for funds in this economy. Relatively inefficient entrepreneurs become inactive and lend their capital in the market. Relatively productive firms lever up and use the funds they raise to finance their use of physical capital. Due to the collateral constraint, returns are not equalized across firms, as the more productive firms have positive profits (a *Ricardian*

[1] For a recent discussion and taxonomy of pecuniary externalities, see Dávila and Korinek (2017).

[2] See, for instance, Lorenzoni (2008).

rent) and are capable of generating a positive excess return. Capital is misallocated in this economy and aggregate total factor productivity (TFP) depends on the distribution of net worth. Importantly, this implies that ex-ante return risk is endogenous and depends on the initial portfolio decisions.

Absent interventions, the equilibrium in this economy is constrained inefficient. In order to see this result, consider first an economy subject only to idiosyncratic shocks for example. For high levels of risk aversion, entrepreneurs attempt to hedge their productivity risk by shifting resources to (ex-post) less productive firms. In doing so, they reduce aggregate productivity and wages. This wage reduction increases proportionally more the profits of bigger and more productive firms, increasing (ex-ante) the rate of return risk.[3] Since agents do not internalize the effect of their portfolio decisions on the risk borne by other agents, we call this effect a *risk externality*. This is an example of a *fallacy of composition*, as an attempt of firms to reduce their individual risk will make the economy riskier for everyone else. While the magnitude of the externality depends on preferences, its presence and direction does not. A policy that internalizes this effect can improve welfare by inducing firms to shift resources to more productive units and achieve significant increases in total output.

This externality has also implications for the level and riskiness of initial investment decisions. The economy is subject to under-investment. Entrepreneurs do not internalize that, by saving more ex-ante, wages are increased ex-post, reducing proportionally more the return of very productive firms and the amount of return risk in the economy.

Last, consider now an economy with aggregate shocks, where average productivity of the degree of idiosyncratic uncertainty can vary with the state of the world. We show that the risk externality is more severe in downturns, characterized by low average productivity and high uncertainty, as the dispersion of returns tends to increase in bad times. Hence, from the perspective of the planner, the economy is overexposed to aggregate risk, as not enough resources are directed to bad states of the world.

In order to correct for this externality, the planner needs to subsidize savings, stimulate capital reallocation, but tax aggregate risk-taking. The treatment of idiosyncratic risk depends on the actual market structure. If markets for idiosyncratic risk exist, the planner should distort idiosyncratic portfolio decisions. The planner will want to subsidize the allocation of resources to more productive firms. Moreover, the optimal portfolio subsidy is discontinuous with a jump at the point entrepreneurs become active. These two features are meant to correct the intensive and extensive margin effects of the externality. If markets are initially incomplete and entrepreneurs can trade aggregate risk, then the optimal policy can be implemented by a combination of ex-ante macro-prudential policies regarding savings and aggregate risk and ex-post interventions targeting reallocation across firms. Ex-post intervention can be implemented by a combination of a borrowing subsidy to firms and a non-linear corporate income tax schedule.

We explore quantitatively the gains from interventions. Using an empirical distribution firm-level productivity from the World Bank Enterprise Survey, gains from policy interventions that stimulate risk-taking and capital reallocation are shown to be large. Output is increased by approximately 12%, while welfare is raised by approximately 3% in terms of certainty equivalent consumption. We also study how optimal policies depend on the business cycle. In particular, we study how they respond to

[3] A similar externality operates through the extensive margin: shifting resources to inactive firms depress interest rates, increasing the difference in returns between inactive and active (levered) entrepreneurs.

three shocks that are associated with economic downturns: a negative mean shock affecting all firms, an uncertainty shock that raises dispersion in profitability, and a skewness contraction that makes extremely positive results less likely. The general lesson is that policy responses should be anti-cyclical and facilitate reallocation further in downturns.

Literature. A recent literature has emphasized how pecuniary externalities may lead to under-investment in economies with financial frictions. Asriyan (2015) shows how under-investment can emerge in an economy with fire-sales and information and trading frictions. Kurlat (2018) shows a similar result for an economy with private information. Dávila and Korinek (2017) provides a general taxonomy of pecuniary externalities and show that distributive externalities may lead to over- or under-investment.[4] We provide a more explicit characterization of how individual risk-taking decisions can affect endogenous aggregate risk in the economy and provide an empirical quantification of the externality.

There has been also the growth of a parallel literature that aims to understand how the rich behavior of firm-level shocks might ultimately influence aggregate fluctuations.[5] In particular, focus has been given to the importance of a heavy tail of large firms (Gabaix, 2011), the varying level of uncertainty and productivity dispersion (**?**), and more recently, to the cyclical properties of skewness (Salgado et al., 2016). Despite these recent advances on the positive implications of firm-level heterogeneity, much less is known about its normative consequences. We study how the combination of firm-level uncertainty with frictions that hinder the allocation of labor and capital causes distortions in investment and risk taking, and the nature of the best partial remedies.

2 Model

We study a finite horizon economy with three dates, $t = 0, \frac{1}{2}, 1$. Consumption occurs at dates $t = 0$ and $t = 1$. The interim period $t = \frac{1}{2}$ corresponds to the moment in which uncertainty is realized, financial contracts are settled, and factors of production for date $t = 1$ are hired. This economy is populated by two groups of agents: workers and entrepreneurs. Workers only have a relevant role at dates $t = \frac{1}{2}$ when they supply their labor and at date t=1 when they consume.

The population of entrepreneurs consists of a unit-mass continuum of ex-ante identical agents, who live for three periods. At date $t = 0$, these agents make consumption and saving decisions. Uncertainty has both an aggregate and an idiosyncratic component. In particular, at t=$\frac{1}{2}$ the aggregate state $s \in \boldsymbol{S} = \{1, ..., S\}$ is revealed, with $p_s > 0$ representing the probability of each state occurring. Entrepreneurs also learn their productivity parameter $\theta \in \mathbb{R}$, drawn from at state $s \in \mathbf{S}$ from the cumulative distribution $F_s(\theta)$, with density $f_s(\theta)$. Both states become public information.

Entrepreneurs also have access to production functions with their individual-specific productivity. As a consequence, each agent owns a potential firm, which he or she can choose to operate or leave inactive. When this firm is operational, it rents capital from savers at a rental rate r_s and hires workers in a spot market at wages w_s.

[4]Our risk externality is similar to their distributive externalities, but notice they do not formally consider the case of idiosyncratic risk, so our environment does not fit exactly into their formulation.

[5]See, for instance, Acemoglu et al. (2012), Baqaee (2016), Baqaee and Farhi (2017), Carvalho and Gabaix (2013), Carvalho and Grassi (2015), Di Giovanni and Levchenko (2012), Gabaix (2011), Stella (2015), and Yeh (2017).

All potential firms solve the following problem, which determines its decision of whether to operate and its factor demands.

Firms' problem

A firm with known TFP of θ and a net worth of $a_s(\theta)$ rents capital (k) and hires workers (l) to solve

$$\Pi(a_s(\theta), \theta, s) \equiv \max_{k,l} (\theta k)^{\alpha} l^{1-\alpha} - w_s l - (r_s + \tau_s^k + \delta)k \tag{1}$$

subject to

$$0 \leq k \leq \lambda_s a_s(\theta), \tag{2}$$

where τ_s^k is a per-unit tax on capital .[6]

Equation 2 includes a non-negativity constraint on capital plus a collateral constraint, which requires a fraction $\frac{1}{\lambda_s}$ of the capital stock to be financed by the owner's own resources , $a_s(\theta)$. This form of constraint, common in the literature on misallocation and credit frictions, is typically motivated through a limited commitment problem.[7]

The first-order conditions for the firm's problem leads to a simple labor demand function,

$$l_s(\theta) = \left[\frac{(1-\alpha)}{w_s}\right]^{\frac{1}{\alpha}} \theta k_s(\theta), \tag{3}$$

showing that effective (productivity-adjusted) capital-labor ratios are equalized across firms.

As a consequence of constant returns to scale, the profit function becomes linear in net worth and can be written as

$$\Pi(a_s(\theta), \theta, s) = \pi_s(\theta) a_s(\theta) \tag{4}$$

where

$$\pi_s(\theta) = \lambda_s \max\left\{\alpha \left(\frac{1-\alpha}{w_s}\right)^{\frac{1-\alpha}{\alpha}} \theta - (r_s + \tau_s^k + \delta),\ 0\right\}. \tag{5}$$

It naturally follows that if we define a marginal firm $\hat{\theta}_s$ that has zero profits, implicitly in $\pi_s\left(\hat{\theta}_s\right) = 0$, all infra-marginal firms ($\theta < \hat{\theta}_s$) refrain from operating and all supra-marginal firms ($\theta > \hat{\theta}_s$) earn positive profits (*Ricardian rents*) and operate with the fullest possible leverage.

Aggregate Production

Using the equalization of effective capital-labor ratios implied by the labor demand in equation 3, we get

$$\frac{\theta k_s(\theta)}{l_s(\theta)} = \frac{\Theta_s K_s}{L_s} \tag{6}$$

[6]We assume that the tax on capital rental incides per unit of k and not over the rental cost $r_s k$. This simplifies the algebra of optimal taxes in Section 4 without any additional consequence, as alternative rental rates and taxes can be obtained from the current formulation after a change of variables. We do not allow an output tax, as output expropriation circumvents the collateral constraints and allows the costless implementation of the first-best. We discuss its role in more detail in a later section.

[7]See Buera et al. (2011), for instance.

where

$$K_s = \int_0^\infty k_s(\theta)\, dF_s(\theta), \qquad L_s = \int_0^\infty l_s(\theta) dF_s(\theta), \tag{7}$$

$$\Theta_s = \int_0^\infty \theta \frac{k_s(\theta)}{K_s} dF_s(\theta), \tag{8}$$

represent respectively the aggregate capital demand, aggregate labor demand, and aggregate TFP in state $s \in \mathbf{S}$. Notice that aggregate TFP is then a weighted average of the individual TFPs, in which the weights are given by the share of the total capital supply that is operated by each type of entrepreneur.

We can then compute aggregate output as

$$Y_s = \int_0^\infty (\theta k_s(\theta))^\alpha l_s(\theta)^{1-\alpha} dF_s(\theta) = (\Theta_s K_s)^\alpha L_s^{1-\alpha}. \tag{9}$$

Entrepreneur's consumption at $t = 1$

Entrepreneurs, unlike workers, have no labor endowment at date $t = 1$.[8] They do have assets $a_s(\theta)$, which are consequence of their previous saving and hedging decisions, to be soon described.

Given the optimal firm decision, consumption in state $s \in S$ for an agent of type θ and assets $a_s(\theta)$ can simply be written as

$$c_s(a_s(\theta), \theta) = R_s(\theta)\, a_s(\theta), \tag{10}$$

where

$$R_s(\theta) = 1 + r_s + \pi_s(\theta),$$

is the return that type θ obtains on own savings in state s. Notice that because of credit constraints, some entrepreneurs earn rents, $\pi_s(\theta) > 0$, giving rise to dispersion in perceived returns on savings and ultimately to consumption risk.

Workers − at $t = 1$

We impose an inelastic labor supply of a single unit for simplicity. Worker consumption in state s at date $t = 1$ is given by

$$c_s^w = w_s + T_s^w,$$

where T_s^w a net transfer received from the government.

The entrepreneur's problem at $t = 0$

At the original date, agents decide on their savings and might also engage in hedging contracts. Specifically, they solve

$$\max_{c_0, a} u(c_0) + \beta E[u(a\, R)] \tag{11}$$

s.t.

[8]This is assumed for tractability, to ensure the linearity of consumption on asset holdings.

$$c_0 + E[m\ a] \leq a_0 + T_0$$

where $m : \boldsymbol{S} \times \mathbb{R} \rightarrow \mathbb{R}_{++}$ is a stochastic discount factor and $a : \boldsymbol{S} \times \mathbb{R} \rightarrow \mathbb{R}_{+}$ is the agent's net worth allocation function across states. In analogy with the $a_s(\theta)$, the realization of the stochastic discount factor is denoted by $m_s(\theta)$, that depends potentially on both sources of uncertainty and represents the relative cost of acquiring one unit of wealth in aggregate state s and idiosyncratic state θ.

Financial Intermediation

In the saver's problem above, contingent contracts written over $a_s(\theta)$ have a cost density $m_s(\theta)$. These contracts are offered by competitive financial intermediaries subject to a state-contingent tax $\tau_s^a(\theta)$. Intermediaries use the funds from these contracts to finance possibly risky projects, described by a set of linear investment technologies $\boldsymbol{I} = \{1, \ldots, I\}$. An investment of 1 unit of the consumption good in date $t = 0$ in technology $i \in \boldsymbol{I}$ delivers φ_s^i units of a capital good at date $t = \frac{1}{2}$, which can then be allocated to production. Two particular cases will be of interest. First, the case of a single riskless technology, $I = 1$ and $\varphi_s^1 = 1$, $\forall s \in \boldsymbol{S}$. That corresponds to a standard assumption in which one unit of investment (foregone consumption) becomes one unit of productive capital in the next date. Second, the case of multiple risky technologies that pay in a single state, $I = S$ and $\varphi_s^i = \overline{\varphi}_s$ for $s = i$ and zero otherwise. This case will allow us to study the efficiency of risk-taking decisions in the economy.

Intermediaries maximize the present value of possible dividend payouts and finance investment in the different technologies by issuing state contingent claims. They solve

$$\max_{a,x} \left[E\left[(1-\tau^a)\, m\, a\right] - \sum_{i \in \boldsymbol{I}} x^i \right] + \sum_{s \in \boldsymbol{S}} p_s M_s \left[\sum_{i \in \boldsymbol{I}} \varphi_s^i x^i - \int a_s(\theta)\, df_s(\theta) \right]$$

subject to $x^i \geq 0$. The two terms inside brackets are the cash-flows net of financing activities, which can be used for financing possible dividends. Here, the tax schedule $\tau^a : \boldsymbol{S} \times \mathbb{R} \rightarrow \mathbb{R}$, where each entry $\tau_s(\theta)$ can be used by the planner to distort the capital accumulation and allocation in the economy.

A competitive solution requires the discount factor of intermediaries satisfy[9]

$$m_s(\theta)\left[1 - \tau_s^a(\theta)\right] = M_s. \tag{12}$$

Additionally, optimal investment requires, for each $i \in \boldsymbol{I}$,

$$\sum_s p_s M_s \varphi_s^i \leq 1, \tag{13}$$

with equality if positive investment occurs in that technology.

Since part of the uncertainty is idiosyncratic, absent any policy intervention, we would see that $m_s(\theta) = M_s$, i.e., idiosyncratic uncertainty is not priced. Moreover, the price of aggregate risk would be pinned down by the return of the investment technologies. For instance, if $\varphi_s^i = \overline{\varphi}_s$ for $i = s$ and

[9] In the appendix, we show that this is the appropriate stochastic discount factor if they were owned by the entrepreneurs.

zero otherwise, then $p_s M_s = \overline{\varphi}_s^{-1}$. We will show that the planner has incentives to distort the price of idiosyncratic and aggregate uncertainty in this economy.

The government

The government sets tax rates and lump-sum rebates, effectively controlling the prices of financial claims and affecting equilibrium allocations. The budget constraints that need to be respected are

$$\sum_s p_s \int \tau_s^a(\theta) a_s(\theta) dF(\theta) = T_0, \tag{14}$$

for date $t = 0$ and

$$\tau_s^k K_s = T_s^w, \tag{15}$$

for each $s = \{1, ..., S\}$ at date $t = 1$.

Equilibrium

To help with notation, we first revisit the objects involved in the definition of a competitive equilibrium with taxes. First, an *allocation* consists of a list $\{c_0, x, a, c, l, k, L, K, c^w\}$, where $c_0 \in \mathbb{R}_+$ is a scalar describing $t = 0$ consumption, $x \in \mathbb{R}_+^I$ denotes the vector of investments in period $t = 0$, $a, c, l, k : \boldsymbol{S} \times \mathbb{R} \rightarrow \mathbb{R}_+$ respectively describe state-contingent savings, consumption, labor, and capital rental demand, $L, K \in \mathbb{R}_+$ denote aggregate quantities of labor and capital, and $c^w : \boldsymbol{S} \rightarrow \mathbb{R}_+$. Second, *prices* consist of the elements in $\{m, r, w\}$, where $m : \boldsymbol{S} \times \mathbb{R} \rightarrow \mathbb{R}_{++}$ is a stochastic discount factor, that depends potentially on both idiosyncratic and aggregate uncertainty, and $r, w : \boldsymbol{S} \rightarrow \mathbb{R}$ are aggregate-state-contingent rental rates and wages (that is, they show only dependence on aggregate states). Last, a *tax system* $\left\{\tau^a, \tau^k, T_0, T^w\right\}$ lists $\tau^a : \boldsymbol{S} \times \mathbb{R} \rightarrow \mathbb{R}$, that describes a schedule of taxes or subsidies on contingent savings, $\tau^k : \boldsymbol{S} \rightarrow \mathbb{R}$ an ex-post per-unit tax on capital (which does not depend on the identity or type of the user), and lump-sum rebates $T_0 \in \mathbb{R}$ (at date $t = 0$ for savers) and $T^w : \boldsymbol{S} \rightarrow \mathbb{R}$ (at date $t = 1$ for workers).

A *competitive equilibrium with taxes* is defined as an allocation, prices, and a tax system such that:

1. Asset purchases, a, and consumption, $\{c_0, c_1\}$ solve Problem 11 given prices and taxes.

2. Worker consumption in each state $s \in \boldsymbol{S}$ is given by $c_s^w = w_s + T_s^w$.

3. Factor demands $\{l, k\}$ solve Problem 1.

4. Capital rental and labor markets clear at each $s \in \boldsymbol{S}$, i.e.,

$$L_s = \int l_s(\theta) dF(\theta|s) = 1 \tag{16}$$

and

$$K_s = \int k_s(\theta) dF(\theta|s) = \sum_{i \in \boldsymbol{I}} \varphi_s^i x^i \tag{17}$$

5. Consumption goods markets clear, i.e.,

$$c_0 + \sum_{i \in \boldsymbol{I}} x^i = a_0$$

and at each $s \in \boldsymbol{S}$

$$c_w + \int_0^{\infty} a_s(\theta) R_s(\theta) df_s(\theta) = \Theta^{\alpha} K_s^{\alpha} + (1-\delta) K_s.$$

6. Financial markets clear, i.e.,

$$\int a_s(\theta) \, df_s(\theta) = \sum_{i \in \boldsymbol{I}} \varphi_s^i x^i,$$

while financial assets are priced competitively in the presence of taxes, ensuring equations 12 and 13.

7. The government budget is balanced at both dates, satisfying equations 14 and 15.

3 Constrained inefficiency and risk externalities

In this section, we provide a characterization of the laissez-faire equilibrium and show that it is possible to find perturbations of the portfolio decisions of entrepreneurs that achieve a Pareto improvement. We defer the analysis of implementation of these perturbed allocations as equilibria and the characterization of optimal policy for Section 4.

3.1 Laissez-faire equilibrium

For simplicity, we assume in the remainder of the paper that the utility of entrepreneurs is identical across dates and states and also isoelastic,

$$u(c_s) = \begin{cases} \frac{c_s^{1-\sigma}}{1-\sigma} & , if \; \sigma \in \mathbb{R} \backslash \{1\} \\ \log c_s & , if \; \sigma = 1 \end{cases},$$

where σ represents a constant coefficient of relative risk aversion.

The equilibrium in laissez-faire is determined by the following set of conditions.

Factor prices, capital allocation, and productivity. Equilibrium in labor and capital rental markets requires that, for each aggregate state $s \in \boldsymbol{S}$,

$$w_s = (1-\alpha)(\Theta_s K_s)^{\alpha} \tag{18}$$

and

$$r_s = \hat{\theta}_s \alpha (\Theta_s K_s)^{\alpha-1} - \delta. \tag{19}$$

where Θ_s represents aggregate TFP.

Here, $\hat{\theta}_s$ is the productivity of the marginal firm. All firms which are more efficient than this level have positive profits, but are prevented from scaling up by the presence of leverage constraints. All firms with $\theta < \hat{\theta}_s$ do not operate, with their owners renting out any units of capital they own. Therefore, capital allocation is given by[10]

$$k_s(\theta) = \begin{cases} \lambda_s a_s(\theta) & , if\ \theta \geq \hat{\theta}_s \\ 0 & \text{, otherwise} \end{cases}. \tag{20}$$

and the capital stock in state s satisfies

$$K_s = \int a_s(\theta) f_s(\theta) d\theta \tag{21}$$

Aggregate TFP is given by

$$\Theta_s = \lambda_s \int_{\hat{\theta}_s}^{\infty} \theta \omega_s(\theta) d\theta, \tag{22}$$

where $\omega_s(\theta)$ is the wealth density in state $s \in \mathbf{S}$

$$\omega_s(\theta) := \frac{a_s(\theta) f_s(\theta)}{\int a_s(\theta) f_s(\theta) d\theta} \tag{23}$$

The productivity of the marginal firm is determined by the condition that the supply of funds by inactive entrepreneurs must equal the demand of funds by active firm-owners

$$\int_0^{\hat{\theta}_s} \omega_s(\theta) d\theta = 1 - \frac{1}{\lambda_s}, \tag{24}$$

where $1 - \frac{1}{\lambda_s}$ is the fraction of the capital stock financed by external funds.

Equations 22 and 24 illustrate how the portfolio decision of entrepreneurs plays a key role in determining TFP, and therefore factor prices, both through the intensive and extensive margin. A shift in relative wealth from low-productivity active firms to high-productivity active firms increases aggregate TFP. Similarly, a shift in mass from inactive firms to active ones increases the productivity of the marginal firm and aggregate TFP.

Asset and contingent capital prices. In the absence of any interventions, the stochastic discount factor does not depend on the idiosyncratic, so

$$m_s(\theta) = M_s, \forall \theta \in \mathbb{R}. \tag{25}$$

It also follows that total asset expenditures by agents equal total capital investment in the economy, i.e., $E[m\,a] = \sum_{i \in \boldsymbol{I}} x^i$.

Savings and portfolio decisions. Owners of the most efficient firms will earn rents and obtain a return on their own savings which is higher than the marginal agent. In particular,

$$R_s(\theta) = 1 + r_s + \pi_s(\theta) \text{ and } \pi_s(\theta) = \lambda_s(\theta - \hat{\theta}_s)^+ \left(\frac{1-\alpha}{w_s}\right)^{\frac{1-\alpha}{\alpha}}. \tag{26}$$

[10]To resolve an inconsequential indeterminacy, we assume the marginal firm uses full leverage.

The second equation in 26 characterizes the nature of Ricardian rents. Returns for an active firm increases with the leverage parameter λ_s, with the productivity differential relative to the marginal firm, and it is decreasing in wages.

The solution to the savers' portfolio problem (in 11) in the absence of interventions implies

$$M_s = \beta \left(\frac{c_s(\theta)}{c_0} \right)^{-\sigma} R_s(\theta) = \beta \left(\frac{K_s}{a_0 - \sum_{i \in \boldsymbol{I}} x^i} \right)^{-\sigma} \left[\int R_s(\theta)^{\frac{1-\sigma}{\sigma}} f_s(\theta)\, d\theta \right]^{\sigma}. \tag{27}$$

One of the general implications of the equation above is that asset holdings will all be the same for types that are inactive in production at state $s \in S$, i.e., $a_s\left(\theta'\right) = a_s\left(\theta''\right)$, $\forall \theta', \theta'' < \hat{\theta}_s$.

We have now enough to combine equations and obtain the full characterization of the equilibrium. Manipulation and integration of equation 27 over the set of agents allows us to obtain the relative wealth density, represented by

$$\omega_s(\theta) = \frac{R_s(\theta)^{\frac{1-\sigma}{\sigma}} f_s(\theta)}{\int R_s(\theta)^{\frac{1-\sigma}{\sigma}} f_s(\theta)\, d\theta}. \tag{28}$$

Equation 28 above makes clear the role of the hedging demand, and how the wealth distribution gets distorted towards low or high productivity idiosyncratic states. In particular, whenever $\sigma > 1$ (resp. $\sigma < 1$), low productivity entrepreneurs become relatively richer (poorer). With logarithmic preferences ($\sigma = 1$), the wealth and the substitution effects exactly cancel out, making the hedging demand disappear.

By using using explicit substitutions of $R_s(\theta) = 1 + \alpha(\underline{\theta}_s + \lambda(\theta - \underline{\theta}_s)^+)\,(\Theta_s K)^{\alpha-1} - \delta$ and the pricing kernel equation 27, we can summarize the equilibrium with the following system involving four types of equation: equation 22 determines the TFP for each state, equation 28 determines the state-contingent wealth distribution, equation 24 determines the marginal firm, and 13 determines capital accumulation at date $t = 0$.

3.2 Risk externalities

An important aspect of the laissez-faire equilibrium is that the distribution of risk faced by an entrepreneur is endogenous, as it is a function of the factor prices (w_s, r_s). These prices are, in their turn, directly determined by the equilibrium quantities $(K_s, \Theta_s, \hat{\theta}_s)$. Whenever making their privately optimal decisions, entrepreneurs fail to take into account their equilibrium consequences on the distribution of risk born by other agents. In this section, we illustrate the nature of the risk externalities and show that their risk-taking and investment decisions under laissez-faire are inefficient.

Welfare. Let us compute the welfare of entrepreneurs for an alternative portfolio allocation $a_s(\theta)$ instead of the laissez-faire allocation $a_s^{lf}(\theta)$. We proceed by directly choosing portfolio perturbations, while leaving a more careful discussion of implementation and the presentation of the details that support this approach for Section 4.1. We focus on alternatives that keep the consumption of workers constant by adjusting the tax on capital, when necessary, so any improvements in the welfare of entrepreneurs are sure to represent a Pareto improvement. In particular, the tax on capital satisfies

$$\tau_s^k(\Theta_s, K_s) = \frac{(1-\alpha)(\Theta_s^{lf} K_s^{lf})^\alpha - (1-\alpha)(\Theta_s K_s)^\alpha}{K_s},$$

where (Θ_s, K_s) and $(\Theta_s^{lf}, K_s^{lf})$ denote TFP and the capital stock at state $s \in \mathbf{S}$ under the alternative allocation $a_s(\theta)$ and laissez-faire $a_s^{lf}(\theta)$, respectively.

Take any feasible vector of state-contingent capital supply K_s.[11] By feasibility, there exists a vector $q \in \mathbb{R}^S$ such that the required initial investment can be written as $\sum_i x^i = \sum_s q_s K_s$. We can then write the welfare of an entrepreneur as follows

$$W(a) = u(a_0 - \sum_s q_s K_s) + \beta \sum_s p_s \int u\left(a_s(\theta) R_s(\theta)\right) f_s(\theta) d\theta,$$

where $(K_s, \Theta_s, \hat{\theta}_s)$ are given by conditions 21, 22, and 24, respectively, and the return for type θ is given by

$$R_s(\theta) = 1 + \alpha(\hat{\theta}_s + \lambda_s(\theta - \hat{\theta}_s)^+)\left(\Theta_s K_s\right)^{\alpha-1} - \delta - \tau_s^k(\Theta_s, K_s).$$

To measure welfare gains from perturbation in date $t = 0$ consumption units, let $V(a) \equiv W(a)/u'(c_0)$.

Insufficient idiosyncratic risk-taking. Consider a perturbed allocation that increases idiosyncratic risk-taking in a given state $s \in \mathbf{S}$. Formally, the perturbation moves an infinitesimal amount Δ_s of net worth from agents from agents of type θ_l to a higher type $\theta_h > \theta_l$. We now state and discuss the following proposition.

Proposition 1. *Consider a perturbation that sets* $a_s(\theta_h) = a_s^{lf}(\theta_h) + \frac{\Delta_s}{f_s(\theta_h)}$, $a_s(\theta_l) = a_s^{lf}(\theta_l) - \frac{\Delta_s}{f_s(\theta_l)}$, *and leaves* $a(\theta) = a_s^{lf}(\theta)$ *for all remaining types.*

1. *Intensive margin:* If $\theta_h > \theta_l > \hat{\theta}_s$, then

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} = p_s Cov_s\left(\beta \frac{u'(c_s)}{u'(c_0)}, \frac{\partial R_s}{\partial \Theta_s} a_s\right) \frac{\partial \Theta}{\partial \Delta_s} > 0,$$

   where $\frac{\partial \Theta_s}{\partial \Delta_s} = \frac{\lambda_s}{K_s}(\theta_h - \theta_l)$.

2. *Extensive margin:* If $\theta_h = \hat{\theta}_s > \theta_l$, then

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} = p_s Cov_s\left(\beta \frac{u'(c_s)}{u'(c_0)}, \frac{\partial R_s}{\partial \hat{\theta}_s} a_s\right) \frac{d\hat{\theta}_s}{d\Delta_s} > 0,$$

   where $\frac{\partial \hat{\theta}_s}{\partial \Delta_s} = \left[a_s(\hat{\theta}_s) f_s(\hat{\theta}_s)\right]^{-1}$.

Proof. In the appendix. □

Whenever $\theta_h > \theta_l > \hat{\theta}_s$, the net worth reallocation occurs entirely along the intensive margin: resources are reallocated across active entrepreneurs, from a least productive to a most productive one. By construction, this perturbation increases aggregate TFP while keeping both the aggregate capital stock and the productivity of the marginal firm constant. The impact on welfare of a small increase in Δ_s is then given by

[11] Formally, $K_s = \sum_i \varphi_s^i x^i$ for some $x^i \geq 0$.

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} = p_s\left[\underbrace{\frac{\beta u^{'}\left(c_s(\theta_h)\right)}{u^{'}\left(c_0\right)}R_s(\theta_h) - \frac{\beta u^{'}\left(c_s(\theta_l)\right)}{u^{'}\left(c_0\right)}R_s(\theta_l)}_{\text{private risk-return trade-off}} + \underbrace{E_s\left[\frac{\beta u^{'}\left(c_s\right)}{u^{'}\left(c_0\right)}\frac{\partial R_s(\theta)}{\partial \Delta_s}a_s(\theta)\right]}_{\text{risk externality}}\right].$$

The first term captures the private trade-off between risk and return. By shifting resources to the high-type θ_h, the entrepreneur is able to enjoy higher returns, but at states with smaller marginal utility of consumption. From optimality in savings (condition 27), this term is equal to zero, as this trade-off is fully internalized by entrepreneurs. The second term, however, captures the effect that is not internalized by private agents and is fully mediated by (after tax) factor prices, being a consequence of pecuniary externalities. Interestingly, it can be easily shown that

$$E_s\left[\frac{\partial R_s(\theta)}{\partial \Theta_s}a_s(\theta)\right] = 0,$$

so the externality amounts only to a reallocation of resources among agents with different marginal valuations. From this fact, it becomes possible to rewrite the risk externality in a covariance form.

Hence, even though the perturbation does not affect the average (weighted) return, it affects its riskness. This results in a form of *fallacy of composition,* as the attempt of each individual entrepreneur to reduce her own exposure to risk ends up increasing the risk of all other entrepreneurs. The logic is the following: whenever entrepreneurs try to hedge return risk, they shift resources to less productive firms. This will reduce aggregate productivity in the economy and depress wages. However, since more productive firms use more labor per unit of capital (see equation 3), the difference in profitability between high and low productivity firms increases, amplifying the dispersion of returns and ex-ante risk faced by entrepreneurs. While the increase in return risk is mediated by prices, it is not internalized by individual entrepreneurs. This results in excessive hedging or insufficient idiosyncratic risk-taking.[12]

We refer to the mechanism in which agents do not internalize the impact of their actions on the risk faced by others as a *risk externality.* The second part of Proposition 1, which studies reallocation along an extensive margin, shows another example of this externality. A reallocation of resources from inactive entrepreneurs to any active agent tightens the capital rental market, displacing the marginal firm. In a limit in which this reallocation is received by the marginal agent, it has no effect on aggregate productivity (nor wages). Once again, the private part of trade-off is fully accounted for in laissez-faire and cancels out. However, an increase in the rental rate reallocates consumption from renters, active entrepreneurs with high returns and have lower marginal utility, toward lenders, inactive savers who have lower returns and higher marginal utility.

One particular distinction between the two cases of the perturbation above lies in the fact that, while a manipulation of the extensive margin acts purely through an increase in the rental rate, a change in

[12] In Section 5.2, we explore the sign and magnitude of the hedging demand and discuss how strategic complementarities amplify aggregate distortions. It is worth pointing out the direction of the inefficiency identified in Proposition 1 does not depend on the sign of the hedging demand. In particular, it is present in the case where preferences have unit elasticities of relative risk aversion and no hedging occurs under laissez-faire.

the intensive margin acts through a sum of changes in wages, capital taxes, and the rental rate. The representation of the welfare gain as a combination of each of these pieces is straightforward.[13]

A sufficient statistic approach. We can also use Proposition 1 for deriving approximate formulas for the measurement of welfare gains from policy changes in terms of observables . We first define a *coefficient of variation of returns* in the cross-section of agents at state $s \in \boldsymbol{S}$ as

$$CV_s\left(R_s\right) \equiv \frac{\sqrt{Var_s^{\omega}(R_s)}}{\mathbb{E}_s^{\omega}[R_s]},$$

where the variance and expectation are computed using the wealth density ω_s (Eq. 28). Then, by means of a first-order approximation, we provide the following.

Corollary 1. *The welfare gains from the interventions studied in Proposition 1 can be approximated as*

1. *Intensive margin ($\theta_h > \theta_l \geq \hat{\theta}_s$):*

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} \approx \gamma_s^{int} CV_s\left(R_s\right)^2 \frac{\partial \log \Theta_s}{\partial \Delta_s},$$

where $\gamma_s^{int} \equiv (1-\alpha) p_s M_s K_s$.

2. *Extensive margin ($\theta_h = \hat{\theta}_s > \theta_l$):*

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} \approx \gamma_s^{ext} \frac{\mathbb{E}_s^{\omega}[R_s] - \mathbb{E}_s^{\omega}[R_s|\, \theta < \hat{\theta}_s]}{\mathbb{E}_s^{\omega}[R_s]^2} \frac{\partial r_s}{\partial \Delta_s},$$

where $\gamma_s^{ext} \equiv \left(\lambda_s - 1\right) p_s M_s K_s$.

Corollary 1 illustrates how the magnitude of the externality depends on the level of idiosyncratic return risk faced by entrepreneurs. The externality along the intensive margin is increasing in the coefficient of variation of returns. Along the extensive margin, the externality depends on the difference of returns between active and inactive firms. In both cases, the externality would disappear in the absence of idiosyncratic return risk, as it would be the case without financial frictions or firm heterogeneity.

Underinvestment and excessive aggregate risk-taking. Consider now the optimality of the level and composition of investment in the economy.

Proposition 2. *Suppose that the following technologies are available and used in the laissez-faire equilibrium: i) a riskless technology that pays the same amount in all states $s \in \mathbf{S}$; ii) technologies that payoff φ_s on state s and zero otherwise, for $s \in \{l, h\}$.*

1. *Underinvestment:* Consider a perturbation that sets $a_s\left(\theta\right) = a_s^{lf}(\theta)\left(1 + \frac{\Delta}{K_s^{lf}}\right)$ for all $s \in \mathbf{S}$ and all types θ.

$$\left.\frac{\partial V}{\partial \Delta}\right|_{\Delta=0} = \beta \sum_s p_s Cov_s\left(u'(c_s), \frac{\partial R_s}{\partial K_s} a_s\right) \approx \left(1-\alpha\right) E\left[M_s CV_s(R_s)^2\right] > 0.$$

[13] Notice that $\frac{\partial V_0}{\partial r_s} = \beta p_s Cov_s\left(u^{'}\left(c_s\right), \left(1 - \lambda \mathbf{1}_{\theta \geq \hat{\theta}_1}\right) a_s\right)$, $\frac{\partial V_0}{\partial w_1} = -\beta p_s E_s\left[u^{'}\left(c_1\left(\theta\right)\right) \lambda_s \theta \mathbf{1}_{\theta \geq \hat{\theta}_s} a\right] \frac{1}{\Theta K}$ and $\frac{\partial V_0}{\tau_s^k} = -\beta p_s E_s\left[u^{'}\left(c_1\left(\theta\right)\right) a\right]$and the price consequences of perturbations in allocations are easy to compute.

2. *Excessive aggregate risk-taking:* Consider a perturbation that sets $a_l(\theta) = a_l^{lf}(\theta)\left(1 + \frac{\Delta}{K_l^{lf}}\varphi_l\right)$, $a_h(\theta) = a_h^{lf}(\theta)\left(1 - \frac{\Delta}{K_h^{lf}}\varphi_h\right)$, and $a_s(\theta) = a_s^{lf}(\theta)$ for the remaining states. If $\Theta_l \leq \Theta_h$, $K_l \leq K_h$, $Var_l^{\omega}\left((\theta - \hat{\theta}_s)^+\right) \geq Var_h^{\omega}\left((\theta - \hat{\theta}_s)^+\right)$, with at least one strict inequality, then

$$\left.\frac{\partial V}{\partial \Delta}\right|_{\Delta=0} = p_l \varphi_l Cov_l\left(\beta \frac{u'(c_l)}{u'(c_0)}, \frac{\partial R_l}{\partial K_l} a_l\right) - p_h \varphi_h Cov_h\left(\beta \frac{u'(c_h)}{u'(c_0)}, \frac{\partial R_h}{\partial K_h} a_h\right)$$
$$\approx (1-\alpha)\left[CV_l(R_l)^2 - CV_h(R_h)^2\right] > 0$$

Proof. In the appendix. □

Proposition 2 shows that there is *underinvestment* in the economy. This is in sharp contrast with the literature on fire-sales externality that emphasizes overinvestment. The reason for the different result is that there is an interaction between the level of capital and the amount of return risk when we allow for firm heterogeneity. A larger investment leads to higher wages next period, reducing the difference in profitability between high productivity and low productivity firms as well as ex-ante return risk. Since entrepreneurs do not internalize the impact of additional investment in the return risk faced by others, investment is inefficiently depressed from an aggregate perspective.

The proposition also shows that there is *excessive aggregate risk-taking*, meaning that a share larger than it is socially desirable is being invested in projects that payoff more in good than bad states. In order to understand this result, consider the effect of increasing by one unit investment in a technology that pays off in state s. The increase in capital stock reduces idiosyncratic return risk, and it increases welfare by $\kappa CV_s(R_s)^2$. Hence, shifting resources to states where the variance of returns is already high or the average is low is welfare improving. The theory predicts that the coefficient of variation is decreasing in capital and aggregate productivity and it is increasing in the dispersion of TFP of active firms, so it will tend to be higher in bad times. We can also test this prediction directly by looking at firm-level data, which we now turn to.

3.3 Testing the efficiency of aggregate risk-taking

We analyze standardized firm-level data from the World Bank Enterprise Survey. As far as we know, this is the most comprehensive publicly available dataset including standardized measures of input and output for the manufacturing and service sectors. We discard low income countries according to the World Bank classification and focus on firms in the manufacturing sector for comparability.

We are interested in comparing the distribution of productivity and returns during periods of economic expansion and economic contraction. The survey provides firm-level TFP measures obtained from estimation of production functions at the two-digit ISIC code sector level. We remove country fixed effects and split firm-years in two events: 2008-9, which are the years of the Great Recession, and remaining years. For brevity, we call the first set the contraction years and the remainder the expansion years.

Table 1, below, reports some key statistics of the data. At an aggregate level, all occurrences of

Table 1: Summary Statistics

	Years	Country-years Surveyed	Country-years with GDP/capita contraction	Firms
Expansion	2006-7, 2010-16	27	2	13230
Contraction	2008, 2009	5	5	2840
		Firm-level log-TFP measures		
	Mean	Variance	Skewness	Kurtosis
Expansion	1.6005	0.5121	0.1883	2.1549
Contraction	1.5134	0.5528	0.1555	2.2809

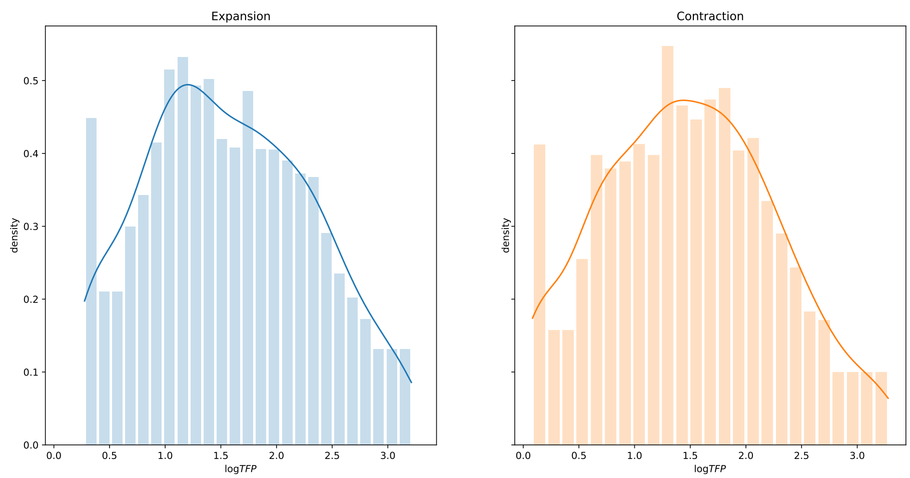


Figure 1: The distribution of log-TFP

negative GDP growth (3 cases) in the data occur in the contraction years. The maximum GDP growth rate registered in one of these years (1.2% for Kazakhstan, 2009) is still below the minimal growth rate recorded in the expansion years (2.6% for Sweden, 2014). Also, all country-year pairs classified as contraction years have negative per-capita GDP growth. In contrast, among those classified as expansion years, only two countries ever register negative per-capita GDP growth.[14]

To be conservative with regards to measurement error, we discard the top and bottom five percent of the TFP distribution for each country-year pair, and pool observations within each of the two events. The bottom panel of Table 1 displays key statistics of the pooled cross-sectional distributions. Figure 1 displays the density estimates for the distribution of firm-level TFP (in logarithms) across the expansion and contraction events from the data.

As can be seen from table 1, there are substantial differences in the distribution of firm-level TFP between expansion and contraction periods. Contractions are associated with a decline in average TFP and an increase in dispersion. This is consistent with the evidence on countercyclical (micro)

[14]These two outliers are Jordan and Lebanon in 2013, which despite growth rates above 2%, display per-capita contractions due to the population inflow from the Syrian refugee crisis.

Table 2: Welfare gains of perturbations

	Contraction	Expansion	Difference
Welfare gains	0.66	0.48	0.17

uncertainty in Bloom (2014); Campbell et al. (2001). Our results in proposition 2 shows how one can use these variations in uncertainty to evaluate the efficiency of aggregate risk-taking. Recently, changes in higher moments of the firm size distribution have attracted attention, as in the work of Salgado et al. (2016). By taking a higher-order approximation, we can express the welfare impact of reducing aggregate risk-taking in terms of higher moments of the return distribution.

Corollary 2. *Suppose technologies that payoff φ_s on state s and zero otherwise is available and used in the laissez-faire equilibrium. Consider a perturbation that sets $a_s(\theta) = a_s^{lf}(\theta)\left(1 + \frac{\Delta_s}{K_s^{lf}}\varphi_s\right)$. Then,*

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} \approx (1-\alpha)\left[CV_s(R_s)^2 - Skew_s(R_s) + \frac{1}{2}Kur_s(R_s)\right],$$

where $Skew_s(R_s) = \mathbb{E}_s^{\omega}\left[\left(\frac{R_s - \mathbb{E}_s^{\omega}[R_s]}{\mathbb{E}_s^{\omega}[R_s]}\right)^3\right]$ and $Kur_s(R_s) = \mathbb{E}_s^{\omega}\left[\left(\frac{R_s - \mathbb{E}_s^{\omega}[R_s]}{\mathbb{E}_s^{\omega}[R_s]}\right)^4\right]$.

Corollary 2 shows the welfare gains of investing one extra unit on a project that pays off on state s. Hence, the difference $\frac{\partial V}{\partial \Delta_l} - \frac{\partial V}{\partial \Delta_h}$ captures the gain of reallocating one unit of investment from the technology that pays off in the good state h to the bad state l. These derivatives are measured in terms of utils, but if we divide by the marginal utility of consumption in zero, they can be expressed in terms of permanent consumption changes. Table 2 shows the gains of increasing by one unit investiment in a technology that pays off in a contraction or an expansion as well as the difference between the two using the expression in corollary 2. The difference measures the welfare gains of reducing aggregate risk-taking, i.e., shifting resources from expansions to contractions. First, notice this difference is positive, an indication of excessive aggregate risk-taking. The magnitude of the welfare gains of this perturbation can interpreted as following: a reduction in aggregate risk-taking of one percent of total wealth leads to a welfare gain equivalent to an increase in permanent consumption of 0.17%. Similarly, an average of the welfare gains of increasing capital in contractions and expansions give the welfare gains of increasing the aggregate capital stock.

The results on table 2 indicates that there are potentially large welfare gains of regulating the aggregate risk-taking in the economy as well as the overall level of investment. However, these perturbations can only provide an estimate of small perturbations around the laissez-faire. In the next section, we derive the optimal regulation in this economy and compute the welfare gains obtained by the optimal policy.

4 Optimal investment and risk-taking policy

4.1 Implementation

When we consider the set of taxes described, the characterization of the equilibrium follows through with minor changes relative to laissez-faire. First, worker compensation and consumption satisfy

$$c_s^w = w_s + \tau_s^k K = (1-\alpha)\left(\Theta_s K_s\right)^{\alpha} + \tau_s^k K_s \tag{29}$$

and rental rates are given by

$$r_s = \hat{\theta}_s \left(\Theta_s K_s\right)^{\alpha-1} - \delta - \tau_s^k. \tag{30}$$

Capital demand as a function of wealth, profits and Ricardian rents are unchanged, and still follow equation 20 and 20. The characterization of TFP as a function of the wealth distribution still follows 22. The key equilibrium condition for the rental market, still satisfies 20.

The main change lie in the Euler equations, that take into account the presence of contingent taxes. The saving Euler equation is now given by

$$M_s = \left[1 - \tau_s^a(\theta)\right] \beta \left(\frac{c_s(\theta)}{c_0}\right)^{-\sigma} R_s(\theta), \tag{31}$$

where $R_s(\theta) = 1 + \alpha(\hat{\theta}_s + \lambda_s(\theta - \hat{\theta}_s)^+)\left(\Theta_s K_s\right)^{\alpha-1} - \delta - \tau_s^k$ and $c_s(\theta) = \omega_s(\theta) K_s R_s(\theta) / f_s(\theta)$, when written as a function of the wealth density.

After taking into account market-clearing conditions for contingent assets and the zero-profit condition for the sellers of these securities, we obtain the set of equations that are analogous to the equation that determines of the shape of the wealth distribution (Eq. 28) and capital accumulation in laissez-faire. When taxes are present, these equations become, respectively:

$$\omega_s(\theta) = \frac{\left[1 - \tau_s^a(\theta)\right]^{\frac{1}{\sigma}} \left[R_s(\theta)\right]^{\frac{1-\sigma}{\sigma}} f_s(\theta)}{\int \left[1 - \tau_s^a(\theta)\right]^{\frac{1}{\sigma}} \left[R_s(\theta)\right]^{\frac{1-\sigma}{\sigma}} df_s(\theta)}, \tag{32}$$

for each $\theta \in \mathbb{R}$ and $s \in \boldsymbol{S}$, and the investment equation (Eq. 13).

An allocation is *implementable* if there exists an associated tax system and set of prices which jointly form a competitive equilibrium with taxes.

Proposition 3. *An allocation is implementable, if and only if, the following conditions are verified:*

1. $x_i \geq 0, \forall i \in \boldsymbol{I}$, $a_0 = c_0 + \sum x_i$ *and* $K_s = \sum_i \varphi_s^i x_s^i$.
2. $\exists M \in \mathbb{R}_{++}^S$ *such that, for each* $i \in \boldsymbol{I}$,

$$\begin{aligned} \sum_s p_s M_s \varphi_s^i &= 1, \textit{if } x_i > 0 \\ \sum_s p_s M_s \varphi_s^i &\leq 1, \textit{otherwise.} \end{aligned} \tag{33}$$

3. For each $s \in \boldsymbol{S}$, $\omega_s(\bullet) : \mathbb{R} \to \mathbb{R}_+$ defined in equation 23 constitutes a density, that is, it satisfies

$$\int \omega_s(\theta)\, d\theta = 1.$$

4. Consumption satisfies

$$c_s(\theta) = \frac{\omega_s(\theta)\, K_s}{f_s(\theta)} R_s(\theta),$$

where

$$R_s(\theta) := 1 + \alpha(\underline{\theta}_s + \lambda_s(\theta - \hat{\theta}_s)^+)\, (\Theta_s\, K_s)^{\alpha-1} - \delta - [c_s^w - (1-\alpha)\, (\Theta_s K_s)^\alpha]\, K_s^{-1},$$

with Θ_s given by equation 22 and $\hat{\theta}_s$ implicitly defined in equation 24.

Proof. Verifying that a competitive equilibrium with taxes satisfies these equations is easy and has its main steps done in the text of Section 4.1. We will prove the converse through construction. First, we set

$$\tau_s^k = [c_s^w - \alpha\, (\Theta K)^\alpha]\, K^{-1}.$$

Take any $\exists M \in \mathbb{R}_{++}^S$ that satisfies condition 33 above. Then, optimality in savings implies

$$M_s = [1 - \tau_s^a(\theta)]\, \beta \left(\frac{c_s(\theta)}{c_0}\right)^{-\sigma} R_s(\theta) \implies \tau_s^a(\theta) = \frac{\beta c_s(\theta)^{-\sigma} R_s(\theta) - M_s c_0^{-\sigma}}{\beta c_s(\theta)^{-\sigma} R_s(\theta)}.$$

All additional conditions for a competitive equilibrium with taxes can then be easily verified. □

4.2 Characterization of optimality

Using the implementation result obtained in Proposition 3, it is possible to write the constrained Pareto problem as

$$\max_{\{K,\omega,\hat{\theta},\tau^k\}} u\left(a_0 - \sum x^i\right) + \beta E\left[u(\frac{\omega}{f}\, R \sum_i \varphi x^i)\right] \tag{34}$$

subject to

$$\sum_s p_s u\left((1-\alpha)(\Theta_s \sum_i \varphi_s^i x^i)^\alpha + \tau_s^k \sum_i \varphi_s^i x^i\right) \geq u_w, \tag{35}$$

for each $s \in \boldsymbol{S}$,

$$1 = \lambda_s \int_{\hat{\theta}_s}^{\infty} \omega_s(\theta)\, d\theta \tag{36}$$

and

$$1 = \int \omega_s(\theta)\, d\theta, \tag{37}$$

where the two definitions $R_s(\theta) = 1 + \alpha(\hat{\theta}_s + \lambda_s(\theta - \hat{\theta}_s)^+)\left(\Theta_s \sum_i \varphi x^i\right)^{\alpha-1} - \delta - \tau_s^k$and $\Theta_s = \lambda_s \int_{\hat{\theta}_s}^{\infty} \theta \omega_s(\theta)\, d\theta$ can be used explicitly.

Equation 35 indicates the worker utility needs to achieve a minimum level. The level itself, u_w, is a free parameter than can be varied to trace out the Pareto Frontier. One particular case of interest is when this utility level is set to the same value that workers would achieve under laissez-faire. Equation 36 can be seen as a constraint that implicitly determines the marginal firm, originating from the rental market clearing requirement. Equation 37 originates from market clearing in contingent claims, but can be more simply interpreted as a constraint imposing that the relative wealth distribution has a well-defined density.

The first-order conditions and some additional analysis are provided in the appendix. The key conclusions are summarized in the next proposition. Let $E_{\omega_s}[x] = \int x\,\omega_s(\theta)\,d\theta$ denote a mean using the relative wealth distribution as the weighting measure and, analogously, $Cov_{\omega_s}(x, y) = E_{\omega_s}[x\,y] - E_{\omega_s}[x]\,E_{\omega_s}[y]$ describe a covariance against that measure.

Proposition 4. *A constrained optimal allocation needs to satisfy the following set of intertemporal conditions:*

1. *Among all inactive entrepreneurs in each state $s \in \boldsymbol{S}$, that is $\forall \theta < \hat{\theta}_s$, the marginal rate of substitution (MRS) is equalized*
$$\frac{\beta u'(c_s(\theta))\, R_s(\theta)}{u'(c_0)} = \mu_s, \tag{38}$$
implying also the equalization of consumption and asset holdings, since $R_s(\theta)$ is constant in that set.

2. *Active entrepreneurs, that is $\theta \geq \hat{\theta}_s$, face an affine schedule of additional intertemporal wedges, with their MRS satisfying*
$$\frac{\beta u'(c_s(\theta))\, R_s(\theta)}{u'(c_0)} = \mu_s - \frac{\Psi_s^{\hat{\theta}}}{\omega_s\left(\hat{\theta}_s\right)} - \Psi_s^{\Theta}\left(\theta - \hat{\theta}_s\right), \tag{39}$$
where
$$\Psi_s^{\hat{\theta}} := Cov_{\omega_s}\left(\frac{\beta u'(c)}{u'(c_0)}, \frac{\partial R}{\partial \hat{\theta}_s}\right) K_s \quad \text{and} \quad \Psi_s^{\Theta} := Cov_{\omega_s}\left(\frac{\beta u'(c)}{u'(c_0)}, \frac{\partial R}{\partial \Theta_s}\right) K_s. \tag{40}$$

3. *The capital investment Euler equation for technology i is given by*
$$1 \geq \beta E\left[\frac{u'(c)\left(R + \tau^k\right)}{u'(c_0)} \frac{a}{K} \varphi^i\right] + E\left[\Psi^k \varphi^i\right], \tag{41}$$
with equality whenever $x_i > 0$, where Ψ^k has realizations $\Psi_s^k := Cov_{\omega_s}\left(\frac{\beta u'(c)}{u'(c_0)}, \frac{\partial R}{\partial K}\right) K_s$.

Proof. In Appendix A.2. □

The defining feature of the constrained optimal allocation is the emergence of an affine schedule of wedges in the saving Euler equations, as indicated by equations 38 and 39. An extensive margin wedge appears in the intercept $\Psi_s^{\hat{\theta}}/\omega_s\left(\hat{\theta}_s\right)$. It is evident in the comparison between the marginal rates of substitution of the marginal $\left(\theta = \hat{\theta}_s\right)$ and the inactive entrepreneurs ($\theta < \hat{\theta}_s$). An additional, intensive margin wedge, is linear in productivity and emerges between any two active entrepreneurs. Let us focus first on the intuition behind the extensive margin wedge.

First, the rates of return earned by both the marginal and inactive entrepreneurs are the same, as neither obtains any Ricardian rents. In the absence of interventions, they would save the same and have the same net worth at $t = 1/2$. There is however an external effect from the determination of the marginal firm that would not be taken into account in private decisions. When additional resources are allocated to active agents, the capital rental market tightens and the marginal agent is displaced: a slightly more productive entrepreneur takes his or her place and rental rates increase. This increase, in its turn compresses Ricardian rents, reducing the rate of return risk in the economy. With a positive value for $\Psi_s^{\hat{\theta}}$, rents accrue to agents with lower marginal rates of substitution and there is an insurance gain from resource allocation toward active firms. At the optimal allocation, the marginal gains from manipulating this margin are exactly offset by the marginal losses from saving distortions.

The optimal wedge schedule also features a slope term, which is originates from an analogous effect working through the labor market. When net worth is reallocated among active firms, from the less to the more productive, aggregate productivity is increased and the labor market tightens. Therefore, the aggregate wage bill increases and, as high productivity firms hire more labor, they bear relatively more of the increase. Consequently, an increase in aggregate productivity also compresses exceptional profits and reduces rate of return dispersion in the economy. This linear piece of the wedge is increasing in marginal productivity, because resources allocated to higher productivity firms have bigger impacts on aggregate TFP.

Additionally, Equation 41 shows that the planner implements two forms of wedges on aggregate savings. First, as indicated by the presence of τ^k, the planner internalizes the value of the revenue raised from capital taxes. Optimal ex ante interventions take into account how the presence of $t = 1$ would distort incentives to save and offset that effect. Second, with a positive value for Ψ_s^k, the planer attempts to increase capital accumulation through an additional saving wedge. Again, increased savings serve to reduce the marginal product of capital and, ultimately, compress exceptional returns. Again, agents with higher productivity bear relatively more of the reduction of returns on capital.

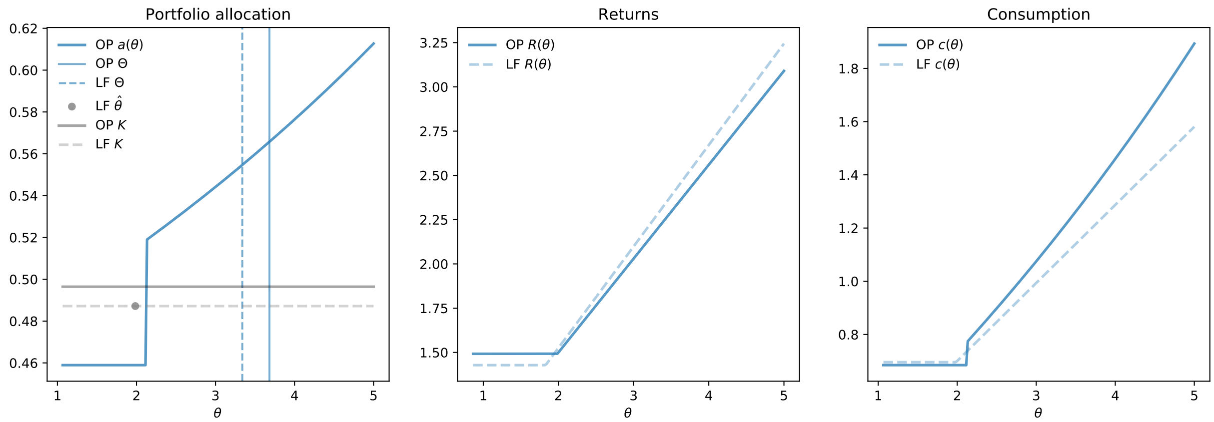
4.3 Optimal Policy – An illustration

We now illustrate the key features of the optimal policy. We use a set-up featuring logarithmic preferences ($\sigma = 1$), a single capital investment technology ($\boldsymbol{I} = \{1\}$), and two states of the world. The distribution of firm-level TFP is a (truncated) generalized Pareto distribution, which allows for simple shocks to key moments.[15] We first focus on features of the allocations that are shared across all states of world, while using the high output state for the purpose of illustration. In all three panels of Figure 2 , dashed lines represent the laissez-faire benchmark, while solid lines represent the outcomes after optimal interventions.

On the left-most panel, we have optimal asset holdings. They can be easily compared to the dashed horizontal line in gray, which denotes the constant asset holdings that occur in laissez-faire. Notice that, under the optimal policy, all inactive savers have asset holdings that lie below the laissez-faire level. At

[15] See Appendix B for the analysis of equilibrium allocations in the absence of intervention.

Figure 2: Portfolio, returns, and consumption



Note: OP stands for *optimal policy* and LF stands for *laissez-faire*.

the marginal agent, there is a large discrete upward jump in asset holdings, while a continuous increase follows for higher productivity agents. The gray dot denotes the marginal agent under laissez-faire and, by inspecting the point at which the upward jump in assets occurs under the optimal policy, it is easy to verify an increase in the marginal type. The vertical lines describe aggregate TFP, which shows a significant increase as we compare its value under laissez-faire (the vertical dashed line in blue) with its value under the optimal policy (the solid vertical line in blue).

The rationale for the intervention is made explicit in the intermediate panel. By distorting risk-taking and wealth accumulation, the planner can change the profile of risky returns savers face. When we compare the optimal policy (the solid line) with returns under laissez-faire (the dashed line), we see that, after optimal interventions, returns obtained by inactive agents are higher and also that returns obtained by active producers increase less steeply with individual productivity. The key consequence of optimal interventions is that a risk externality, working through factor prices and Ricardian rents, is corrected.

The right-most panel plots consumption. More dispersed asset holdings under the optimal policy interact with less dispersed returns. Interestingly, in the final outcome, consumption is riskier, but the gains from the correction of the externality and increased level are sufficient to ensure all agents are made better off from an ex-ante perspective.

Appendix B.1 studies the cyclical properties of optimal policies, by focusing on differences across states of the world under different scenarios. There are three of these scenarios: a situation in which the two states of the world differ by the mean of of the TFP distribution, its dispersion, or its skewness. The main conclusion is that the policy response is stronger for: lower mean, higher variance, and more left-skewed productivity distributions. Each of these three features is typically associated with macroeconomic contractions, as we have revisited in the literature review. While the comparisons of responses to moment shocks are done in isolation in that extension, for analytical clarity, the results indicate a complementarity: policies should be strongly anti-cyclical because optimal responses to the cyclical behavior of the first three moments point in the same direction.

Table 3: Laissez-faire and optimal policy comparison

	Expansion			Contraction		
	Laissez-faire	Optimal Policy	Ratio	Laissez-faire	Optimal Policy	Ratio
K	0.431	0.439	1.018	0.431	0.439	1.018
Θ_s	1.0	1.381	1.381	0.935	1.39	1.486
$\hat{\theta}_s$	0.205	0.402	1.955	0.169	0.392	2.31
Y_s	0.755	0.846	1.12	0.739	0.848	1.148
$R_{gross}\left(\hat{\theta}_s\right)$	1.02	1.087	1.066	1.004	1.087	1.083
τ^k	0.0	-0.139	·	0.0	-0.148	·
$a_s\left(\overline{\theta}_s\right)$	0.431	1.482	3.437	0.431	2.456	5.696
$a_s\left(\hat{\theta}_s\right)$	0.431	0.445	1.033	0.431	0.449	1.041
$a_s\left(\underline{\theta}_s\right)$	0.431	0.363	0.842	0.431	0.356	0.825

5 Extensions and additional analysis

5.1 The optimal policy in the data

In Table 3, we display a comparison of the defining characteristics of the laissez-faire and the optimal policies. We see different margins of intervention being used. First, there is a small increase in the total level of savings, of about 2%. A rationale is offered in the previous analysis of optimal policies: an increase in capital stocks tightens the labor market, compressing exceptional returns and Ricardian rents. It is noteworthy that in light of the other forms of interventions, the savings increase is quantitatively modest, and has a small role in explaining an output increase of mode than 12% across the the two states world.

This output increase is brought about by a large aggregate TFP gain (the capital augmenting value Θ increases by 38.1% in the expansion event and by 48.6% in the contraction event). There are noticeable displacements of the marginal firm and optimal reallocation of net worth toward the most productive firms. Notice also the policy is more active in the contraction event, meaning that it more aggressively induces capital reallocation. As there are relevant increases in total output, the pre-tax wage is largely increased (by the same magnitude, 12% in the expansion event and 14.8% in a contraction). This potentially large wage increase would mean a large redistribution from entrepreneurs to workers, which is offset by a labor tax and substantial $\tau_s^k < 0$ rebate to entrepreneurs. In terms of certainty equivalent consumption, the optimal policy raises welfare by 3.2%.

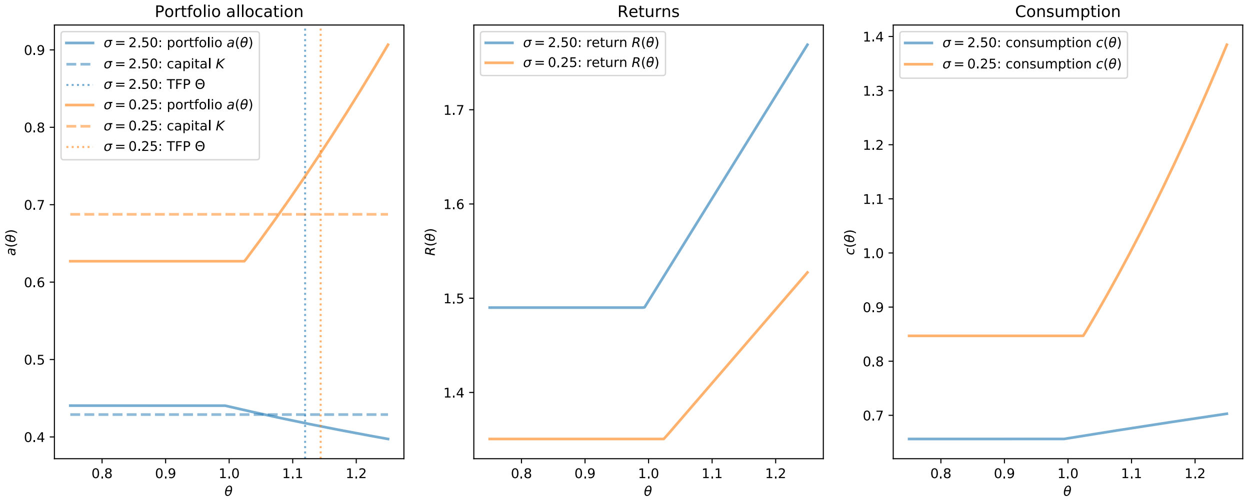
5.2 Portfolio choice and risk aversion

The case of log preferences is particularly tractable but it is also rather special. From the optimality condition for $a_s(\theta)$, one obtains the marginal benefit of having one extra unit of goods in period 1, $(a_s(\theta)R_s(\theta))^{-\sigma} R_s(\theta)$,is equalized across states. This implies that the portfolio choice satisfies

$$\log a_s(\theta) = \log a_s(\theta_o) - \frac{\sigma - 1}{\sigma} \left[\log R_s(\theta) - \log R_s(\theta_o)\right],$$

for any idiosyncratic states θ and θ_o.

Figure 3: Portfolio, returns, and consumption: different risk aversion



Hence, the entrepreneur will shift resources to the high-return states if, and only, if $\sigma < 1$. Whenever $\sigma > 1$, then entrepreneurs tilt their portfolio toward low-return states. The different behavior is the result of opposing income and substitution effects. The substitution effect induces the entrepreneur to shift resources to the high return state, while the income effect induces the entrepreneur to shift resources to the low consumption states. In the case of log utility, $\sigma = 1$, income and substitution effects exactly cancel out and the portfolio choice is independent of returns.

In equilibrium, the hedging behavior of entrepreneurs will have implications for the determination of returns and aggregate productivity. Figure 5.2 shows the portfolio choice in a case where the substitution effects dominates ($\sigma = 0.25$) and one where the income effect dominates ($\sigma = 2.5$).

Consider, as an illustration, the in which case $\sigma > 1$. Assets $a_s(\theta)$ and consumption are equalized across inactive firms, $\theta < \hat{\theta}$. For $\theta \geq \hat{\theta}$, the amount of assets falls with productivity θ due to the dominant income effect. Entrepreneurs decide to hedge their productivity risk by shifting resources from good to bad states, so the perceived variability in their consumption is reduced. However, the more intense is the reallocation of resources toward less productive firms, the lower is aggregate productivity. A lower TFP, in its turn, tends to increase the variability of returns and induces the entrepreneurs to shift even more resources to less productive firms. We see, therefore, a strategic complementarity at work. This strategic complementarity amplifies misallocation and the dispersion in asset returns. It is stronger the higher is the coefficient of relative risk aversion, but it is present in all economies.

Despite of the potential impact of hedging behavior on aggregate productivity, entrepreneurs internalize the fact that they will produce less by sending resources to low-return states. The desire to reduce consumption volatility though is strong enough to compensate for the reduction in production when the income effect in portfolio choice dominates. However, entrepreneurs do not take into account the implications of their actions on equilibrium prices and how prices can act as natural hedges, by affecting differently firms with different productivity levels. We will see that there will be a role for policy interventions even when entrepreneurs have access to a rich menu of assets.

6 Conclusion

We have studied the interactions of firm-level uncertainty and risk taking and identified a pecuniary externality that affects the distribution of risk in the economy. Credit constraints create exceptional profits (Ricardian rents), depress output, and create dispersion in asset returns. Optimal taxes stimulate risk taking and resource reallocation, mitigating the externalities, and increasing total output. In a quantitative exploration, the gains from interventions are shown to be potentially large: about 12% in terms of output and 3.2% in terms of certainty equivalent consumption.

The environment features endogenous savings and wealth distributions, despite its static essence. Some paths for further research include the exploration of consequences for firm dynamics, entry, exit, and the shaping of the economy's sectoral composition.

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A Proofs

A.1 Proofs of results from Section 3

Proof of proposition 1.

Proof. Consider the following perturbation of the laissez-faire allocation a^{lf}:

$$a_s(\theta) = \begin{cases} a_s^{lf}(\theta_h) + \frac{\Delta_s}{f_s(\theta_h)} & \text{for } \theta = \theta_h \\ a_s^{lf}(\theta_l) - \frac{\Delta_s}{f_s(\theta_l)} & \text{for } \theta = \theta_l \\ a_s^{lf}(\theta) & \textit{otherwise} \end{cases}$$

Intensive margin: suppose $\theta_h > \theta_l > \hat{\theta}_s^{lf}$. Notice that K_s and $\hat{\theta}_s$ are the same as in the laissez-faire allocation, but Θ_s will be different under the perturbed allocation. The derivative of V with respect to Δ_s is given by:

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} = \beta p_s \left[\underbrace{c_s(\theta_h)^{-\sigma} R_s(\theta_h) - c_s(\theta_l)^{-\sigma} R_s(\theta_l)}_{\text{private risk-return trade-off}} + \underbrace{\int c_s(\theta)^{-\sigma} \frac{\partial R_s(\theta)}{\partial \Theta_s} \frac{\partial \Theta_s}{\partial \Delta_s} a_s(\theta) f_s(\theta) d\theta}_{\text{risk externality}} \right]$$

where

$$\frac{\partial R_s(\theta)}{\partial \Theta_s} = -\alpha(1-\alpha) \left[\frac{\hat{\theta}_s + \lambda_s(\theta - \hat{\theta}_s)^+}{\Theta_s} - 1 \right] (\Theta_s K_s)^{\alpha-1},$$
$$\frac{\partial \Theta_s}{\partial \Delta_s} = \lambda_s \frac{\theta_h - \theta_l}{K_s}.$$

Using the optimality condition for the entrepreneur's portfolio problem, we obtain

$$\beta \frac{c_s(\theta)^{-\sigma}}{c_0^{-\sigma}} R_s(\theta) = M_s \Rightarrow c_s(\theta) = \left(\frac{\beta R_s(\theta)}{M_s} \right)^{\sigma} c_0.$$

The condition above implies the private risk-return trade-off is equal to zero. Notice that in the absence of differences in marginal utility, the second term is equal to zero

$$\int \frac{\partial R_s(\theta)}{\partial \Theta_s} \frac{\partial \Theta_s}{\partial \Delta_s} a_s(\theta) f_s(\theta) d\theta = -\alpha(1-\alpha)\lambda_s(\theta_h - \theta_l)(\Theta_s K_s)^{\alpha-1} \int \left[\frac{\hat{\theta}_s + \lambda_s(\theta - \hat{\theta}_s)^+}{\Theta_s} - 1 \right] \omega_s(\theta) d\theta = 0.$$

This allow us to rewrite the expression above for the welfare as a covariance

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} = \beta p_s Cov_s \left(u'(c_s), \frac{\partial R_s}{\partial \Theta_s} a_s \right) \frac{\partial \Theta_s}{\partial \Delta_s} > 0,$$

since $u'(c_s(\theta))$ and $\frac{\partial R_s}{\partial \Theta_s}$ are decreasing in θ.

Extensive margin. Suppose now $\theta_h > \hat{\theta}_s > \theta_l$. In this case, $\hat{\theta}_s$ will also be affected

$$\frac{\partial \hat{\theta}_s}{\partial \Delta_s} = \frac{1}{a_s(\hat{\theta}_s) f_s(\hat{\theta}_s)}.$$

The effect on productivity is given by

$$\begin{aligned}\frac{\partial \Theta_s}{\partial \Delta_s} &= \frac{\lambda_s}{K_s}\left[\theta_h - \hat{\theta}_s a_s(\hat{\theta}_s) f_s(\hat{\theta}_s) \frac{\partial \hat{\theta}_s}{\partial \Delta_s}\right] \\ &= \frac{\lambda_s}{K_s}\left[\theta_h - \hat{\theta}_s\right].\end{aligned}$$

If $\theta_h \to \hat{\theta}_s$, then the perturbation has no effect on TFP, only on $\hat{\theta}_s$. Focusing on this case, we obtain

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} = \beta p_s \int c_s(\theta)^{-\sigma} \frac{\partial R_s(\theta)}{\partial \hat{\theta}_s} \frac{\partial \hat{\theta}_s}{\partial \Delta_s} a_s(\theta) f_s(\theta) d\theta,$$

where

$$\frac{\partial R_s(\theta)}{\partial \hat{\theta}_s} = \alpha \left(1 - \lambda_s \mathbf{1}_{\theta \geq \hat{\theta}_s}\right) (\Theta_s K_s)^{\alpha - 1}.$$

As before, the expression can be written as a covariance, since the following term is equal to zero

$$\int \left(1 - \lambda_s \mathbf{1}_{\theta \geq \hat{\theta}_s}\right) a_s(\theta) f_s(\theta) d\theta = 0.$$

This allow us to write the derivative of V as follows

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} = \beta p_s Cov_s\left(u'(c_s), \frac{\partial R_s}{\partial \hat{\theta}_s} a_s\right) \frac{\partial \hat{\theta}_s}{\partial \Delta_s}.$$

□

Proof of corollary 1.

Proof. **Intensive margin.**

Using the optimality condition for the entrepreneur to eliminate the marginal utility of consumption, we obtain

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} = -p_s M_s c_0^{-\sigma} (1-\alpha) Cov_s^{\omega}\left(R_s^{-1}, R_s\right) \lambda_s \frac{\theta_h - \theta_l}{\Theta_s}.$$

Using a first-order expansion, we obtain

$$R_s^{-1} \approx \mathbb{E}_s^w[R_s]^{-1} - \frac{R_s - \mathbb{E}_s^w[R_s]}{\mathbb{E}_s^w[R_s]^2}.$$

Plugging the approximation in the expression for the derivative of V:

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} \approx \kappa_s^{int} CV_s\left(R_s\right)^2,$$

where $\kappa_s^{int} \equiv c_0^{-\sigma} p_s M_s (1-\alpha)\lambda_s \frac{\theta_h - \theta_l}{\Theta_s}$.

Extensive margin. The derivative of V with respect to Δ_s can be written as follows

$$\begin{aligned}\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} &= c_0^{-\sigma} p_s M_s \alpha \left(\Theta_s K_s\right)^{\alpha-1} \int \frac{1}{R_s(\theta)} \left(1 - \lambda_s \mathbf{1}_{\theta \geq \hat{\theta}_s}\right) \frac{a_s(\theta) f_s(\theta)}{a_s(\hat{\theta}_s) f_s(\hat{\theta}_s)} d\theta \\ &= c_0^{-\sigma} \frac{p_s M_s \alpha \left(\Theta_s K_s\right)^{\alpha-1}}{\omega_s(\hat{\theta}_s)} \left[\mathbb{E}_s^{\omega}\left[\frac{1}{R_s}\right] - \mathbb{E}_s^{\omega}\left[\frac{1}{R_s}\middle| \theta \geq \hat{\theta}_s\right]\right],\end{aligned}$$

where we used the fact that $\lambda_s \omega_s(\theta)$ corresponds to the density of θ conditional on $\theta \geq \hat{\theta}_s$. Using the approximation for R_s^{-1}, we obtain

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} \approx c_0^{-\sigma} \frac{p_s M_s \alpha \left(\Theta_s K_s\right)^{\alpha-1}}{\omega_s(\hat{\theta}_s)} \frac{\mathbb{E}_s^{\omega}\left[R_s \middle| \theta \geq \hat{\theta}_s\right] - \mathbb{E}_s^{\omega}\left[R_s\right]}{\mathbb{E}_s^{\omega}\left[R_s\right]^2}.$$

The unconditional average return can be written as follows

$$\mathbb{E}_s^{\omega}\left[R_s\right] = \left(1 - \frac{1}{\lambda_s}\right) \mathbb{E}_s^{\omega}\left[R_s \middle| \theta < \hat{\theta}_s\right] + \frac{1}{\lambda_s} \mathbb{E}_s^{\omega}\left[R_s \middle| \theta \geq \hat{\theta}_s\right].$$

Rearranging

$$\frac{1}{\lambda_s}\left(\mathbb{E}_s^{\omega}\left[R_s \middle| \theta \geq \hat{\theta}_s\right] - \mathbb{E}_s^{\omega}\left[R_s\right]\right) = \left(1 - \frac{1}{\lambda_s}\right)\left(\mathbb{E}_s^{\omega}\left[R_s\right] - \mathbb{E}_s^{\omega}\left[R_s \middle| \theta < \hat{\theta}_s\right]\right).$$

This allow us to obtain the expression

$$\begin{aligned}\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} &\approx c_0^{-\sigma} p_s M_s \frac{\lambda_s - 1}{\omega_s(\hat{\theta}_s)} \alpha \left(\Theta_s K_s\right)^{\alpha-1} \frac{\mathbb{E}_s^{\omega}\left[R_s\right] - \mathbb{E}_s^{\omega}\left[R_s \middle| \theta < \hat{\theta}_s\right]}{\mathbb{E}_s^{\omega}\left[R_s\right]^2} \\ &= \kappa_s^{ext} \frac{\mathbb{E}_s^{\omega}\left[R_s\right] - \mathbb{E}_s^{\omega}\left[R_s \middle| \theta < \hat{\theta}_s\right]}{\mathbb{E}_s^{\omega}\left[R_s\right]^2},\end{aligned}$$

where $\kappa_s^{ext} \equiv c_0^{-\sigma} p_s M_s \frac{\lambda_s - 1}{\omega_s(\hat{\theta}_s)} \alpha Y_s^{-\frac{1-\alpha}{\alpha}}$. □

Proof of proposition 2.

Proof. **Under-investment:** Suppose that a riskless technology is available and it is used in the laissez-faire equilibrium. Consider the following perturbation $a_s(\theta) = a_s^{lf}(\theta)\left(1 + \frac{\Delta_s}{K_s^{lf}}\right)$. This imply that $\omega_s(\theta)$ is the same for all $s \in \mathbf{S}$, so Θ_s and $\hat{\theta}_s$ are the same as in the laissez-faire allocation, but $K_s = K_s^{lf} + \Delta$ for all $s \in \mathbf{S}$.

The impact on welfare of the perturbation is given by

$$\sum_s \frac{\partial V}{\partial \Delta_s}\bigg|_{\Delta_s=0} = \beta \sum_s p_s \left[\int c_s(\theta)^{-\sigma} R_s(\theta) \frac{a_s(\theta) f_s(\theta)}{K_s} d\theta + \int c_s(\theta)^{-\sigma} \frac{\partial R_s(\theta)}{\partial K_s} a_s(\theta) f_s(\theta) d\theta \right] - c_0^{-\sigma} \sum_s q_s$$

where

$$\frac{\partial R_s(\theta)}{\partial K_s} = -\left[\frac{\hat{\theta}_s + \lambda_s(\theta - \hat{\theta}_s)^+}{\Theta_s} - 1 \right] \alpha(1-\alpha)\Theta_s^\alpha K_s^{\alpha-2}$$

The optimality condition for technology 1 can be written as

$$\sum_s \beta \int \frac{c_s(\theta)^{-\sigma}}{c_0^{-\sigma}} R_s(\theta)\omega_s(\theta) d\theta = \frac{1}{\varphi}$$

Since $\sum_s q_s = \frac{1}{\varphi}$ when a riskless technology is available, the first term in the previous expression is equal to zero:

$$\frac{\partial V}{\partial \Delta}\bigg|_{\Delta=0} = \beta \sum_s p_s Cov_s \left(u'(c_s), \frac{\partial R_s}{\partial K_s} a_s \right)$$

using the fact that $\int \frac{\partial R_s(\theta)}{\partial K_s} \omega_s(\theta) d\theta = 0$.

Using the optimality condition for the entrepreneur's portfolio choice, we obtain

$$\frac{\partial V}{\partial \Delta}\bigg|_{\Delta=0} = -c_0^{-\sigma} \sum_s p_s M_s (1-\alpha) Cov_s^\omega \left(R_s^{-1}, R_s \right) > 0$$

Using the approximation of $R_s(\theta)^{-1}$, we obtain

$$\frac{\partial V}{\partial \Delta}\bigg|_{\Delta=0} \approx \sum_s \kappa_s CV_s(R_s)^2$$

where $\kappa_s \equiv c_0^{-\sigma} p_s M_s (1-\alpha)$.

Excessive aggregate risk-taking: Suppose that a technology paying off φ_s at state s and zero at other states, for $s = l$ and $s = h$, exists and they are used in the laissez-faire equilibrium. Consider the following perturbation:

$$a_s(\theta) = \begin{cases} a_l^{lf}(\theta) \left(1 + \frac{\Delta}{K_l^{lf}} \varphi_l \right) & \text{if } s = l \\ a_h^{lf}(\theta) \left(1 - \frac{\Delta}{K_h^{lf}} \varphi_h \right) & \text{if } s = h \\ a_s^{lf}(\theta) & \text{otherwise} \end{cases}$$

This perturbation will increase K_l by $\Delta\varphi_l$ and reduce K_h by $\Delta\varphi_h$. This will keep initial savings fixed and only change the composition of the capital stock.

The impact on welfare of the perturbation is given by

$$\left.\frac{\partial V}{\partial \Delta}\right|_{\Delta=0} = \beta p_l \left[\int c_l(\theta)^{-\sigma} R_l(\theta)\frac{a_l(\theta)f_l(\theta)}{K_l}\varphi_l d\theta + \int c_l(\theta)^{-\sigma}\frac{\partial R_l(\theta)}{\partial K_l}\varphi_l a_l(\theta)f_l(\theta)d\theta\right] +$$
$$- \beta p_h \left[\int c_h(\theta)^{-\sigma} R_h(\theta)\frac{a_h(\theta)f_h(\theta)}{K_h}\varphi_h d\theta + \int c_h(\theta)^{-\sigma}\frac{\partial R_h(\theta)}{\partial K_h}\varphi_h a_h(\theta)f_h(\theta)d\theta\right]$$

From the pricing equation for the two technologies, we can cancel out the direct effect and it remains only the externality

$$\left.\frac{\partial V}{\partial \Delta}\right|_{\Delta=0} = \beta\left[p_l\varphi_l Cov_l\left(u'(c_l), \frac{\partial R_l(\theta)}{\partial K_l}a_l\right) - p_h\varphi_h Cov_h\left(u'(c_h), \frac{\partial R_h(\theta)}{\partial K_h}a_h\right)\right]$$

using $p_s\varphi_s\beta c_s(\theta)^{-\sigma} = \frac{c_0^{-\sigma}}{R_s(\theta)}$, we obtain

$$\left.\frac{\partial V}{\partial \Delta}\right|_{\Delta=0} = -c_0^{-\sigma}(1-\alpha)\left[Cov_l^{\omega}\left(R_l^{-1}, R_l\right) - Cov_h^{\omega}\left(R_h^{-1}, R_h\right)\right]$$

Finally, using the approximation for $R_s(\theta)^{-1}$, we obtain

$$\left.\frac{\partial V}{\partial \Delta}\right|_{\Delta=0} \approx c_0^{-\sigma}(1-\alpha)\left[CV_l\left(R_l\right)^2 - CV_h\left(R_h\right)^2\right]$$

Notice that the mean and variance of returns can be written as

$$\mathbb{E}_s^w[R_s] = 1 + \alpha\Theta_s^{\alpha}K_s^{\alpha-1} - \delta$$
$$Var_s^{\omega}(R_s) = Var_s^{\omega}((\theta-\hat{\theta}_s)^+)\left[\alpha\lambda_s\left(\Theta_s K_s\right)^{\alpha-1}\right]^2$$

The coefficient of variation can then be written as

$$CV_s(R_s) = \sqrt{Var_s^{\omega}((\theta-\hat{\theta}_s)^+)}\frac{\alpha\lambda_s\left(\Theta_s K_s\right)^{\alpha-1}}{1+\alpha\Theta_s^{\alpha}K_s^{\alpha-1}-\delta}$$
$$= \sqrt{Var_s^{\omega}((\theta-\hat{\theta}_s)^+)}\frac{\alpha\lambda_s}{(1-\delta)\left(\Theta_s K_s\right)^{1-\alpha}+\alpha\Theta_s}$$

Notice that CV_s is decreasing in Θ_s and K_s, so $CV_l > CV_h$. □

Proof of proposition x.

Consider the following approximation of R_s^{-1}

$$\frac{1}{x} = \frac{1}{\overline{x}} - \frac{1}{\overline{x}^2}$$

$$R_s^{-1} \approx \mathbb{E}_s^w[R_s]^{-1} - \frac{R_s - \mathbb{E}_s^w[R_s]}{\mathbb{E}_s^w[R_s]^2} + \frac{\left(R_s - \mathbb{E}_s^w[R_s]\right)^2}{\mathbb{E}_s^w[R_s]^3} - \frac{1}{2}\frac{\left(R_s - \mathbb{E}_s^w[R_s]\right)^3}{\mathbb{E}_s^w[R_s]^4}$$

The covariance can then be written as

$$Cov_s^{\omega}\left(R_s^{-1}, R_s\right) \approx -\left[Var_s^{\omega}\left(\tilde{R}_s\right) - Skew_s^{\omega}\left(\tilde{R}_s\right) + \frac{1}{2}Kur_s^{\omega}\left(\tilde{R}_s\right)\right]$$

where $\tilde{R}_s \equiv \frac{R_s}{\mathbb{E}_s^w[R_s]}$,$Skew_s^{\omega}(\tilde{R}_s) \equiv \mathbb{E}_s^w\left[\left(\tilde{R}_s - \mathbb{E}_s^w[\tilde{R}_s]\right)^3\right]$, and $Kur_s^{\omega}(\tilde{R}_s) \equiv \mathbb{E}_s^w\left[\left(\tilde{R}_s - \mathbb{E}_s^w[\tilde{R}_s]\right)^4\right]$.

Consider the perturbation that increases capital in state s by $\Delta_s \varphi_s$, where φ_s is the payoff of a technology that pays off only at state s: $a_s(\theta) = a_s^{lf}(\theta)\left(1 + \frac{\Delta_s \varphi_s}{K_s^{lf}}\right)$. The impact on welfare of a change in Δ_s is given by

$$\begin{aligned}\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} &= \beta p_s \varphi_s Cov_s\left(u'(c_s), \frac{\partial R_s(\theta)}{\partial K_s} a_s\right) \\ &= -c_0^{-\sigma}(1-\alpha) Cov_s^{\omega}\left(R_s^{-1}, R_s\right)\end{aligned}$$

Using the approximation derived above, we obtain

$$\left.\frac{\partial V}{\partial \Delta_s}\right|_{\Delta_s=0} \approx \kappa\left[Var_s^{\omega}\left(\tilde{R}_s\right) - Skew_s^{\omega}\left(\tilde{R}_s\right) + \frac{1}{2}Kur_s^{\omega}\left(\tilde{R}_s\right)\right]$$

where $\kappa = c_0^{-\sigma}(1-\alpha)$.

Hence, the externality is larger in periods where the variance of $\tilde{R}_s$ is larger, the skewness is smaller, and the kurtosis is larger. Notice this result generalizes the previous expression of the sufficient statistic, since

$$Var_s^{\omega}\left(\tilde{R}_s\right) = CV_s\left(R_s\right)^2$$

A.2 Necessary first order conditions of the planning problem and Proof of Proposition 4

We first take the first order conditions associated with Problem 34, associating multipliers $\mu^w, p_s\eta_s, -p_s\mu_s$ respective with constraints, 35, 36 and 37.

We obtain, for each $\omega_s(\theta)$:

$$\hat{\mu}_s = \begin{cases} \beta u'\left(c_s(\theta)\right) R_s(\theta) K_s, & if\, \theta < \hat{\theta}_s, \\ \beta u'\left(c_s(\theta)\right) R_s(\theta) K_s + \hat{\Psi}_s^{\Theta} \lambda_s \theta + \lambda_s \eta_s, & if\, \theta \geq \hat{\theta}_s, \end{cases} \tag{42}$$

where$\hat{\Psi}_s^{\Theta} := E\left[\beta u'\left(c_1\right) \frac{\omega}{f} \frac{\partial R}{\partial \Theta} | s\right] K_s + \mu^w u'\left(c_s^w\right) \frac{\partial c_s^w}{\partial \Theta}$.

Additionally, for each τ_s^k:

$$\mu^w u'\left(c_s^w\right) = \beta E\left[u'(c_1)\frac{\omega}{f}|s\right]; \tag{43}$$

From the first-order condition for $\hat{\theta}_s$, we obtain the value of the Lagrange multiplier

$$\lambda_s \eta_s = \frac{\hat{\Psi}_s^{\hat{\theta}}}{\omega_s\left(\hat{\theta}_s\right)} - \hat{\Psi}_s^{\Theta} \lambda_s \hat{\theta}_s, \tag{44}$$

where$\hat{\Psi}_s^{\hat{\theta}} := \beta E\left[u^{'}(c_1)\frac{\omega}{f}\frac{\partial R}{\partial \hat{\theta}_s}|s\right] K_s$ and $\frac{\partial R}{\partial \hat{\theta}_s} = (1 - \lambda_s \mathbf{1}_{\theta > \hat{\theta}_s})\alpha\left(\Theta_s K_s\right)^{\alpha-1}$.

Last, the first-order condition for x^i is given by

$$u^{'}\left(c_0\right) = \beta E\left[u^{'}(c_1)\frac{\omega}{f}R\varphi^i\right] + \beta E\left[u^{'}(c_1)\frac{\omega}{f}\frac{\partial R}{\partial K}K\ \varphi^i\right] + \mu^w E\left[u^{'}\left(c_s^w\right)\frac{\partial c_1^w}{\partial K}\varphi^i\right]. \tag{45}$$

Using condition 43, we can rewrite $\hat{\Psi}_s^{\Theta} = E\left[u^{'}\left(c_1\right)\frac{\omega}{f}\frac{\partial R}{\partial \Theta}|s\right] K_s + \beta E\left[u^{'}(c_1)\frac{\omega}{f}|s\right]\frac{\partial c_1^w}{\partial \Theta}$. With some simple algebra, it can be verified that $\frac{\partial c_s^w}{\partial \Theta} = -E\left[\frac{\partial R}{\partial \Theta}\frac{\omega}{f}|s\right] K_s = -E_{\omega_s}\left[\frac{\partial R}{\partial \Theta}|s\right] K_s$, illustrating that an increase in aggregate TFP generates a pecuniary transfer from entrepreneurs to workers, with a heterogeneous burden on entrepreneurs. It follows that

$$\hat{\Psi}_s^{\Theta} = Cov_{\omega_s}\left(\beta u^{'}\left(c_1\right), \frac{\partial R}{\partial \Theta}\right) K_s.$$

Also, rental market clearing guarantees that

$$E_{\omega_s}\left[\frac{\partial R}{\partial \hat{\theta}_s}\right] = E_{\omega_s}\left[1 - \lambda_s \mathbf{1}_{\theta > \hat{\theta}_s}\right]\alpha\left(\Theta_s K\right)^{\alpha-1} = 0,$$

which ensures that we can rewrite

$$\hat{\Psi}_s^{\hat{\theta}} = \beta E\left[u^{'}(c_1)\frac{\omega}{f}\frac{\partial R}{\partial \hat{\theta}_s}|s\right] K_s = Cov_{\omega_s}\left(\beta u^{'}(c_1), \frac{\partial R}{\partial \hat{\theta}_s}\right) K_s.$$

Analogously, we can verify that $\frac{\partial c_1^w}{\partial K} = -E\left[\frac{\partial R}{\partial K}K\right] + \tau_s^k$ and define $\hat{\Psi}_s^k = Cov_{\omega_s}\left(\beta u^{'}(c_1), \frac{\omega}{f}\frac{\partial R_s}{\partial K_s}\right) K_s$. This allows us to rewrite equation 45 as

$$u^{'}\left(c_0\right) = \beta E\left[u^{'}(c_1)\frac{\omega}{f}\left(R + \tau^k\right)\varphi^i\right] + E\left[\hat{\Psi}^k\ \varphi^i\right].$$

Last, the statement of the proposition normalizes $\hat{\mu}_s$, $\hat{\Psi}_s^{\Theta}$, $\hat{\Psi}_s^{\hat{\theta}}$ and $\hat{\Psi}_s^k$ by the marginal utility at $t = 0$.

B A simple canonical case

The laissez-faire equilibrium can be solved in a simple-closed form when $\sigma = 1$ and preferences take the logarithmic form. In this case, the most extensively used in the literature on financial friction and misallocation due to its ease of aggregation, the wealth and the substitution effect cancel out. The hedging demand vanishes and savings become independent of returns and, therefore, constant across states.

As a consequence, the equations related to optimal savings simplify to

$$\omega_s(\theta) = f_s(\theta), \text{or equivalently, } a_s(\theta) = K, \tag{46}$$

for each $s \in \boldsymbol{S}$, $\theta \in \mathbb{R}$, and

$$K = \frac{\beta}{1+\beta}. \tag{47}$$

For particular cases of the conditional distribution of productivity, closed-form solutions are obtained for the allocation. We resort to a a Generalized Pareto distribution, with cumulative distribution function and density, respectively, given by

$$F(\theta|s) = 1 - \left(\frac{\theta - m_s}{x_s}\right)^{-\nu_s} \qquad f(\theta|s) = \nu_s \frac{x_s^{\nu_s}}{(\theta - m_s)^{(\nu_s+1)}} \tag{48}$$

with support $[m_s + x_s, \infty)$. Here m_s is a location parameter, x_s is a scale parameter, and ν_s is a shape parameter that plays a key role in determining the skewness of the distribution.[16] The main advantage of the Generalized Pareto is that the distribution of TFP of firms that operate also belongs to that class, with the same location and shape parameters, but a scale parameter of $\hat{\theta}_s - m_s$, where $\hat{\theta}_s$ is still to be determined in equilibrium. Empirically, this distribution shows a good fit to the tail distribution of firm size (Gabaix, 2009, and 2016, for instance).

Equilibrium in the rental market, in its turn, requires

$$\lambda_s[1 - f_s(\theta)] = 1 \implies \hat{\theta}_s = m_s + x_s \lambda_s^{\frac{1}{\nu s}}, \tag{49}$$

pinning down the scale parameter of the distribution of productivity for active firms.

Additionally, aggregate TFP is given by

$$\Theta_s = \mathbb{E}[\theta_s|\theta \geq \hat{\theta}_s] = m_s + \frac{x_s \lambda_s^{\frac{1}{\nu_s}}}{1 - \nu_s^{-1}} \tag{50}$$

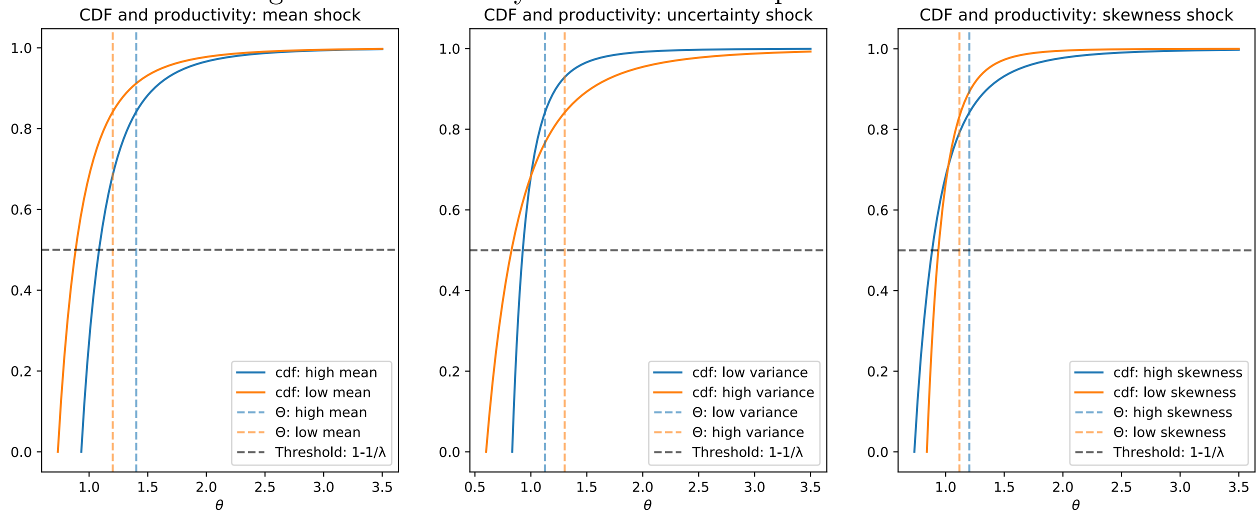
$$= \mathbb{E}[\theta|s] + \frac{x_s(\lambda_s^{\frac{1}{\nu_s}} - 1)}{1 - \nu_s^{-1}}, \tag{51}$$

the last line calls attention to the role of capital reallocation which is induced by leverage and rental of external capital, that is, $\lambda_s > 0$. The last term in that equation is an interaction term: high leverage (high λ_s) and heavier tails (low ν_s) are complements in their effect on aggregate TFP. In other words, the presence of a higher mass of extremely productive firms increases TFP and output the higher are leverage and reallocation possibilities. The scale term x_s which increases variance without affecting the skewness of the distribution also complements leverage and reallocation possibilities.

Figure 4 plots the consequences of three moment shocks that have been documented to play key roles along the business cycle. In particular, we consider three movements that are typically associated

[16] The mean is given by $\mathbb{E}[\theta|s] = m_s + \frac{x_s}{1-\nu_s^{-1}}$, for $\nu_s > 1$, the variance is given by $\mathbb{V}[\theta|s] = \frac{x_s^2 v_s^{-2}}{(1-v_s^{-1})^2(1-2\nu_s^{-1})}$, for $\nu_s > 2$, while the skewness is $\mathbb{S}[\theta|] = 2\frac{1+\nu_s}{\nu_s-3}\sqrt{1-2\nu_s^{-1}}$, for $\nu_s > 3$. When these restrictions on ν_s are not satisfied, these moments are not finite.

Figure 4: Productivity distribution and equilibrium outcomes



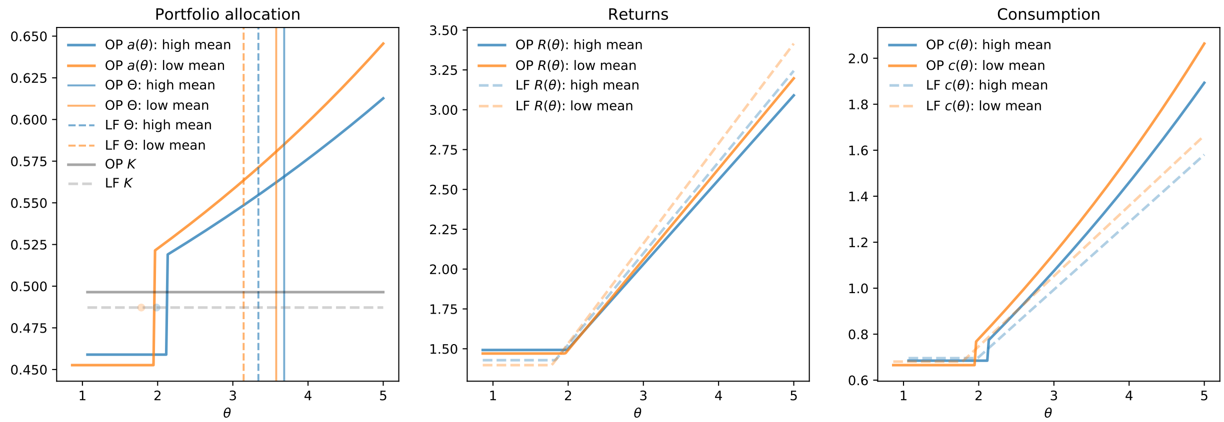
with economic downturns: a negative shock to the first moment, associated with worse prospects for all firms, a positive shock to the second moment, associated to increased productivity dispersion and uncertainty, and a negative shock the skewness, associated with less frequent extremely positive results.

The left-most panel displays an additive productivity decrease, a negative first-moment shock affecting all potential producers uniformly. The leverage parameter is set to $\lambda_s = 2$ and market clearing in the rental market ensures that the most productive half of the distribution of firms is active and operate the totality of the capital stock. In the panel, rental market equilibrium is represented by the intersection of the horizontal dashed line at the quantile 0.5 and the cumulative distribution function, which determines the marginal firm $\hat{\theta}_s$. As expected, this first-moment contraction decreases the marginal firm and TFP by the same magnitude of the change in each firm's TFP .

The panel in the middle of Figure 4 displays a mean-preserving spread of the distribution of potential firm productivity, which can be interpreted as an uncertainty shock. TFP always increases as long as $\lambda_s > 1$ allowing some capital reallocation, as indicated by equation 50, and so will wages. We can see that these dispersion shocks, when combined with capital reallocation, have consequences that help mitigate the loss of TFP that a concurrent first moment shock typically brings. The consequences over the marginal firm are theoretically ambiguous, nonetheless. If leverage is sufficiently low, as in the numerical example illustrated, the marginal firm actually drops, since a mean preserving spread reduces TFP of the median firm. Higher leverage economies (for instance, $\lambda_s = 5$) have their marginal firms at higher percentiles of the productivity distribution (in this example, at the eightieth percentile), and these are percentiles that increase with a mean preserving spread. In other words, if the marginal firm is relatively in the left-tail of the productivity distribution, a mean preserving spread lowers it even further.

Last, the right-most panel brings in a change in skewness, conditional on an unchanged mean. Recent work has emphasized the extent of the skewness of the firm size distribution vary over the cycle (see Salgado et al., 2016). Skewness has been shown to be procyclical, so it tends to fall in bad times. An increase in the parameter ν_s generates a negative skewness shock, that moves mass from right to

Figure 5: Portfolio, returns, and consumption: mean shock



Note: OP stands for *optimal policy* and LF stands for *laissez-faire*. The two colored dots denote marginal firms in laissez-faire: the dot in blue, for the high mean case, while the one in orange, for the low mean case.

left in the distribution of firm productivity. TFP and wage unambiguously fall, and the more so, the higher is leverage. The effects on the marginal firm again are theoretically ambiguous: the marginal firm increases for low leverage economies and decreases for high leverage ones.

B.1 Optimal Policies and the business cycle

Figure 2 also serves as an illustration of how policies should react to the business cycle by illustrating aggregate states that have different realizations of a first-moment shock. In orange, we have an aggregate downturn that can be compared to a (higher mean) benchmark state, in blue.

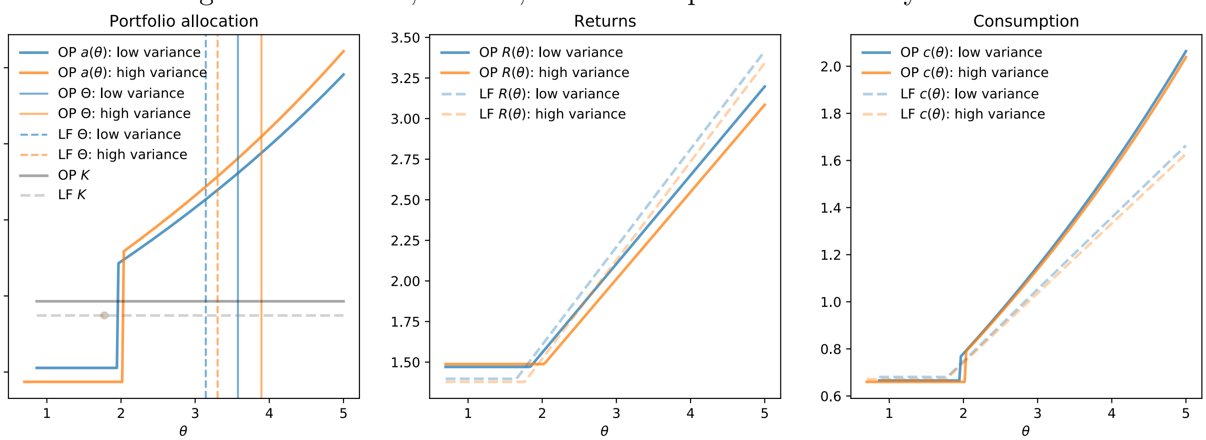
The main lesson is that policies are stronger in downturns. The reallocation of net worth across firms is more pronounced both in the extensive margin, as the larger jump in asset holdings at the marginal firm indicates, and in the intensive margin, with a higher slope of optimal asset holdings with respect to realized productivity. The outcomes of these intensified policies are a stronger change in marginal firm and larger aggregate TFP gain from the policy intervention. The main reason is that a lower aggregate TFP implies higher return volatility, which invites a larger policy response.

This intuition follows through when we evaluate uncertainty and skewness shocks. In Figure 6, we study the consequences of a mean-preserving spread that makes productivity more dispersed. Although the aggregate TFP increases with an uncertainty shock, the productivity of firms is itself more dispersed, giving rise to large dispersion of returns on wealth. Again, this invites a larger policy response in both the extensive and intensive margins, as indicated by the panel in the left-hand. This stronger policy response leads to higher displacement of firms, larger capital reallocation, and a larger relative gain of TFP. Last, Figure 7 studies skewness shock and shows that the conclusions are very similar.[17]

Therefore, all three key forms moment shocks understood to be associated with downturns generate similar policy conclusions. The intensity of the use of the policy instruments that target reallocation of resources is pro-cyclical.

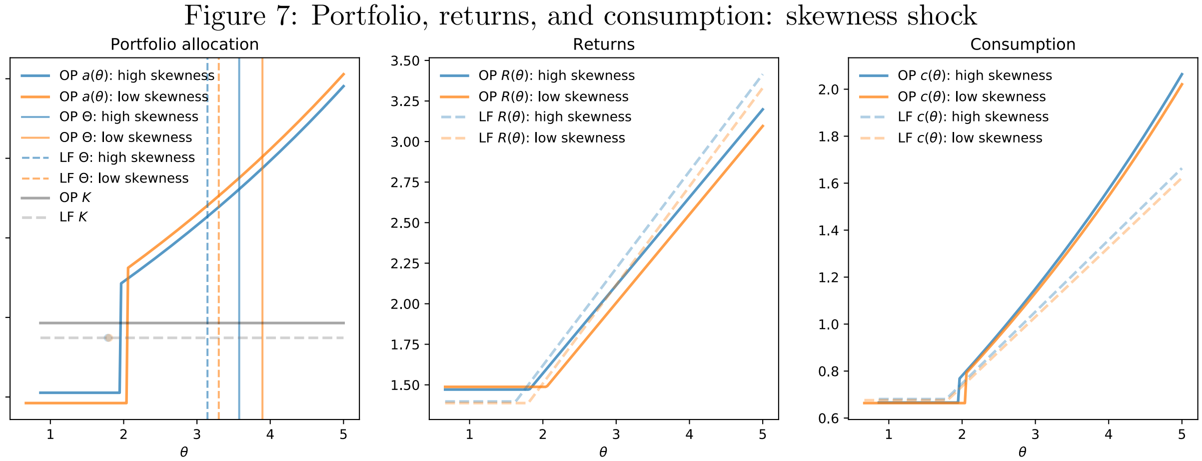
[17] An intriguing feature for future study is the fact that a skewness shock seems to generate a policy response concentrated on the extensive margin.

Figure 6: Portfolio, returns, and consumption: uncertainty shock



Note: OP stands for *optimal policy* and LF stands for *laissez-faire*.

Figure 7: Portfolio, returns, and consumption: skewness shock



Note: OP stands for *optimal policy* and LF stands for *laissez-faire*.

Figure 8: Optimal portfolio taxes

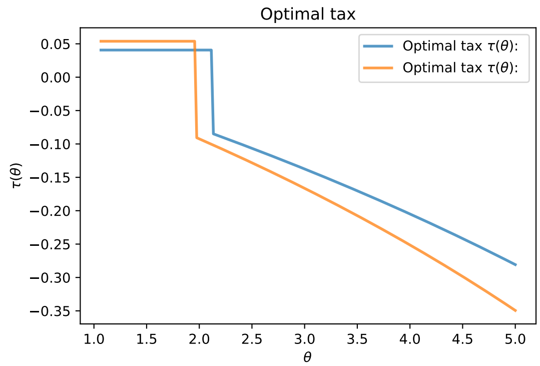


Figure 8, in its turn, displays portfolio taxes that implement the optimal allocation. The implications are consistent across the three moment shocks: there is constant tax on the savings of inactive entrepreneurs, there is a subsidy that targets active firms and is increasing in the firm's productivity, and last, the magnitude of these instruments is counter-cyclical. Additionally, as Proposition 4, optimal taxes are such that on average they stimulate aggregate savings and capital accumulation.